Preface

Ergodic theory, as a mathematical discipline, refers to the analysis of asymptotic or long-range behaviour of a dynamical system, that is, a map or flow on a state space, using measure-theoretic or probabilistic methods. A close cousin to smooth dynamics (the study of differentiable actions on a smooth manifold) and to topological dynamics (comprising a continuous action on a topological space), there is a well-established and rich synergy between the three fields. Indeed, many important applications bring tools from all the three fields to bear in the study of a particular dynamical system.

As a quickly maturing mathematical field, both theoretical developments and applications in the physical sciences, engineering, and computer science are flourishing within the arena of modern research in ergodic theory. Driven by these new theoretical tools and a growing breadth of natural applications, computational aspects are now a central challenge to researchers in the field.

An open dynamical system is a natural extension of the traditional (closed) dynamical system. In an open system, the state space is no longer deemed to be invariant under the dynamical action, but some orbits are allowed to 'escape' depending on location and time. An everyday example is the dynamics of a ball on a billiard table; when the ball falls in a hole in the table, the orbit is terminated. As introduced by Yorke and Pianigiani in the late 1970s, the abstract concept of an open system leads immediately to the notion of a conditionally invariant measure and escape rate along with a host of detailed questions about how mass escapes or fails to escape from the system under time evolution.

Perhaps ironically, concepts from open systems have recently been used to analyse traditional, closed systems. For example, in many closed systems, relaxation to equilibrium is by no means uniform throughout the state space. There may be regions that remain 'almost invariant' for long periods of time, mixing with the rest of the space at quantifiably slower rates than the other parts of the system. These 'almost invariant sets' become key features determining the asymptotics of the system. One particularly fruitful idea is to study almost invariant sets as open subsystems of the larger closed dynamical system, wherein the escape rate determines the rate of mixing and relaxation to equilibrium.

In some realistic applications, time-varying parameters governing the flow or transformation on the state space necessitate modelling by a non-autonomous system. While the ergodic theory of non-autonomous systems parallels that of autonomous dynamics in many ways, there are important differences. Stable and unstable foliations, a foundation of geometric analysis for an autonomous map or a flow, become equivariant, time-dependent structures. Other dynamical objects such as Lyapunov exponents and Oseledets subspaces can be used in alternative ways to describe non-autonomous or random dynamics. Invariant or almost invariant objects arising in autonomous dynamics have non-autonomous analogues called coherent structures. These are features that move around in the state space under time evolution but that may still represent barriers to mixing and relaxation to 'equilibrium', a concept that also has to be reinterpreted compared to the autonomous setting.

From April 9 through April 15, 2012 a group of more than 40 researchers in ergodic theory gathered at the Banff International Research Station in Banff, Alberta, Canada to exchange cutting-edge developments in the field.[1] Thirty-five research talks were given during the course of the workshop, covering theoretical, applied, and computational aspects of both open and closed, autonomous and non-autonomous dynamics. After the workshop, a number of participants volunteered to expand on their presentations and contribute chapters to this volume. Each contribution was rigorously peer-reviewed before inclusion in this volume. We briefly outline the resulting contributions:

- Balasuriya considers time-dependent flows where the time dependence enters as a perturbation of an autonomous flow. He describes how to analytically estimate the perturbed stable and unstable manifolds, which may be regarded as Lagrangian coherent structures. He then uses Melnikov theory to quantify flux across these perturbed manifolds.
- Bandtlow and Jenkinson consider the spectrum of transfer operators of real-analytic expanding maps acting on holomorphic functions of the interval and other finite-dimensional spaces. They particularly consider the open setting where mass is leaving the phase space and prove bounds for each spectral point of the corresponding open operators.
- Bandtlow, Jenkinson, and Pollicott specialise the previous chapter to the setting of piecewise real-analytic expanding Markov maps of the interval, with escape through a Markov hole. They show that the leading spectral point of the transfer operator, which quantifies the escape rate, can be approximated using derivative information from all periodic points of increasing period. The invariant measure on the survivor set is also estimated.
- Basnayake and Bollt describe a method of extracting a flow field from a movie of observations. Under the assumption of smooth time dependence of the flow field,

[1] Materials from this workshop, including abstracts and some videos of the presentations, are available at the BIRS website, www.birs.math.ca; search on workshop code 12w5050.

they introduce a multi-step method to enforce smooth behaviour. As an example, they extract a flow field from a movie of sea surface temperature and calculate Lagrangian coherent structures in the form of finite-time Lyapunov exponent fields.

- Bose and Murray extend their earlier work on estimating absolutely continuous invariant measures (ACIMs) to the open dynamics setting. In general, an open system may support a continuum of escape rates and an infinity of absolutely continuous conditional invariant measures (ACCIMs). Their approach, based on maximum entropy and convex optimisation, allows one to prescribe the desired escape rate and find the corresponding ACCIM.
- Bruin studies a map on a Euclidean $(d-1)$-dimensional triangle that arises from a simple d-dimensional subtractive algorithm. For $d = 2$ the map becomes the well-known Farey map on $[0, 1]$; for $d = 3$, Bruin shows that the map is dissipative but at the same time ergodic with respect to two-dimensional Lebesgue measure and even exact. This paper contributes to a long and historically important development of number theoretic applications of ergodic theory dating back to Renyi in the 1950s with his foundational work on the continued fraction expansion. At the same time, the tools used are up to date, bringing modern notions such as distortion, Schwarzian derivative, and random walks to bear on the problem.
- Bunimovich and Webb consider piecewise differentiable expanding Markov maps of the interval with a Markov hole. Their focus is on estimating the survival and escape probabilities after a finite number of iterations of the open system. They provide explicit upper and lower bounds for these probabilities in terms of eigenvalues of the transition matrices of induced Markov chains.
- Demers studies billiard dynamical systems with a variety of holes. Using transfer operator and Young tower techniques, he proves the existence of a natural escape rate and corresponding absolutely continuous conditional invariant measure (ACCIM). He then considers the question of stability as the size of the hole goes to zero and shows the limiting ACCIM is the SRB (or physical) measure for the closed system. Finally he shows that the escape rate also arises via a variational principle.
- Froyland and Padberg-Gehle give an overview of transfer operator methods for finite-time almost-invariant and coherent sets. Their chapter unifies the autonomous and time-dependent methodologies and then focuses on three aspects, namely the flow direction, the flow duration, and the level of diffusion present. They show that the coherent structures produced by the transport-based transfer operator approach are very natural from a geometric dynamical point of view.
- Haydn, Winterberg, and Zweimüller consider a general ergodic process and the return time and hitting time distributions corresponding to a sequence of sets of decreasing size. They first show that as the size of the sets approaches zero, limiting return and hitting time distributions exist. Further, they show that if one induces the original ergodic process via return times to a fixed set of positive

measure, the limiting distributions of the return and hitting times of the original and induced systems coincide.

The contributions to this book represent a broad cross section of the topics represented at the April 2012 workshop and, in turn, make a fine collection of sample papers for researchers who may be looking to broaden their outlook in modern aspects of the field. The editors wish to thank all the workshop participants for their contributions, but especially those participants who took the time to write up their work as a submission to this book and the referees who helped to hone the author's contributions into the high-quality research papers you will find in the following pages.

Finally, none of this would have been possible without the remarkable support offered by the Banff International Research Station and its staff. BIRS is indeed one of only a handful of first-class mathematical research venues in the world; if you have a chance to go there, do not hesitate!

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