Chapter 2
Incompressible Navier–Stokes Equations

Abstract We aim to derive the incompressible Navier–Stokes equations from classical mechanics. We define Lagrange and Euler coordinates and the mass density within the framework of measure theory. This yields a mathematical statement that expresses the mass conservation principle, which allows to derive the mass conservation equation. We introduce the incompressible flows and focus on their kinematic, starting with the deformation tensor and the vorticity and then the local deformations of a ball of fluid in an incompressible flow by standard ODEs. We introduce the fluid motion equation for Newtonian fluids through appropriate measures, based on the fundamental law of classical mechanics and the expression of the stress tensor in terms of the deformation tensor. The mass conservation equation coupled to the fluid motion equation yields the incompressible Navier–Stokes equations. This chapter ends with a comprehensive list of boundary conditions associated with the Navier–Stokes equations.

2.1 Introduction

The aim of this chapter is to lay the foundations for basic fluid mechanics, in order to prepare the ground for the mathematical modeling of turbulent flows performed in Chaps. 3, 4, and 5.

We consider a fluid, liquid, or gas, moving in a domain Ω included in $\mathbb{R}^3$. We aim to find a mathematical description of this motion, which is a difficult task since this is a nonlinear physical phenomenon involving many unknowns. The main unknowns are the mass density, the pressure, the velocity, and the temperature, but the list may be longer depending on the particular case being studied.

In this chapter, we derive from physical and mathematical considerations the incompressible Navier–Stokes equations for Newtonian fluids:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} - \nabla \cdot (2\nu D\mathbf{v}) + \nabla p = \mathbf{f}, \tag{2.1}$$

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$$\nabla \cdot \mathbf{v} = 0, \tag{2.2}$$

where $\mathbf{v}$ is the velocity of the flow, $D\mathbf{v} = (1/2)(\nabla\mathbf{v} + \nabla\mathbf{v}^t)$ its deformation tensor, and p its pressure. The momentum equation (2.1) is inherited from Newton's law, while equation (2.2) is the mass conservation equation for incompressible flows.

We define the mass density of the fluid ρ and the velocity $\mathbf{v}$ in Sect. 2.2. To do so, we outline the Lagrangian and Eulerian descriptions of the motion, although we shall only deal with the Euler description after Sect. 2.2. We shall also prove some useful abstract results.

In Sect. 2.3, we obtain the general mass conservation equation satisfied by $\mathbf{v}$ and ρ:

$$\partial_t \rho + \nabla \cdot (\mathbf{v}\rho) = 0. \tag{2.3}$$

Two procedures are developed to derive this equation. One is heuristic and shows the physical features of the mass balance. The other is mathematically rigorous, making use of the abstract results of Sect. 2.2.

We describe in Sect. 2.4 various approximations in the mass conservation equation, which leads to the notion of incompressibility and how equation (2.2) is deduced from equation (2.3) within this framework.

Section 2.5 is concerned with the kinematics of incompressible flows. We study the transformations of an infinitesimal fluid body δV in the flow, over an infinitesimal time period $\delta P = [t, t + \delta T]$. This yields the introduction on the one hand of the deformation tensor $D\mathbf{v}$, which governs the stability of δV over δP, and on the other hand of the vorticity $\boldsymbol{\omega}$, being the angular velocity of δV.

In Sect. 2.6, we perform the analysis of the internal forces acting on the fluid during its motion. The dynamic pressure p is introduced at this stage. This analysis yields the derivation of the momentum equation (2.1) from Newton's law, and finally the incompressible Navier–Stokes equations, presented in their various forms at the end of Sect. 2.6.

A comprehensive list of boundary conditions is presented in Sect. 2.7, describing some examples that are often studied, depending on different flow geometries as well as different approximations.

2.2 General Framework

2.2.1 Aim of the Section

A fluid is a continuous medium that can be continually subdivided into infinitesimal particles of fluid material having a mass. Each particle of fluid sits on an abstract point $\mathbf{x}$ in Ω at time $t \geq 0$. The measure $dm(t, \mathbf{x})$ is the mass of the particle that sits on $\mathbf{x}$ at time t, which will be defined by the end of Sect. 2.2.3.

The measure dm is absolutely continuous with respect to the Lebesgue's measure $d\mathbf{x}$, $dm = \rho d\mathbf{x}$, where $\rho = \rho(t, \mathbf{x})$ is the mass density at a time t and a point $\mathbf{x}$, that defines the mass of fluid per unit volume, expressed in kilograms per meter cubed. The total mass of fluid contained in a fixed subdomain ω of Ω at time t, $m(t, \omega)$, is given by

$$m(t, \omega) = \int_{\omega} \rho(t, \mathbf{x}) d\mathbf{x}, \tag{2.4}$$

provided that $\rho(t, \cdot)$ is locally integrable on Ω, which we shall assume to be the case.

To begin with, we define the Lagrange and Euler coordinate systems. The Lagrange description helps initially, but after this section we shall only use the Euler description. We refer to [7, 8] and [11] for further details about the Lagrange description.

We define the Lagrange and Euler velocities $\mathbf{V}$ and $\mathbf{v}$ in Sect. 2.2.2. The technical lemma 2.1 in Sect. 2.2.3.1 below points out how $\nabla \cdot \mathbf{v}$ is involved in volume variations during the motion. The local volume dv, being a form on the tangent space at a given point $(t, \mathbf{x})$, and the associate mass $dm = \rho dv$ are defined in Sect. 2.2.3.

2.2.2 Euler and Lagrange Coordinates and Velocities

Let us consider a particle of fluid sitting on a point $\mathbf{X} = (X_1, X_2, X_3) \in \Omega$ at time $t = 0$. Assume that this particle moves to a point $\mathbf{x} = (x_1, x_2, x_3) \in \Omega$ at a given time t and that there exists a map

$$F : \mathbb{R}_+ \times \Omega \to \Omega$$

such that

$$\mathbf{x} = F(t, \mathbf{X}) = (F_1(t, \mathbf{X}), F_2(t, \mathbf{X}), F_3(t, \mathbf{X})). \tag{2.5}$$

Assumption 2.1. *We assume that for each fixed $t \geq 0$, the restricted map $F(t, \cdot)$ is a C^1 diffeomorphism on Ω.*

Thus the relation $\mathbf{x} = F(t, \mathbf{X})$ can be inverted to give

$$\mathbf{X} = G(t, \mathbf{x}).$$

We shall say that $\mathbf{X}$ is the Lagrangian coordinate of the particle whereas $\mathbf{x}$ is its Eulerian coordinate.

Definition 2.1. The *Lagrangian velocity* at point $(t, \mathbf{X}) \in \mathbb{R}_+^\star$ of the fluid is the field $\mathbf{V} = \mathbf{V}(t, \mathbf{X})$:

$$\mathbf{V}(t, \mathbf{X}) = \partial_t F(t, \mathbf{X}), \quad \mathbf{V} = (V_1, V_2, V_3), \tag{2.6}$$

where $\partial_t = \partial/\partial t$ is the time derivative. The *Eulerian velocity* at a point $(t, \mathbf{x}) \in \mathbb{R}_+^\star$ is the field $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$:

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{V}(t, G(t, \mathbf{x})), \quad \mathbf{v} = (v_1, v_2, v_3). \tag{2.7}$$

For a better understanding of the Eulerian velocity, let us consider a fixed point $\mathbf{x}$ in Ω and a physical fluid particle going past $\mathbf{x}$ at time t with an Eulerian velocity $\mathbf{v}$. Therefore this particle moves approximately to the point $\mathbf{x} + \mathbf{v}\delta t$ at the time $t + \delta t$ for some small $\delta t > 0$. We note that at a time $t' \neq t$, there is a chance that another physical particle goes past the same point $\mathbf{x}$.

2.2.3 *Volume and Mass*

2.2.3.1 Fundamental Kinematic Relation

We show the fundamental relation (2.12) linking $\nabla_{\mathbf{x}} \cdot \mathbf{v}$ and $\det \nabla_{\mathbf{X}} F$, where

$$\nabla_{\mathbf{X}} F = \left(\frac{\partial F_i}{\partial X_j} \right)_{1 \leq i, j \leq 3}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i}. \tag{2.8}$$

Let δ_{ij} denote the Kronecker tensor:

$$\delta_{ij} = 1 \text{ if } i = j, \quad \delta_{ij} = 0 \text{ if } i \neq j. \tag{2.9}$$

Let ε_{ijk} denote the Levi-Civita tensor that is fully characterized by

$$\varepsilon_{123} = 1; \varepsilon_{ijk} \textit{ is antisymmetric against the indices.}$$

On one hand, let $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{F} = (F_1, F_2, F_3)$ be two given vector fields. Then the i[th] component of their cross product $\mathbf{E} \times \mathbf{F}$ is given by

$$(\mathbf{E} \times \mathbf{F})_i = \varepsilon_{ijk} E_j F_k, \tag{2.10}$$

where the Einstein summation convention is used. On the other hand, the determinant of any 3×3 matrix $A = (a_{ij})_{1 \leq i, j \leq 3}$ is equal to

$$\det A = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} a_{ip} a_{jq} a_{ir}, \tag{2.11}$$

Lemma 2.1. *Assume that F defined by (2.5) is of class C^2 on $\mathbb{R}_+ \times \Omega = Q$. Then*

$$\partial_t (\det \nabla_{\mathbf{X}} F) = (\nabla_{\mathbf{x}} \cdot \mathbf{v}) \det \nabla_{\mathbf{X}} F, \tag{2.12}$$

Proof. From (2.11), we have

$$\det \nabla_{\mathbf{X}} F = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\frac{\partial F_i}{\partial X_p}\frac{\partial F_j}{\partial X_q}\frac{\partial F_k}{\partial X_r}.$$

Therefore

$$\partial_t \det \nabla_{\mathbf{X}} F = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\left[\frac{\partial^2 F_i}{\partial t \partial X_p}\frac{\partial F_j}{\partial X_q}\frac{\partial F_k}{\partial X_r} + \frac{\partial F_i}{\partial X_p}\frac{\partial^2 F_j}{\partial t \partial X_q}\frac{\partial F_k}{\partial X_r} + \frac{\partial F_i}{\partial X_p}\frac{\partial F_j}{\partial X_q}\frac{\partial^2 F_k}{\partial t \partial X_r}\right]. \tag{2.13}$$

Since F is of class C^2, the Schwarz theorem applies [5] and we have

$$\frac{\partial^2 F}{\partial t \partial X_\ell} = \frac{\partial^2 F}{\partial X_\ell \partial t},$$

regardless of ℓ. Moreover, all indices play the same role in the formula (2.13). Therefore, we may exchange their positions, reorder the terms, and use the antisymmetry of ε_{ijk}, which yields

$$\partial_t \det \nabla_{\mathbf{X}} F = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{pqr}\frac{\partial^2 F_i}{\partial t \partial X_p}\frac{\partial F_j}{\partial X_q}\frac{\partial F_k}{\partial X_r}. \tag{2.14}$$

From the definition (2.6), we have

$$\frac{\partial^2 F}{\partial X_p \partial t} = \frac{\partial V_i}{\partial X_p}$$

at every point $(t, \mathbf{X})$. Moreover, formula (2.7) can also be written as

$$\mathbf{v}(t, F(t, \mathbf{X})) = V(t, \mathbf{X}), \tag{2.15}$$

leading to the following identity:

$$\frac{\partial V_i}{\partial X_p} = \frac{\partial v_i}{\partial x_\alpha}\frac{\partial F_\alpha}{\partial X_p}, \tag{2.16}$$

regardless of i and p. We insert the formula (2.16) in the equality (2.14), leading to

$$\partial_t \det \nabla_{\mathbf{X}} F = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{pqr}\frac{\partial v_i}{\partial x_\alpha}\frac{\partial F_\alpha}{\partial X_p}\frac{\partial F_j}{\partial X_q}\frac{\partial F_k}{\partial X_r}. \tag{2.17}$$

In writing carefully the six nonvanishing terms of ε_{ijk}, which take values in $\{-1, 1\}$, we find that for any fixed indices α, j, and k,

$$\varepsilon_{\alpha jk} \det \nabla_{\mathbf{X}} F = \varepsilon_{pqr} \frac{\partial F_\alpha}{\partial X_p} \frac{\partial F_j}{\partial X_q} \frac{\partial F_k}{\partial X_r}.$$

Then, the equality (2.17) becomes

$$\partial_t \det \nabla_{\mathbf{X}} F = \frac{1}{2} \varepsilon_{\alpha jk} \varepsilon_{ijk} \frac{\partial v_i}{\partial x_\alpha} \det \nabla_{\mathbf{X}} F. \tag{2.18}$$

Using the relation

$$\varepsilon_{\alpha jk} \varepsilon_{ijk} = 2\delta_{\alpha i}, \tag{2.19}$$

we have

$$\frac{1}{2} \varepsilon_{\alpha jk} \varepsilon_{ijk} \frac{\partial v_i}{\partial x_\alpha} = \delta_{\alpha,i} \frac{\partial v_i}{\partial x_\alpha} = \frac{\partial v_i}{\partial x_i} = \nabla_{\mathbf{x}} \cdot \mathbf{v}. \tag{2.20}$$

We combine (2.18) and (2.20), which yields

$$\partial_t \det \nabla_{\mathbf{X}} F = (\nabla_{\mathbf{x}} \cdot \mathbf{v}) \det \nabla_{\mathbf{X}} F,$$

concluding the proof of Lemma 2.1. □

To better understand this result, recall that $\det \nabla_{\mathbf{X}} F$ is involved in the change of variables in integral calculus. Indeed, let $g : \Omega \to \mathbb{R}$ be an integrable function, $\omega_0 \subset\subset \Omega$ be a measurable set, $\omega_t = F(t, \omega_0)$. Then

$$\int_{\omega_t} g(\mathbf{x}) d\mathbf{x} = \int_{\omega_0} g(F(t, \mathbf{X})) |\det \nabla_{\mathbf{X}} F(t, \mathbf{X})| d\mathbf{X}.$$

Roughly speaking, $\det \nabla_{\mathbf{X}} F(t, \cdot)$ measures how the diffeomorphism transforms the Lebesgue measure at time t. Formula (2.12) links its time evolution to $\nabla_{\mathbf{x}} \cdot \mathbf{v}$, which will later appear to be the indicator of the fluid's capacity to perform volume variations, such as compressions or decompressions.

2.2.3.2 Volume Form and Mass Measure

Let $\mathbf{X} \in \Omega$ be a given point and $T_{\mathbf{X}}$ the tangent space at $\mathbf{X}$, which is isomorphic to $\mathbb{R}^3$ in this case [1]. The volume form dv_0 at $\mathbf{X}$ is defined by

$$\forall\, (\zeta_1, \zeta_2, \zeta_3) \in T_{\mathbf{X}}^3, \quad dv_0(\mathbf{X})(\zeta_1, \zeta_2, \zeta_3) = \det(\zeta_1, \zeta_2, \zeta_3). \tag{2.21}$$

Let $t \in \mathbb{R}_+$ be fixed, $\mathbf{x} = F(t, \mathbf{X})$, $T_{t,\mathbf{x}}$ be the tangent space at $(t, \mathbf{x})$, also isomorphic to $\mathbb{R}^3$. Since $F(t, \cdot)$ is a diffeomorphism on Ω,

$$\nabla_{\mathbf{X}} F(t, \mathbf{X}) : T_{\mathbf{X}} \to T_{t,\mathbf{x}}$$

is an isomorphism. Therefore, for each $(\eta_1, \eta_2, \eta_3) \in T^3_{t,\mathbf{x}}$, there exists $(\zeta_1, \zeta_2, \zeta_3) \in T^3_{\mathbf{X}}$ such that

$$(\eta_1, \eta_2, \eta_3) = (\nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_1, \nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_2, \nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_3). \tag{2.22}$$

This allows us to define a local volume form on $T_{t,\mathbf{x}}$ denoted by $dv(t, \mathbf{x})$ and defined by

$$\begin{aligned} \forall\, (\eta_1, \eta_2, \eta_3) \in T^3_{t,\mathbf{x}},\ dv(t, \mathbf{x})(\eta_1, \eta_2, \eta_3) = det(\eta_1, \eta_2, \eta_3) = \\ \det(\nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_1, \nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_2, \nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_3). \end{aligned} \tag{2.23}$$

The following relation holds true:

$$dv(t, \mathbf{x}) = \det \nabla_{\mathbf{X}} F(t, \mathbf{X}) dv_0(\mathbf{X}), \tag{2.24}$$

following the classical determinant theory [4].

Since parallelepipeds generate a Borel algebra on $T_{t,\mathbf{x}}$, this allows the mass of a particle of fluid that sits on $\mathbf{x}$ at time t, $dm(t, \mathbf{x})$, to be defined as a measure on $T_{\mathbf{X}}$ by the formula

$$dm(t, \mathbf{x}) = \rho(t, \mathbf{x}) dv(t, \mathbf{x}). \tag{2.25}$$

This is in accordance with the definition (2.4) since when t is fixed, $dv(t, \mathbf{x}) = d\mathbf{x}$.

This may seem to be unnecessarily complicated at a first glance. However, we shall see in Sect. 2.3.4 how it simplifies significantly the mathematical derivation of the mass conservation equation (2.26) below.

2.3 Mass Conservation Equation

2.3.1 Aim of the Section

This section is devoted to the derivation of the mass conservation equation

$$\partial_t \rho + \nabla \cdot (\mathbf{v}\rho) = 0, \tag{2.26}$$

satisfied at every $(t, \mathbf{x}) \in \mathbb{R}_+ \times \Omega$, where we have set

$$\forall\, \mathbf{E} = (E_1, E_2, E_3) = \mathbf{E}(t, \mathbf{x}) \in C^1(\mathbb{R}_+ \times \Omega)^3, \quad \nabla \cdot \mathbf{E} = \frac{\partial E_i}{\partial x_i}, \quad \partial_t \rho = \frac{\partial \rho}{\partial t}.$$

From now on, each tensor field depends on $\mathbf{x}$ in space and no longer on $\mathbf{X}$, and we shall no longer explicitly specify $\mathbf{x}$ or $\mathbf{X}$ subscripts when writing derivatives with respect to the space variable.

Equation (2.26) is called a "conservation law," the mass being preserved during the motion. Although it is easy to claim "the mass is preserved," this is difficult to define rigorously.

We derive equation (2.26) using two different procedures. One is heuristic, based on rough approximations that are not rigorously true. Nevertheless, this procedure presents the great advantage that we can simply derive the equation, which enables a better understanding of the physics. This is the aim of Sect. 2.3.2.

The other procedure is based on the relation (2.12) and the definition (2.25) of dm. This allows a rigorous definition of the mass conservation by imposing that the total derivative of dm is equal to zero at each point $(t, \mathbf{x}) \in Q$. Therefore, we must first define the total derivative, which we do in Sect. 2.3.3, that specifies how to derive in time along the flow trajectories. The rest of the program is implemented in Sect. 2.3.4.

Throughout this chapter, we assume

Assumption 2.2. *The fields ρ and $\mathbf{v}$ are of class C^1 on $\mathbb{R}_+ \times \Omega = Q$,*

without stating it systematically.

2.3.2 Heuristical Considerations

Let $\omega \subset\subset \Omega$ be a fixed open set strictly included in Ω, $\Gamma = \partial\omega$ its boundary, $\mathbf{n} = \mathbf{n}(\mathbf{x})$ the outward-pointing unit normal vector at any point $\mathbf{x} \in \Gamma$.

Let $t \in \mathbb{R}_+$, $\delta t > 0$ be an infinitesimal time. We count how much mass of fluid leaves ω through Γ over the time period $[t, t + \delta t]$, a quantity denoted by $\Delta m(t, \delta t, \omega)$. Of course, some mass of fluid might also enter ω through Γ over the same period. This will still be considered as leaving, counted with a nonpositive sign.

The calculation of $\Delta m(t, \delta t, \omega)$ is made in two different ways. We first expand $\Delta m(t, \delta t, \omega)$ around $\delta t = 0$, using the expression (2.4). We secondly carry out a local analysis at Γ to express $\Delta m(t, \delta t, \omega)$ as an integral over Γ, and we use the Stokes formula to transform it into an integral over ω. We obtain two distinct integrals over ω that we equate, thus expressing that the mass budget is balanced and deriving equation (2.26).

Since we require an algebraic loss in mass, the quantity we aim at computing is

$$\Delta m(t, \delta t, \omega) = m(t, \omega) - m(t + \delta t, \omega).$$

Using the Taylor formula and the definition (2.4), we have

$$\begin{aligned} \Delta m(t, \delta t, \omega) &= -\delta t \frac{d}{dt} \int_\omega \rho(t, \mathbf{x}) d\mathbf{x} + O(\delta t^2) \\ &= -\delta t \int_\omega \partial_t \rho(t, \mathbf{x}) d\mathbf{x} + O(\delta t^2). \end{aligned} \tag{2.27}$$

We hold equality (2.27) in reserve for the moment, and we turn to a local analysis at the boundary Γ.

Let $\mathbf{x} \in \Gamma$ be a fixed point, $\delta S \subset \Gamma$ be an infinitesimal part of Γ, whose gravity center is $\mathbf{x}$, and $\mathbf{n}(\mathbf{x})$ be the outward-pointing unit normal vector at $\mathbf{x}$. We assume that the field $(\mathbf{v}, \rho)$ is constant in the vicinity of $(t, \mathbf{x})$ and equal to $(\mathbf{v}(t, \mathbf{x}), \rho(t, \mathbf{x}))$, which is of course not satisfied exactly but is reasonable to first order.

We must characterize the particles that can leave ω through δS over the time period $[t, t + \delta t]$.

From the assumption we made on $\mathbf{v}$ at $(t, \mathbf{x})$, a physical particle of fluid that sits on $\mathbf{y}$, a point near $\mathbf{x}$ at time t, moves to $\mathbf{y} + \mathbf{v}(t, \mathbf{x})\delta t$ at $t + \delta t$. Thus, the particles we are looking for are those contained in the volume

$$\delta V = \mathbf{v}(t, \mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, \delta S\, \delta t$$

at time t. This represents a mass of fluid $\delta m = \rho(t, \mathbf{x})\delta V$. We have to sum δm over $\partial\omega$ to compute the total mass of fluid going out ω over the time period $[t, t + \delta t]$. Hence, we have

$$\Delta m(t, \delta t, \omega) = \delta t \int_{\partial\omega} \rho(t, \mathbf{x})\, \mathbf{v}(t, \mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, dS. \tag{2.28}$$

We apply the Stokes formula (see in [5, 27]) to the right-hand side of (2.28), leading to

$$\Delta m(t, \delta t, \omega) = \delta t \int_\omega \nabla \cdot (\rho(t, \mathbf{x})\, \mathbf{v}(t, \mathbf{x})) d\mathbf{x}. \tag{2.29}$$

We combine the equalities (2.27) and (2.29), divide by $\delta t > 0$, and let it tend to zero. Then we obtain the following relation:

$$\int_\omega [\partial_t \rho(t, \mathbf{x}) d\mathbf{x} + \nabla \cdot (\rho(t, \mathbf{x})\, \mathbf{v}(t, \mathbf{x}))] d\mathbf{x} = 0 \tag{2.30}$$

which holds at every $t \geq 0$ and for every subdomain ω of Ω. As we assume $\mathbf{v}$ and ρ of class C^1, we deduce from the results of integration theory [26] that the relation (2.30) yields the mass conservation equation (2.26). □

2.3.3 Total Derivative

In the previous analysis, we considered that a particle sitting on a point $\mathbf{x}$ at time t moves to the point $\mathbf{x}+\delta\mathbf{x} = \mathbf{x}+\delta t\ \mathbf{v}(t,\mathbf{x})$ at time $t+\delta t$. It is as if the point $\mathbf{x} = \mathbf{x}(t)$ was moving along a trajectory of the ordinary differential equation

$$\mathbf{x}'(t) = \mathbf{v}(t,\mathbf{x}(t)), \tag{2.31}$$

where $\mathbf{x}'(t)$ denotes the standard derivative with respect to t.

Let $\mathbf{E} = \mathbf{E}(t,\mathbf{x})$ be any tensor field of class C^1 on Q. As the point $\mathbf{x}$ also depends on t, so does

$$\mathbf{G}(t) = \mathbf{E}(t,\mathbf{x}(t)).$$

The question arises of how to express the time derivative $\mathbf{G}'(t)$ in terms of $\partial_t\mathbf{E}$ and $\nabla\mathbf{E}$.

Lemma 2.2. *Assume that* $\mathbf{E}$ *is of class* C^1 *on* Q. *We have*

$$\mathbf{G}'(t) = \partial_t\mathbf{E} + \mathbf{v}\cdot\nabla\mathbf{E}. \tag{2.32}$$

The field $\mathbf{G}'(t)$ *is denoted by* $\dfrac{D\mathbf{E}}{Dt}$ *and is called the total derivative of* E.

Proof. We expand $\mathbf{G}$ around $\delta t = 0$,

$$\mathbf{G}(t+\delta t) = \mathbf{E}(t+\delta t,\mathbf{x}(t+\delta t)) = \mathbf{E}(t+\delta t,\mathbf{x}+\delta t\mathbf{v}(t,\mathbf{x})+o(\delta t)), \tag{2.33}$$

leading to

$$\mathbf{G}(t+\delta t) = \mathbf{G}(t) + \delta t(\partial_t\mathbf{E}(t,\mathbf{x}) + \mathbf{v}(t,\mathbf{x})\cdot\nabla\mathbf{E}(t,\mathbf{x})) + o(\delta t), \tag{2.34}$$

which is valid because $\mathbf{E}$ is of class C^1 on Q. We finally find

$$\mathbf{G}'(t) = \lim_{\delta t\to 0}\frac{G(t+\delta t)-G(t)}{\delta t} = \partial_t\mathbf{E}(t,\mathbf{x}) + \mathbf{v}(t,\mathbf{x})\cdot\nabla\mathbf{E}(t,\mathbf{x}), \tag{2.35}$$

hence the result follows. □

Remark 2.1. The total derivative satisfies the usual derivative rules, namely

$$\frac{D(\mathbf{E}+\mathbf{F})}{Dt} = \frac{D\mathbf{E}}{Dt} + \frac{D\mathbf{F}}{Dt}, \quad \frac{D(\mathbf{E}\cdot\mathbf{F})}{Dt} = \frac{D\mathbf{E}}{Dt}\cdot\mathbf{F} + \mathbf{E}\cdot\frac{D\mathbf{F}}{Dt}. \tag{2.36}$$

Remark 2.2. The mass conservation equation (2.26) can be rewritten in terms of a total derivative. Indeed, we have

$$\nabla \cdot (\mathbf{v}\rho) = \mathbf{v} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{v}, \tag{2.37}$$

which allows equation (2.26) to be rewritten as

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{v}. \tag{2.38}$$

This form of the mass conservation equation may sometimes be useful.

2.3.4 Rigorous Derivation of the Mass Conservation Equation

Let us consider a fluid particle that sits on $\mathbf{x} = \mathbf{x}(t)$ at time t and which moves to $\mathbf{x}(t + \delta t)$ at time $t + \delta t$, almost along a trajectory of the ODE (2.31). Recall that the mass $dm = dm(t, \mathbf{x}) = \rho(t, \mathbf{x})dv(t, \mathbf{x})$ of the particle at $(t, \mathbf{x})$ was defined by (2.25).

The principle of mass conservation is that the mass of the particle remains constant along the trajectory, which is expressed by the following equation:

$$\frac{D(dm)}{Dt} = 0. \tag{2.39}$$

We show in what follows that equation (2.39) is precisely the mass conservation equation (2.26).

According to the definitions (2.25) and (2.32), we have

$$\frac{D(dm)}{Dt} = (\partial_t\rho + \mathbf{v} \cdot \nabla\mathbf{v})dv + \rho\frac{D(dv)}{Dt}. \tag{2.40}$$

We must compute the total derivative $\frac{D(dv)}{Dt}$.

We insert the identity (2.12) of Lemma 2.1 in the relation (2.24) that expresses dv in terms of dv_0 and the Jacobian determinant, by noting that $dv_0(\mathbf{X})$ is totally time independent. The point $\mathbf{X}$ denotes the Lagrangian coordinate of the particle, which means its position at time $t = 0$. Therefore, we find the relation

$$\frac{D(dv)}{Dt} = (\nabla \cdot \mathbf{v})\, dv, \tag{2.41}$$

satisfied at every point $(t, \mathbf{x}) \in \mathbb{R}_+ \times \Omega$. We combine equations (2.39), (2.40), and (2.41). Since $dv \neq 0$, we obtain

$$\partial_t\rho + \mathbf{v} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{v} = 0, \tag{2.42}$$

which is the mass conservation equation (2.26), following relation (2.37). $\square$

2.4 Incompressibility

2.4.1 *Basic Definition*

Compressibility and incompressibility are natural physical notions that we all are familiar with from everyday life. Generally speaking, we know that the volume occupied by a fixed mass of gas can be reduced, but that in the case of a liquid, the mass density remains more or less constant during motion. In the first case, we say that the flow is compressible while in the second case it is incompressible, and so the state equation can be written as

$$\rho = \rho(t, \mathbf{x}) = \rho_0 \quad \text{on} \quad Q = \mathbb{R}_+ \times \Omega, \tag{2.43}$$

for some constant $\rho_0 > 0$. In this case, the mass conservation equation (2.26) becomes

$$\nabla \cdot \mathbf{v} = 0. \tag{2.44}$$

The nature of equation (2.44) is kinematic. Moreover, experimental data [10] indicate that there are flow motions, the velocity of which still satisfy equation (2.44), but whose density is not constant. This suggests the global definition:

Definition 2.2. Any fluid flow on Q with $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ as velocity field is incompressible on Q if and only if $\mathbf{v}$ satisfies equation (2.44) at each point $(t, \mathbf{x}) \in Q$.

Incompressible flows preserve the volumes, according to Formula (2.12). Incompressibility refers to the nature of the motion. This is why the term of incompressibility is applied to the flow rather than the physical nature of the fluid. Some gas motions might be considered as incompressible flows, depending on the scales involved. Even if it is more difficult to conceptualize, some liquid motions may be considered as compressible.

In the remainder of the section, we consider the example of oceanic flow which is the typical example of an incompressible flow with a variable density. We then evoke the Mach number, closely linked to the question of compressibility/incompressibility.

2.4.2 *Incompressible Flow with Variable Density: The Example of the Ocean*

The density of the ocean varies by about 2 % around a mean value $\rho_0 = 1035\,\text{Kg.m}^{-3}$ ([10, 21]), so that

$$\left|\frac{D\rho}{\rho}\right| \approx \left|\frac{\rho-\rho_0}{\rho_0}\right| \leq 2.10^{-3}. \tag{2.45}$$

This density change is mainly due to salt, which is not evenly distributed in the water, as well as to temperature variations. As a result, the density ρ of the ocean satisfies a state equation $\rho = \rho(S, \theta)$, where θ denotes the temperature and S the salinity (the mass of salt per unit of volume). This equation is nonlinear and may vary according to the place of study [10]. In some situations, the pressure p may also be involved. Simplified mathematical models use the linearized state equation:

$$\rho = \rho_0 + \alpha_S(S - S_0) + \alpha_\theta(\theta_0 - \theta), \tag{2.46}$$

for $\alpha_\theta > 0$, $\alpha_S > 0$, $S_0 > 0$, and $\theta_0 > 0$ constant.

Nevertheless, the bound (2.45) allows us to consider the ocean's motion as incompressible. To see this, we introduce a typical velocity magnitude U, a typical time magnitude T, and a typical length magnitude L. Those values may change, according to the case we focus on. Their choice usually fixes the parameters for numerical simulations: the time step is related to T while the mesh size is related to L. We assume

$$U = LT^{-1}. \tag{2.47}$$

We examine the magnitude of each term in the mass conservation equation (2.38). We denote by $[\mathbf{E}]$ the magnitude of any given field $\mathbf{E}$. Therefore,

$$[\nabla \cdot \mathbf{v}] = \frac{U}{L} = T^{-1}. \tag{2.48}$$

Similarly,

$$\left[\frac{1}{\rho}\frac{D\rho}{Dt}\right] = T^{-1}\left[\frac{D\rho}{\rho}\right] = 2.10^{-3}T^{-1}. \tag{2.49}$$

Therefore, in equation (2.38), the magnitude of the right-hand side (r.h.s.) differs from that of the left-hand side (l.h.s.) by a coefficient $\varepsilon = 10^{-3}$. This is as if the equation was written as

$$\varepsilon E = F, \tag{2.50}$$

where $O(E) = O(F) = 1$ and $\varepsilon = o(1)$. This is a standard situation in asymptotic analysis [3], and the result is that both E and F must vanish to satisfy equation (2.50), which yields

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \nabla \cdot \mathbf{v} = 0,$$

and hence we can conclude that the flow is incompressible according to the definition 2.2.

2.4.3 Incompressible Limit

An observer swimming in the sea with his head underwater may hear the sound of a boat engine apparently close by, although the boat is actually a large distance away. It seems that the speed of the sound in the water is infinite.

The study of sound propagation in fluids [20] led E. Mach to introduce in 1887 the dimensionless number

$$M = \frac{\mathbf{U}}{c},$$

where c is the speed of the sound. This number is now called the Mach number.

It can be shown that when M goes to zero, which implies an infinite speed of sound, then the corresponding limit of the velocity satisfies the incompressible equation (2.44). This is the incompressible limit, which can be derived from asymptotic expansions in the compressible Navier–Stokes equations [22]. An analysis based on physical arguments at small scales yields the same results [2].

Throughout the rest of this book, we assume the following:

Assumption 2.3. *The flow specified by the vector field* $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ *is incompressible, that is,* $\nabla \cdot \mathbf{v} = 0$.

2.5 Kinematic Features of Incompressible Flows

2.5.1 Aim of the Section

This aim of this section is to study the transformations of an infinitesimal body of fluid δV during its motion in an incompressible flow, over an infinitesimal time period. We assume the following:

Assumption 2.4. *The body* δV *can be identified to an open set* $\omega \subset \Omega$ *at a given time* t *and to* $\omega_\tau \subset \Omega$ *at time* $t + \tau$, $\tau \in [0, \delta T]$ *for some* $\delta T > 0$. *We also assume that* ω_τ *has a boundary* Γ_τ *of class* C^1 *for all* $\tau \in [0, \delta T]$. *We denote by* $\boldsymbol{n}_\tau$ *the outward-pointing unit normal vector on* Γ_τ.

Recall that during its motion, the total volume δV is constant, thanks to the incompressibility assumption.

The local analysis carried out in Sect. 2.5.2 below reveals that the tensor field $\nabla \mathbf{v}(t, \mathbf{x}) = \nabla \mathbf{v}$ defined by

$$\nabla \mathbf{v} = \left(\frac{\partial v_i}{\partial x_j}(t, \mathbf{x}) \right)_{1 \le i, j \le 3} \tag{2.51}$$

governs the first-order transformations of δV over a time period $\delta P = [t, t + \delta T]$, where we assume that over δP, the point $\mathbf{x}$ remains the gravity center of δV, $\delta T = o(1)$ as well as $\text{diam}(\delta V) = o(1)$.

We write $\nabla \mathbf{v}$ at $(t, \mathbf{x})$ in the form

$$\nabla \mathbf{v} = D\mathbf{v} + \nabla^{\mathrm{a}} \mathbf{v}, \tag{2.52}$$

designating by $\nabla^{\mathrm{a}} \mathbf{v}$ the antisymmetrical part of $\nabla \mathbf{v}$ and $D\mathbf{v}$ the symmetric part of $\nabla \mathbf{v}$, namely

$$D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t), \quad \nabla^{\mathrm{a}} \mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} - \nabla \mathbf{v}^t),$$

where A^t denotes the transpose of any matrix A. The tensor $D\mathbf{v}$ is called the deformation tensor.

We show in Sect. 2.5.3 that the spectral analysis of $D\mathbf{v}$ determines in what directions δV remains stable and how it might deform. The incompressibility assumption 2.3 makes the study of stability easy, since the trace of $D\mathbf{v}$ is equal to zero in this case. This analysis also explains why $D\mathbf{v}$ is so important for expressing the internal forces acting on the fluid, performed in Sect. 2.6.

Turning to the tensor $\nabla^{\mathrm{a}} \mathbf{v}$, we show (Lemma 2.3 in Sect. 2.5.4) that it is fully specified through the vorticity vector

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}, \quad \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3). \tag{2.53}$$

In Sect. 2.5.5, we study the contribution of the vorticity $\boldsymbol{\omega}$ in the transformations of δV.

In the light of the decomposition (2.52), we distinguish three cases:

(i) $D\mathbf{v}$ is large compared to $\boldsymbol{\omega}$, $||\boldsymbol{\omega}|| << ||D\mathbf{v}||$,
(ii) $\boldsymbol{\omega}$ is large compared to $D\mathbf{v}$, $||D\mathbf{v}|| << ||\boldsymbol{\omega}||$,
(iii) they are of the same magnitude, $||D\mathbf{v}|| \approx ||\boldsymbol{\omega}||$,

where by default,

$$\forall\, \mathbf{E} = (E_{ijk\ell\ldots})_{1 \le ijk\ell\ldots \le 3}, \quad ||\mathbf{E}|| = ||\mathbf{E}||_2 = \Big(\sum_{1 \le ijk\ell\ldots \le 3} E_{ijk\ell\ldots}^2 \Big)^{\frac{1}{2}},$$

or any equivalent norm. The conclusions of this section are the following.

In case (i), if δV has the form of a football at time t, it is then transformed into a rugby ball.

In case (ii), δV behaves like a rotating solid body, whose angular velocity is $(1/2)\boldsymbol{\omega}$. In regions where (ii) holds, small-scale vortices may be observed.

Case (iii) is more difficult. The transformation of δV might be anything because both effects compensate. To illustrate this, we sketch out in Sect. 2.5.6 the typical example of a shear flow, for which such a compensation occurs.

2.5.2 Local Role of $\nabla \mathbf{v}$ and Fundamental ODE

Let $\mathbf{x}$ be the gravity center of δV at time t and $\mathbf{y} \in \delta V$ be any other point. Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ denote the position of two particles sitting on $\mathbf{x}$ and $\mathbf{y}$ at time t. Let us consider

$$\boldsymbol{\xi} = \boldsymbol{\xi}(t) = \mathbf{x}(t) - \mathbf{y}(t), \tag{2.54}$$

Assume that the particles move to $\mathbf{x} + \delta\mathbf{x}$ and $\mathbf{y} + \delta\mathbf{y}$ respectively at time $t + \delta t$ for some $\delta t > 0$. Let us set $\delta\boldsymbol{\xi} = \delta\mathbf{x} - \delta\mathbf{y}$.

We perform the asymptotic expansions:

$$\delta\mathbf{x} = \mathbf{v}(t, \mathbf{x})\delta t + o(\delta t), \tag{2.55}$$

$$\delta\mathbf{y} = \mathbf{v}(t, \mathbf{y})\delta t + o(\delta t), \tag{2.56}$$

$$\mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, \mathbf{y}) = \nabla\mathbf{v} \cdot \boldsymbol{\xi} + o(||\boldsymbol{\xi}||), \tag{2.57}$$

where $\nabla\mathbf{v} = \nabla\mathbf{v}(t, \mathbf{x})$. We combine (2.55), (2.56) and (2.57) and we find

$$\frac{\delta\boldsymbol{\xi}}{\delta t} = \nabla\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi} + o(\delta t + ||\boldsymbol{\xi}||). \tag{2.58}$$

We take the limit in equation (2.58) as δt goes to 0, which yields the following differential equation:

$$\boldsymbol{\xi}' = \nabla\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi} + o(||\boldsymbol{\xi}||). \tag{2.59}$$

This suggests the following local ODE, which corresponds to the first-order term in equation (2.59):

$$\boldsymbol{\xi}'(t + \tau) = \nabla\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi}(t + \tau), \tag{2.60}$$

where now $\boldsymbol{\xi} = \boldsymbol{\xi}(t + \tau)$ is only time dependent, t and $\boldsymbol{\xi}(t)$ are given and fixed, $\tau > 0$, and $\boldsymbol{\xi}'$ denotes the derivative with respect to τ.

Although equation (2.60) makes sense around τ=0 from the physical viewpoint, as a linear ODE it possesses a unique global solution defined on $\mathbb{R}$, for any initial datum $\boldsymbol{\xi}$ [5].

However, without any specific information about the matrix $\nabla\mathbf{v}(t, \mathbf{x}) = \nabla\mathbf{v}$, it is difficult to easily picture the overall appearance of the solutions to (2.60). We can only get qualitative stability properties, by using the incompressibility assumption. However, following the decomposition (2.52), it is natural to split the ODE (2.60) into

$$\boldsymbol{\xi}' = D\mathbf{v} \cdot \boldsymbol{\xi}, \tag{2.61}$$

$$\boldsymbol{\xi}' = \nabla^{\mathrm{a}}\mathbf{v} \cdot \boldsymbol{\xi}, \tag{2.62}$$

In what follows, we study (2.61) and (2.62) separately after having first analyzed the stability properties of equation (2.60).

2.5.3 Deformation Tensor

2.5.3.1 Stability

The question is whether $\boldsymbol{\xi}(t+\tau)$ goes to zero as τ goes to infinity, for a given initial data $\boldsymbol{\xi}(t) \neq 0$. If yes, we say that equation (2.60) is stable, otherwise we say it is unstable. We follow the theory developed by A.M. Lyapunov in 1892 (see the details in [13]). To do so, we take the inner product by $\boldsymbol{\xi}$ with both sides of (2.60), which yields

$$(\boldsymbol{\xi}', \boldsymbol{\xi}) = (\nabla\mathbf{v} \cdot \boldsymbol{\xi}, \boldsymbol{\xi}). \tag{2.63}$$

According to a well-known result of linear algebra,

$$(\nabla\mathbf{v} \cdot \boldsymbol{\xi}, \boldsymbol{\xi}) = (D\mathbf{v} \cdot \boldsymbol{\xi}, \boldsymbol{\xi}),$$

regardless of $\boldsymbol{\xi}$. Therefore, equation (2.63) may be written as

$$\frac{1}{2}\frac{d||\boldsymbol{\xi}||^2}{dt} = (D\mathbf{v} \cdot \boldsymbol{\xi}, \boldsymbol{\xi}). \tag{2.64}$$

The inner product $(D\mathbf{v} \cdot \boldsymbol{\xi}, \boldsymbol{\xi})$ is a Lyapunov function for the ODE (2.60) which specifies local stability properties near the point $(t, \mathbf{x})$.

To check the stability properties of the flow, we use the symmetry of $D\mathbf{v}$, which is therefore orthogonal diagonalizable [4]. Moreover, incompressibility yields

$$\mathrm{tr} D\mathbf{v} = 2\nabla \cdot \mathbf{v} = 0.$$

Assume first that $D\mathbf{v} = 0$. Then the velocity is locally constant around $\mathbf{x}$ and particles move along straight lines.

Assume next that $D\mathbf{v} \neq 0$. Because of incompressibility, $D\mathbf{v}$ has at least one strictly negative eigenvalue, denoted by λ_1, and one strictly positive, denoted by λ_2. The ODE is stable along the eigendirection associated with λ_1 and unstable along the one associated with λ_2.

In particular, let $\xi(t) \neq 0$ be an initial datum that is an eigenvector associated with λ_1, then $\xi(t+\tau)$ goes to zero when τ goes to infinity. Let $\xi(t) \neq 0$ be an initial datum that is an eigenvector associated with λ_2, then $||\xi(t+\tau)||$ goes to infinity when τ goes to infinity. If $\lambda_3 \neq 0$, its sign determines if stability holds along a plane or along a line only.

2.5.3.2 When Footballs Become Rugby Balls

Assume that $D\mathbf{v}$ is large compared to $\nabla^{\mathrm{a}}\mathbf{v}$ at $(t, \mathbf{x})$. At that point, the solutions to the fundamental equation (2.60) are very close to those of equation (2.61) that we solve in the eigen-coordinate system of $D\mathbf{v}$.

Let $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ be an orthogonal eigenbasis of the tangent space $T_{t,\mathbf{x}}$ for $D\mathbf{v}$, associated with the eigenvalues λ_1, λ_2, λ_3. From now, we assimilate $T_{t,\mathbf{x}}$ to $\mathbb{R}^3$ for simplicity. We write

$$\boldsymbol{\xi}(t + \tau) = \xi_i(t + \tau)\mathbf{a}_i, \tag{2.65}$$

where each ξ_i satisfies

$$\xi_i' = \lambda_i \xi_i. \tag{2.66}$$

Therefore, the solution to equation (2.61) is

$$\boldsymbol{\xi}(t + \tau) = e^{\lambda_i \tau}\xi_i(t)\mathbf{a}_i. \tag{2.67}$$

To picture what the solution looks like, assume that δV is a ball of radius $r = o(1)$ centered on $\mathbf{x}$ at time t. Then δV instantaneously becomes a rugby ball, an ellipsoid whose axes are defined by the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ and its shape determined by the sign of the eigenvalues.

2.5.4 Vorticity

If the body δV were a solid body with $\mathbf{x}$ as its center of gravity, then the velocity of each $\mathbf{y} \in \delta V$ would be expressed by the law [15]:

$$\mathbf{v}(t, \mathbf{y}) = \mathbf{v}(t, \mathbf{x}) - \boldsymbol{\Omega} \times \boldsymbol{\xi}, \tag{2.68}$$

for some angular vector $\boldsymbol{\Omega}$ to be determined.

We rewrite the asymptotic expansion (2.57) using the decomposition (2.52) as follows:

$$\mathbf{v}(t, \mathbf{y}) = \mathbf{v}(t, \mathbf{x}) - D\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi} - \nabla^{\mathrm{a}}\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi} + o(||\boldsymbol{\xi}||). \tag{2.69}$$

We assume that $D\mathbf{v}$ is negligible against $\nabla^{\mathrm{a}}\mathbf{v}$ at $(t, \mathbf{x})$. Consequently, (2.68) is similar to (2.69), provided that $\nabla^{\mathrm{a}}\mathbf{v} \cdot \boldsymbol{\xi}$ can be written in the form $\boldsymbol{\Omega} \times \boldsymbol{\xi}$.

The appropriate vector field is the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v} = (\omega_1, \omega_2, \omega_3)$, as proved by the following.

Lemma 2.3. *Let*

$$\zeta_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

be the general term of $\nabla^{\mathrm{a}}\mathbf{v}$. *Then the following relation holds:*

$$\zeta_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k, \tag{2.70}$$

for all $1 \leq i, j \leq 3$. *Furthermore,*

$$\nabla^{\mathrm{a}}\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi} = \frac{1}{2}\boldsymbol{\omega}(t, \mathbf{x}) \times \boldsymbol{\xi}, \tag{2.71}$$

regardless of $\boldsymbol{\xi}$.

Proof. From now on and if no risk of confusion occurs, we shall write

$$\partial_i = \frac{\partial}{\partial x_i}, \tag{2.72}$$

for every $i = 1, 2, 3$. Following the formula (2.10) and using the antisymmetry of the Levy-Civita tensor, we have

$$\omega_k = \varepsilon_{pqk}\partial_p v_q, \tag{2.73}$$

which yields $\varepsilon_{ijk}\omega_k = \varepsilon_{ijk}\varepsilon_{pqk}\partial_p v_q$. The relation

$$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \tag{2.74}$$

shows

$$\varepsilon_{ijk}\omega_k = \partial_i v_j - \partial_j v_i = -2\zeta_{ij},$$

which proves the relation (2.70) as well as the identity (2.71) following (2.10), combined with (2.70). □

As a consequence of the identity (2.71), the expansion (2.69) becomes

$$\mathbf{v}(t, \mathbf{y}) = \mathbf{v}(t, \mathbf{x}) - D\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi} - \frac{1}{2}\boldsymbol{\omega}(t, \mathbf{x}) \times \boldsymbol{\xi} + o(||\boldsymbol{\xi}||), \tag{2.75}$$

where we recall that $\boldsymbol{\xi} = \mathbf{x} - \mathbf{y}$.

Remark 2.3. By explaining the formula (2.73), we find

$$\boldsymbol{\omega} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix},$$

which is the practical expression of the vorticity.

2.5.5 Vortices

When we compare the expansion (2.75) to the expression (2.68) by taking $\boldsymbol{\Omega} = (1/2)\boldsymbol{\omega}$, we observe that the vorticity characterizes the instantaneous rotation of δV, provided $\boldsymbol{\omega}$ is large compared to $D\mathbf{v}$. In this case, the fundamental equation (2.60) is very close to equation (2.62), which we rewrite as

$$\boldsymbol{\xi}' = \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{\xi}, \tag{2.76}$$

using the relation (2.71), in which $\boldsymbol{\omega} = \boldsymbol{\omega}(t, \mathbf{x})$ for a fixed $(t, \mathbf{x})$. Let us solve equation (2.76).

If $\boldsymbol{\omega} = 0$, then $\boldsymbol{\xi}(t+\tau) = \boldsymbol{\xi}(t)$ regardless of τ. Let us assume that $\boldsymbol{\omega} \neq 0$, and let us consider

$$\mathbf{b}_1 = \frac{\boldsymbol{\omega}}{||\boldsymbol{\omega}||}.$$

Let $\mathbf{b}_2$ and $\mathbf{b}_3$ be such that $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is an orthogonal basis of $\mathbb{R}^3$ that in particular satisfies

$$\mathbf{b}_1 \times \mathbf{b}_2 = \mathbf{b}_3, \quad \mathbf{b}_2 \times \mathbf{b}_3 = \mathbf{b}_1, \quad \mathbf{b}_3 \times \mathbf{b}_1 = \mathbf{b}_2. \tag{2.77}$$

We write

$$\boldsymbol{\xi} = \tilde{\xi}_i \mathbf{b}_i. \tag{2.78}$$

Using the relations (2.77) and setting $z = \tilde{\xi}_2 + i\tilde{\xi}_3$, we get

$$\tilde{\xi}_1' = 0, \quad z' = i\frac{||\boldsymbol{\omega}||}{2}z, \tag{2.79}$$

which yields

$$\tilde{\xi}_1(t+\tau) = \tilde{\xi}_1(t), \quad z(t+\tau) = e^{i\frac{||\boldsymbol{\omega}||}{2}\tau}z(t). \tag{2.80}$$

Therefore, trajectories rotate around the axis spanned by $\boldsymbol{\omega}$ with a frequency equal to $||\boldsymbol{\omega}||/2$.

The resolution of the ODE (2.76) explains why the vorticity is closely linked to the notion of vortex (also called eddy) which plays a central role in the study of turbulent flows.

However, the fact that $\boldsymbol{\omega} \neq 0$ in a given flow does not imply that an observer will see vortices in the flow, essentially because the analysis carried above is local in time as well as in space and therefore only makes sense for small scales. Moreover, the deformation tensor effects may balance the vorticity effects when $\boldsymbol{\omega}$ and $D(\mathbf{v})$ are of the same magnitude, as shown in the example discussed in Sect. 2.5.6.

The question of defining mathematically a "vortex" as we picture it at large scales is hard. The simplest and popular criterium that is used in practical simulations to locate where there may be vortices is the Q-criterium which says that vortices are located in the set

$$\{(t, \mathbf{x}) \in \mathbb{R}_+ \times \Omega, \quad Q(t, \mathbf{x}) = \frac{1}{2}[|\boldsymbol{\Omega}(t, \mathbf{x})|^2 - |D\mathbf{v}(t, \mathbf{x})|^2] > 0\}. \tag{2.81}$$

2.5.6 A Typical Example of a Shear Flow

In this example, we work in a dimensionless framework for simplicity, dimensional analysis being detailed in Sect. 3.2.

Let $\mathbf{v} = (v_1, v_2, v_3)$ be the stationary vector field defined by

$$\forall\, \mathbf{x} = (x, y, z), \quad v_1(x, y, z) = z, \quad v_2(x, y, z) = v_3(x, y, z) = 0, \tag{2.82}$$

where $\mathbf{x} = x\,\mathbf{e}_1 + y\,\mathbf{e_2} + z\,\mathbf{e}_3$. The field $\mathbf{v}$ satisfies $\nabla \cdot \mathbf{v} = 0$. Basic calculations yield

$$\nabla \mathbf{v} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D\mathbf{v} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\omega} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{2.83}$$

We notice that

$$||\boldsymbol{\omega}||_1 = ||D\mathbf{v}||_1 = 1. \tag{2.84}$$

Let $\boldsymbol{\xi} = (x, y, z)^{\mathrm{T}}$ be given at time t. The solution to equation (2.60), denoted by $\boldsymbol{\xi}_\tau$ at time $t + \tau$ and such that $\boldsymbol{\xi}_0 = \boldsymbol{\xi}$, is equal to

$$\boldsymbol{\xi}_\tau = \begin{pmatrix} x + \tau z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tau \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \boldsymbol{\xi} = R_\tau \cdot \boldsymbol{\xi}. \tag{2.85}$$

For a given $\tau > 0$, the matrix R_τ is the matrix of a transvection [4]. Let us describe carefully the transformation of δV in the flow when $\delta V = B(0, r)$ is a ball centered at the origin. Three cases occur:

(a) Points in $\delta V \cap \{z = 0\}$ remain steady.
(b) Points $\mathbf{x} = (x, y, z) \in \delta V \cap \{z > 0\}$ are such that $x \to \infty$ while y and z remain constant when $\tau \to \infty$.
(c) Points $\mathbf{x} = (x, y, z) \in \delta V \cap \{z < 0\}$ are such that $x \to -\infty$ while y and z also remain constant when $\tau \to \infty$.

The ball δV is sheared in the region $\{\sup(|y|, |z|) \leq 1\}$ and fluid particles contained in δV at time t are mixed in the whole strip, which is a typical process in turbulent flows.

Transformations of bodies totally included in the region $\{z > 0\}$ or $\{z < 0\}$, both stable through R_τ, are sheared along the x-axis.

We study equations (2.61) and (2.62) one by one. Equation (2.62) is already solved by the general formula (2.80). The solution is a rotation whose frequency is equal to $1/2$, around the line spanned by $\boldsymbol{\omega}$, that is, the line $\{x = z = 0\}$.

We turn to the ODE (2.61). The spectrum of $D\mathbf{v}$ is the set $\{-1/2, 1/2, 0\}$. The eigenspace associated with the eigenvalue $-1/2$ is the line $\{x = -z,\ y = 0\}$, spanned by $\mathbf{e}_1 + \mathbf{e}_3$, which is a stability direction according to Sect. 2.5.3.1. The eigenspace associated with the eigenvalue $1/2$ is the line $\{x = z,\ y = 0\}$, spanned by $\mathbf{e}_1 - \mathbf{e}_3$, which is an unstable direction. Finally, the eigenspace associated with the eigenvalue 0 is the y-axis that also coincides with the line spanned by $\boldsymbol{\omega}$. The general solution is then given by

$$\boldsymbol{\xi}_\tau = (x \cosh(\tau) - z \sinh(\tau))\, \mathbf{e}_1 + y\, \mathbf{e}_2 + (-x \sinh(\tau) + z \cosh(\tau))\, \mathbf{e}_3, \tag{2.86}$$

by noting that $\boldsymbol{\omega}$ is neither a stable nor an unstable direction.

The overall impression is that the resolution of (2.61) and (2.62) does not allow the transformations of δV to be pictured in this specific case. This is more easily done by solving the fundamental equation (2.60), which is fortunately straightforward. The solutions are highly unstable, especially when considering bodies initially in both $\{z > 0\}$, $\{z < 0\}$.

Nevertheless, we suspect the fact that $D\mathbf{v}$ has an eigenvalue equal to zero, whose eigenspace is spanned by $\boldsymbol{\omega}$, together with the result (2.84), may explain some of the features of this example. No more can really be said, apart from noting the great importance of shears in turbulent flows.

2.6 The Equation of Motion and the Navier–Stokes Equation

2.6.1 Aim of the Section

We turn to the momentum equation, based on Newton's law:

$$\text{mass} \times \text{acceleration} = \text{total applied forces}, \tag{2.87}$$

which a priori applies over a time period $[t, t + \delta T]$ to a body δV satisfying assumption 2.4.

We start by modeling the total forces applied on $\delta V = \omega_\tau$ at time $t + \tau$. We distinguish two different types of forces:

(i) The body forces, applied at distance on δV, such as gravity, electromagnetic forces, and so on,
(ii) The "internal forces" $F(\tau, \delta V)$, that are those the rest of the fluid applies on δV at time $t + \tau$.

As the internal forces are those that are hard to model, particular attention will be paid to them. The appropriate tool is the stress tensor

$$\boldsymbol{\sigma} = (\sigma_{ij})_{1 \le i,j \le 3} = \boldsymbol{\sigma}(t, \mathbf{x})$$

(see [2, 7, 8, 11, 14]) that is symmetric and such that

$$F(\tau, \delta V) = \int_{\Gamma_\tau} \boldsymbol{\sigma} \cdot \mathbf{n}_\tau, \tag{2.88}$$

which results in an internal force density equal to $(\nabla \cdot \boldsymbol{\sigma})dv$, according to Stokes' formula. Therefore, we must specify $\boldsymbol{\sigma}$ by making some reasonable assumptions. This is the aim of Sect. 2.6.2, where the dynamic pressure is introduced through the relation $p = -(1/3)\text{tr}\boldsymbol{\sigma}$. Furthermore, we introduce the definition of a Newtonian fluid together with the notion of dynamic viscosity μ.

Next we consider a local point of view, just as we did when studying the mass conservation energy in Sect. 2.3. Indeed, we prefer to apply Newton's law to a particle of fluid that sits on $\mathbf{x}$ at time t, rather than to a body δV, which has been useful in finding $\nabla \cdot \boldsymbol{\sigma}$. Once the acceleration of $\mathbf{x}$ at time t has been calculated in Sect. 2.6.3, we present the momentum equation for incompressible flows in as general terms as possible.

We introduce the kinematic viscosity ν in Sect. 2.6.4 and we take the opportunity to present various forms of the incompressible Navier–Stokes equations, each form being useful for theoretical investigations or for practical simulations.

We conclude with Sect. 2.6.5, where we derive the equations satisfied by the vorticity $\boldsymbol{\omega}$ and the pressure p.

2.6.2 Stress Tensor

2.6.2.1 Physical Evidences and Some History

The goal is the determination of $F(\tau, \delta V)$. Before doing any mathematics, we recall some historic physical considerations which manifest such internal forces.

The well-known Archimedes' principle (approximately year 250 B.C, see in [7]) may be stated as follows:

> "Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object."

This law characterizes the internal force in the fluid at rest, which is the hydrostatic pressure.

The famous experience carried out much later by E. Torricelli in 1644 (see in [7]), who constructed the first mercury barometer, has highlighted existence of atmospheric pressure, which varies depending on the weather.

Therefore, the first internal force exerted on any flow that comes to mind is the pressure, although this was initially considered for steady fluids. This led L. Euler to derive in 1757 a momentum equation based on Newton's law. In Euler's equations (see in [2, 7, 8, 14, 22]), which couple the incompressibility equation to the momentum equation, the pressure is the only internal force, which is treated as an unknown of the equation together with the velocity. A fluid governed by Euler's equations is called a perfect fluid. According to legend, D. Bernoulli is supposed to have said a short time after Euler's work:

> "If a perfect fluid would exist, then the birds would not fly."

Indeed, any body moving in a fluid faces a drag that yields an energy dissipation. Moreover, Le Rond d'Alembert [19], shortly after Euler's work, showed that the drag in a perfect fluid is zero, highlighting what was regarded as a paradox at that time. Therefore, something was missing in Euler's model, though it still remains a very exciting mathematical objet.

In the light of this, fluid dynamics was subject to intensive research, especially experimentally. The notion of viscosity that quantifies the concept of drag in flows rapidly emerged, leading in 1822 to the famous model due to Navier [24], who added a term in the Euler equations to model the viscosity effects and the loss of energy by dissipation during the motion. Stokes [28] made a significant contribution (1842–1846), notably in studying the flow around a rigid sphere, that yields the Stokes law.

2.6.2.2 Constitutive Law for Newtonian Flows

The concept of stress tensor, as expressed by (2.88) above, exists for any material that touches continuum mechanics. The stress tensor is often determined by experiments and its expression varies depending on the material under study.

It is convenient to split $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbb{D}, \tag{2.89}$$

where p is the dynamic pressure and $\mathbb{D}$ the deviatoric part of $\boldsymbol{\sigma}$. We have the fundamental relation

$$p = -\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}. \tag{2.90}$$

It must be stressed that many experiments indicate that the dynamic pressure agrees with the static pressure [2], so far that a static fluid is considered to be a fluid in motion with a velocity equal to 0.

It remains to specify $\mathbb{D}$, which is responsible of the shear in the flow. The fluids we are interested in are water and air because they are involved in oceanography, meteorology, and climate, which are the applications we have in mind. For both water and air, experiments indicate that $\mathbb{D}$ is a linear function of $\nabla\mathbf{v}$, thus defining Newtonian fluids. The following definition holds.

Definition 2.3. Every fluid whose deviatoric tensor is a linear function of its velocity gradient is called a Newtonian fluid.

In addition to air and water, most organic solvents and mineral oils are also Newtonian fluids. Their main physical property consists in filling the space instantaneously when they are poured into some cavity. In contrast, fluids such as paints, mustard, and ketchup do not behave in the same way and therefore are not Newtonian fluids.

Throughout the book we assume the following:

Assumption 2.5. *The fluid is Newtonian.*

The Newtonian assumption 2.5 leads us to write $\mathbb{D} = (d_{ij})_{1\le i,j\le 3}$ in the form

$$d_{ij} = A_{ijk\ell}\partial_\ell u_k, \tag{2.91}$$

where $(A_{ijk\ell})_{1\le i,j,k,\ell\le 3}$ remains to be specified. We assume that $\mathbb{D} = (d_{ij})_{1\le i,j\le 3}$ is isotropic, which means that it is invariant under coordinates changes. This implies that the tensor $\mathbf{A} = (A_{ijk\ell})_{1\le i,j,k,\ell\le 3}$ is also isotropic. Because of this isotropic assumption 2.3, we know that $\mathbf{A}$ is of the form [12]

$$A_{ijk\ell} = \mu\delta_{ik}\delta_{jl} + \mu'\delta_{i\ell}\delta_{jk} + \mu''\delta_{ij}\delta_{k\ell}, \tag{2.92}$$

where μ, μ', and μ'' are real numbers.

From the incompressibility assumption, we have

$$\delta_{ij}\delta_{k\ell}\partial_\ell u_k = \partial_k u_k = \nabla \cdot \mathbf{v} = 0. \tag{2.93}$$

Therefore, (2.91), (2.92), and (2.93) yield

$$d_{ij} = \mu\delta_{ik}\delta_{jl}\partial_\ell u_k + \mu'\delta_{i\ell}\delta_{jk}\partial_\ell u_k = \mu\partial_j v_i + \mu'\partial_i v_j. \tag{2.94}$$

Furthermore, since $\boldsymbol{\sigma}$ is symmetric, so is $\mathbb{D}$. Therefore we have $\mu = \mu'$ and

$$\mathbb{D} = 2\mu D\mathbf{v}. \tag{2.95}$$

Consequently

$$\boldsymbol{\sigma} = 2\mu D\mathbf{v} - p\mathbf{I}. \tag{2.96}$$

The coefficient μ is the dynamic viscosity, a typical unit of which is Pascal $\times$ seconds. Since viscous effects are known from experiments to be dissipative, we have $\mu > 0$. The dynamic viscosity varies depending on the temperature θ. For air and many other gases, μ satisfies Sutherland's law:

$$\mu = \mu(\theta) = \mu_0 \left(\frac{\theta}{\theta_0}\right)^{\frac{3}{2}} \frac{\theta_0 + C}{\theta + C}, \tag{2.97}$$

where μ_0, θ_0, and C are constants that must be fixed from experiments. For water and many other liquids, μ satisfies the exponential law

$$\mu = \mu(\theta) = \mu_0 e^{-b\theta}, \tag{2.98}$$

for some constants μ_0 and b.

To conclude this subsection, we notice that when we apply the Stokes formula to (2.96), we obtain

$$F(\tau, \delta V) = \int_{\omega_\tau} \nabla \cdot \boldsymbol{\sigma}(t, \mathbf{x}) d\mathbf{x}. \tag{2.99}$$

Therefore, the quantity $\nabla \cdot \boldsymbol{\sigma}$ can be understood as the density function of internal strength. Therefore, we can define the force $d\mathbf{f}_{int}(t, \mathbf{x})$ exerted by the rest of the fluid on a particle at $\mathbf{x} \in \Omega$ as a measure on Ω for a fixed t, by the formula

$$d\mathbf{f}_{int}(t, \mathbf{x}) = (\nabla \cdot \boldsymbol{\sigma}(t, \mathbf{x}))\, dv(t, \mathbf{x}), \tag{2.100}$$

where dv was defined by the formula (2.24). This point of view will be useful in what follows.

2.6.3 The Momentum Equation

We apply Newton's law (2.87) to a given fluid particle that sits on $\mathbf{x}$ at time t. We recall that $\mathbf{v}$ satisfies the regularity assumption 2.2, that is, $\mathbf{v}$ is of class C^1 with respect to t and $\mathbf{x}$.

We assume that the external forces exerted on $\mathbf{x}$ at time t by the fluid can be described by a density function $\mathbf{f}_{ext}(t, \mathbf{x})$. This is completely true for gravity, where $\mathbf{f}_{ext}(t, \mathbf{x}) = \rho(t, \mathbf{x})\mathbf{g}$, $\mathbf{g}$ being the gravitational acceleration. Then the "sum of applied forces" is equal to

$$d\mathbf{f}_{int}(t, \mathbf{x}) + \mathbf{f}_{ext}(t, \mathbf{x}) dv(t, \mathbf{x}) = (\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_{ext}) dv. \tag{2.101}$$

We aim at computing the acceleration of the particle. This particle moves to $\mathbf{x} + u(t, \mathbf{x})\delta t + o(\delta t)$ at time $t + \delta t$, where its velocity is equal to $\mathbf{v}(t + \delta t, \mathbf{x} + u(t, \mathbf{x})\delta t + o(\delta t))$. Therefore its acceleration, denoted by $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, is equal to

$$\boldsymbol{\gamma}(t, \mathbf{x}) = \lim_{\delta \to 0} \frac{\mathbf{v}(t + \delta t, \mathbf{x} + u(t, \mathbf{x})\delta t + o(\delta t)) - \mathbf{v}(t, \mathbf{x})}{\delta t} = \frac{D\mathbf{v}}{Dt}(t, \mathbf{x}). \tag{2.102}$$

Following the arguments for proving formula (2.32) in Sect. 2.3.3, we find component by component

$$\boldsymbol{\gamma}_i(t, \mathbf{x}) = \partial_t \mathbf{v}_i + \mathbf{v} \cdot \nabla \mathbf{v}_i = \partial_t \mathbf{v}_i + \mathbf{v}_j \partial_j \mathbf{v}_i. \tag{2.103}$$

The vector, whose coordinates are $(\mathbf{v} \cdot \nabla \mathbf{v}_1, \mathbf{v} \cdot \nabla \mathbf{v}_2, \mathbf{v} \cdot \nabla \mathbf{v}_3)$ and which appears in the expression of γ, is denoted by $(\mathbf{v} \cdot \nabla)\, \mathbf{v}$.

From (2.101), Newton's law applied to our particle at $(t, \mathbf{x})$ yields

$$\rho \boldsymbol{\gamma} dv = (\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_{ext}) dv. \tag{2.104}$$

We divide each side of this equation by $dv \neq 0$. We find the momentum equation

$$\rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v}) = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_{ext}. \tag{2.105}$$

By using formulas (2.89) and (2.95), this equation becomes

$$\rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v}) - \nabla \cdot (2\mu D\mathbf{v} - p\mathbf{I}) = \mathbf{f}_{ext}. \tag{2.106}$$

When we combine equation (2.106) with the incompressibility condition (2.44), we get the Navier–Stokes equations in their initial form.

2.6.4 The Navier–Stokes Equations: Various Forms

2.6.4.1 Basic Form

It is commonly accepted that any variations of the density ρ are negligible in the momentum equation for incompressible flows ([2, 14]). Therefore, we take $\rho = \rho_0$ in equation (2.106), where ρ_0 is a constant. For instance, $\rho_0 = 1035\,\mathrm{Kg.m}^{-3}$ for the ocean ([10, 21]).

We divide equation (2.106) by ρ_0, and we still denote by p the ratio p/ρ_0, which becomes a "density of pressure per unit of mass and volume." Consistent with usual practice, we still call this new variable "the pressure."

We denote by $\mathbf{f}$ the ratio $\mathbf{f}_{ext}/\rho_0$, still called the "external forcing." Finally, we put

$$\nu = \frac{\mu}{\rho_0}, \tag{2.107}$$

which defines the kinematic viscosity, a typical unit of which is the square meter per second ($m^2 s^{-1}$).

We combine the momentum equation and the mass conservation equation to obtain the main usual form of the incompressible Navier–Stokes equations (NSE in the remainder) while noting

$$\nabla \cdot (p\mathbf{I}) = \nabla p, \tag{2.108}$$

in assuming p to be of class C^1. We find

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} - \nabla \cdot (2\nu D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0. \end{cases} \tag{2.109}$$

The unknowns are the pressure term $p = p(t, \mathbf{x})$ and the velocity $\mathbf{v}(t, \mathbf{x})$. The external forcing $\mathbf{f} = \mathbf{f}(t, \mathbf{x})$ and the initial value $\mathbf{v}_0 = \mathbf{v}_0(\mathbf{x}) = \mathbf{v}(0, \mathbf{x})$ are given.

Note that the pressure is not a prognostic variable, and so knowledge of its initial value is not required.

2.6.4.2 The Nonlinear Term in Divergence Form

The i^{th} component of the vector $(\mathbf{v} \cdot \nabla)\,\mathbf{v}$ is $v_j \partial_j v_i$. Because of the incompressibility condition, we have

$$v_j \partial_j v_i = \partial_j (v_i v_j). \tag{2.110}$$

The term $\partial_j(v_i v_j)$ is the i^{th} component of the vector $\nabla \cdot (\mathbf{v} \otimes \mathbf{v})$, where $\mathbf{v} \otimes \mathbf{v}$ denotes the tensor $\mathbf{v} \otimes \mathbf{v} = (v_i v_j)_{1 \le i,j \le 3}$. This allows the NSE to be written as follows:

$$\begin{cases} \partial_t \mathbf{v} + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nabla \cdot (2\nu D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{2.111}$$

or equivalently

$$\begin{cases} \partial_t \mathbf{v} + \nabla \cdot (\mathbf{v} \otimes \mathbf{v} - 2\nu D\mathbf{v} + p\mathbf{I}) = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0. \end{cases} \tag{2.112}$$

This last form might be interesting, especially when $\mathbf{f}$ is a restoring force, $\mathbf{f} = \nabla \cdot V$, such as gravity. Hence, the NSE can be considered as a conservative law of the form

$$\begin{cases} \partial_t \mathbf{v} + \nabla \cdot P(\mathbf{v}, p) = 0, \\ \nabla \cdot \mathbf{v} = 0. \end{cases} \tag{2.113}$$

2.6.4.3 Form with the Vorticity

We note that

$$(\mathbf{v} \cdot \nabla)\, \mathbf{v} = \nabla \mathbf{v} \cdot \mathbf{v}, \tag{2.114}$$

where the r.h.s. above is the product of the matrix $\nabla \mathbf{v}$ by the vector $\mathbf{v}$ (see in [4]). Using the decomposition (2.52) combined with (2.71), we find

$$(\mathbf{v} \cdot \nabla)\, \mathbf{v} = D\mathbf{v} \cdot \mathbf{v} + \frac{1}{2}\boldsymbol{\omega} \times \mathbf{v}, \tag{2.115}$$

leading to

$$(\mathbf{v} \cdot \nabla)\, \mathbf{v} = \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) + \boldsymbol{\omega} \times \mathbf{v}. \tag{2.116}$$

The NSE then take the form

$$\begin{cases} \partial_t \mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} - \nabla \cdot (2\nu D\mathbf{v}) + \nabla \left(p + \dfrac{|\mathbf{v}|^2}{2} \right) = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0. \end{cases} \tag{2.117}$$

2.6.4.4 Rotating Fluids

Up to now, we have calculated the acceleration of a particle using formulas (2.102) and (2.103), which require the coordinate system to be Galilean.

In the case of "rotating fluids" such as the atmosphere and the ocean, the acceleration is computed in a local system that turns with the earth, with an angular velocity $\boldsymbol{\Omega}$. Then we have ([10, 15, 21])

$$\boldsymbol{\gamma} = \frac{D\mathbf{v}}{Dt} - 2\boldsymbol{\Omega} \times \mathbf{v}.$$

Therefore, for such flows the NSE become

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} - 2\boldsymbol{\Omega} \times \mathbf{v} - \nabla \cdot (2\nu D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{2.118}$$

which is the form primarily used to model the motion of the ocean. The term $-2\boldsymbol{\Omega} \times \mathbf{v}$ is commonly considered as a force, because any observer in a rotating reference frame feels an eastward deflection, which is of prime importance in meteorology and oceanography.

Although this effect has been known since Galileo, it is called the "Coriolis Force," because of G. Coriolis who formalized it in 1835 [6].

2.6.4.5 Case of a Constant Viscosity

We consider an adiabatic flow, whose viscosity ν remains constant. For the record:

- $\nu = 1.006.10^{-6}\,\mathrm{m^2 s^{-1}}$ for the water at 20° C.
- $\nu = 15.6.10^{-6}\,\mathrm{m^2 s^{-1}}$ for the air at 25° C.

In such case, we have

$$\nabla \cdot (2\nu D\mathbf{v}) = \nu \nabla \cdot (D\mathbf{v}) = \nu(\Delta \mathbf{v} + \nabla(\nabla \cdot \mathbf{v})) = \nu \Delta \mathbf{v}, \tag{2.119}$$

because $D\mathbf{v} = (1/2)(\nabla \mathbf{v} + \nabla \mathbf{v}^t)$ with $\nabla \cdot \mathbf{v} = 0$. Hence, the NSE become

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0. \end{cases} \tag{2.120}$$

2.6.5 Equations for the Vorticity and the Pressure

Throughout this section, we assume that the field $\mathbf{v}$ is of class C^3 and p is of class C^2, with respect to t and $\mathbf{x}$, while the source term $\mathbf{f}$ is of class C^1. This regularity assumption is being made to justify the formal calculus carried out in this section. We also assume that the viscosity ν is constant for simplicity.

2.6.5.1 Vorticity Equation

To find the equation satisfied by $\boldsymbol{\omega}$, we take the curl of the NSE in its form (2.117) when applying (2.119). We check each term carefully.

On one hand, we observe that for any scalar field E, $\nabla \times \nabla E = 0$. On the other hand, the formula (2.74) gives the general rule

$$\nabla \times (\mathbf{E} \times \mathbf{F}) = \nabla \cdot (\mathbf{E} \otimes \mathbf{F}) - \nabla \cdot (\mathbf{F} \otimes \mathbf{E}) = (\mathbf{E} \cdot \nabla)\, \mathbf{F} - (\mathbf{F} \cdot \nabla)\, \mathbf{E} \tag{2.121}$$

satisfied by any vector field $\mathbf{E}$ and $\mathbf{F}$ with free divergence. Due to the regularity assumption, we can write

$$\nabla \cdot \boldsymbol{\omega} = \varepsilon_{ijk} \partial_i \partial_j u_k = -\varepsilon_{jik} \partial_j \partial_i u_k = -\nabla \cdot \boldsymbol{\omega} = 0, \tag{2.122}$$

by using the antisymmetry of the Levy-Civita tensor. Hence, the general rule (2.121) applies to $\mathbf{v}$ and $\boldsymbol{\omega}$. Furthermore, regularity allows the Schwarz theorem to be applied:

$$\nabla \times \Delta \mathbf{v} = \Delta(\nabla \times \mathbf{v}) = \Delta \boldsymbol{\omega}, \quad \nabla \times \partial_t \mathbf{v} = \partial_t (\nabla \times \mathbf{v}) = \partial_t \boldsymbol{\omega}.$$

Accordingly, by taking the curl of (2.117) with ν constant, we find

$$\partial_t \boldsymbol{\omega} + (\mathbf{v} \cdot \nabla)\, \boldsymbol{\omega} - \nu \Delta \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\, \mathbf{v} + \nabla \times \mathbf{f}. \tag{2.123}$$

The term $(\boldsymbol{\omega} \cdot \nabla)\, \mathbf{v}$ is called the vortex-stretching term. It is worth noting that in the two-dimensional case, where things are simplest, this term does not appear in the vorticity equation. This may be of relevance in the study of stratified flows such as large-scale motions in the ocean or in the atmosphere, for example, cyclones and anticyclones, which present some two-dimensional structure.

2.6.5.2 Pressure Equation

We take the divergence of the NSE in its form (2.120) and study each term separately. We have

$$\nabla \cdot (\nabla p) = \partial_i (\partial_i p) = \Delta p. \tag{2.124}$$

Applying the Schwarz theorem together with the incompressibility assumption, we obtain

$$\nabla \cdot \partial_t \mathbf{v} = \partial_t (\nabla \cdot \mathbf{v}) = 0, \quad \nabla \cdot (\Delta \mathbf{v}) = \Delta(\nabla \cdot \mathbf{v}) = 0,$$

which yields

$$\Delta p = \nabla \cdot ((\mathbf{v} \cdot \nabla)\, \mathbf{v}) + \nabla \cdot \mathbf{f}. \tag{2.125}$$

Furthermore, note that

$$\nabla \cdot ((\mathbf{v} \cdot \nabla)\, \mathbf{v}) = \partial_i v_j \partial_j v_i = \nabla \mathbf{v} : \nabla \mathbf{v}^{\mathrm{t}}.$$

Equation (2.125) therefore takes the form

$$\Delta p = \nabla \mathbf{v} : \nabla \mathbf{v}^{\mathrm{t}} + \nabla \cdot \mathbf{f}. \tag{2.126}$$

2.7 Boundary Conditions

We derived the NSE from mathematical principles combined with experimental observations. The equations are based on conservation, dynamics, and dissipation principles, which are of course essential features of the flow. However, emphasis must also be given to the role played by the boundary conditions, which are crucial in order to provide a full mathematical description of the flow, which is our main aim.

The boundary conditions describe macroscopic as well as microscopic effects that can be considered as engines of the motion. For example, movements in the air and the sea are essentially due to the heating by the sun, which supplies energy to the sea/air system. This energy is converted into kinetic energy and dissipation. Moreover, the air and sea exchange energy all the time, some of which is dissipated during the transaction.

The energy process sketched above and many others are described through boundary conditions (BC in what follows). They are often hard to model with mathematics, and there are many possible ways of describing the same thing. The choice may vary depending on the specific case under study. However, boundary conditions may also be suggested—and even imposed—by numerical or purely mathematical constraints. Moreover, some boundary conditions are simply mathematical artifacts but relevant for a better understanding of the local nature of the NSE.

In this section, we examine the following BC: periodic BC, the case of a full space, no-slip BC, Navier BC, friction BC, and air/sea interface.

2.7.1 Periodic Boundary Conditions

The periodic BC are certainly the least physical ones of all, but they still remain very popular because they have the great advantage that Fourier analysis can be used to study the NSE, especially when ν is constant. This helps in getting a better understanding of the interaction between small and large scales and the balance between the convection term $(\mathbf{v} \cdot \nabla)\, \mathbf{v}$ and the diffusion term $\nu \Delta \mathbf{v}$ in the NSE, either in the form (2.120) or in the form (2.111).

Let $[0, L]^3$ be a given box, for some $L > 0$. The "set of wave vectors" is defined as the quotient set

$$\mathscr{T}_3 = \frac{2\pi \mathbb{Z}^3}{L}.$$

The domain of study is the torus

$$\mathbb{T}_3 = \frac{[0, L]^3}{\mathscr{T}_3}, \tag{2.127}$$

within which the velocity $\mathbf{v}$ and the pressure p can both be decomposed into Fourier series,

$$\mathbf{v}(t, \mathbf{x}) = \sum_{\mathbf{k} \in \mathscr{T}_3} \hat{\mathbf{v}}_{\mathbf{k}}(t) e^{i \mathbf{k} \cdot \mathbf{x}}, \quad p(t, \mathbf{x}) = \sum_{\mathbf{k} \in \mathscr{T}_3} p_{\mathbf{k}}(t) e^{i \mathbf{k} \cdot \mathbf{x}}, \quad (t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{T}_3. \tag{2.128}$$

2.7.2 The Full Space

In this case, the flow domain is $\mathbb{R}^3$. It is assumed that the fluid is at rest at infinity, which is not so unreasonable. Rather than forcing $\mathbf{v}$ to be zero at infinity, we impose the integrability condition

$$\forall\, t \in \mathbb{R}_+ \quad \mathbf{v}(t, \cdot) \in L^2(\mathbb{R}^3), \tag{2.129}$$

We emphasize that we do not require p to satisfy any boundary condition.

Remark 2.4. Leray [18] and Oseen [25], who pioneered the mathematical analysis of the NSE, considered this type of BC, with ν constant. However, we impose $\mathbf{v}(t, \cdot) \in L^2(\mathbb{R}^3)$ at all times because of the continuity assumption 2.2 that holds for local time solutions such as those studied by C. Oseen. J. Leray obtained a global time solution to the NSE in this case, which he called a "turbulent solution" (see also Sect. 3.4.2 in Chap. 3), but we do not know if this satisfies assumption 2.2 or not, when $\mathbf{v}_0 \in L^2(\mathbb{R}^3)$ is continuous on $\mathbb{R}^3$. Therefore, the right BC should be "at almost all $t \in \mathbb{R}_+$, $\mathbf{v}(t, \cdot) \in L^2(\mathbb{R}^3)$" as introduced in [18]. The same applies to the other BC below, where "at all" should be replaced by "at almost all."

2.7.3 No-Slip Condition

Let Ω denote the flow domain with a boundary Γ. The three typical cases are:

1. Ω is a half space and Γ is a plane.
2. $\Omega = \mathbb{R}^3 \setminus V$, where V is a bounded smooth set in $\mathbb{R}^3$ and $\Gamma = \partial V$.
3. Ω is a bounded domain in $\mathbb{R}^3$.

In case 1, the plane Γ may be the fixed bottom of an infinite ocean, which has meaning for an observer very deep in the sea.

Case 2 models a body moving in a fluid, such as a plane flying in the air, a fishing net being pulled in the ocean, and many similar examples, in particular the sphere which is the most studied since the initial work by Stokes [28]. The body's velocity is known and denoted by $\mathbf{U}$. We aim to describe the flow structure around the body and to calculate the constrains exerted on it by the fluid. It is more convenient to consider that the body is at rest and that the velocity of the fluid is equal to $-\mathbf{U}$ at infinity.

Case 3 models a flow in a closed cavity, such as fuel in an engine.

The "no-slip condition" is of the form

$$\forall\, (t, \mathbf{x}) \in \mathbb{R}_+ \times \Gamma, \quad \mathbf{v}(t, \mathbf{x}) = 0, \tag{2.130}$$

or more simply

$$\mathbf{v}|_\Gamma = 0. \tag{2.131}$$

Here too, no special condition on the pressure is required at Γ. We sometimes say the "homogeneous Dirichlet BC" instead of "no-slip condition," in line with the terms used in the study of partial differential equations.

The argument that yields the condition (2.131) is based on a microscale observation. Indeed, even if a physical surface Γ may seem very smooth at a macroscale, a closer examination at a microscale reveals many irregularities, which are however very large in comparison with the scale of the fluid at which the NSE hold. Hence, the fluid particles are stuck in the surface's irregularities, leading to the no-slip condition.

2.7.4 Navier Boundary Condition

Although the no-slip condition has been popular for a long time, it has also been very controversial. We may imagine that the fluid slips on the boundary while considering the possibility of friction, for example, a body experiencing drag when it moves in the fluid. The Navier condition represents a balance between slip and friction.

Let $\mathbf{w} = \mathbf{w}(\mathbf{x})$ be any vector field defined on Γ. We introduce the tangential part of $\mathbf{w}(\mathbf{x})$ at $\mathbf{x} \in \Gamma$, denoted by $\mathbf{w}_\tau(\mathbf{x})$:

$$\mathbf{w}_\tau(\mathbf{x}) = \mathbf{w}(\mathbf{x}) - (\mathbf{w}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}))\, \mathbf{n}(\mathbf{x}). \tag{2.132}$$

Let $t \in \mathbb{R}_+$ be a fixed time, $\mathbf{x} \in \Gamma$. We note $\mathbf{n} = \mathbf{n}(x)$, $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ for simplicity. We assume that Γ is not porous, such that no fluid particle crosses Γ, which means $\mathbf{v} = \mathbf{v}_\tau$ almost everywhere on Γ or in other words

$$\mathbf{v} \cdot \mathbf{n}|_\Gamma = 0. \tag{2.133}$$

The calculation of the friction at the boundary is based on the principles introduced in Sect. 2.6.2. Taking the view that the force applied by the fluid on Γ is equal to $\boldsymbol{\sigma} \cdot \mathbf{n}$, we take as friction the corresponding tangential part denoted by $(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau$.

The Navier-slip condition is based on the observation that the fluid is slowed by the frictional force at Γ, resulting in the relation

$$\mathbf{v}_\tau = \mathbf{v} = -\alpha(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau, \tag{2.134}$$

for some $\alpha > 0$. In conclusion, Navier BC are

$$\mathbf{v} \cdot \mathbf{n}|_\Gamma = 0, \quad (\mathbf{v} + \alpha(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau)|_\Gamma = 0, \quad \alpha > 0. \tag{2.135}$$

Note that when α goes to zero, the Navier condition (2.135) converges to the no-slip condition (2.131), at least formally. When α goes to infinity, we find $(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau|_\Gamma = 0$, which is the total slip condition, which only holds in the case of a perfect fluid.

2.7.5 Friction Law

We show in this subsection another way of computing the force exerted by the fluid on a given body V moving with a constant velocity $\mathbf{U}$. Equating the result with $\sigma \cdot \mathbf{n}$ yields another type of BC.

Let G be the center of gravity of V and S its effective area. Assume that at time t, G sits on $\mathbf{x}$. At time $t + \delta t$, G sits on $\mathbf{x} + \mathbf{U}\delta t$. Therefore, the total volume of fluid displaced is equal to $S|\mathbf{U}|\delta t$, the mass of which is equal to $\delta m = \rho S|\mathbf{U}|\delta t$. The momentum carried by the sphere, denoted by $\delta \mathbf{p}_S$, is equal to

$$\delta \mathbf{p}_S = \mathbf{U}\delta m = \rho S\mathbf{U}|\mathbf{U}|\delta t. \tag{2.136}$$

The fluid slides with friction on the body. This suggests that only one part of the momentum of the sphere is transmitted to the fluid. Therefore, the momentum of the displaced fluid is equal to

$$\delta \mathbf{p} = C\rho S\mathbf{U}|\mathbf{U}|\delta t, \tag{2.137}$$

where $C \in]0, 1[$ is a constant that is determined by experiment. Therefore, the force applied on the body by the fluid is equal to

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{p}}{\delta t} = \boldsymbol{\sigma} \cdot n = C\rho S\mathbf{U}|\mathbf{U}|. \tag{2.138}$$

This law was firstly stated by Gauckler [9], then redeveloped by Manning [23], and therefore is often called the Gauckler–Manning law. Engineers also call it the Plotter–Landweber law [16], depending on the context in which they use it. Anyway, this law is in agreement with experiments and is used in numerical simulations (see in [17], for instance).

A natural general BC based on (2.138), which is used, for instance, in the modelization of the ocean–atmosphere interface considered in the next subsection, is

$$\mathbf{v} \cdot \mathbf{n}|_{\Gamma} = 0, \quad (\boldsymbol{\sigma} \cdot n)_{\tau}|_{\Gamma} = C(\mathbf{U}_0 - \mathbf{v}_{\tau})|\mathbf{U}_0 - \mathbf{v}_{\tau}|, \tag{2.139}$$

for some given $\mathbf{U}_0$. Moreover, we shall derive the same law from the turbulence modeling process carried out in Sect. 5.3 in Chap. 5. In this case, we call it the wall law, to which we shall pay attention from Chap. 6.

2.7.6 Ocean–Atmosphere Interface

We conclude this section with the ocean–atmosphere coupling. The usual assumption, known as the rigid lid assumption, is that the interface between the ocean and the atmosphere is a fixed surface, denoted by Γ.

Although this assumption is not very realistic, it is commonly used. Indeed, many highly complicated physical effects occur at the mixing layer between both media. Because of this complexity, we prefer to replace the physical mixing layer by an averaged thin layer called the rigid lid, especially when considering large scales. The energy processes between air and water are then modeled through suitable boundary conditions.

The processes involved in the air/sea coupling are dynamic as well as thermodynamic. We will only briefly outline the dynamic part in this subsection. The BC that we obtain is based on the law (2.139), considering friction between air and water.

For simplicity, we sit on a local earth coordinate frame. Let $\mathbf{k}$ be the vertical unit vector and $(\mathbf{i}, \mathbf{j})$ the unit vectors spanning Γ viewed as a plane in $\mathbb{R}^3$. The coordinates are denoted by (x, y, z).

Note that $\mathbf{k}$ is the outward-pointing unit normal vector $\mathbf{n}^w$ of the ocean at Γ, while $-\mathbf{k}$ is the outward-pointing unit normal vector $\mathbf{n}^a$ of the atmosphere at Γ.

Let $\mathbf{v}^w$ and $\mathbf{v}^a$ denote the water velocity and the air velocity, respectively. We split these into a horizontal part and a vertical part:

$$\mathbf{v}^w = (\mathbf{v}_h^w, w^w), \quad \mathbf{v}^a = (\mathbf{v}_h^a, w^a), \quad \mathbf{v}_h^w = (u^w, v^w), \quad \mathbf{v}_h^a = (u^a, v^a). \tag{2.140}$$

The rigid lid assumption yields

$$w^w|_\Gamma = w^a|_\Gamma = 0. \tag{2.141}$$

This is consistent with the condition (2.133) above, since

$$w^w|_\Gamma = \mathbf{v}^w \cdot \mathbf{n}^w, \quad w^a|_\Gamma = \mathbf{v}^a \cdot \mathbf{n}^a.$$

For the moment, we focus on the ocean. We shall adapt the condition (2.139), where we have to take into account the relative velocity at Γ equal to $\mathbf{v}^a - \mathbf{v}^w$. Note that $|\mathbf{v}^a - \mathbf{v}^w| = |\mathbf{v}_h^a - \mathbf{v}_h^w|$ at Γ due to (2.141).

Let us compute $\boldsymbol{\sigma}^w \cdot \mathbf{n}^w = \boldsymbol{\sigma}^w \cdot \mathbf{k}$, where $\boldsymbol{\sigma}^w = 2\mu_w D(\mathbf{v}^w) - p^w \mathbf{I}$. Applying the basic definitions yields

$$\boldsymbol{\sigma}^w \cdot \mathbf{k} = \begin{pmatrix} \partial_z u^w + \partial_x w^w \\ \partial_z v^w + \partial_y w^w \\ 2\partial_z w^w + p \end{pmatrix}. \tag{2.142}$$

We consider just the two first components of $\boldsymbol{\sigma}^w \cdot \mathbf{k}$. From (2.139), we find that on Γ

$$2\mu_w \frac{\partial \mathbf{v}_h^w}{\partial z} = C(\mathbf{v}^a - \mathbf{v}^w)|\mathbf{v}^a - \mathbf{v}^w|. \tag{2.143}$$

The same analysis holds for the atmosphere by the action and reaction principle, by using $\mathbf{v}^w - \mathbf{v}^a$ instead of $\mathbf{v}^a - \mathbf{v}^w$. Notice that the third component in the relation (2.142) is useless.

In summary, the BC on Γ is given by (2.141), together with

$$2\mu_w \frac{\partial \mathbf{v}_h^w}{\partial z} = C_1(\mathbf{v}^a - \mathbf{v}^w)|\mathbf{v}^a - \mathbf{v}^w|, \quad 2\mu_a \frac{\partial \mathbf{v}_h^a}{\partial z} = C_2(\mathbf{v}^w - \mathbf{v}^a)|\mathbf{v}^w - \mathbf{v}^a|, \tag{2.144}$$

where C_1 and C_2 are two constants that must be fixed from observations [10].

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