Chapter 2
A Short Tutorial on Game Theory

Game theory [23, 25, 44–46] is a collection of analytical tools designed to study a system of self-interested decision-makers in conditions of strategic interaction. This section briefly reviews important game-theoretic solution concepts.

2.1 Strategic Games

2.1.1 Definition of Strategic Game

In the model of strategic game, there is a finite set of players $\mathbb{N} = \{1, 2, \ldots, n\}$ and, for each player $i \in \mathbb{N}$, a nonempty set Σ_i of all possible (mixed) strategies and a preference relation.

A strategy is a complete contingent plan, or decision rule, that defines the action a player will select in every distinguishable state of the game. A strategy can be either pure (deterministic), or mixed (stochastic). A mixed strategy $\sigma_i \in \Delta(\Sigma_i)$ defines a probability distribution over pure strategies. The set of strategy profiles is $\Sigma = \times_{i \in \mathbb{N}} \Sigma_i$. Each player i chooses a strategy $s_i \in \Sigma_i$. As a notational convention, $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ represents the strategies of all players except player i. Note that $\mathbf{s} = (s_i, s_{-i})$ is a strategy profile, in which player i takes strategy s_i and the other players take strategies s_{-i}.

A player i's preferences can be determined by a utility function $u_i(s)$ of the strategies of all the player. The utility function captures the essential concept of strategic interdependence. In other words, the utility, $u_i(s)$, of player i determines its preferences over its own strategy and the strategies of other players. Player i prefers strategy s_i to s'_i when the other players take s_{-i}, if $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

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To summarize, the definition of the strategic game is the following.

Definition 2.1 (Strategic Game). A strategic game consists of

- a finite set $\mathbb{N}$ of players;
- for each player $i \in \mathbb{N}$ a nonempty set Σ_i of all possible (mixed) strategies;
- for each player $i \in \mathbb{N}$ a utility function $u_i(s)$, which determines the player's preference.

The strategic game models players' rationality in maximizing their individual expected utility. A player will always selects a strategy that maximized its expected utility, given its belief of the others players' strategies.

2.1.2 Nash Equilibrium

The most commonly used solution concept in game theory is *Nash Equilibrium* (NE) [46], which captures a steady state of the strategic game in which each player holds the correct expectation about the other players' behavior and acts rationally.

Definition 2.2 (Nash Equilibrium). A Nash Equilibrium of a strategic game is a profile $s^\star \in \Sigma$ of strategies with the property that for every player $i \in \mathbb{N}$ we have

$$u_i(s_i^\star, s_{-i}^\star) \geq u_i(s_i, s_{-i}^\star), \tag{2.1}$$

for all $s_i \in \Sigma_i$.

Intuitively, for any $s^\star$ to be a Nash Equilibrium, it must be that no player i has an action other than $s_i^\star$ yielding an outcome that is more beneficial to the player, given that every other player j chooses her equilibrium strategy $s_j^\star$. In other words, no player can get more benefit by unilaterally deviating from the Nash Equilibrium.

2.1.3 Dominant Strategy

Although the Nash Equilibrium gives a fundamental solution concept to game theory, it relies on knowing all the other players' strategies and beliefs on the other players, and also loses power in the games where multiple NEs exist. We now introduce a very strong solution concept called *dominant strategy* [46], in which every player has the utility-maximizing strategy for all strategies of the other players.

Definition 2.3 (Dominant Strategy). A dominant strategy of a player is one that maximizes its utility regardless of what strategies other players choose. Specifically, $s\star_i$ is player i's dominant strategy if, for any $s_i' \neq s\star_i$ and any s_{-i},

$$u_i(s\star_i, s_{-i}) \geq u_i(s_i', s_{-i}). \tag{2.2}$$

In other words, a strategy $s\star_i$ is a dominant strategy for a player if it maximizes her expected utility, no matter what strategies the other players take.

Dominant-strategy equilibrium is a very robust solution concept, because it makes no assumptions about the information available to players about each other, and does not require an player to believe that other players will behave rationally to select its own optimal strategy.

People further strengthen the requirement for dominant strategy equilibrium to have a stronger solution concept called *Strongly Dominant Strategy Equilibrium* (SDSE):

Definition 2.4 (Strongly Dominant Strategy Equilibrium). A Strongly Dominant Strategy Equilibrium of a strategic game is a profile $s^\star \in \Sigma$ of strategies with the property that for every player $i \in \mathbb{N}$,

$$\begin{cases} \forall s_{-i} \in \Sigma_{-i}, \forall s_i' \neq s_i^\star,\ u_i(s_i^\star, s_{-i}) \geq u_i(s_i', s_{-i}) \\ \exists s_{-i} \in \Sigma_{-i}, \forall s_i' \neq s_i^\star,\ u_i(s_i^\star, s_{-i}) > u_i(s_i', s_{-i}). \end{cases} \tag{2.3}$$

2.2 Mechanism Design

The mechanism design aims to implement an optimal system-wide solution to a decentralized optimization problem with self-interested players who have private information about their preferences for different outcomes. We call the private information of a player i as *type*, denoted by t_i.

A mechanism consists of the strategies available to the players and the method used to select the final outcome based on players' strategies. The mechanism design problem is to implement the rules of a game. Game theory is usually used to analyze the outcome of a mechanism.

A *direct-revelation* mechanism is a mechanism in which the only actions available to players are to make claims about their preferences to the mechanism. That is, the strategy of player i is reporting type $\hat{t_i} = s_i(t_i)$, based on its actual preferences t_i.

A direct-revelation mechanism is *incentive-compatible* (IC) if reporting truthful information is a dominant strategy for each player. Another important property of a mechanism is *individual-rationality* (IR)—each player can always achieve at least as much expected utility from participation as without participation. Finally, we say a direct-revelation mechanism is *strategy-proof* if it satisfies both IC and IR properties.

Definition 2.5 (Strategy-Proof Mechanism [40, 56]). A mechanism is strategy-proof when it satisfies both incentive-compatibility and individual-rationality.

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