Chapter 2
Function Spaces and Reconstruction

A reconstruction problem is solved when we are able to find a function that models or describes the behavior of data. Although in each problem there will be a particular method to obtain or define this function, many of them are included in the theory of function spaces. In this chapter we will see the alphabet that permit us to understand the language for the rest of the book. The basic idea is to consider functions as simple points that behaves with the properties of Euclidean space, so we begin by generalizing the ideas of distance and orthogonality of three dimensional space to norms and inner products in function spaces. Next we study integration by parts and its application to define distributions and Sobolev spaces. Distribution theory builds a bridge between discrete and continuous processes by extending the concept of differentiability of a function.

2.1 3D Reconstruction

In this book, our main goal is the reconstruction of the form of an object given a set of three dimensional data sampled from a real object. In general we are dealing with the problem of the complete reconstruction of a multivariate function $f:\mathbb{R}^n \to \mathbb{R}$ from sampling values $\{f(x_i)\}_{i=1}^N$.

For the one-variable case a possible answer to this problem is the well-known Shannon sampling theorem [34]. A band limited function f with frequency spectrum limited by the Nyquist frequency, can be reconstructed perfectly from its regularly-spaced (ideal) samples:

Theorem1 (Shannon) Suppose $f \in \mathcal{L}^2$ and $\hat{f}(\xi) = 0$ for $|\xi| \geq \Omega$. Then f is completely determined by its values at the points $t_k = k\pi/\Omega$, $k = 0, \pm 1, \pm 2, \ldots$ by means of

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\Omega}\right) \frac{\sin(\Omega t - k\pi)}{\Omega t - k\pi}.$$

This expression shows that $f(t)$ can be written as translates or convolution with a single function. In the next chapters, we will show that the same situation may happen for functions on $\mathbb{R}^n$. In these cases, it is necessary to deal with multivariable functions and non-uniform sampling in which the location of measurement points

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may be irregular, either because it is not possible to control the measurement process or because some domain needs a particular emphasis. During the last years, this problem has been also widely studied and there exist a good number of results that share some of the features of Shannon's sampling theorem.

This can be generalized by changing the sinc function to an appropriate generating function $\phi(x)$ that results in spaces of the form $\left\{\sum_{k\in\mathbb{Z}} c(k)\phi(x-k)\,|\,c\in\ell^2\right\}$, where $\phi(x)$ does not have to be band-limited [66]. Extensions to the multidimensional irregular sampling of band-limited functions in very general spaces can be found in [34].

Our point of view to the problem will be to consider 3D reconstruction as an inverse problem that can be solved by variational regularization. Well-posedness of an inverse problem depends on the topological properties, therefore it is very important the choice of function spaces in which the optimization problems are going to be formulated and solved. We apply distribution theory as a fundamental tool for modeling and understanding reconstruction problems. Distributional spaces provides an abstract setting for including discrete and continuous function in the same framework. In this way it is possible to obtain explicit expressions for a family of interpolating and smoothing splines. This family includes the well-known cubic spline in one variable and the thin plate spline (TPS).

2.2 Function Spaces and Norms

Function spaces are vector spaces whose elements are functions. The collection of real valued functions u, v on a nonempty set Ω forms a real linear space (or vector space) with respect to the operations of pointwise addition: $(u+v)(x) = u(x)+v(x)$, $\forall x\in\Omega$ and scalar multiplication: $(\alpha u)(x) = \alpha u(x)\ \forall x\in\Omega\ \alpha\in\mathbb{R}$. If Ω is a nonempty open set in $\mathbb{R}^n$, an important example is $C^m(\Omega)$, the vector space of all real functions $f:\Omega\to\mathbb{R}$ that have continuous partial derivatives of orders 0,1,…,m. If $f\in C^m(\Omega)$ for all $m = 0,1,2,\ldots$, then we write $f\in C^\infty(\Omega)$.

Definition 1 If $\mathcal{U}$ is a vector space a **norm** on $\mathcal{U}$ is a real valued function, denoted by $\|\cdot\|$, that satisfy three axioms

(i) $\|u\|>0$ for each nonzero element u in $\mathcal{U}$.
(ii) $\|\alpha u\| = |\alpha|\,\|u\|$ for each $\alpha\in\mathbb{R}$ and each $u\in\mathcal{U}$.
(iii) $\|u+v\|\le\|u\|+\|v\|$ for all $u, v\in\mathcal{U}$. (Triangle Inequality)

A linear space $\mathcal{U}$ is a Banach space when is complete in the norm $\|\bullet\|$. The completeness condition means that every Cauchy sequence in the space converge to an element of the space.

Definition 2 We say a linear space $\mathcal{U}$ has an inner product if there is a symmetric bilinear form $B(u,v) = \langle u, v\rangle$, $\langle\bullet,\bullet\rangle : \mathcal{U}\times\mathcal{U}\to\mathbb{R}$ with the following properties for all u, v, w in $\mathcal{U}$

(i) $\langle u, v\rangle = \langle v, u\rangle$
(ii) $\langle u + v, w\rangle = \langle u.w\rangle + \langle v, w\rangle$
(iii) $\langle \alpha u, v\rangle = \alpha\langle v, u\rangle$
(iv) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ if and only if $u = 0$ (positive definiteness)

Every inner product space $\mathcal{U}$ is a normed space with the norm $\| u \| = \sqrt{\langle u, u\rangle}$ of u. If $\mathcal{U}$ is a complete inner product space is called a *Hilbert Space*. As an important example, the space of real-valued continuous functions on $[0, 1]$ with the inner product

$$\langle u, v\rangle = \int_0^1 u(t)v(t)dt$$

is not complete (see [10] for a proof). Contrary, the set $\mathcal{L}^2[a, b]$ is a Hilbert space. This is the space of all complex-valued Lebesgue measurable functions on $[a, b]$ such that $\int_a^b |u(t)|^2 \, dt < \infty$. In $\mathcal{L}^2[a, b]$ the inner product is

$$\langle u, v\rangle = \int_a^b u(t)\overline{v(t)}dt$$

In an inner product space two vectors u, v are said to be orthogonal if $(u, v) = 0$. We symbolize this by $u \perp v$. A vector u is said to be orthogonal to a set $\mathcal{V}$ if $u \perp v$ for all $v \in \mathcal{V}$, and all the vectors u with this property form the orthogonal complement $\mathcal{V}^{\perp}$ of $\mathcal{V}$. The idea of orthogonality has many consequences in Hilbert spaces, similar to Euclidian geometry.

Definition 3 A real valued function $p(u) = |u|_{\mathcal{U}}$ defined on a linear space $\mathcal{U}$ is called a **seminorm** on $\mathcal{U}$, if the following conditions hold:

i. $p(u + v) \leq p(u) + p(v)$

ii. $p(\alpha u) = |\alpha| \, p(u)$

by this definition, every norm is a seminorm that can be seen as a norm with the requirement of positive definiteness removed. This condition is also removed from an inner product to obtain a **semi-inner product,** with corresponding seminorm $|u|_{\mathcal{U}}^2 = (u, u)$. This means that if $|u|_{\mathcal{U}} = 0$ it may happens $u \neq 0$. As a consequence the null space $\mathcal{N}$ of $|u|_{\mathcal{U}}$ is defined as

$$\mathcal{N} = \{u \in \mathcal{U} : |u|_{\mathcal{U}} = 0\}$$

Every vector space $\mathcal{U}$ with seminorm $|u|_{\mathcal{U}}$ induces a normed space $\mathcal{U}/\mathcal{N}$, called the quotient space. The induced norm on $\mathcal{U}/\mathcal{N}$ is clearly well-defined and is given by:

$$|u + \mathcal{N}|_{\mathcal{U}} = |u|_{\mathcal{U}}$$

Example 1 The thin plate energy

$$J[u] = \int_{\mathbb{R}^2} (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)dydx$$

is an example of seminorm which is not a norm, its null space is $\mathcal{N} = \Pi_1(\mathbb{R}^2)$, the set of polynomials in two variables of degree less or equal than 1, thus $\Pi_1(\mathbb{R}^2) = span\{1, x, y\}$. As we will see in the next chapters, some variational problems of continuum mechanics and spline theory have to deal with optimization of seminorms.

2.3 Operators on Function Spaces

We say that two sets $\mathcal{U}$, $\mathcal{V}$ are connected by a functional dependency $A : \mathcal{U} \to \mathcal{V}$ if to each element $u \in \mathcal{U}$ there corresponds a unique element $v \in \mathcal{V}$. Roughly speaking, this functional dependency is called a *function* if the sets $\mathcal{U}$, $\mathcal{V}$ are sets of numbers; it is called a *functional* if $\mathcal{U}$ is a set of functions and $\mathcal{V}$ is a set of numbers, and it is called an *operator* if both sets are sets of functions [73].

Definition 4 An operator is a mapping $A : \mathcal{U} \to \mathcal{V}$ from one function space into another (or the same) function space. If $\mathcal{U}$ and $\mathcal{V}$ are linear spaces, then the mapping is **linear** if

$$A(u+v) = A(u) + A(v) \text{ and } A(\alpha u) = \alpha A(u)$$

For all u, v in $\mathcal{U}$ and all real α. When an operator does not fulfill these conditions then it is called **nonlinear**

Example 2

1. Taking $\mathcal{U} = \mathcal{V} = \mathbb{R}^n$, any linear transformation represented by a $n \times n$ matrix is a linear operator
2. The Laplace operator $\Delta : C^2(\Omega) \to C(\Omega)$, $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ with $\Omega \subset \mathbb{R}^2$, is a linear differential operator on the space of real valued functions with continuous second order derivatives on a region Ω in the plane into continuous functions in Ω.
3. For $\mathcal{U} = \mathcal{V} = C[0,1]$ and K continuous on $[0,1] \times [0,1]$ the mapping

$$A : C[0,1] \to C[0,1] \text{ with } A(u)(t) = \int_0^1 K(\xi, t) u(\xi) d\xi$$

It is a linear integral operator.

Definition 5 If $\mathcal{U}$ and $\mathcal{V}$ are normed linear spaces, an operator $A : \mathcal{U} \to \mathcal{V}$ is said to be continuous at $u \in \mathcal{U}$ if for every sequence $\{u_k\} \subset \mathcal{U}$ converging to u

$$\lim_{k \to \infty} \| A(u_k) - A(u) \| = 0$$

Definition 6 A linear operator $A : \mathcal{U} \to \mathcal{V}$ is called bounded when there exists a real number B such that $\| A(u) \|_{\mathcal{V}} \leq B \| u \|_{\mathcal{U}}$ for all u in $\mathcal{U}$. The **norm** of a bounded

linear operator is defined as $\| A \| = \sup_{\|u\|=1} \| A(u) \|_{\mathcal{V}}$. It can be shown that a linear transformation acting between normed linear spaces is continuous if and only if it is bounded [8].

Definition 7 The *null space* $\mathcal{N}(A)$ of a linear operator $A : \mathcal{U} \to \mathcal{V}$ is the subspace of $\mathcal{U}$ given by

$$\mathcal{N}(A) = \{u \in \mathcal{U} : Au = 0\},$$

and the *range* $\mathcal{R}(A)$ of A is the subspace of $\mathcal{U}$ given by

$$\mathcal{R}(A) = \{v \in \mathcal{V} : Au = v, \exists u \in \mathcal{U}\}$$

Definition 8 A real valued mapping $J : \mathcal{U} \to \mathbb{R}$ defined on a normed linear space $\mathcal{U}$ is called a **functional**. If the mapping is linear, it is called a **linear functional**. A linear functional can be seen as a special case of a linear operator with $\mathcal{V} = \mathbb{R}$ then we can also speak about bounded functionals with norm $\| J \| = \sup_{\|u\|=1} | J(u) |$.

Example 3

1. A class of very important functionals for modeling data are the evaluation functionals $L_{x_0}(f)$ defined on a function space $\mathcal{U}$. If $f \in \mathcal{U}$ $f : \Omega \to \mathbb{R}$ and $x_0 \in \Omega$, then $L_{x_0}(f) := f(x_0)$
   If $\mathcal{U}$ is a linear space then L_{x_0} is a linear functional since

$$L_{x_0}(\alpha u + v) = (\alpha u + v)(x_0) = \alpha u(x_0) + v(x_0) = \alpha L_{x_0}(u) + L_{x_0}(v)$$

2. If $\mathcal{U}$ is the linear space of integrable functions on $[a,b]$, then

$$J(u) = \int_a^b u(t)dt$$

   is a linear functional on $\mathcal{U}$
3. The functional

$$J(u) = \int_a^b \sqrt{1 + u'(t)^2}\, dt$$

   is a nonlinear functional defined in $C^1[a,b]$ that gives the arc length of the curve u in $[a,b]$.

The set of bounded linear functionals on $\mathcal{U}$ forms a linear space itself $\mathcal{U}'$ called the **dual space** of $\mathcal{U}$. By the Riesz representation theorem, on a Hilbert space the bounded linear functionals have a very simple form.

Theorem 2 (Riesz Representation Theorem) Every continuous linear functional $J : \mathcal{U} \to \mathbb{R}$ defined on a Hilbert space is of the form $J(u) = \langle u, R \rangle$ for an appropriate vector R that is uniquely determined by the given functional [8].

2.4 Distributions

Distribution theory was created mainly by Sobolev and Schwartz [19, 20, 32, 59] to give answers to problems of mathematical physics. However, as usual, after being rigorously formulated in mathematical terms, the theory has developed very far from its initial applications and became useful in other disciplines such as approximation theory. In the following chapters we will show how distribution spaces are especially suited for dealing with inverse problems of 3D reconstruction, providing a variational framework that conducts to a generalization from classical cubic spline to multivariate interpolation and approximation. The results of this approach include the well-known thin plate spline and other radial basis functions.

Although surface reconstruction is an ill-posed inverse problem, distribution theory gives a setting to construct spaces where they become well-posed. Discontinuous functions can be handled as easily as continuous or differentiable functions into a unified framework, making it appropriate for dealing with discrete data. We will show how this approach may serve as a tool for the double task of modelling these data while providing solutions for its reconstruction.

Distributions generalize functions by considering a function as a continuous linear functional on the space $C_0^\infty(\mathbb{R}^n)$ of infinitely differentiable functions with compact support. This setting is adequate for introducing the concept of generalized differentiation, which makes possible the calculus of distributions with all its practical consequences. The **delta functional** $\delta(x)$ contradictorily defined by Dirac as $\int \delta(x)\varphi(x)dx = \varphi(0)$; is a generalized function. The spaces of distributions have shown to be very useful for multivariate approximation and spline theory [37–39].

One important task is to find or construct spaces where approximation problems can be well-posed. Nevertheless, classical spaces like $C^m[a,b]$ may not be adequate for this purpose. For example, there may exist a sequence $\{f_k\}$ of functions in $C^1(\mathbb{R})$ such that they converge uniformly to f, but $f \notin C^1(\mathbb{R})$. Other example is the non invertibility of the order of differentiation, so it may happen that $\frac{\partial f}{\partial x \partial y} \neq \frac{\partial f}{\partial y \partial x}$. Distribution theory provides more versatile viewpoint for treating the problems attached to the operation of differentiation.

Definition 9 A multi-index α is a n-tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$. The order of the multi-index α is the integer $|\alpha| = \sum_{k=1}^{n} \alpha_k$. Differential operators are defined using a multi-index $D^\alpha = \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, on n real variables $x_1, x_2, ..., x_n$. For example if $n = 3$ and $\alpha = (1,0,2)$, then $D^\alpha u = \frac{\partial^3 u}{\partial x_1 \partial x_3^2}$.

The space $C^\infty(\Omega)$ consists of all functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $D^\alpha u \in C(\Omega)$ for each multi-index α. Then, u has continuous partial derivatives of all orders.

The space of **test functions** denoted $\mathcal{D}$, $\mathcal{D}(\Omega)$ or $C_0^\infty(\Omega)$ is formed by the elements φ of $C^\infty(\Omega)$ which have compact support. The support $Supp(\varphi)$ of φ is the closure of $\{x : \varphi(x) \neq 0\}$.

The set $C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$ the set of square integrable functions in the sense of Lebesgue. In other words, any function in $\mathcal{L}^2(\Omega)$, can be approximated by functions in $C_0^\infty(\Omega)$, in the sense that there is a sequence $\{u_k\}$ of functions in $C_0^\infty(\Omega)$, which converges to u in $\mathcal{L}^2(\Omega)$.

Every function $u \in C_0^k(\Omega)$ can be extended to a function of $C_0^k(\mathbb{R}^n)$, in this way, $C_0^k(\Omega)$ can be interpreted as a subspace of $C_0^k(\mathbb{R}^n)$. Thus, given an open set $\Omega \subset \mathbb{R}^n$, the set $C_0^k(\Omega)$ can be seen as the set of elements $u \in C_0^k(\mathbb{R}^n)$ for which $Supp(u) \subset \Omega$.

The space $\mathcal{D}$, of test functions introduces interesting properties in integration by parts formula becoming the basis for further developments in variational calculus [73].

The classic integration by parts formula with $-\infty < a < b < \infty$, states

$$\int_a^b u'\varphi dx = [u\varphi]_a^b - \int_a^b u\varphi' dx \text{ holds for } u, \varphi \in C^1[a,b] \tag{2.1a}$$

If $\varphi \in C_0^\infty(a,b)$ then $\varphi(a) = \varphi(b) = 0$ and $[u\varphi]_a^b = 0$, then

$$\int_a^b u'\varphi dx = -\int_a^b u\varphi' dx \text{ holds for all } u \in C^1(a,b),\ \varphi \in C_0^\infty(a,b) \tag{2.1b}$$

The generalization of these two formulas to higher dimensions use the classical Green's theorem provided Ω is a nonempty bounded set in $\mathbb{R}^n$ that has sufficiently smooth boundary

$$\int_\Omega \frac{\partial u}{\partial x_j}\varphi d\mathbf{x} = \int_{\partial\Omega} u\varphi n_j dS - \int_\Omega u\frac{\partial \varphi}{\partial x_j} d\mathbf{x} \text{ For all } u, \varphi \in C^1(\overline{\Omega}), \tag{2.2a}$$

where n_j is the j-th component of the outward unit normal vector n to the boundary $\partial\Omega$ of the domain Ω. Replacing in (2.2a), by a function φ in $C_0^\infty(\mathbb{R}^n)$ and using the fact $\varphi = 0$ on $\partial\Omega$, the result is

$$\int_\Omega u\frac{\partial \varphi}{\partial x_k} dx = -\int_\Omega \varphi\frac{\partial u}{\partial x_k} dx, \text{ holds for all } u \in C^1(\Omega) \text{ and } \varphi \in C_0^\infty(\Omega). \tag{2.2b}$$

Definition 10 To define distributions we provide to $\mathcal{D}(\mathbb{R}^n)$, the following notion of convergence: If $\{\varphi_k\}$ is a sequence in $\mathcal{D}(\mathbb{R}^n)$, is said that $\varphi_k \to \varphi$ in $\mathcal{D}(\mathbb{R}^n)$ if (i) $\partial^\alpha \varphi_k \to \partial^\alpha \varphi$ uniformly for all multi-indices α and (ii) the φ_k's and φ are all supported in a common compact set (for more details see [8, 9]).

A **distribution** on $\mathbb{R}^n$ is a linear functional $T : C_0^\infty(\mathbb{R}^n) \to \mathbb{R}$, that is continuous in the sense that if $\varphi_k \to \varphi$ in $C_0^\infty(\mathbb{R}^n)$ then $T[\varphi_k] \to T[\varphi]$. The space of distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$ or $\mathcal{D}'$.

A distribution (or generalized function) is characterized by its "actions" $T[\varphi]$ or $\langle T,\varphi\rangle$ ("Duality bracket") over the elements $\varphi \in \mathcal{D}(\mathbb{R}^n)$. For instance $\langle \delta,\varphi\rangle = \varphi(0)$ is the action of the Dirac delta functional over φ. In general each $a \in \mathbb{R}^n$ determines a linear functional $\delta_{(a)} = \delta(x-a)$ on $\mathcal{D}(\mathbb{R}^n)$ by the expression $\langle \delta_{(a)},\varphi\rangle = \varphi(a)$. This is a way to solve the formal inconsistency settled by Dirac's sampling property.

An important thing to note is that any locally integrable function f, will define a distribution by

$$\langle f,\varphi\rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)dx \;\forall \varphi \in C_0^\infty(\mathbb{R}^n) \tag{2.1}$$

These are called **regular** distributions. If this is not the case, they are called **singular** distributions (for example, δ). By abuse of notation singular distributions are also denoted by the symbol $f(x)$ used for ordinary functions, although there is no value at the point x and the effect of an arbitrary distribution f on φ is written as an integral $\int f(x)\varphi(x)dx$. So we commonly write $\int \delta(x)\varphi(x)dx$ as equivalent to $\langle \delta,\varphi\rangle$, and we proceed as in ordinary calculus, for example

$$\int_{\mathbb{R}} \delta(x-a)\varphi(x)dx = \int \delta(\xi)\varphi(\xi+a)d\xi = [\varphi(\xi+a)]_{\xi=0} = \varphi(a)$$

Operations on functions can be extended to distributions (derivatives, convolution, Fourier transform). One key idea is that the definition of operations on distributions should coincide with the definition for regular distributions. For example, from $\int (f\xi)\varphi = \int f(\xi\varphi)$, it follows that the product of a distribution T in $\mathcal{D}'(\mathbb{R}^n)$ and a function $\xi \in C_0^\infty(\mathbb{R}^n)$ is defined as

$$\langle \xi T,\varphi\rangle = \langle T,\xi\varphi\rangle.$$

Nevertheless, this should be done carefully, because it may arise several limitations; for example, it will not be possible to multiply distributions nor define the Fourier transform without making extra assumptions. It is a well-known fact that classical function spaces may not be suitable in order to formulate well-posedness. The derivative concept is in the core of this difficulty, so it is necessary to get a more versatile definition of derivative.

Distributional derivatives are motivated by integration by parts formulas (2.1b)

$$\int_a^b u'\varphi dx = -\int_a^b u\varphi'\,dx \;\text{ holds for all } u \in C^1(a,b),\; \varphi \in C_0^\infty(a,b)$$

Thus, in one variable we may consider the functional $\langle f',\varphi\rangle = \int_{-\infty}^{\infty} f'\varphi$ and applying integration by parts

$$\begin{aligned}\langle f',\varphi\rangle &= \int_{-\infty}^{\infty} f'\varphi = -\int_{-\infty}^{\infty} f\varphi' \\ &= -\langle f,\varphi'\rangle\end{aligned}$$

Then the derivative of the functional $\langle f,\varphi\rangle$ is the functional $-\langle f,\varphi'\rangle$. Thus, given any distribution f, it is defined

$$\left\langle \frac{\partial f}{\partial x_k}, \varphi \right\rangle = -\left\langle f, \frac{\partial \varphi}{\partial x_k} \right\rangle$$

and applying this repeatedly

$$\int_\Omega (\partial^\alpha f)\varphi dx = (-1)^{|\alpha|} \int_\Omega f(\partial^\alpha \varphi) dx.$$

Assuming sufficient conditions of differentiability of u, we can define a regular distribution

$$\langle \partial^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, \partial^\alpha \varphi \rangle \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n) \tag{2.3}$$

This means that the derivative of a non-differentiable function can be defined in terms of relations with smooth functions of compact support. For example, the derivative δ' of the delta distribution it is expressed as $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = \varphi'(0)$. By (2.3), a distribution T has derivatives of all orders, a direct consequence is that T is indefinitely differentiable and always

$$\frac{\partial^2 T}{\partial x_j x_k} = \frac{\partial^2 T}{\partial x_k x_j}$$

This point of view solves the famous problem about quantities such as Dirac delta and Heaviside step function do not have derivatives in the classical sense. Nevertheless, treating them as distributions allows to extend the concept of a derivative in such a way that any number of derivatives can be defined for these quantities and, even, for any distribution. Thus, the classical notion of a derivative is recovered.

Example 4 The derivative of the Heaviside function $H(x) = 0$ if $x < 0$, $H(x) = 1$ if $x > 0$ is $H' = \delta$. This can be seen as

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_0^\infty \varphi'(x) dx = \varphi(0) - \varphi(\infty) = \varphi(0) = \langle \delta, \varphi \rangle.$$

Example 5 The derivative of a function f with jumps can be calculated as a distribution. Suppose f is a piecewise function on $\mathbb{R}$ and differentiable at all $x \neq 0$ but has a jump discontinuity at $x = 0$. Then, calling $f^{[1]}$ the pointwise derivative and f' the distributional derivative, we have

$$\begin{aligned} \langle f', \varphi \rangle &= -\int f(x)\varphi'(x) dx = \int_{-\infty}^0 f\varphi' dx - \int_0^\infty f\varphi' dx \\ &= -f(x)\varphi(x)\,|_{-\infty}^0 + \int_{-\infty}^0 f^{[1]}\varphi dx - f(x)\varphi(x)\,|_0^\infty + \int_0^\infty f^{[1]}\varphi dx \\ &= f(0-)\varphi(0) + f(0+)\varphi(0) + \int_{-\infty}^\infty f^{[1]}\varphi dx, \end{aligned}$$

thus,

$$f' = f^{[1]} + [f(0+) - f(0-)]\delta$$

2.4.1 Convolution

Many important results of applied mathematics can be expressed as convolutions. Given two functions f and u their **convolution product** is a new function $f*u(x)$ such that

$$f*u(x)=\int_{\mathbb{R}^n} f(x-t)u(t)dt$$

This definition can be generalized to distributions [8, 9, 59, 60] and several conditions on f and u are necessary to ensure that the integral exists. Convolution obeys the same algebraic laws of ordinary multiplication (i) $f*(\alpha u+\beta v)=\alpha(f*u)+\beta(f*v)$, (ii) $f*u=u*f$, (iii) $f*(u*v)=(f*u)*v$. One property which convolution does not share with ordinary multiplication is that, on the contrary to $f*1=f$ for all f, there is no function u such that $f*u=f$. This limitation is solved introducing distribution theory, where $f*\delta=f$; because

$$\delta*f(x)=\int\delta(t)f(x-t)dt=f(x-t)|_{t=0}=f(x)$$

Very useful for function approximation is the existence of sequences $\{u_k\}$ such that $f*u_k$ converges to f as $n\to\infty$. In these problems convolution behaves as a continuous superposition of translates of f and $f*u_k$ may be regarded as a smoothed version of f. In the next chapters we will see how splines approximators are expressed as some kind of convolution with fundamental solutions of differential operators.

The convolution of a distribution $T\in D'(\mathbb{R}^n)$ with a test function $\varphi\in\mathcal{D}(\mathbb{R}^n)$ is a C^∞ function $T*\varphi$ on $\mathbb{R}^n$. $T*\varphi$ is called the **regularization** of T. Besides the above properties, convolution has the following properties with Dirac's delta

(i) $\delta*T=T$

(ii) $\delta_{(a)}*T=\tau_a T$

(iii) $\delta'*T=T'$

(iv) $\delta_{(a)}*f(x)=f(x-a)$ (2.4)

In general, if D is a differential operator with constant coefficients in $\mathbb{R}^n$, $D\delta*T=DT$. For example, if D is the Laplacian operator $\Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ in $\mathbb{R}^n$, then $\Delta\delta*T=\Delta T$.

2.4.2 The Schwartz Space and Fundamental Solutions

A very important problem when extending the Fourier transform of a function f

$$\mathcal{F}[f]=\hat{f}(\xi)=\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx,\ \xi=(\xi_1,\xi_2,\ldots,\xi_n),$$



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