Chapter 2
Reinsurance and Investment

In this chapter we present the two main ways to control the insurance risk process: reinsurance and investment. We focus on the classical risk model.

2.1 Reinsurance in the Classical Risk Model

An insurance company can share the risk by a reinsurance contract. We only consider the case in which this contract reduces the impact of each one of the claims, that is, by paying to the reinsurance company some part of the premium; this company covers some predetermined part of the claim. A reinsurance contract has two elements:

- A Borel measurable function $R : \mathbf{R}_+ \to \mathbf{R}_+$ (called *retained loss function*) that satisfies $0 \leq R(\alpha) \leq \alpha$, where $R(\alpha)$ is the part of the claim paid by the insurance company when the size of the claim is α (the reinsurance company covers $\alpha - R(\alpha)$) .
- The premium rate q_R paid to the reinsurance company. So the premium rate left to the insurance company is $p_R = p - q_R$.

The part of the claim paid by the insurance company is the random variable $R(U)$ where U is the claim size. We define

$$F_R(x) = P(R(U) \leq x). \tag{2.1}$$

The two more common examples of reinsurance contracts are *proportional* reinsurance and *excess-of-loss* reinsurance. In the first case the reinsurance company covers a fixed ratio of the claim and therefore the retained loss function is $R(\alpha) = b\alpha$ for some *retained proportion* $b \in [0, 1]$; here $F_R(x) = F(x/b)$ if $b > 0$ and $F_R(x) = 1$ if $b = 0$. In the second case a *retention level* $a \in [0, \infty]$ is fixed in such a way

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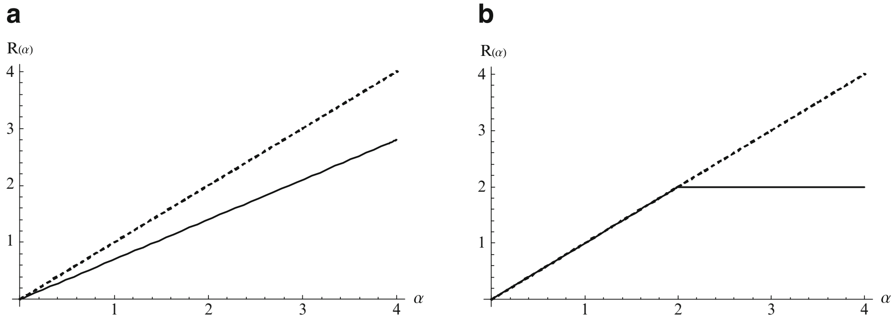

Fig. 2.1 (**a**) Proportional reinsurance. (**b**) Excess-of-loss reinsurance

that, paying to the reinsurance company some part of the premium, the reinsurance company covers the amount of the claim exceeding a; in this case the retained loss function is $R(\alpha) = \min\{\alpha, a\}$ and

$$F_R(x) = F(x)I_{\{x<a\}} + I_{\{x \geq a\}}.$$

We show the graphs of the retained loss function of proportional and excess-of-loss reinsurance contracts corresponding to $b = 0.7$ and $a = 2$ in Fig. 2.1a, b, respectively (the identity function is shown in dotted line).

We assume that the premium rate of the reinsurance company is calculated using the expected value principle with a relative safety loading $\eta_1 \geq \eta > 0$; we obtain that

$$q_R = (1 + \eta_1)\beta E(U_i - R(U_i))$$

and so

$$\begin{aligned} p_R &= (1+\eta)\beta E(U_i) - (1+\eta_1)\beta E(U_i - R(U_i)) \\ &= (1+\eta_1)\beta E(R(U_i)) - (\eta_1 - \eta)\beta E(U_i). \end{aligned} \tag{2.2}$$

The case $\eta_1 = \eta$ is called cheap reinsurance. There are other criteria for computing the premium rate of the reinsurance company; see for instance Teugels [62]. We assume that the premium rate left to the insurance company is $p_R = p - q_R > 0$.

Definition 2.1. Let us call $\mathcal{R}_A$ the family of all the retained loss functions with positive p_R, $\mathcal{R}_P \subset \mathcal{R}_A$ the subfamily of proportional retained loss functions, and $\mathcal{R}_{XL} \subset \mathcal{R}_A$ the subfamily of the excess-of-loss retained loss functions. We denote by $\mathcal{R}_F$ any finite subfamily of retained functions in $\mathcal{R}_A$.

Given any subfamily $\mathcal{R} \subset \mathcal{R}_A$ of retained loss functions, we assume that the manager of the insurance company can choose at any time a reinsurance contract within the family $\mathcal{R}$ and that the premium rate of the reinsurance company is calculated using the expected value principle with relative safety loading $\eta_1 > \eta$.

Consider the filtered probability space $(\Omega, \Sigma, (\mathcal{F}_t)_{t \geq 0}, P)$ introduced in (1.3). A reinsurance control strategy is a collection $\overline{R} = (R_t)_{t \geq 0}$ of functions $R_t : \Omega \rightarrow \mathcal{R}$ for any $t \geq 0$. We say that a reinsurance control strategy is *admissible* if the function $(\omega, \alpha, t) \rightarrow R_t(\omega)(\alpha)$ is $(\Sigma \times$ Borel $\times$ Borel) measurable and the function $\omega \rightarrow R_t(\omega)(\alpha)$ is $\mathcal{F}_{t-}$ measurable for every $t \geq 0$ and $\alpha \geq 0$. Note that this definition means that the process $(R_t(\cdot)(\alpha))_{t \geq 0}$ is predictable for any α. We denote by Π_x^R the set of all the admissible control strategies with initial surplus $x \geq 0$. Note that, for any reinsurance admissible control strategies $\overline{R} \in \Pi_x^R$, the premium process $(p_{R_t})_{t \geq 0}$ is Borel measurable.

Given an admissible control strategy $\overline{R}$, the *controlled risk process* $X_t^{\overline{R}}$ is given by

$$X_t^{\overline{R}} = x + \int_0^t p_{R_s}\, ds - \sum_{i=1}^{N_t} R_{\tau_i}(U_i), \tag{2.3}$$

where τ_i is the time of occurrence of the ith claim. We define the corresponding *ruin time* $\tau^{\overline{R}}$ of the company as

$$\tau^{\overline{R}} = \inf \left\{ t \geq 0 : X_t^{\overline{R}} < 0 \right\}. \tag{2.4}$$

An important class of reinsurance admissible strategies is the one where the decision of the reinsurance contract depends only on the current surplus. The idea is the following: consider a retained loss function $\rho^y \in \mathcal{R}$ for each $y \geq 0$, and define for any initial surplus $x \geq 0$, the process $(X_t)_{t \geq 0}$ obtained by taking ρ^y as retained loss function when the current surplus is y; the process X_t should satisfy

$$X_t = x + \int_0^t p_{\rho^{X_s-}}\, ds - \sum_{i=1}^{N_t} \rho^{X_{\tau_i^-}}(U_i) \tag{2.5}$$

and it should be the controlled reinsurance process associated with the reinsurance strategy $\left(\rho^{X_t-}\right)_{t \geq 0} \in \Pi_x^R$.

We define a stationary reinsurance control as a choice of a retained loss function for each surplus; with the suitable measurability conditions, we obtain that the strategy $\left(\rho^{X_t-}\right)_{t \geq 0}$ is admissible and therefore it belongs to Π_x^R. More precisely:

Definition 2.2. A *stationary reinsurance control in* $\mathcal{R}$ is a Borel measurable function $\rho : \mathbf{R}_+ \times \mathbf{R}_+ \rightarrow \mathbf{R}$ such that $\rho(x, \cdot) = \rho^x \in \mathcal{R}$ for all $x \geq 0$ and $1/p_{\rho^x}$ is locally integrable.

Proposition 2.1. *Given any stationary reinsurance control ρ and any initial surplus $x \in \mathbf{R}_+$, there exists a unique solution $(X_t)_{t \geq 0}$ of the stochastic integral equation (2.5). Moreover, if we define $R_t = \rho^{X_t-}$, then the strategy $(R_t)_{t \geq 0} \in \Pi_x^R$ and its associated surplus process $\left(X_t^R\right)_{t \geq 0}$ coincides with $(X_t)_{t \geq 0}$.*

Proof. In order to see that there exists a unique solution of (2.5), it is enough to show existence a uniqueness for a fix $\omega \in \Omega$ and for t between two claims, that is,

$$X_t = x + \int_0^t p_{\rho^{X_s-}}\, ds. \tag{2.6}$$

Let us define the function

$$G(x) = \int_0^x \frac{1}{p_{\rho^y}} dy;$$

since $p_{\rho^y} \in (0, p]$, the function G is Lipschitz and increasing. The unique solution of (2.6) can be written as

$$X_t = G^{-1}(G(x) + t).$$

On the other hand, since ρ is Borel measurable and the process X_{t^-} is $\mathcal{F}_{t-}$-measurable, we have that the strategy $\left(\rho^{X_{t^-}}\right)_{t\geq 0}$ is admissible. □

Given an initial surplus $x \geq 0$ and any fixed retained function $R \in \mathcal{R}$, we consider the constant admissible strategy $\overline{R} = (R)_{t\geq 0}$, the corresponding controlled surplus process $X_t^{\overline{R}}$, and the ruin time $\tau^{\overline{R}}$. The process $X_{t\wedge\tau^{\overline{R}}}^{\overline{R}}$ is Markov, by (1.11), its infinitesimal generator is $\mathcal{G}(\left(X_{t\wedge\tau^{\overline{R}}}^{\overline{R}}\right)_{t\geq 0}, f)(x) = \mathcal{L}_R(f)(x)$ where

$$\mathcal{L}_R(f)(x) = p_R f'(x) - \beta f(x) + \beta \mathcal{I}_R(f)(x), \tag{2.7}$$

and

$$\mathcal{I}_R(f)(x) = \textstyle\int_0^\infty f(x - R(\alpha))dF(\alpha) = \int_0^x f(x - \alpha)dF_R(\alpha) \tag{2.8}$$

(here f is a continuously differentiable function in $\mathbf{R}_+$ extended as $f = 0$ for $x < 0$). This integral is interpreted in the Lebesgue–Stieltjes sense; in the case that the integral exists in the Riemann–Stieltjes sense, both notions agree. In the case that R is either continuous or it has finitely many discontinuities which do not coincide with the discontinuities of F, this integral exists in the Riemann–Stieltjes sense. See Sect. 12.3 in Royden [52].

2.1.1 Survival Probability and Reinsurance

We assume in this section that within the family $\mathcal{R}$ there exists at least one retained function $\hat{R} \in \mathcal{R}$ that satisfies the *net profit condition*, that is $p_{\hat{R}} > \beta E(\hat{R}(U_i))$.

Given $x \geq 0$, any admissible control strategy $\overline{R} \in \Pi_x^R$ and its controlled risk process $X_t^{\overline{R}}$, we define the corresponding survival probability as

$$\delta^{\overline{R}}(x) = P(\tau^{\overline{R}} = \infty | X_0^{\overline{R}} = x),$$

where $\tau^{\overline{R}}$ is introduced in (2.4). The *optimal survival probability function with reinsurance* is defined as

$$\delta(x) = \sup_{\overline{R} \in \Pi_x^R} \delta^{\overline{R}}(x). \tag{2.9}$$

We have the following property with respect to the behavior of the surplus $X_t^{\overline{R}}$ at infinity, the proof is similar to the one given in Lemma 2.9 of [57].

Proposition 2.2. *Take any admissible strategy $\overline{R} \in \Pi_x^R$; with probability one, either ruin occurs in finite time or $X_t^{\overline{R}}$ diverges to infinity as t goes to infinity.*

Proof. Suppose that the initial surplus is $x \geq 0$. By (2.2), we have that

$$E(R(U)) \geq \varsigma := \frac{(\eta_1 - \eta)\beta\,(E(U))}{2(1+\eta_1)\beta} > 0$$

for all $R \in \mathcal{R}$. We show now that there exists $\gamma > 0$ small enough such that $P(R(U) \geq \gamma) \geq \gamma > 0$ for all $R \in \mathcal{R}$. Suppose that this is not the case, then for each $n \geq 1$ there exists $R_n \in \mathcal{R}$ such that $P(R_n(U) \geq 1/n) \leq 1/n$. Since $E(U)$ is finite and

$$E(R_n(U)) \leq \int_0^\infty \tfrac{1}{n} I_{\{R_n < \frac{1}{n}\}} dF(\alpha) + \int_0^\infty \alpha I_{\{R_n \geq \frac{1}{n}\}} dF(\alpha),$$

we have

$$0 < \varsigma \leq \limsup_{n\to\infty} E(R_n(U)) \leq \lim_{n\to\infty} \frac{1}{n} + \limsup_{n\to\infty} \int_0^\infty \alpha I_{\{R_n \geq \frac{1}{n}\}} dF(\alpha) = 0$$

and this is a contradiction.

Fix $a > 0$, we define recursively the sequence $(t_k)_{k\in\mathbf{N}}$ in the following way

$$t_1 = \inf\{t \geq 0 : X_t^{\overline{R}} < a\}$$

and

$$t_{k+1} = \inf\{t \geq 0 : t \geq t_k + 1 \text{ and } X_t^{\overline{R}} < a\}.$$

In the case that $\liminf_{t\to\infty} X_t^{\overline{R}} < a$, all the $t_k' s$ are finite. In the case that $X_t^{\overline{R}} \geq a$ for all $t \geq 0$, we put $t_n = \infty$ for $n \geq 1$, and in the case that $X_t^{\overline{R}} \geq a$ for all $t \geq t_k + 1$, we put $t_n = \infty$ for $n \geq k+1$. Let us consider the case $t_k < \infty$; we define Σ_k as the σ-algebra generated by $\left(X_{t\wedge t_k}^{\overline{R}}\right)_{t\geq 0}$. Then, if $\omega \in \Omega$ satisfies $t_k(\omega) < \infty$, we have for any $\varepsilon > 0$ that

$$
\begin{aligned}
& E(I_{\{t_k<\infty \text{ and } X_{t_k+1}^{\overline{R}} \leq a-\varepsilon\}} | \Sigma_k)(\omega) \\
& \geq P(X_{t_k+1}^{\overline{R}} \leq a-\varepsilon | X_{t_k}^{\overline{R}} = a) \\
& = P(\textstyle\sum_{\tau_i \in (t_k, t_k+1]} R_{\tau_i}(U_i) \geq \int_{t_k}^{t_k+1} p_{R_s} ds + \varepsilon) \geq P_0 > 0,
\end{aligned}
$$

because $P(R_{\tau_i}(U_i) \geq \gamma) \geq \gamma > 0$, R_t is predictable, U_i are i.i.d, and N_t is a Poisson process independent on both the arrival claim times and the size of the claims. Hence,

$$
E(I_{\{t_k<\infty \text{ and } X_{t_k+1}^{\overline{R}} \leq a-\varepsilon\}} - P_0 I_{\{t_k<\infty\}} | \Sigma_k) \geq 0. \tag{2.10}
$$

Let us define

$$
W_k = I_{\{t_k<\infty \text{ and } X_{t_k+1}^{\overline{R}} \leq a-\varepsilon\}}, \; Z_k = P_0 I_{\{t_k<\infty\}} \text{ and } A_n = \sum_{k=1}^{n} (W_k - Z_k).
$$

Using that $A_1, A_2, \ldots, A_{n-1}$ are Σ_n-measurable and (2.10), we have that A_n is a submartingale, and so from Lemma 2.1 of Niemiro and Pokarowski [48], we obtain that

$$
P\left(\textstyle\sum_{k=1}^{\infty} Z_k = \infty, \sum_{k=1}^{\infty} W_k < \infty\right) = 0.
$$

We conclude that if $\liminf_{t\to\infty} X_t^{\overline{R}} < a$, then $\liminf_{t\to\infty} X_t^{\overline{R}} < a - \varepsilon$, except possibly for a set of probability zero. So either $X_t^{\overline{R}}$ diverges to infinity or $\liminf_{t\to\infty} X_t^{\overline{R}} = -\infty$. In the last case, ruin occurs in finite time. □

The following lemma is similar to Lemma 1.1, and so is its proof.

Lemma 2.1. *Given an initial surplus* $x \geq 0$, *let us consider* $\overline{R} \in \Pi_x^R$, *then* $\delta^{\overline{R}}\left(X_{t\wedge\tau_{\overline{R}}}^{\overline{R}}\right)$ *is a martingale .*

The next proposition gives the dynamic programming principle for δ and it will be used to find the HJB equation of this problem.

Proposition 2.3. *Given any initial surplus $x \geq 0$, we have that*

$$\delta(x) = \sup_{\overline{R}\in\Pi_x^R} E_x(\delta(X^{\overline{R}}_{\tau\wedge\tau^{\overline{R}}}))$$

for any stopping time τ with $P(\tau = \infty) = 0$.

Proof. It is enough to prove this proposition for the case that the stopping time is constant, that is $\tau(\omega) = T$ for all $\omega \in \Omega$. We call

$$\vartheta(x,T) = \sup_{\overline{R}\in\Pi_x^R} E_x(\delta(X^{\overline{R}}_{T\wedge\tau^{\overline{R}}})).$$

Let us prove first that $\delta(x) \leq \vartheta(x,T)$. Take any admissible strategy $\overline{R} = (R_t)_{t\geq 0} \in \Pi_x^R$. We define for any $\omega \in \Omega$ the strategy $\overline{R}_1 = (R_t^1)_{t\geq 0} \in \Pi^R_{X_T^{\overline{R}}}$ as $R_t^1(\omega) = R_{t+T}(\omega)$. The strategy $\overline{R}_1$ is admissible and

$$\begin{aligned}
\delta^{\overline{R}}(x) &= E\left(E\left(I_{\{\tau^{\overline{R}}=\infty\}}\middle|\mathcal{F}_T\right)\right) \\
&= E\left(I_{\{\tau^{\overline{R}}=\infty\}}(\delta^{\overline{R}_1}(X_T^{\overline{R}})\right) \\
&\leq E\left(I_{\{\tau^{\overline{R}}=\infty\}}(\delta(X_T^{\overline{R}})\right) \\
&\leq E_x(\delta(X^{\overline{R}}_{T\wedge\tau^{\overline{R}}})).
\end{aligned}$$

From (2.9), we get the result.

Let us prove now that $\delta(x) \geq \vartheta(x,T)$. Given any $\varepsilon > 0$, take an admissible strategy $\overline{R} = (R_t) \in \Pi_x^R$ such that

$$E_x(\delta(X^{\overline{R}}_{T\wedge\tau^{\overline{R}}})) \geq \vartheta(x,T) - \varepsilon/2. \tag{2.11}$$

Since δ is Lipschitz, we can find points $0 = x_0 < x_1 < x_2 < \cdots$ such that if $x \in [x_i, x_{i+1})$, then

$$\delta(x) - \delta(x_i) < \frac{\varepsilon}{4}. \tag{2.12}$$

Let us call $A_i = [x_i, x_{i+1})$. Take admissible strategies $\overline{R}_i = (R_t^i)_{t\geq 0} \in \Pi^R_{x_i}$ such that $\delta(x) - \delta^{\overline{R}_i}(x_i) \leq \varepsilon/4$ for all i. We define a new strategy $\overline{R}_* = (R_t^*)_{t\geq 0}$ in the following way: For $t \leq T$, take $R_t^* = R_t$, so $T \wedge \tau^{\overline{R}} = T \wedge \tau^{\overline{R}_*}$. In the case that $\tau^{\overline{R}} > T$, take i_0 such that $X^{\overline{R}}_{T\wedge\tau^{\overline{R}}} \in A_{i_0}$ and follow strategy $R^*_{T+s} = R_s^{i_0}$ for all $s \geq 0$; note that $\tau^{\overline{R}_*} \geq T + \tau^{\overline{R}_{i_0}}$. We have

$$
\begin{aligned}
\delta(x) &\geq \delta^{\overline{R}*}(x) \\
&= E\left(E\left(I_{\left\{\tau^{\overline{R}*}=\infty\right\}}\middle|\mathcal{F}_T\right)\right) \\
&\geq E_x\left(\sum_i I_{A_i}(X^{\overline{R}*}_{T\wedge\tau^{\overline{R}*}})\delta^{\overline{R}_i}(x_i)\right) \\
&\geq E_x\left(\sum_i I_{A_i}(X^{\overline{R}*}_{T\wedge\tau^{\overline{R}*}})\delta(X^{\overline{R}*}_{T\wedge\tau^{\overline{R}*}})\right) - \frac{\varepsilon}{2} \\
&\geq \vartheta(x,T) - \varepsilon,
\end{aligned}
$$

and so we have the result. □

We want now to derive heuristically the HJB equation. Given an initial surplus $x \geq 0$ and any fixed retained function $R \in \mathcal{R}$, we consider the constant admissible strategy $\overline{R} = (R)_{t\geq 0}$ and the controlled surplus process

$$X^R_t := x + p_R t - \sum_{i=1}^{N_t} R(U_i).$$

with ruin time τ^R. By Lemma 2.1 we have that $\delta(x) \geq E\left(\delta(X^R_{t\wedge\tau^R})\right)$ and so assuming that δ is differentiable at $x \geq 0$, we obtain from (2.7),

$$\sup_{R\in\mathcal{R}} \mathcal{L}_R(\delta)(x) \leq 0.$$

The HJB equation of the optimal problem of survival probability with reinsurance would be

$$\sup_{R\in\mathcal{R}} \mathcal{L}_R(\delta)(x) = 0. \tag{2.13}$$

In Chap. 3, we will show that δ is a viscosity solution of this equation, and in Chap. 6, we will show examples in which δ is not differentiable.

The next proposition is similar to Lemma 2.10 in [57].

Proposition 2.4. *We have that* $0 < \delta(x) < 1$ *for all* $x \geq 0$*,* $\lim_{x\to\infty}\delta(x) = 1$*,* δ *is increasing, and it is Lipschitz with Lipschitz constant* $K = \beta/\sup_{R\in\mathcal{R}} p_R$.

Proof. Take the survival probability function corresponding to the constant admissible strategy $\overline{R} = (\hat{R})_{t\geq 0}$, where $\hat{R}$ satisfies the net-profit condition. So we obtain from Lemma 1.1 that the corresponding survival probability function satisfies $\delta^{\overline{R}}(x) > 0$ for all $x \geq 0$ and that $\lim_{x\to\infty}\delta^{\overline{R}}(x) = 1$. So $\delta(x) > 0$ and $\lim_{x\to\infty}\delta(x) = 1$.

Let us prove first that $\delta < 1$; take any $\overline{R} \in \Pi^R_x$; from the proof of Proposition 2.2, we have that taking any $\varepsilon < -p_{R_0}/2$, we get that $P(X_1 \leq x - \varepsilon) \geq P_0 > 0$ where P_0 does not depend on x. So we obtain $P(\inf_{t\geq 0} X^{\overline{R}}_t < 0) \geq P_0^{(x/\varepsilon)+1}$ and so

$\delta(x) \leq 1 - P_0^{(x/\varepsilon)+1} < 1$. Let us show now that δ is nondecreasing; given $x_0 < x_1$, take any $\overline{R} = (R_t)_{t\geq 0} \in \Pi_{x_0}^R$ and consider $\overline{R}_1 = (R_t^1)_{t\geq 0} \in \Pi_{x_1}^R$ defined as $R_t^1 = R_t$ for $t \geq 0$; we obtain that $\tau^{\overline{R}} \leq \tau^{\overline{R}_1}$ and so $\delta^{\overline{R}_1}(x_1) \geq \delta^{\overline{R}}(x_0)$.

Let us prove now that δ is increasing; suppose that $\delta(x_0) = \delta(x_1)$, for any $\gamma > 0$ small enough, take $\overline{R} = (R_t)_{t\geq 0} \in \Pi_{x_0}^R$ such that $\delta^{\overline{R}}(x_0) \geq \delta(x_0) - \gamma\delta(x_0)$. Let us consider

$$\tau_{x_1} = \inf\left\{t \geq 0 : X_t^{\overline{R}} = x_1\right\}$$

and the strategy $\overline{R}_1 \in \Pi_{x_1}^R$ defined as follows: For $t \leq \tau_{x_1}$, follow strategy $\overline{R}$ and for $t > \tau_{x_1}$ follow an strategy $\overline{R}_2 \in \Pi_{2x_1-x_0}^R$ such that $\delta^{\overline{R}_2}(2x_1 - x_0) \geq \delta(2x_1 - x_0) - \gamma$. We have from Proposition 2.2 that $P(\tau_{x_1} = \infty) = 0$, so from Lemma 2.1, $\delta^{\overline{R}_1}(X_{t\wedge\tau_{x_1}}^{\overline{R}_1})$ is a martingale. Hence,

$$\delta^{\overline{R}_1}(x_1) = E_{x_1}(\delta^{\overline{R}_1}(X_{t\wedge\tau_{x_1}}^{\overline{R}_1})) = \delta^{\overline{R}_1}(2x_1 - x_0)P(\tau_{x_1} < \tau^{\overline{R}_1}).$$

Since $\delta(x_0) = \delta(x_1)$, we obtain

$$\begin{aligned}
\delta(x_0) - \gamma\delta(x_0) &\leq \delta^{\overline{R}}(x_0) \\
&= E_{x_0}(\delta^{\overline{R}}(X_{t\wedge\tau_{x_1}}^{\overline{R}})) \\
&= \delta^{\overline{R}}(x_1)P(\tau_{x_1} < \tau^{\overline{R}}) \\
&\leq \delta(x_1)P(\tau_{x_1} < \tau^{\overline{R}}) \\
&= \delta(x_0)P(\tau_{x_1} < \tau^{\overline{R}}),
\end{aligned}$$

and so $P(\tau_{x_1} < \tau^{\overline{R}}) \geq 1 - \gamma$. Then,

$$\delta(x_1) \geq \delta^{\overline{R}_1}(x_1) \geq (\delta(2x_1 - x_0) - \gamma)(1 - \gamma)$$

for any small enough $\gamma > 0$ and so $\delta(x_1) \geq \delta(2x_1 - x_0)$, but since δ is nondecreasing and $2x_1 - x_0 > x_1$ we obtain $\delta(x_0) = \delta(x_1) = \delta(2x_1 - x_0)$. Iterating this procedure and using that δ is nondecreasing we obtain $\delta(x) = \delta(x_0)$ para $x \geq x_0$, and this is a contradiction because $\delta(x_0) < 1$ and $\lim_{x\to\infty}\delta(x) = 1$.

Let us prove now that δ is Lipschitz. Take $\hat{R} \in \mathcal{R}$ such that $p_{\hat{R}} > \beta E(\hat{R}(U_i)) > 0$. Consider $x_0 \leq x_1$ and any $\varepsilon > 0$ and take an strategy $\overline{R}_1 = (R_t^1)_{t\geq 0} \in \Pi_{x_1}^R$ such that $\delta^{\overline{R}_1}(x_1) \geq \delta(x_1) - \varepsilon$. Let us define $\overline{R} = (R_t)_{t\geq 0} \in \Pi_{x_1}^R$ as follows: $R_t = \hat{R}$ for all $t \leq \tau_{x_1} = \inf\left\{t \geq 0 : X_t^{\overline{R}} = x_1\right\}$ and $R_t = R_{t-\tau_{x_1}}^1$ for $t > \tau_{x_1}$. In the event of no claims the process $X_t^{\overline{R}}$ with initial surplus x_0 reaches x_1 at time $h = (x_1 - x_0)/p_R$. So we have

$$\delta(x_0) \geq \delta^{\overline{R}}(x_0) \geq \delta^{\overline{R}}(x_1)P(h < \tau_1) = \delta^{\overline{R}}(x_1)e^{-\beta h} \geq (\delta(x_1) - \varepsilon)\,e^{-\beta h}.$$

Then, $\delta(x_0) \geq \delta(x_1)e^{-\beta h}$ and since δ is bounded by 1, we conclude that

$$0 \leq \delta(x_1) - \delta(x_0) \leq \delta(x_1)(1 - e^{-\beta(x_1 - x_0)/p_R}) \leq \frac{\beta}{p_{\hat{R}}}(x_1 - x_0). \qquad \square$$

Remark 2.1. Unlike the uncontrol case, the survival probability with initial surplus zero depends on the family of retained functions $\mathcal{R}$; the natural boundary condition for the optimal survival probability function is $\lim_{x\to\infty} \delta(x) = 1$.

2.1.2 Dividends and Reinsurance

In this section, we consider the problem of maximizing the cumulative expected discounted dividend payouts in the case that the insurer can control the risk by reinsurance within a family of retained functions $\mathcal{R}$. We do not assume here the existence of a retained function $R \in \mathcal{R}$ with the net-profit condition $p_R > \beta E(R(U))$.

A *dividend and reinsurance strategy* is a pair $(\overline{L}, \overline{R})$ where $\overline{L} = (L_t)_{t\geq 0}$ is a dividend strategy and $\overline{R} = (R_t)_{t\geq 0}$ is a reinsurance control strategy. Given a dividend and reinsurance strategy $(\overline{L}, \overline{R})$, we define the *controlled surplus process* as

$$X_t^{\overline{L},\overline{R}} = X_t^{\overline{R}} - L_t \tag{2.14}$$

and the *ruin time* as

$$\tau^{\overline{L},\overline{R}} = \inf\{t \geq 0 : X_t^{\overline{L},\overline{R}} < 0\}.$$

The strategy $(\overline{L}, \overline{R})$ is *admissible* if the reinsurance control strategy $\overline{R}$ is admissible and the dividend strategy $\overline{L}$ is nondecreasing, càglàd, and predictable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and verifies $L_0 = 0$ and

$$L_t \leq X_t^{\overline{R}} = x + \int_0^t p_{R_s} ds - \sum_{i=1}^{N_t} R_{\tau_i}(U_i)$$

for $0 \leq t < \tau^{\overline{L},\overline{R}}$. As in the case without reinsurance, we extend the definition of the admissible dividend process as $L_t = L_{\tau^{\overline{L},\overline{R}}}$ for $t \geq \tau^{\overline{L},\overline{R}}$. We denote by $\Pi_x^{L,R}$ the set of all admissible dividend and reinsurance strategies with initial surplus $x \geq 0$. Given an initial surplus $x \geq 0$ and an admissible strategy $(\overline{L}, \overline{R}) \in \Pi_x^{L,R}$, the cumulative expected discounted dividends is defined as

$$V_{\overline{L},\overline{R}}(x) = E_x\left(\int_0^{\tau^{\overline{L},\overline{R}}} e^{-cs} dL_s\right), \tag{2.15}$$

where $c > 0$ is a discount factor. The optimal value function of this problem is defined as

$$V(x) = \sup\{V_{\overline{L},\overline{R}}(x) \text{ with } (\overline{L},\overline{R}) \in \Pi_x^{L,R}\} \text{ for } x \geq 0. \tag{2.16}$$

The proofs of the following two propositions are similar to the ones of Propositions 1.2 and 1.3.

Proposition 2.5. *The optimal value function V is well defined and satisfies*

$$x + \frac{\overline{p}}{c+\beta} \leq V(x) \leq x + \frac{\overline{p}}{c} \text{ for } x \geq 0.$$

where $\overline{p} := \sup_{R\in\mathcal{R}} p_R$.

Note that the above proposition implies in particular that $V(0) \geq \overline{p}/(c+\beta) > 0$.

Proposition 2.6. *The optimal value function V is increasing and locally Lipschitz in* $[0,+\infty)$ *and satisfies*

$$y - x \leq V(y) - V(x) \leq \beta \frac{V(x)}{\overline{p}} (y-x)$$

for $y > x \geq 0$. *So V is Lipschitz on compact sets and absolutely continuous with* $1 \leq V' \leq (\beta/\overline{p})\, V$ *a.e..*

As in the case of optimizing dividend payments with no reinsurance, in order to obtain the HJB equation, we need to use a DPP. The proof is similar to the one of Lemma 1.2; the only difference is that here we consider admissible strategies in $\Pi_x^{L,R}$ instead of Π_x^L.

Lemma 2.2. *For any $x \geq 0$ and any stopping time $\overline{\tau}$, we can write*

$$V(x) = \sup_{(\overline{L},\overline{R})\in\Pi_x^{L,R}} E_x\left(\int_0^{\overline{\tau}\wedge\tau^{\overline{L},\overline{R}}} e^{-cs}\,dL_s + e^{-c\left(\tau\wedge\tau^{\overline{L}}\right)} V(X^{\overline{L},\overline{R}}_{\tau\wedge\tau^{\overline{L},\overline{R}}})\right).$$

Assume that V is continuously differentiable at x. Given any $l \geq 0$ and any $R \in \mathcal{R}$, let us consider the admissible strategy $\left(\overline{L},\overline{R}\right) = \left((lt)_{t\geq 0},(R)_{t\geq 0}\right)$ which pays dividends at constant rate l and takes reinsurance with constant retained function R. Let us call the corresponding controlled surplus process $X_t^{\overline{L},\overline{R}} = X_t^{\overline{R}} - lt$ and the corresponding ruin time τ. The surplus process $X_{\tau\wedge t}^{\overline{L},\overline{R}}$ stopped at the ruin time is a Markov process, so as in (1.11) and Remark 1.7, we get

$$\tilde{\mathcal{G}}\left(X^{\overline{L},\overline{R}}_{t\wedge\tau}, V\right)(x) = \begin{cases} (p_R - l)\, V'(x) - (\beta + c)\, V(x) + \beta\mathcal{I}_R(V)(x) & \text{if } l \leq p_R \\ (p_R - l)\, V'(x) - (\beta + c)\, V(x) + \beta\mathcal{I}_R(V)(x^-) & \text{if } l > p_R, \end{cases} \tag{2.17}$$

where $\tilde{\mathcal{G}}$ is the discounted infinitesimal generator defined in (1.2) and $\mathcal{I}_R(V)$ is the operator defined in (2.8). As in (1.19), but using Lemma 2.2, we obtain the inequality

$$\sup_{l \geq 0, R \in \mathcal{R}} \left\{ l + \tilde{\mathcal{G}} \left(X_{t \wedge \tau}^{\overline{L},\overline{R}}, V \right) (x) \right\} \leq 0.$$

The HJB equation of this optimization problem is

$$\sup_{l \geq 0, R \in \mathcal{R}} \left\{ l + \tilde{\mathcal{G}} \left(X_{t \wedge \tau}^{\overline{L},\overline{R}}, V \right) (x) \right\} = 0. \tag{2.18}$$

As in Sect. 1.5.2, we obtain that the HJB equation of this problem can be rewritten as

$$\max\{1 - V'(x), \sup_{R \in \mathcal{R}} \tilde{\mathcal{L}}_R(V)(x)\} = 0, \tag{2.19}$$

where

$$\tilde{\mathcal{L}}_R(V)(x) = p_R V'(x) - (c + \beta) V(x) + \beta \mathcal{I}_R(V)(x). \tag{2.20}$$

2.2 Investments in the Classical Risk Model

In this control problems, the management of an insurance company has the possibility to invest a fraction of the surplus in the financial market. For simplicity, we assume that the claim-size distribution has bounded density. The financial market is described as a classical Black–Scholes model that consists on a risk-free asset with price process B_t and a risky asset with price process P_t satisfying

$$\begin{cases} dB_t = r_0 B_t dt \\ dP_t = r P_t dt + \sigma P_t dW_t, \end{cases}$$

where $\sigma > 0$, $r > r_0 \geq 0$, and W_t is a standard Brownian motion independent to the probability space (Ω, Σ, P) defined in (1.3); without loss of generality, we consider here $r_0 = 0$.

Let us denote by $(\Omega_3, \Sigma_3, \left(\mathcal{F}_t^3\right)_{t \geq 0}, P_3)$ the filtered probability space of the Brownian motion W_t. Let us define the filtered probability space $(\overline{\Omega}, \overline{\Sigma}, \left(\overline{\mathcal{F}}_t\right)_{t \geq 0}, \overline{P})$ as the product of probability spaces

$$(\overline{\Omega}, \overline{\Sigma}, \overline{P}) = (\Omega, \Sigma, P) \times (\Omega_3, \Sigma_3, , P_3) \tag{2.21}$$

with filtration $\overline{\mathcal{F}}_t$ generated by $\mathcal{F}_t$ and $\mathcal{F}_t^3$.

We fix the set $\Gamma \subset \mathbf{R}$ of all the fractions of the surplus which could be invested in stocks. For instance, $\Gamma = [0, 1]$ means that neither short-selling of stocks nor borrowing money to buy stocks is allowed, $\Gamma = \mathbf{R}_+$ means that it is allowed borrowing money to buy stocks but short-selling of stocks is not allowed, and $\Gamma = \mathbf{R}$ means that both borrowing money and short-selling of stocks are allowed.

An investment strategy is a process $\overline{\gamma} = (\gamma_t)_{t\geq 0}$ where $\gamma_t \in \Gamma \subset \mathbf{R}$ is the fraction of the surplus invested in stocks. Given an investment strategy $\overline{\gamma} = (\gamma_t)_{t\geq 0}$, the *controlled risk process* $X_t^{\overline{\gamma}}$ should be a solution of the equation

$$X_t^{\overline{\gamma}} = x + pt - \sum_{i=1}^{N_t} U_i + \int_0^t \gamma_s X_s^{\overline{\gamma}} \left(r ds + \sigma \, d W_s \right). \tag{2.22}$$

The first three terms comes from the classical risk model and the integral term corresponds to the change of the surplus due to the investment. As before, we define the *ruin time* as

$$\tau^{\overline{\gamma}} = \inf\{t \geq 0 : X_t^{\overline{\gamma}} < 0\}.$$

An investment strategy is *admissible* if the process $(\gamma_t)_{t\geq 0}$ is predictable with respect to the filtration $\left(\mathcal{F}_t\right)_{t\geq 0}$ and there exists a unique strong solution $X_t^{\overline{\gamma}}$ of (2.22). We denote by Π_x^{γ} the set of all the admissible investment strategies with initial value x.

Remark 2.2. We introduce the process $Y_t^{\overline{\gamma}}$ as the solution of

$$Y_t^{\overline{\gamma}} = x + pt + \int_0^t \gamma_s Y_s^{\overline{\gamma}} \left(r ds + \sigma \, d W_s \right). \tag{2.23}$$

Note that $Y_t^{\overline{\gamma}}$ can be viewed as the controlled risk process without claims and that the processes $X_t^{\overline{\gamma}}$ and $Y_t^{\overline{\gamma}}$ coincide up to the first claim τ_1.

As in the case of reinsurance admissible strategies, we define a stationary investment control as the one where the investment decision depends only on the current surplus: consider a fraction $g(y) \in \Gamma$ for each $y \geq 0$, and define for any initial surplus $x \geq 0$ the surplus process $\left(X_t\right)_{t\geq 0}$ obtained by investing a fraction $g(y)$ when the current surplus is y; the process X_t should satisfy

$$X_t = x + pt - \sum_{i=1}^{N_t} U_i + \int_0^t g(X_{s}-) X_s \left(r ds + \sigma \, d W_s \right), \tag{2.24}$$

and it should be the controlled investment process associated with the investment strategy $\left(g(X_{t}-)\right)_{t\geq 0}$ with initial surplus $x \geq 0$.

Definition 2.3. A *stationary investment control in* Γ is a function $g : \mathbf{R}_+ \to \Gamma$ which satisfies that the function $g(x)x$ is Lipschitz.

Remark 2.3. As is pointed out in Theorem 1.19 of Øksendal and Sulem [49], the following result holds: Given any stationary investment control g and any initial surplus $x \in \mathbf{R}_+$ there exists a unique càdlàg solution X_t^g of the stochastic integral equation (2.24). Moreover, if we define $\gamma_t = g(X_{t-}^g)$, then the strategy $\overline{\gamma}^g = (\gamma_t)_{t\geq 0} \in \Pi_x^\gamma$ and its associated surplus process $\left(X_t^{\overline{\gamma}}\right)_{t\geq 0}$ defined in 2.22 coincides with $(X_t^g)_{t\geq 0}$. The global Lipschitz condition on $g(x)x$ is only used to ensure the existence of the process $\left(X_t^g\right)_{t\geq 0}$.

Given $\gamma_0 \in \Gamma$, let us consider the constant investment strategy $\overline{\gamma} = (\gamma_0)_{t\geq 0}$; let $X_t^{\gamma_0}$ and $Y_t^{\gamma_0}$ be the processes defined in (2.22) and (2.23) and τ^{γ_0} the ruin time of the surplus process $X_t^{\gamma_0}$. Then $(X_{t\wedge\tau^{\gamma_0}}^{\gamma_0})_{t\geq 0}$, the controlled surplus process stopped at the ruin time, is a Markov process. We now compute formally its infinitesimal generator. Assume that f is twice continuously differentiable, bounded, and with bounded derivatives up to order two in $\mathbf{R}_+$ extended as $f = 0$ for $x < 0$. Defining $A_0 = \{\tau_1 > t\}$, $A_1 = \{\tau_1 \leq t, \tau_2 > t\}$ and $A_2 = \{\tau_2 \leq t\}$. We have as in Sect. 1.4, that $P(A_0) = e^{-\beta t}$, $P(A_1) = \beta t e^{-\beta t}$ and that $P(A_2) = 1 - (1+\beta t)\, e^{-\beta t} = o(t)$. So,

$$
\begin{aligned}
E\left(f(X_{t\wedge\tau^{\gamma_0}}^{\gamma_0})\right) &= E_x\left(f(X_{t\wedge\tau^{\gamma_0}}^{\gamma_0}) I_{A_0}\right) + E_x\left(f(X_{t\wedge\tau^{\gamma_0}}^{\gamma_0}) I_{A_1}\right) + E_x\left(f(X_{t\wedge\tau^{\gamma_0}}^{\gamma_0}) I_{A_2}\right) \\
&= e^{-\beta t}\, E_x\left(f(Y_t^{\gamma_0})\right) + \beta \int_0^t \left(\int_0^\infty E_x(f(Y_s^{\gamma_0} - \alpha)) dF(\alpha)\right) e^{-\beta s} ds \\
&\quad + o(t)
\end{aligned}
$$

because f is bounded. Hence,

$$
\begin{aligned}
\frac{E_x\left(f(X_{t\wedge\tau^{\gamma_0}}^{\gamma_0})\right) - f(x)}{t} &= e^{-\beta t}\left(\frac{E_x(f(Y_t^{\gamma_0})) - f(x)}{t}\right) + \frac{(e^{-\beta t} - 1)}{t} f(x) \\
&\quad + \frac{\beta}{t}\int_0^t \left(\int_0^\infty E_x(f(Y_s^{\gamma_0} - \alpha)) dF(\alpha)\right) e^{-\beta s} ds + \frac{o(t)}{t},
\end{aligned}
$$

and then

$$
\mathcal{G}(X_{t\wedge\tau^{\gamma_0}}^{\gamma_0}, f)(x) = \mathcal{G}(Y_t^{\gamma_0}, f)(x) - \beta f(x) + \beta \mathcal{I}(f)(x).
$$

Since f is twice continuously, we get from Itô's formula

$$
\begin{aligned}
f(Y_t) - f(x) &= \int_0^t f'(Y_s^{\gamma_0}) dY_s^{\gamma_0} + \frac{\sigma^2\gamma_0^2}{2}\int_0^t f''(Y_s^{\gamma_0})\left(Y_s^{\gamma_0}\right)^2 ds \\
&= \int_0^t \left(f'(Y_s^{\gamma_0})\left(p + r\gamma_0 Y_s^{\gamma_0}\right) + \frac{\sigma^2\gamma_0^2}{2} f''(Y_s^{\gamma_0})\left(Y_s^{\gamma_0}\right)^2\right) ds \qquad (2.25) \\
&\quad + \int_0^t f'(Y_s^{\gamma_0})\sigma\gamma_0 Y_s^{\gamma_0} dW_s.
\end{aligned}
$$

So, since the last term of (2.25) is a martingale with zero expectation, we obtain that

$$\mathcal{G}(Y_t^{\gamma_0}, f)(x) = \frac{\sigma^2 \gamma_0^2 x^2}{2} f''(x) + (p + r\gamma_0 x)\, f'(x). \tag{2.26}$$

Therefore,

$$\mathcal{G}(X_{\tau\wedge t}^{\gamma_0}, f)(x) = \tfrac{\sigma^2 \gamma_0^2 x^2}{2} f''(x) + (p + r\gamma_0 x)\, f'(x) - \beta f(x) + \beta \mathcal{I}(f)(x). \tag{2.27}$$

2.2.1 *Survival Probability and Investments*

Given an admissible investment strategy $\overline{\gamma}$, we define the survival probability function as

$$\delta^{\overline{\gamma}}(x) = P(\tau^{\overline{\gamma}} = \infty | X_0^{\overline{\gamma}} = x)$$

and the optimal survival probability function as

$$\delta(x) = \sup_{\overline{\gamma}\in\Pi_x^\gamma} \delta^{\overline{\gamma}}(x). \tag{2.28}$$

As in Sect. 2.1.1, we have the following three results.

Proposition 2.7. *Take any admissible strategy $\overline{\gamma} \in \Pi_x^\gamma$, with probability one, either $X_t^{\overline{\gamma}}$ diverges to infinity as t goes to infinity or ruin occurs in finite time.*

Lemma 2.3. *Given an initial surplus $x \geq 0$, let us consider $\overline{\gamma} \in \Pi_x^\gamma$, then $\delta^{\overline{\gamma}}\left(X_{t\wedge\tau^{\overline{\gamma}}}^{\overline{\gamma}}\right)$ is a martingale .*

Proposition 2.8. *Given any $x \geq 0$, we have that*

$$\delta(x) = \sup_{\overline{\gamma}\in\Pi_x^\gamma} E_x(\delta(X_{\tau\wedge\tau^{\overline{\gamma}}}^{\overline{\gamma}}))$$

for any stopping time τ with $P(\tau = \infty) = 0$.

The argument of the proof of Proposition 2.7 is similar to the one of Proposition 2.2; the complete proof can be found in Lemma 2.18 in [57]. The proofs of Lemma 2.3 and of Proposition 2.8 are like the ones of Lemma 2.1 and Proposition 2.3.

We can now derive heuristically the HJB equation associated to this problem. Given $\gamma \in \Gamma$, consider the constant investment strategy $\overline{\gamma} = (\gamma)_{t\geq 0}$. By Proposition 2.8, we have that

$$\delta(X_{t\wedge\tau^{\overline{\gamma}}}^{\overline{\gamma}}) - \delta(x) \leq 0$$

and so $\mathcal{G}(\left(X_{t\wedge\tau}^{\overline{\gamma}}\right)_{t\geq 0}, \delta)(x) \leq 0$. Then by (2.27) we get

$$\mathcal{L}_{\gamma}(f)(x) = \tfrac{\sigma^2\gamma^2 x^2}{2} f''(x) + (p + r\gamma x)\, f'(x) - \beta f(x) + \beta\mathcal{I}(f)(x) \leq 0. \quad (2.29)$$

The HJB equation of the problem of survival probability with investment is

$$\sup_{\gamma\in\Gamma} \mathcal{L}_{\gamma}(\delta)(x) = 0. \quad (2.30)$$

Remark 2.4. We can rewrite

$$\mathcal{L}_{\gamma}(f)(x) = (\tfrac{\sigma^2\gamma^2 x^2}{2} f''(x) + r\gamma x f'(x)) + \mathcal{L}_0(f)(x)$$

where $\mathcal{L}_0$ is defined in (1.13).

The following proposition gives some elementary properties of the optimal survival probability function with investment; the proof is similar to the one of Proposition 2.4

Proposition 2.9. *Assume that* $0 \in \Gamma$. *We have that* $0 < \delta(x) < 1$ *for all* $x \geq 0$, $\lim_{x\to\infty} \delta(x) = 1$ *and that* δ *is Lipschitz and increasing.*

2.2.2 Dividends and Investments

In this section, we consider the problem of maximizing the cumulative expected discounted dividend payouts in the case that the insurer can control the risk by investing a fraction of the surplus in the financial market.

Let us fix the set $\Gamma \subset \mathbf{R}$; a *dividend and investment strategy* is a process $\left(\overline{L}, \overline{\gamma}\right) = (L_t, \gamma_t)_{t\geq 0}$ where $\overline{\gamma} = (\gamma_t)_{t\geq 0}$ is an investment strategy with $\gamma_t \in \Gamma$ and $\overline{L}$ is a dividend strategy. Given a dividend and investment strategy $\left(\overline{L}, \overline{\gamma}\right)$, the *controlled risk process* $X_t^{\overline{L},\overline{\gamma}}$ is given by

$$X_t^{\overline{L},\overline{\gamma}} = x + pt + r\int_0^t X_s^{\overline{L},\overline{\gamma}} \gamma_s ds + \sigma \int_0^t \gamma_s X_s^{\overline{L},\overline{\gamma}} dW_s - \sum_{i=1}^{N_t} U_i - L_t \quad (2.31)$$

and the *ruin time* is defined as $\tau^{\overline{L},\overline{\gamma}} = \inf\{t \geq 0 : X_t^{\overline{L},\overline{\gamma}} < 0\}$. The dividend and investment strategy $(\overline{L}, \overline{\gamma})$ is *admissible* if the investment strategy $\overline{\gamma}$ is admissible and the dividend strategy $\overline{L}$ is nondecreasing, càglàd, and predictable with respect to the filtration $\left(\mathcal{F}_t\right)_{t\geq 0}$ and verifies $L_0 = 0$ and

$$L_t \leq X_t^{\overline{\gamma}} = x + pt - \sum_{i=1}^{N_t} U_i + \int_0^t \gamma_s X_s^{\overline{\gamma}} (rds + \sigma\, dW_s)$$

for $0 \leq t < \tau^{\overline{L},\overline{\gamma}}$. As in the case without investments, we extend the definition of the admissible dividend process as $L_t = L_{\tau^{\overline{L},\overline{\gamma}}}$ for $t \geq \tau^{\overline{L},\overline{\gamma}}$. We denote by $\Pi_x^{L,\gamma}$ the set of all the dividend and investment admissible strategies with initial surplus x and the value function $V_{\overline{L},\overline{\gamma}}(x)$ as the cumulative expected discounted dividends with initial surplus $x \geq 0$ that corresponds to the predictable admissible control strategy $\left(\overline{L}, \overline{\gamma}\right)$. We can write $V_{\overline{L},\overline{\gamma}}(x)$ as

$$V_{\overline{L},\overline{\gamma}}(x) = E_x\left(\int_0^{\tau^{\overline{L},\overline{\gamma}}} e^{-cs} dL_s\right). \tag{2.32}$$

where $c > 0$ is a discount factor. The optimal dividend function is defined as

$$V(x) = \sup\{V_{\overline{L},\overline{\gamma}}(x) \text{ with } \left(\overline{L}, \overline{\gamma}\right) \in \Pi_x^{L,\gamma}\} \text{ for } x \geq 0. \tag{2.33}$$

In this problem, we assume that $0 \in \Gamma$ and that

$$0 < \hat{\gamma} := \sup \Gamma < c/r \tag{2.34}$$

and show that under this assumption V is finite. We will show in Remark 2.6 that if $\Gamma = [0, \hat{\gamma}]$ with $\hat{\gamma} > c/r$, then $V(x) = \infty$ for all $x \geq 0$.

We first state some results of a related controlled continuous risk process without the downward jumps.

Lemma 2.4. *Given $x \geq 0$, any $m \in \mathbf{R}$ and any admissible investment strategy $\overline{\gamma} \in \Pi_x^{\gamma}$, consider (with a slightly abuse of notation) the process Y_t defined in (2.23), but putting m instead of p. We have that:*

(a) If $m \geq 0$, then $E_x\left(Y_t e^{-ct}\right) \leq e^{-(c-r\hat{\gamma})t}\left(x + m(1 - e^{-r\hat{\gamma}t})/\left(r\hat{\gamma}\right)\right)$.
(b) If $x > 0$ and $\tilde{\tau} = \inf\{t : Y_t < 0\}$, then $\lim_{h\to 0} P(\tilde{\tau} < h) = 0$.
(c) If $\gamma_t \equiv \gamma_0 \in \Gamma \setminus \{0\}$ for all $t \geq 0$, then

$$E_x\left(Y_t e^{-ct}\right) = e^{-(c-r\gamma_0)t}\left(x + m(1 - e^{-r\gamma_0 t})/\left(r\gamma_0\right)\right).$$

Proof. (a) Since the process

$$U_t = \exp\left(\int_0^t (r\gamma_u - \frac{\sigma^2}{2}\gamma_u^2)du + \int_0^t \sigma\gamma_u dW_u\right) \tag{2.35}$$

is the solution of the stochastic equation

$$dU_t = U_t(r\gamma_t dt + \sigma\gamma_t dW_t) \text{ with } U_0 = 1$$

and Y_t is the solution of

$$dY_t = (m + Y_t r\gamma_t)dt + \sigma\gamma_t Y_t dW_t \text{ with } Y_0 = x,$$

we can write

$$Y_t = xU_t + U_t \int_0^t mU_s^{-1} ds. \tag{2.36}$$

Let us define

$$U_{ts} = \exp\left(\int_s^t (r\gamma_u - \frac{\sigma^2}{2}\gamma_u^2)du + \int_s^t \sigma\gamma_u dW_u\right), \tag{2.37}$$

then

$$Y_t = xU_{t0} + \int_0^t mU_{ts} ds. \tag{2.38}$$

We have that

$$A_{ts} = e^{-\int_s^t r\gamma_u du} U_{ts} \tag{2.39}$$

is a martingale; see for instance Karatzas and Shreve [39]. We conclude from (2.37) to (2.39) that

$$\begin{aligned} E_x\left(Y_t e^{-ct}\right) &= E_x(e^{-ct} x e^{\int_0^t r\gamma_u du} A_{t0} + e^{-ct} \int_0^t m e^{\int_s^t r\gamma_u du} A_{ts} ds) \\ &\leq E_x(e^{-(c-r\hat{\gamma})t} x A_{t0} + e^{-ct}\, m \int_0^t e^{r\hat{\gamma}(t-s)} A_{ts} ds) \\ &= e^{-(c-r\hat{\gamma})t} x + e^{-(c-r\hat{\gamma})t}\, m \int_0^t e^{-r\hat{\gamma}s} ds \\ &= e^{-(c-r\hat{\gamma})t}\left(x + \tfrac{m}{r\hat{\gamma}}(1 - e^{-r\hat{\gamma}t})\right). \end{aligned}$$

(b) This result is standard for linear diffusion processes; see Borodin and Salminen [17].

(c) Follows from the proof of (a). □

Remark 2.5. Given any $(\overline{L}, \overline{\gamma}) \in \Pi_x^{L,\gamma}$, consider the controlled process $X_t^{\overline{L},\overline{\gamma}}$ and the process Y_t introduced in Lemma 2.4, with $m = p$ and investment strategy $\overline{\gamma} = (\gamma_s)_{s\geq 0}$, then we have that $X_t^{\overline{L},\overline{\gamma}} \leq Y_t$ for all $t \geq 0$. We can use the following argument to see this result: $X_t^{\overline{L},\overline{\gamma}} = Y_t$ for $t < \tau_1$, where τ_1 is the arrival time of the first claim, $X_{\tau_1}^{\overline{L},\overline{\gamma}} < X_{\tau_1 -}^{\overline{L},\overline{\gamma}} = Y_{\tau_1}$. If there exists $t_0 \in (\tau_1, \tau_2)$ such that $X_{t_0}^{\overline{L},\overline{\gamma}} = Y_{t_0}$, then by definition $X_t^{\overline{L},\overline{\gamma}} = Y_t$ for $t \in (t_0, \tau_2)$; if this were not the case, $X_t^{\overline{L},\overline{\gamma}} < Y_t$ for $t \in (\tau_1, \tau_2)$ because the trajectories are continuous in this interval. Then $X_{\tau_2}^{\overline{L},\overline{\gamma}} < X_{\tau_2^-}^{\overline{L},\overline{\gamma}} \leq Y_{\tau_2}$ and the same argument applies again.

Remark 2.6. In the case that $\Gamma = [0, \hat{\gamma}]$ with $\hat{\gamma} > c/r$, the value function V is infinite. We can assume that $x > x_0 := \left((\beta\mu - p)^+ + 1\right)/r > 0$ because, if the initial surplus x is smaller than x_0, there is a positive probability that the surplus surpasses the level x_0 (take for instance the strategy which pays no dividends and keeps all the surplus in bonds up to time $T = (x_0 - x)/p + 1$).

Given $t_0 > 0$, consider the following admissible strategy $\left(\overline{L}^{t_0}, \overline{\gamma}^{t_0}\right) \in \Pi_x^{L,\gamma}$: divide the company in two departments; one of them deals only with the investment and the payment of dividends and the other with the insurance business. The investment department starts with capital x, invests a fraction $\hat{\gamma}$ of its surplus on risky assets, and diverts to the insurance department a constant flow $p_0 = (\beta\mu - p)^+ + 1$ up to time $t_0 \wedge \tilde{\tau}_1$ when the whole surplus is paid as dividends. Here $\tilde{\tau}_1$ is the first time the surplus of the investment department reaches zero. Let $X_t^{(1)}$ be the surplus process of the investment department and Y_t be the process described in Lemma 2.4(c) with $m = -p_0$. We have that $X_{t\wedge\tilde{\tau}_1}^{(1)} = Y_t$ for $t \leq \tilde{\tau}_1$ and $X_{t\wedge\tilde{\tau}_1}^{(1)} = 0 > Y_t$ for $t > \tilde{\tau}_1$. The insurance department starts with no surplus, pays no dividends, and receives a constant flow $p_0 + p > \beta\mu$ up to time $t_0 \wedge \tilde{\tau}_1 \wedge \tilde{\tau}_2$, where $\tilde{\tau}_2$ is the ruin time of the insurance department (assuming that the insurance department keeps always receiving the constant flow $p_0 + p$). Note that $\gamma_t^{t_0} \in \Gamma$ because

$$0 \leq \gamma_t^{t_0} = \frac{\hat{\gamma} X_t^{(1)}}{X_{t-}^{\overline{L}^{t_0}, \overline{\gamma}^{t_0}}} \leq \frac{\hat{\gamma} X_t^{(1)}}{X_t^{(1)}} = \hat{\gamma}$$

for $t < t_0 \wedge \tilde{\tau}_1 \wedge \tilde{\tau}_2$ and that the stopping time $\tilde{\tau}_2$ is independent of both $\tilde{\tau}_1$ and the process Y_t. Call $\tau = t_0 \wedge \tilde{\tau}_1 \wedge \tilde{\tau}_2$, the value function of this admissible strategy satisfies

$$\begin{aligned} V_{\overline{L}^{t_0}, \overline{\gamma}^{t_0}}(x) &\geq E_x(X_\tau^{(1)} e^{-c\tau} I_{\{\tilde{\tau}_1 \geq t_0, \tilde{\tau}_2 \geq t_0\}}) \geq E_x(Y_{t_0} e^{-ct_0} I_{\{\tilde{\tau}_1 \geq t_0, \tilde{\tau}_2 \geq t_0\}}) \\ &= E_x(Y_{t_0} e^{-ct_0} I_{\{\tilde{\tau}_1 \geq t_0\}}) P(\{\tilde{\tau}_2 \geq t_0\}) \\ &\geq E_x(Y_{t_0} e^{-ct_0}) P(\{\tilde{\tau}_2 = \infty\}). \end{aligned}$$

As we have seen in Remark 1.3, the survival probability of the insurance department $P(\{\tilde{\tau}_2 = \infty\}) = 1 - \beta\mu/(p_0 + p) > 0$. So, from Lemma 2.4(c), we conclude that $V(x) \geq \lim_{t_0 \to \infty} V_{\overline{L}^{t_0}, \overline{\gamma}^{t_0}}(x) = \infty$.

In the next two propositions, we prove that V has linear growth and we give bounds on the increments of V using the value functions of some simple admissible strategies.

Proposition 2.10. *The optimal value function V is well defined and satisfies*

$$x + p/(\beta + c) \leq V(x) \leq rx\hat{\gamma}/(c - r\hat{\gamma}) + p/(c - r\hat{\gamma}) \text{ for } x \geq 0.$$

Proof. Consider an initial surplus $x \geq 0$. Given any $(\overline{L}, \overline{\gamma}) \in \Pi_x^{L,\gamma}$, consider the controlled process $X_t^{\overline{L},\overline{\gamma}}$ for $t \geq 0$ and define $X_t^{\overline{L},\overline{\gamma}} = 0$ for $t < 0$. Then,

$$\begin{aligned}\tilde{L}_s = L_s - \sigma \int_0^s X_u^{\overline{L},\overline{\gamma}} \gamma_u dW_u \leq x + ps + r \int_0^s X_u^{\overline{L},\overline{\gamma}} \gamma_u du - \sum_{i=1}^{N_s} U_i \\ \leq x + ps + r \int_0^s X_u^{\overline{L},\overline{\gamma}} \gamma_u du.\end{aligned}$$

Consider the process Y_t defined as in Lemma 2.4, with $m = p$ and the investment strategy $\overline{\gamma} = (\gamma_s)_{s \geq 0}$. Since, by Remark 2.5, we have that $X_t^{\overline{L},\overline{\gamma}} \leq Y_t$, we obtain from Lemma 2.4(a) that

$$E_x \left(X_t^{\overline{L},\overline{\gamma}} e^{-ct} \right) \leq e^{-(c-r\hat{\gamma})t} \left(x + p(1 - e^{-r\hat{\gamma}t}) / (r\hat{\gamma}) \right).$$

Since $r\hat{\gamma} < c$ and e^{-cs} is a positive and decreasing function, we have that

$$\begin{aligned} V_{\overline{L},\overline{\gamma}}(x) &= E_x(\int_0^\tau e^{-cs} d\tilde{L}_s) \\ &\leq E_x(\int_0^\infty e^{-cs} d(x + ps + r \int_0^s X_u^{\overline{L},\overline{\gamma}} \gamma_u du)) \\ &\leq \int_0^\infty e^{-cs} p ds + r\hat{\gamma} \int_0^\infty E_x(e^{-cs} X_s^{\overline{L},\overline{\gamma}}) ds \\ &\leq \tfrac{p}{c} + r\hat{\gamma} \int_0^\infty (e^{-(c-r\hat{\gamma})s} \left(x + p \tfrac{1-e^{-r\hat{\gamma}s}}{r\hat{\gamma}} \right)) ds \\ &= \tfrac{rx\hat{\gamma}+p}{c-r\hat{\gamma}}. \end{aligned}$$

So $V(x)$ is finite and satisfies the second inequality.

Let us prove now the first inequality. Given an initial surplus $x \geq 0$, consider the admissible strategy $(\overline{L}, 0)$ which pays immediately the whole surplus x and then pays the incoming premium p as dividends with no investment in the risky assets until the first claim, which in this strategy means ruin. Define τ_1 as the time arrival of the first claim, we have

$$V_{\overline{L},0}(x) = x + pE_x(\int_0^{\tau_1} e^{-ct} \, dt) = x + p/(\beta + c),$$

but by definition $V(x) \geq V_{\overline{L},0}(x)$, so we get the result. □

Proposition 2.11. *If $y > x \geq 0$, the function V satisfies*

(a) $V(y) - V(x) \geq y - x$
(b) $V(y) - V(x) \leq \left(e^{(c+\beta)(y-x)/p} - 1 \right) V(x)$

Proof. (a) Given $\varepsilon > 0$, consider an admissible strategy $(\overline{L}, \overline{\gamma}) \in \Pi_x^{L,\gamma}$ with $V_{\overline{L},\overline{\gamma}}(x) \geq V(x) - \varepsilon$. We define a new strategy in $(\overline{L}^1, \overline{\gamma}^1) \in \Pi_y^{L,\gamma}$ in the following way: pay immediately $y - x$ as dividends and then follow the strategy $(\overline{L}, \overline{\gamma})$; this new strategy is admissible. We have that

$$V(y) \geq V_{\overline{L}^1,\overline{\gamma}^1}(y) = V_{\overline{L},\overline{\gamma}}(x) + (y - x) \geq V(x) - \varepsilon + (y - x),$$

and we obtain the result.

(b) Given $\varepsilon > 0$, take an admissible strategy $(\overline{L}, \overline{\gamma}) \in \Pi_y$ such that $V_{\overline{L},\overline{\gamma}}(y) \geq V(y) - \varepsilon$. Let us define the strategy $(\overline{L}^1, \overline{\gamma}^1) \in \Pi_x^{L,\gamma}$ that starting at x, pay no dividends and invest all the surplus in bonds if $X_t^{\overline{L}^1,\overline{\gamma}^1} < y$ and follow strategy $\left(\overline{L}, \overline{\gamma}\right)$ when the current surplus reaches y. This strategy is admissible. If there is no claim up to time $t_0 = (y - x)/p$, the surplus $X_{t_0}^{\overline{L}^1,\overline{\gamma}^1} = y$. The probability of reaching y before the first claim is $e^{-\beta t_0}$, so we obtain

$$V(x) \geq V_{\overline{L}^1,\overline{\gamma}^1}(x) \geq V_{\overline{L},\overline{\gamma}}(y)e^{-(c+\beta)t_0} \geq (V(y) - \varepsilon)\, e^{-(c+\beta)(y-x)/p},$$

and we get the result. □

As a direct consequence of the previous proposition we have that V is increasing and locally Lipschitz in $[0, +\infty)$; this implies that V is absolutely continuous, that $V'(x)$ exists a.e., and that $1 \leq V'(x) \leq V(x)(c + \beta)/p$ at the points where the derivative exists.

Remark 2.7. We will prove later that the linear growth condition given by Proposition 2.10 can be improved to $V(x) \leq x + p/c$ for $x \geq 0$.

As in Lemmas 1.2 and 2.2, there is a DPP for this optimization problem.

Lemma 2.5. *For any $x \geq 0$ and any stopping time $\overline{\tau}$, we can write*

$$V(x) = \sup_{(\overline{L},\overline{\gamma})\in\Pi_x^{L,\gamma}} E_x\left(\int_0^{\overline{\tau}\wedge\tau^{\overline{L},\overline{\gamma}}} e^{-cs}dL_s + e^{-c\left(\tau\wedge\tau^{\overline{L},\overline{\gamma}}\right)}V(X^{\overline{L},\overline{\gamma}}_{\tau\wedge\tau^{\overline{L},\overline{\gamma}}})\right).$$

Assume that V is continuously differentiable at x. Given any $l \geq 0$ and any $\gamma \in \Gamma$, let us consider the admissible strategy $(\overline{L}, \overline{\gamma})$ which pays dividends at constant rate l and invest a constant fraction γ of the surplus in the financial market. Let us call the corresponding controlled surplus process $X_t^{\overline{L},\overline{\gamma}}$ and the corresponding ruin time τ. The surplus process $X_{\tau\wedge t}^{\overline{L},\overline{\gamma}}$ stopped at the ruin time is a Markov process, so by (1.14) and (2.27), we get

$$\tilde{\mathcal{G}}\left(X_{t\wedge\tau}^{\overline{L},\overline{\gamma}}, V\right)(x) = \tfrac{\sigma^2\gamma^2x^2}{2}V''(x) + (p - l + r\gamma x)\, V'(x) - (\beta + c)V(x) + \beta\mathcal{I}(V)(x). \tag{2.40}$$

Using Lemma 2.5, we have

$$V(x) \geq E_x(\int_0^{\tau\wedge t} e^{-c\,s}\, l\, ds) + E_x\left(e^{-c\,(\tau\wedge t)} V(X_{\tau\wedge t}^{\overline{L},\overline{\gamma}})\right),$$

and then, we obtain the inequality

$$\sup_{l\geq 0,\gamma\in\Gamma}\left\{l + \tilde{\mathcal{G}}\left(X_{t\wedge\tau}^{\overline{L},\overline{\gamma}}, V\right)(x)\right\} \leq 0.$$

The HJB equation of this optimization problem is

$$\sup_{l\geq 0,\gamma\in\Gamma}\left\{l + \tilde{\mathcal{G}}\left(X_{t\wedge\tau}^{\overline{L},\overline{\gamma}}, V\right)(x)\right\} = 0. \tag{2.41}$$

Therefore, as in Sects. 1.5.2 and 2.1.2, we obtain that this equation can be written as

$$\max\{1 - V'(x), \sup_{\gamma\in\Gamma} \tilde{\mathcal{L}}_\gamma(V)(x)\} = 0, \tag{2.42}$$

where

$$\tilde{\mathcal{L}}_\gamma(V)(x) = \tfrac{\sigma^2\gamma^2x^2}{2} V''(x) + (p + r\gamma x)\, V'(x) - (\beta + c)V(x) + \beta\mathcal{I}(V)(x). \tag{2.43}$$

2.3 Ito´s Lemma and Infinitesimal Generators

The results of this section are technical and will be used to relate the composition of a function with a controlled surplus process and the corresponding infinitesimal generator. We consider nonnegative smooth enough functions $u : \mathbf{R}_+ \to \mathbf{R}$ and we extend the definition of u in $(-\infty, 0)$ as any nonnegative constant.

Proposition 2.12. *Let $Z = (Z_t)_{t\geq 0}$ be the surplus process defined either in (1.1) or in (2.3) or in (2.22) with initial value x; let τ be the corresponding ruin time, then we can write for any finite stopping time $\tau^* \leq \tau$*

$$u(Z_{\tau^*}) - u(x) = \int_0^{\tau^*} \mathcal{L}\,(u)\,(Z_{s^-})ds + M_{\tau^*},$$

where $\mathcal{L}$ is the operator defined either in (1.13) or in (2.7) or in (2.29). M_t is a martingale with zero expectation in the first two cases and a local martingale with zero expectation in the third case.

Proof. Let us assume first that Z is the surplus process $(X_t)_{t\geq 0}$ defined in (1.1). Take a nonnegative continuously differentiable function u in $\mathbf{R}_+$; using the change of variables formula for finite variation processes, we can write

$$
\begin{aligned}
&u(X_{\tau^*}) - u(x) \\
&= \int_0^{\tau^*} u'(X_{s}-)dX_s + \sum_{\substack{X_{s}- \neq X_s \\ s \leq \tau^*}} (u(X_s) - u(X_{s}-) - u'(X_{s}-)(X_s - X_{s}-)) \\
&= \int_0^{\tau^*} u'(X_{s}-)pds + \sum_{\substack{X_{s}- \neq X_s \\ s \leq \tau^*}} (u(X_s) - u(X_{s}-)) \\
&= \int_0^{\tau^*} \mathcal{L}_0(u)(X_{s}-)ds + M^0_{\tau^*},
\end{aligned}
$$

where $\mathcal{L}_0$ is the operator defined in (1.13) and

$$
\begin{aligned}
M^0_t = &\sum_{\substack{X_{s}- \neq X_s \\ s \leq t}} (u(X_s) - u(X_{s}-)) \\
&- \beta \int_0^t \int_0^\infty (u(X_{s}- - \alpha) - u(X_{s}-))\, dF(\alpha)ds
\end{aligned}
\tag{2.44}
$$

is a martingale with zero expectation because $0 \leq u(X_s) \leq \max_{y \in [0, x+pt]} u(y)$ for $s \leq t$.

In the case that Z is the surplus process $\left(X^{\overline{R}}_t\right)_{t \geq 0}$ defined in (2.3), we also have that $0 \leq u(X^{\overline{R}}_s) \leq \max_{y \in [0, x+pt]} u(y)$ for $s \leq t$, and so we obtain a similar formula with the following zero-expectation martingale:

$$
\begin{aligned}
M^{\overline{R}}_t = &\sum_{\substack{X_{s}- \neq X_s \\ s \leq t}} \left(u(X^{\overline{R}}_s) - u(X^{\overline{R}}_{s-})\right) \\
&- \beta \int_0^t \int_0^\infty \left(u(X^{\overline{R}}_{s-} - \alpha) - u(X^{\overline{R}}_{s-})\right) dF_{R_s}(\alpha)ds.
\end{aligned}
\tag{2.45}
$$

Finally, in the case that Z is the surplus process $\left(X^{\overline{\gamma}}_t\right)_{t \geq 0}$ defined in (2.22), take u a nonnegative twice continuously differentiable function in $\mathbf{R}_+$; we can write, using the Itô's formula,

$$
\begin{aligned}
&u(X^{\overline{\gamma}}_{\tau^*}) - u(x) \\
&= \int_0^{\tau^*} d\left(u(X^{\overline{\gamma}}_s)\right) \\
&= \int_0^{\tau^*} u'(X^{\overline{\gamma}}_{s-})dX^{\overline{\gamma}}_s + \int_0^{\tau^*} \frac{\left(r\gamma_s X^{\overline{\gamma}}_{s-}\right)^2}{2} u''(X^{\overline{\gamma}}_{s-})ds \\
&\quad + \sum_{\substack{X_{s}- \neq X_s \\ s \leq \tau^*}} \left(u(X^{\overline{\gamma}}_s) - u(X^{\overline{\gamma}}_{s-}) - u'(X^{\overline{\gamma}}_{s-})(X^{\overline{\gamma}}_s - X^{\overline{\gamma}}_{s-})\right) \\
&= \int_0^{\tau^*} u'(X^{\overline{\gamma}}_{s-})\left(p + rX^{\overline{\gamma}}_{s-}\gamma_{s-}\right) ds + \sigma \int_0^{\tau^*} \gamma_{s-} X^{\overline{\gamma}}_{s-} u'(X^{\overline{\gamma}}_{s-})dW_s \\
&\quad + \int_0^{\tau^*} \frac{\left(r\gamma_s X^{\overline{\gamma}}_{s-}\right)^2}{2} u''(X^{\overline{\gamma}}_{s-})ds + \sum_{\substack{X_{s}- \neq X_s \\ s \leq \tau^*}} \left(u(X^{\overline{\gamma}}_s) - u(X^{\overline{\gamma}}_{s-})\right) \\
&= \int_0^{\tau^*} \mathcal{L}_{\gamma_{s-}}(u)(X^{\overline{\gamma}}_{s-})ds + M^{\gamma}_{\tau^*},
\end{aligned}
$$

where

$$M_t^\gamma = \sum_{\substack{X_{s-} \neq X_s \\ s \leq t}} \left(u(X_s^{\overline{\gamma}}) - u(X_{s-}^{\overline{\gamma}}) \right) - \beta \int_0^t \int_0^\infty \left(u(X_{s-}^{\overline{\gamma}} - \alpha) - u(X_{s-}^{\overline{\gamma}}) \right) dF(\alpha) ds$$
$$+\sigma \int_0^t \gamma_{s-} X_{s-}^{\overline{\gamma}} u'(X_{s-}^{\overline{\gamma}}) dW_s$$

is a local martingale with zero expectation; to see that take for instance the sequence of stopping times $\tau_n = \min\{t : X_t \leq n\}$, then $M_{t \wedge \tau_n}^\gamma$ is a martingale with zero expectation for all $n \geq 1$. Note that this local martingale is a sum of a two local martingales, one coming from the compound Poisson process (similar to M_t^0) and the other coming from the Brownian motion. □

Remark 2.8. Taking expectation in the result of Proposition 2.12, we get the Dynkin formula for the process Z (see for instance Sect. 1.3 in [49]).

Proposition 2.13. *Let $\overline{Z} = (Z_t)_{t \geq 0}$ be the controlled surplus process with dividends defined either in (1.7) or in (2.14) or in (2.31) with initial value x; let τ be the corresponding ruin time, then we can write for any finite stopping time $\tau^* \leq \tau$*

$$\begin{aligned} e^{-c\tau^*} u(Z_t) - u(x) = & \int_0^{\tau^*} \tilde{\mathcal{L}}(u)(Z_{s-}) e^{-cs} ds - \int_0^{\tau^*} e^{-cs} dL_s \\ & + \int_0^{\tau^*} (1 - u'(Z_{s-})) e^{-cs} dL_s^c \\ & + \sum_{\substack{L_{s+} \neq L_s \\ s < \tau^*}} \left(\int_0^{L_{s+} - L_s} (1 - u'(Z_s - \alpha)) \, d\alpha \right) + \tilde{M}_{\tau^*} \end{aligned}$$

where $\tilde{\mathcal{L}}$ is the operator defined either in (1.22) or in (2.20) or in (2.43). $\tilde{M}_t$ is a martingale with zero expectation in the first two cases and a local martingale with zero expectation in the third case.

Proof. Let us assume first that $\overline{Z}$ is the surplus process $\left(X_t^{\overline{L}} \right)_{t \geq 0}$ defined in (1.7). Since L_t is nondecreasing and left continuous, it can be written as

$$L_t = \int_0^t dL_s^c + \sum_{\substack{X_{s+} \neq X_s \\ s < t}} (L_{s+} - L_s) \tag{2.46}$$

where L_s^c is a continuous and nondecreasing function. Take a nonnegative continuously differentiable function u in $\mathbf{R}_+$. Since the function $e^{-ct} u(x)$ is continuously differentiable in $\mathbf{R}_+$, using the expression (2.46) and the change of variables formula for finite variation processes (see for instance [51]), we can write

$$\begin{aligned}
u(X^{\overline{L}}_{\tau^*})e^{-c\tau^*} - u(x) &= \int_0^{\tau^*} e^{-cs} d\left(u(X^{\overline{L}}_s)\right) - c\int_0^{\tau^*} u(X^{\overline{L}}_{s-})e^{-cs}ds \\
&= \int_0^{\tau^*} u'(X^{\overline{L}}_{s-})e^{-cs}pds - c\int_0^{\tau^*} u(X^{\overline{L}}_{s-})e^{-cs}ds \\
&\quad + \sum_{\substack{X^{\overline{L}}_{s-}\neq X^{\overline{L}}_s \\ s\leq\tau^*}} \left(u(X^{\overline{L}}_s) - u(X^{\overline{L}}_{s-})\right)e^{-cs} \\
&\quad - \int_0^{\tau^*} u'(X^{\overline{L}}_{s-})e^{-cs}dL^c_s + \sum_{\substack{X^{\overline{L}}_{s+}\neq X^{\overline{L}}_s \\ s<\tau^*}} \left(u(X^{\overline{L}}_{s+}) - u(X^{\overline{L}}_s)\right)e^{-cs}.
\end{aligned} \tag{2.47}$$

But $X^{\overline{L}}_{s+} \neq X^{\overline{L}}_s$ only at the jumps of L_s, then $X^{\overline{L}}_{s+} - X^{\overline{L}}_s = -(L_{s+} - L_s)$ and

$$\begin{aligned}
&-\int_0^{\tau^*} u'(X^{\overline{L}}_{s-})e^{-cs}dL^c_s + \sum_{\substack{X^{\overline{L}}_{s+}\neq X^{\overline{L}}_s \\ s<\tau^*}} \left(u(X^{\overline{L}}_{s+}) - u(X^{\overline{L}}_s)\right)e^{-cs} \\
&= -\int_0^{\tau^*} u'(X^{\overline{L}}_{s-})e^{-cs}dL^c_s - \sum_{\substack{L_{s+}\neq L_s \\ s<\tau^*}} \left(\int_0^{L_{s+}-L_s} u'(X^{\overline{L}}_s - \alpha)d\alpha\right)e^{-cs}.
\end{aligned} \tag{2.48}$$

$$\begin{aligned}
&= -\int_0^{\tau^*} e^{-cs}dL_s + \int_0^{\tau^*}(1 - u'(X^{\overline{L}}_{s-}))e^{-cs}dL^c_s \\
&\quad + \sum_{\substack{L_{s+}\neq L_s \\ s<\tau^*}} \left(\int_0^{L_{s+}-L_s}\left(1 - u'(X^{\overline{L}}_s - \alpha)\right)d\alpha\right)e^{-cs}.
\end{aligned}$$

On the other hand, $X^{\overline{L}}_s \neq X^{\overline{L}}_{s-}$ only at the arrival of a claim, so

$$\begin{aligned}
\tilde{M}^0_t = &\sum_{\substack{X^{\overline{L}}_{s-}\neq X^{\overline{L}}_s \\ s\leq t}} \left(u(X^{\overline{L}}_s) - u(X^{\overline{L}}_{s-})\right)e^{-cs} \\
&-\beta\int_0^t e^{-cs}\int_0^\infty \left(u(X^{\overline{L}}_{s-} - \alpha) - u(X^{\overline{L}}_{s-})\right)dF(\alpha)ds
\end{aligned}$$

is a martingale with zero expectation because $0 \leq u(X^{\overline{L}}_s) \leq \max_{y\in[0,x+pt]} u(y)$ for $s \leq t$. So we have the result for this case.

In the case that the controlled surplus process $\overline{Z}$ is the one defined in (2.14), the proof is similar and the corresponding zero-expectation martingale is

$$\tilde{M}_t^{\overline{R}} = \sum_{\substack{X_{s-}^{\overline{L},\overline{R}} \neq X_s^{\overline{L},\overline{R}} \\ s \leq t}} \left(u(X_s^{\overline{L},\overline{R}}) - u(X_{s-}^{\overline{L},\overline{R}}) \right) e^{-cs}$$

$$-\beta \int_0^t e^{-cs} \int_0^\infty \left(u(X_{s-}^{\overline{L},\overline{R}} - \alpha) - u(X_{s-}^{\overline{L},\overline{R}}) \right) dF_{R_x}(\alpha) ds.$$

In the case that the controlled surplus process $\overline{Z}$ is the one defined in (2.31), the corresponding zero-expectation local martingale turns to be

$$\tilde{M}_t^{\overline{\gamma}} = \sum_{\substack{X_{s-}^{\overline{L},\overline{\gamma}} \neq X_s^{\overline{L},\overline{\gamma}} \\ s \leq t}} \left(u(X_s^{\overline{L},\overline{\gamma}}) - u(X_{s-}^{\overline{L},\overline{\gamma}}) \right) e^{-cs}$$

$$-\beta \int_0^t e^{-cs} \int_0^\infty \left(u(X_{s-}^{\overline{L},\overline{\gamma}} - \alpha) - u(X_{s-}^{\overline{L},\overline{\gamma}}) \right) dF(\alpha) ds$$

$$+\sigma \int_0^t e^{-cs} \gamma_{s-} X_{s-}^{\overline{L},\overline{\gamma}} u'(X_{s-}^{\overline{L},\overline{\gamma}}) dW_s.$$

The last result can be proved with the same argument used in the previous proposition for the case of investment and in this proposition for dividend payments. □

Remark 2.9. Propositions 2.12 and 2.13 imply that $u(Z_{t\wedge\tau})$ is a martingale for any smooth solution u of the equation $\mathcal{L}(u) = 0$ and that $u(Z_{t\wedge\tau}^{\overline{L}})$ is a supermartingale for any function u which satisfies $\max\{1-u', \tilde{\mathcal{L}}(u)\} \leq 0$ in the first two cases. In the third case, the result is the same but with local martingale and local supermartingale.

2.4 Comments and References

Let us first give some references on the problem of optimal survival probability in the classical risk model. The case of reinsurance control was addressed by Schmidli [55] and Hipp and Vogt [34]. The case of investment control was considered by Hipp and Plum [32] for $\Gamma = \mathbf{R}$ and by Azcue and Muler [10] for $\Gamma = [0, \hat{\gamma}]$. Investment control with state-dependent constraints was studied by Edalati and Hipp [26]. Hipp and Taksar [33] and Schmidli [56] considered the combination of investment and reinsurance controls. In all the cases, the claim-size distributions were continuous. In this book we allow, for the problems with reinsurance control, general claim-size distributions.

Let us now give some references on optimal dividend payments in the classical risk model. The reinsurance control case was addressed in [9] and by Mnif and Sulem [47]. Azcue and Muler [11] considered the investment control problem for bounded-density claim-size distributions .

Optimal survival probability and optimal dividend payments with investment control can also be studied in the limit diffusion setting; the HJB equation for the optimal survival probability can be written as

$$\sup_{\gamma\in\Gamma}\left(\frac{\sigma^2\gamma^2x^2}{2}\delta''(x)+r\gamma x\delta'(x)\right)+\mathcal{L}_D(\delta)(x)=0, \tag{2.49}$$

(compare with Remark 2.4) and the HJB equation for the optimal dividend payments problem as

$$\max\left\{\sup_{\gamma\in\Gamma}\left(\frac{\sigma^2\gamma^2x^2}{2}V''(x)+r\gamma xV'(x)\right)+\tilde{\mathcal{L}}_D(V)(x),1-V'(x)\right\}=0, \tag{2.50}$$

where $\mathcal{L}_D$ and $\tilde{\mathcal{L}}_D$ are defined in (1.24) and (1.27), respectively. In this diffusion setting, Browne [18] studied the problem of optimal survival probability with investment control and Højgaard and Taksar [35] considered the one of dividend payments with investment control (they also included an option to reduce risk exposure by reinsurance).

Reinsurance control in the diffusion setting was considered by Schmidli [55] for optimal survival probability and by Asmussen et al. [5] for optimal dividend payments.

Some of these problems were studied in the book of Schmidli [57].

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