

Contents

1	L^p Spaces and Interpolation	1
1.1	L^p and Weak L^p	1
1.1.1	The Distribution Function	3
1.1.2	Convergence in Measure	6
1.1.3	A First Glimpse at Interpolation	9
	Exercises	11
1.2	Convolution and Approximate Identities	17
1.2.1	Examples of Topological Groups	18
1.2.2	Convolution	19
1.2.3	Basic Convolution Inequalities	21
1.2.4	Approximate Identities	25
	Exercises	30
1.3	Interpolation	33
1.3.1	Real Method: The Marcinkiewicz Interpolation Theorem . . .	33
1.3.2	Complex Method: The Riesz–Thorin Interpolation Theorem	36
1.3.3	Interpolation of Analytic Families of Operators	40
	Exercises	45
1.4	Lorentz Spaces	48
1.4.1	Decreasing Rearrangements	48
1.4.2	Lorentz Spaces	52
1.4.3	Duals of Lorentz Spaces	56
1.4.4	The Off-Diagonal Marcinkiewicz Interpolation Theorem . . .	60
	Exercises	74
2	Maximal Functions, Fourier Transform, and Distributions	85
2.1	Maximal Functions	86
2.1.1	The Hardy–Littlewood Maximal Operator	86
2.1.2	Control of Other Maximal Operators	90

2.1.3	Applications to Differentiation Theory	93
	Exercises	98
2.2	The Schwartz Class and the Fourier Transform	104
2.2.1	The Class of Schwartz Functions	105
2.2.2	The Fourier Transform of a Schwartz Function	108
2.2.3	The Inverse Fourier Transform and Fourier Inversion	111
2.2.4	The Fourier Transform on $L^1 + L^2$	113
	Exercises	116
2.3	The Class of Tempered Distributions	119
2.3.1	Spaces of Test Functions	119
2.3.2	Spaces of Functionals on Test Functions	120
2.3.3	The Space of Tempered Distributions	123
	Exercises	131
2.4	More About Distributions and the Fourier Transform	133
2.4.1	Distributions Supported at a Point	134
2.4.2	The Laplacian	135
2.4.3	Homogeneous Distributions	136
	Exercises	143
2.5	Convolution Operators on L^p Spaces and Multipliers	146
2.5.1	Operators That Commute with Translations	146
2.5.2	The Transpose and the Adjoint of a Linear Operator	150
2.5.3	The Spaces $\mathcal{M}^{p,q}(\mathbf{R}^n)$	151
2.5.4	Characterizations of $\mathcal{M}^{1,1}(\mathbf{R}^n)$ and $\mathcal{M}^{2,2}(\mathbf{R}^n)$	153
2.5.5	The Space of Fourier Multipliers $\mathcal{M}_p(\mathbf{R}^n)$	155
	Exercises	159
2.6	Oscillatory Integrals	161
2.6.1	Phases with No Critical Points	161
2.6.2	Sublevel Set Estimates and the Van der Corput Lemma	164
	Exercises	169
3	Fourier Series	173
3.1	Fourier Coefficients	173
3.1.1	The n -Torus \mathbf{T}^n	174
3.1.2	Fourier Coefficients	175
3.1.3	The Dirichlet and Fejér Kernels	178
	Exercises	182
3.2	Reproduction of Functions from Their Fourier Coefficients	183
3.2.1	Partial sums and Fourier inversion	183
3.2.2	Fourier series of square summable functions	185
3.2.3	The Poisson Summation Formula	187
	Exercises	191
3.3	Decay of Fourier Coefficients	192
3.3.1	Decay of Fourier Coefficients of Arbitrary Integrable Functions	193
3.3.2	Decay of Fourier Coefficients of Smooth Functions	195

3.3.3	Functions with Absolutely Summable Fourier Coefficients	200
	Exercises	202
3.4	Pointwise Convergence of Fourier Series	204
3.4.1	Pointwise Convergence of the Fejér Means	204
3.4.2	Almost Everywhere Convergence of the Fejér Means	207
3.4.3	Pointwise Divergence of the Dirichlet Means	210
3.4.4	Pointwise Convergence of the Dirichlet Means	212
	Exercises	214
3.5	A Tauberian theorem and Functions of Bounded Variation	216
3.5.1	A Tauberian theorem	216
3.5.2	The sine integral function	218
3.5.3	Further properties of functions of bounded variation	219
3.5.4	Gibbs phenomenon	221
	Exercises	225
3.6	Lacunary Series and Sidon Sets	226
3.6.1	Definition and Basic Properties of Lacunary Series	227
3.6.2	Equivalence of L^p Norms of Lacunary Series	229
3.6.3	Sidon sets	235
	Exercises	237
4	Topics on Fourier Series	241
4.1	Convergence in Norm, Conjugate Function, and Bochner–Riesz Means	241
4.1.1	Equivalent Formulations of Convergence in Norm	242
4.1.2	The L^p Boundedness of the Conjugate Function	246
4.1.3	Bochner–Riesz Summability	250
	Exercises	253
4.2	A. E. Divergence of Fourier Series and Bochner–Riesz means	255
4.2.1	Divergence of Fourier Series of Integrable Functions	255
4.2.2	Divergence of Bochner–Riesz Means of Integrable Functions	261
	Exercises	270
4.3	Multipliers, Transference, and Almost Everywhere Convergence	271
4.3.1	Multipliers on the Torus	271
4.3.2	Transference of Multipliers	275
4.3.3	Applications of Transference	280
4.3.4	Transference of Maximal Multipliers	281
4.3.5	Applications to Almost Everywhere Convergence	285
4.3.6	Almost Everywhere Convergence of Square Dirichlet Means	287
	Exercises	289

4.4	Applications to Geometry and Partial Differential Equations	292
4.4.1	The Isoperimetric Inequality	292
4.4.2	The Heat Equation with Periodic Boundary Condition	294
	Exercises	298
4.5	Applications to Number theory and Ergodic theory	299
4.5.1	Evaluation of the Riemann Zeta Function at even Natural numbers	299
4.5.2	Equidistributed sequences	302
4.5.3	The Number of Lattice Points inside a Ball	305
	Exercises	308
5	Singular Integrals of Convolution Type	313
5.1	The Hilbert Transform and the Riesz Transforms	313
5.1.1	Definition and Basic Properties of the Hilbert Transform . . .	314
5.1.2	Connections with Analytic Functions	317
5.1.3	L^p Boundedness of the Hilbert Transform	319
5.1.4	The Riesz Transforms	324
	Exercises	329
5.2	Homogeneous Singular Integrals and the Method of Rotations	333
5.2.1	Homogeneous Singular and Maximal Singular Integrals . . .	333
5.2.2	L^2 Boundedness of Homogeneous Singular Integrals	336
5.2.3	The Method of Rotations	339
5.2.4	Singular Integrals with Even Kernels	341
5.2.5	Maximal Singular Integrals with Even Kernels	347
	Exercises	353
5.3	The Calderón–Zygmund Decomposition and Singular Integrals	355
5.3.1	The Calderón–Zygmund Decomposition	355
5.3.2	General Singular Integrals	358
5.3.3	L^r Boundedness Implies Weak Type $(1, 1)$ Boundedness . . .	359
5.3.4	Discussion on Maximal Singular Integrals	362
5.3.5	Boundedness for Maximal Singular Integrals Implies Weak Type $(1, 1)$ Boundedness	366
	Exercises	371
5.4	Sufficient Conditions for L^p Boundedness	374
5.4.1	Sufficient Conditions for L^p Boundedness of Singular Integrals	375
5.4.2	An Example	378
5.4.3	Necessity of the Cancellation Condition	379
5.4.4	Sufficient Conditions for L^p Boundedness of Maximal Singular Integrals	380
	Exercises	384
5.5	Vector-Valued Inequalities	385
5.5.1	ℓ^2 -Valued Extensions of Linear Operators	386
5.5.2	Applications and ℓ^r -Valued Extensions of Linear Operators	390

5.5.3	General Banach-Valued Extensions	391
	Exercises	398
5.6	Vector-Valued Singular Integrals	401
5.6.1	Banach-Valued Singular Integral Operators	402
5.6.2	Applications	408
5.6.3	Vector-Valued Estimates for Maximal Functions	411
	Exercises	414
6	Littlewood–Paley Theory and Multipliers	419
6.1	Littlewood–Paley Theory	419
6.1.1	The Littlewood–Paley Theorem	420
6.1.2	Vector-Valued Analogues	426
6.1.3	L^p Estimates for Square Functions Associated with Dyadic Sums	426
6.1.4	Lack of Orthogonality on L^p	431
	Exercises	434
6.2	Two Multiplier Theorems	437
6.2.1	The Marcinkiewicz Multiplier Theorem on \mathbf{R}	439
6.2.2	The Marcinkiewicz Multiplier Theorem on \mathbf{R}^n	441
6.2.3	The Mihlin–Hörmander Multiplier Theorem on \mathbf{R}^n	445
	Exercises	450
6.3	Applications of Littlewood–Paley Theory	453
6.3.1	Estimates for Maximal Operators	453
6.3.2	Estimates for Singular Integrals with Rough Kernels	455
6.3.3	An Almost Orthogonality Principle on L^p	459
	Exercises	461
6.4	The Haar System, Conditional Expectation, and Martingales	463
6.4.1	Conditional Expectation and Dyadic Martingale Differences	464
6.4.2	Relation Between Dyadic Martingale Differences and Haar Functions	465
6.4.3	The Dyadic Martingale Square Function	469
6.4.4	Almost Orthogonality Between the Littlewood–Paley Operators and the Dyadic Martingale Difference Operators	471
	Exercises	474
6.5	The Spherical Maximal Function	475
6.5.1	Introduction of the Spherical Maximal Function	475
6.5.2	The First Key Lemma	478
6.5.3	The Second Key Lemma	479
6.5.4	Completion of the Proof	481
	Exercises	481
6.6	Wavelets and Sampling	482
6.6.1	Some Preliminary Facts	483
6.6.2	Construction of a Nonsmooth Wavelet	485

6.6.3	Construction of a Smooth Wavelet	486
6.6.4	Sampling	490
	Exercises	494
7	Weighted Inequalities	499
7.1	The A_p Condition	499
7.1.1	Motivation for the A_p Condition	500
7.1.2	Properties of A_p Weights	503
	Exercises	511
7.2	Reverse Hölder Inequality for A_p Weights and Consequences	514
7.2.1	The Reverse Hölder Property of A_p Weights	514
7.2.2	Consequences of the Reverse Hölder Property	518
	Exercises	521
7.3	The A_∞ Condition	525
7.3.1	The Class of A_∞ Weights	525
7.3.2	Characterizations of A_∞ Weights	527
	Exercises	530
7.4	Weighted Norm Inequalities for Singular Integrals	532
7.4.1	Singular Integrals of Non Convolution type	532
7.4.2	A Good Lambda Estimate for Singular Integrals	533
7.4.3	Consequences of the Good Lambda Estimate	539
7.4.4	Necessity of the A_p Condition	543
	Exercises	545
7.5	Further Properties of A_p Weights	546
7.5.1	Factorization of Weights	546
7.5.2	Extrapolation from Weighted Estimates on a Single L^{p_0}	548
7.5.3	Weighted Inequalities Versus Vector-Valued Inequalities	554
	Exercises	558
A	Gamma and Beta Functions	563
A.1	A Useful Formula	563
A.2	Definitions of $\Gamma(z)$ and $B(z, w)$	563
A.3	Volume of the Unit Ball and Surface of the Unit Sphere	565
A.4	Computation of Integrals Using Gamma Functions	565
A.5	Meromorphic Extensions of $B(z, w)$ and $\Gamma(z)$	566
A.6	Asymptotics of $\Gamma(x)$ as $x \rightarrow \infty$	567
A.7	Euler's Limit Formula for the Gamma Function	568
A.8	Reflection and Duplication Formulas for the Gamma Function	570
B	Bessel Functions	573
B.1	Definition	573
B.2	Some Basic Properties	573
B.3	An Interesting Identity	576
B.4	The Fourier Transform of Surface Measure on \mathbf{S}^{n-1}	577
B.5	The Fourier Transform of a Radial Function on \mathbf{R}^n	577

B.6	Bessel Functions of Small Arguments	578
B.7	Bessel Functions of Large Arguments	579
B.8	Asymptotics of Bessel Functions	580
B.9	Bessel Functions of general complex indices	582
C	Rademacher Functions	585
C.1	Definition of the Rademacher Functions	585
C.2	Khinchine's Inequalities	586
C.3	Derivation of Khinchine's Inequalities	586
C.4	Khinchine's Inequalities for Weak Type Spaces	589
C.5	Extension to Several Variables	589
D	Spherical Coordinates	591
D.1	Spherical Coordinate Formula	591
D.2	A Useful Change of Variables Formula	592
D.3	Computation of an Integral over the Sphere	593
D.4	The Computation of Another Integral over the Sphere	593
D.5	Integration over a General Surface	594
D.6	The Stereographic Projection	594
E	Some Trigonometric Identities and Inequalities	597
F	Summation by Parts	599
G	Basic Functional Analysis	601
H	The Minimax Lemma	603
I	Taylor's and Mean Value Theorem in Several Variables	607
I.1	Multivariable Taylor's Theorem	607
I.2	The Mean value Theorem	608
J	The Whitney Decomposition of Open Sets in \mathbb{R}^n	609
J.1	Decomposition of Open Sets	609
J.2	Partition of Unity adapted to Whitney cubes	611
	Glossary	613
	References	617
	Index	633

Classical Fourier Analysis

Grafakos, L.

2014, XVII, 638 p. 14 illus., 2 illus. in color., Hardcover

ISBN: 978-1-4939-1193-6