Chapter 2
Hardy Spaces, Besov Spaces, and Triebel–Lizorkin Spaces

The main function spaces we study in this chapter are Hardy spaces which measure smoothness within the realm of rough distributions. Hardy spaces also serve as a substitute for L^p when $p < 1$. We also take a quick look at Besov–Lipschitz and Triebel–Lizorkin spaces, which provide an appropriate framework that unifies the subject of function spaces.

One of the main achievements of this chapter is the characterization of these spaces using Littlewood–Paley theory. Another major accomplishment of this chapter is the atomic characterization of these function spaces. This is obtained from the Littlewood–Paley characterization of these spaces in a single way for all of them.

2.1 Hardy Spaces

The Hardy spaces $H^p(\mathbf{R}^n)$, $0 < p < \infty$, are spaces of distributions which become more singular as p decreases. These function spaces have remarkable similarities to L^p and, in many ways, serve as a substitute for L^p when $p < 1$. In previous sections, we have been able to characterize L^p spaces, Sobolev spaces, and Lipschitz spaces using Littlewood–Paley theory, and it should not come as a surprise that a similar characterization is available for the Hardy spaces as well.

There exists an abundance of equivalent characterizations for Hardy spaces, of which only a few representative ones are discussed in this section. A reader interested in going through the material quickly may define the Hardy space H^p as the space of all tempered distributions f modulo polynomials for which

$$\big\|f\big\|_{H^p} = \bigg\|\Big(\sum_{j\in\mathbf{Z}}|\Delta_j(f)|^2\Big)^{\frac{1}{2}}\bigg\|_{L^p} < \infty \tag{2.1.1}$$

whenever $0 < p \le 1$. An atomic decomposition for Hardy spaces can be obtained from this definition (see Section 2.3), and once this is available, the analysis of these spaces is significantly simplified. For historical reasons, however, we choose

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to define Hardy spaces using a more classical approach, and, as a result, we have to go through a considerable amount of work to obtain the characterization in (2.1.1).

2.1.1 Definition of Hardy Spaces

To give the definition of Hardy spaces on $\mathbf{R}^n$, we need some background. We say that a tempered distribution v is *bounded* if $\varphi * v \in L^\infty(\mathbf{R}^n)$ whenever φ is in $\mathscr{S}(\mathbf{R}^n)$. We observe that if v is a bounded tempered distribution and $h \in L^1(\mathbf{R}^n)$, then the convolution $h*v$ can be defined as a distribution via the convergent integral

$$\langle h*v, \varphi\rangle = \langle \widetilde{\varphi}*v, \widetilde{h}\rangle = \int_{\mathbf{R}^n} (\widetilde{\varphi}*v)(x)\widetilde{h}(x)\,dx,$$

where φ is a Schwartz function and $\widetilde{\varphi}(x) = \varphi(-x)$, $\widetilde{h}(x) = h(-x)$.

The Poisson kernel P is the function

$$P(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}\,. \tag{2.1.2}$$

For $t>0$, let $P_t(x) = t^{-n}P(t^{-1}x)$. If v is a bounded tempered distribution, then $P_t * v$ is a well-defined distribution, since P_t is in L^1. We claim that $P_t * v$ can be identified with a well-defined bounded function. To see this, write $1 = \widehat{\varphi}(\xi) + 1 - \widehat{\varphi}(\xi)$, where $\widehat{\varphi} \in \mathscr{S}(\mathbf{R}^n)$ is equal to 1 in a neighborhood of the origin. Then $\delta_0 = \varphi + (\delta_0 - \varphi)$ and

$$P_t * v = P_t * (\varphi * v) + P_t * (\delta_0 - \varphi) * v\,.$$

Since P_t lies in L^1 and $\varphi * v$ in L^∞, it follows that $P_t * (\varphi * v)$ is a bounded function. Also the Fourier transform of $P_t * (\delta_0 - \varphi)$ is $e^{-2\pi t|\xi|}(1-\widehat{\varphi}(\xi))$ which is a Schwartz function. Thus, $P_t * (\delta_0 - \varphi)$ is also a Schwartz function, and since v is a bounded distribution, it follows that $P_t * (\delta_0 - \varphi) * v$ is a bounded function. These observations prove that $P_t * v$ is a bounded function, whenever v is a bounded distribution.

An important property of bounded tempered distributions f is that

$$P_t * f \to f \qquad \text{in } \mathscr{S}'(\mathbf{R}^n) \text{ as } t \to 0. \tag{2.1.3}$$

For this, see Exercise 2.1.4.

Definition 2.1.1. Let f be a bounded tempered distribution on $\mathbf{R}^n$ and let $0 < p < \infty$. We say that f lies in the *Hardy space* $H^p(\mathbf{R}^n)$ if the *Poisson maximal function*

$$M(f;P)(x) = \sup_{t>0} |(P_t * f)(x)| \tag{2.1.4}$$

lies in $L^p(\mathbf{R}^n)$. If this is the case, we set

$$\big\|f\big\|_{H^p} = \big\|M(f;P)\big\|_{L^p}.$$

It is quite easy to see that the Dirac mass δ_0 does not belong in any Hardy space; indeed, $\delta_0 * P_t = P_t$ and $\sup_{t>0} P_t(x)$ is comparable to $|x|^{-n}$ which does not lie in $L^p(\mathbf{R}^n)$ for any p. However, the difference of Dirac masses $\delta_1 - \delta_{-1}$ lies in $H^p(\mathbf{R})$ for $1/2 < p < 1$. To see this, notice that

$$\sup_{t>0}\Big|(\delta_1 * P_t)(x) - (\delta_{-1} * P_t)(x)\Big| = \sup_{t>0} \frac{4|x|}{\pi}\frac{t}{(t^2+|x-1|^2)(t^2+|x+1|^2)}\,. \tag{2.1.5}$$

Suppose that $|x+1| < |x-1|$, i.e., $x < 0$. Then we have

$$\sup_{t\le|x+1|}\frac{t\,|x|}{(t^2+|x-1|^2)(t^2+|x+1|^2)} \approx \sup_{t\le|x+1|}\frac{t\,|x|}{|x-1|^2|x+1|^2} = \frac{|x|}{|x-1|^2|x+1|}\,.$$

Also,

$$\sup_{|x+1|\le t\le|x-1|}\frac{t\,|x|}{(t^2+|x-1|^2)(t^2+|x+1|^2)} \approx \sup_{|x+1|\le t\le|x-1|}\frac{t\,|x|}{|x-1|^2t^2} = \frac{|x|}{|x-1|^2|x+1|}\,,$$

while

$$\sup_{t\ge|x-1|}\frac{t\,|x|}{(t^2+|x-1|^2)(t^2+|x+1|^2)} \approx \sup_{t\ge|x-1|}\frac{t\,|x|}{t^4} = \frac{|x|}{|x-1|^3}\,.$$

Thus (2.1.5) is comparable to $\frac{|x|}{|x-1|^2|x+1|}$ for $x < 0$ and analogously to $\frac{|x|}{|x+1|^2|x-1|}$ for $x > 0$. Consequently, (2.1.5) lies in $L^p(\mathbf{R})$ if and only if $1/2 < p < 1$.

At this point we don't know whether the H^p spaces coincide with any other known spaces for some values of p. In the next theorem we show that this is the case when $1 < p < \infty$.

Theorem 2.1.2. *(a) Let $1 < p < \infty$. Then every bounded tempered distribution f in H^p is an element of L^p. Moreover, there is a constant $C_{n,p}$ such that for all such f we have*

$$\big\|f\big\|_{L^p} \le \big\|f\big\|_{H^p} \le C_{n,p}\big\|f\big\|_{L^p}\,,$$

and therefore $H^p(\mathbf{R}^n)$ coincides with $L^p(\mathbf{R}^n)$.
(b) When $p = 1$, every element of H^1 is an integrable function. In other words, $H^1(\mathbf{R}^n) \subseteq L^1(\mathbf{R}^n)$ and for all $f \in H^1$ we have

$$\big\|f\big\|_{L^1} \le \big\|f\big\|_{H^1}\,. \tag{2.1.6}$$

Proof. (a) Let $f \in H^p(\mathbf{R}^n)$ for some $1 < p < \infty$. The set $\{P_t * f:\ t > 0\}$ lies in a multiple of the unit ball of $L^p(\mathbf{R}^n)$, which is the dual space of the separable Banach space $L^{p'}(\mathbf{R}^n)$, and hence it is sequentially compact by the Banach–Alaoglu theorem. Therefore, there exists a sequence $t_j \to 0$ such that $P_{t_j} * f$ converges to some L^p function f_0 in the weak* topology of L^p. On the other hand, in view of (2.1.3), $P_{t_j} * f$ in $\mathscr{S}'(\mathbf{R}^n)$ as $t_j \to 0$, and thus the bounded tempered distribution f coincides with the L^p function f_0. Since the family $\{P_t\}_{t>0}$ is an approximate identity, Theorem 1.2.19 in [156] gives that

$$\big\|P_t * f - f\big\|_{L^p} \to 0 \qquad \text{as } t \to 0,$$

from which it follows that

$$\|f\|_{L^p} \le \Big\| \sup_{t>0} |P_t * f| \Big\|_{L^p} = \|f\|_{H^p}. \tag{2.1.7}$$

The converse inequality is a consequence of the fact that

$$\sup_{t>0} |P_t * f| \le M(f),$$

where M is the Hardy–Littlewood maximal operator. (See Corollary 2.1.12 in [156].)

(b) The case $p = 1$ requires only a small modification of the case $p > 1$. We embed L^1 in the space of finite Borel measures $\mathscr{M}$ which is the dual of the separable space $C_{00}^{\infty}(\mathbf{R}^n)$ of all continuous functions on $\mathbf{R}^n$ that vanish at infinity. By the Banach-Alaoglu theorem, the unit ball of $\mathscr{M}$ is weak* sequentially compact, and we can extract a sequence $t_j \to 0$ such that $P_{t_j} * f$ converges to some measure μ in the topology of measures. In view of (2.1.3), it follows that the distribution f can be identified with the measure μ.

It remains to show that μ is absolutely continuous with respect to Lebesgue measure, which would imply that it coincides with some L^1 function. We show that μ is absolutely continuous with respect to Lebesgue measure by showing that for all subsets E of $\mathbf{R}^n$ we have $|E| = 0 \implies |\mu(E)| = 0$. Since $\sup_{t>0} |P_t * f|$ lies in $L^1(\mathbf{R}^n)$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any measurable subset F of $\mathbf{R}^n$ we have

$$|F| < \delta \implies \int_F \sup_{t>0} |P_t * f|\, dx < \varepsilon.$$

Given E with $|E| = 0$, we can find an open set U such that $E \subseteq U$ and $|U| < \delta$. Let us denote by $\mathscr{C}_{00}(U)$ the space of continuous functions $g(x)$ that are supported in U and tend to zero as $|x| \to \infty$. Then for any g in $\mathscr{C}_{00}(U)$ we have

$$\begin{aligned}
\left| \int_{\mathbf{R}^n} g\, d\mu \right| &= \lim_{j\to\infty} \left| \int_{\mathbf{R}^n} g(x)\, (P_{t_j} * f)(x)\, dx \right| \\
&\le \|g\|_{L^\infty} \int_U \sup_{t>0} |(P_t * f)(x)|\, dx \\
&< \varepsilon \|g\|_{L^\infty}.
\end{aligned}$$

Let $|\mu|$ be the total variation of μ. Then we have (see [190] (20.49))

$$|\mu|(U) = \int_U 1\, d|\mu| = \sup \left\{ \left| \int_{\mathbf{R}^n} g\, d\mu \right| :\ g \in \mathscr{C}_{00}(U), \quad \|g\|_{L^\infty} \le 1 \right\},$$

which implies $|\mu|(U) < \varepsilon$. Since ε was arbitrary, it follows that $|\mu|(E) = 0$ and thus $\mu(E) = 0$; hence μ is absolutely continuous with respect to Lebesgue measure. Finally, (2.1.6) is a consequence of (2.1.7), which is also valid for $p = 1$. □

We may wonder whether H^1 coincides with L^1. We show in Corollary 2.4.8 that elements of H^1 have integral zero; thus H^1 is a proper subspace of L^1.

2.1.2 Quasi-norm Equivalence of Several Maximal Functions

We now obtain some characterizations of these spaces.

Definition 2.1.3. Let $a,b>0$. Let Φ be a Schwartz function and let f be a tempered distribution on $\mathbf{R}^n$. We define the *smooth maximal function of f with respect to Φ* as

$$M(f;\Phi)(x)=\sup_{t>0}|(\Phi_t*f)(x)|\,.$$

We define the *nontangential maximal function (with aperture a) of f with respect to Φ* as

$$M_a^*(f;\Phi)(x)=\sup_{t>0}\sup_{\substack{y\in\mathbf{R}^n\\ |y-x|\le at}}|(\Phi_t*f)(y)|\,.$$

We also define the *auxiliary maximal function*

$$M_b^{**}(f;\Phi)(x)=\sup_{t>0}\sup_{y\in\mathbf{R}^n}\frac{|(\Phi_t*f)(x-y)|}{(1+t^{-1}|y|)^b}\,, \tag{2.1.8}$$

and we observe that

$$M(f;\Phi)\le M_a^*(f;\Phi)\le(1+a)^b M_b^{**}(f;\Phi) \tag{2.1.9}$$

for all $a,b>0$. We note that if Φ is merely integrable, for example, if Φ is the Poisson kernel, the maximal functions $M(f;\Phi)$, $M_a^*(f;\Phi)$, and $M_b^{**}(f;\Phi)$ are well defined only for bounded tempered distributions f on $\mathbf{R}^n$.

For a fixed positive integer N and a Schwartz function φ we define the quantity

$$\mathfrak{N}_N(\varphi)=\int_{\mathbf{R}^n}(1+|x|)^N\sum_{|\alpha|\le N+1}|\partial^\alpha\varphi(x)|\,dx\,. \tag{2.1.10}$$

We now define

$$\mathscr{F}_N=\Big\{\varphi\in\mathscr{S}(\mathbf{R}^n):\ \mathfrak{N}_N(\varphi)\le 1\Big\}\,, \tag{2.1.11}$$

and we also define the *grand maximal function of f (with respect to N)* as

$$\mathscr{M}_N(f)(x)=\sup_{\varphi\in\mathscr{F}_N}M_1^*(f;\varphi)(x)\,.$$

It is a fact that all the maximal functions of the preceding subsection have comparable L^p quasi-norms for all $0<p<\infty$. This is the essence of the following theorem.

Theorem 2.1.4. *Let $0 < p < \infty$. Then the following statements are valid:*
(a) There exists a Schwartz function Φ^o with $\int_{\mathbf{R}^n} \Phi^o(x)\,dx = 1$ such that

$$\big\|M(f;\Phi^o)\big\|_{L^p} \le 500\,\big\|f\big\|_{H^p} \tag{2.1.12}$$

for all bounded distributions $f \in \mathscr{S}'(\mathbf{R}^n)$.
(b) For every $a > 0$ and every Φ in $\mathscr{S}(\mathbf{R}^n)$ there exists $C_2(n,p,a,\Phi) < \infty$ such that

$$\big\|M_a^*(f;\Phi)\big\|_{L^p} \le C_2(n,p,a,\Phi)\big\|M(f;\Phi)\big\|_{L^p} \tag{2.1.13}$$

for all distributions $f \in \mathscr{S}'(\mathbf{R}^n)$.
(c) For every $a > 0$, $b > n/p$, and every Φ in $\mathscr{S}(\mathbf{R}^n)$ there exists a constant $C_3(n,p,a,b,\Phi) < \infty$ such that

$$\big\|M_b^{**}(f;\Phi)\big\|_{L^p} \le C_3(n,p,a,b,\Phi)\big\|M_a^*(f;\Phi)\big\|_{L^p} \tag{2.1.14}$$

for all distributions $f \in \mathscr{S}'(\mathbf{R}^n)$.
(d) For every $b > 0$ and every Φ in $\mathscr{S}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n} \Phi(x)\,dx = 1$ there exists a constant $C_4(b,\Phi) < \infty$ such that if $N = [b]+1$ we have

$$\big\|\mathscr{M}_N(f)\big\|_{L^p} \le C_4(b,\Phi)\big\|M_b^{**}(f;\Phi)\big\|_{L^p} \tag{2.1.15}$$

for all distributions $f \in \mathscr{S}'(\mathbf{R}^n)$.
(e) For every positive integer N there exists a constant $C_5(n,N)$ such that every tempered distribution f with $\big\|\mathscr{M}_N(f)\big\|_{L^p} < \infty$ is a bounded distribution and satisfies

$$\big\|f\big\|_{H^p} \le C_5(n,N)\big\|\mathscr{M}_N(f)\big\|_{L^p}, \tag{2.1.16}$$

that is, it lies in the Hardy space H^p.

Choosing $\Phi = \Phi^o$ in parts (b), (c), and (d), $\frac{n}{p} < b < [\frac{n}{p}]+1$, and $N = [\frac{n}{p}]+1$, we conclude that for bounded distributions f we have

$$\big\|f\big\|_{H^p} \approx \big\|\mathscr{M}_N(f)\big\|_{L^p}.$$

Moreover, for any Schwartz function Φ with $\int_{\mathbf{R}^n} \Phi(x)\,dx = 1$ and any bounded distribution f in $\mathscr{S}'(\mathbf{R}^n)$, the following quasi-norms are equivalent

$$\big\|f\big\|_{H^p} \approx \big\|M(f;\Phi)\big\|_{L^p},$$

with constants that depend only on Φ, n, p.

Before we begin the proof of Theorem 2.1.4, we state and prove a useful lemma.

Lemma 2.1.5. *Let $m \in \mathbf{Z}^+$ and let Φ in $\mathscr{S}(\mathbf{R}^n)$ satisfy $\int_{\mathbf{R}^n} \Phi(x)\,dx = 1$. Then there exists a constant $C_0(\Phi,m)$ such that for any Ψ in $\mathscr{S}(\mathbf{R}^n)$, there are Schwartz functions $\Theta^{(s)}$, $0 \le s \le 1$, with the properties*

$$\Psi(x) = \int_0^1 (\Theta^{(s)} * \Phi_s)(x)\,ds \tag{2.1.17}$$

and

$$\int_{\mathbf{R}^n}(1+|x|)^m|\Theta^{(s)}(x)|\,dx \le C_0(\Phi,m)\,s^m\,\mathfrak{N}_m(\Psi). \tag{2.1.18}$$

Proof. We start with a smooth function ζ supported in $[0,1]$ that satisfies

$$\begin{aligned}
0 \le \zeta(s) &\le \frac{2\,s^m}{m!} && \text{for all } 0 \le s \le 1\,,\\
\zeta(s) &= \frac{s^m}{m!} && \text{for all } 0 \le s \le \frac{1}{2}\,,\\
\frac{d^r\zeta}{dt^r}(1) &= 0 && \text{for all } 0 \le r \le m+1\,.
\end{aligned}$$

We define

$$\Theta^{(s)} = \Xi^{(s)} - \frac{d^{m+1}\zeta}{ds^{m+1}}(s)\,\overbrace{\Phi_s * \cdots * \Phi_s}^{m+1\text{ terms}} * \Psi\,, \tag{2.1.19}$$

where

$$\Xi^{(s)} = (-1)^{m+1}\zeta(s)\frac{d^{m+1}}{ds^{m+1}}\Big(\overbrace{\Phi_s * \cdots * \Phi_s}^{m+2\text{ terms}}\Big) * \Psi\,,$$

and we claim that (2.1.17) holds for this choice of $\Theta^{(s)}$. To verify this assertion, we apply $m+1$ integration by parts to write

$$\begin{aligned}
\int_0^1 \Theta^{(s)} * \Phi_s\,ds = \int_0^1 \Xi^{(s)} * \Phi_s\,ds + \frac{d^m\zeta}{ds^m}(0)\lim_{s\to 0+}\big(\overbrace{\Phi * \cdots * \Phi}^{m+2\text{ terms}}\big)_s * \Psi\\
-(-1)^{m+1}\int_0^1 \zeta(s)\,\frac{d^{m+1}}{ds^{m+1}}\Big(\overbrace{\Phi_s * \cdots * \Phi_s}^{m+2\text{ terms}}\Big) * \Psi\,ds\,,
\end{aligned}$$

noting that all the boundary terms vanish except for the term at $s=0$ in the first integration by parts. The first and the third terms in the previous expression on the right add up to zero, while the second term is equal to Ψ, since Φ has integral one. This implies that the family $\{(\Phi * \cdots * \Phi)_s\}_{s>0}$ is an approximate identity as $s \to 0$. Therefore, (2.1.17) holds.

We now prove estimate (2.1.18). Let Ω be the $(m+1)$-fold convolution of Φ. For the second term on the right in (2.1.19), we note that the $(m+1)$st derivative of $\zeta(s)$ vanishes on $\big[0,\frac{1}{2}\big]$, so that we may write

$$\begin{aligned}
&\int_{\mathbf{R}^n}(1+|x|)^m\left|\frac{d^{m+1}\zeta(s)}{ds^{m+1}}\right||\Omega_s * \Psi(x)|\,dx\\
&\qquad\le C_m\,\chi_{[\frac{1}{2},1]}(s)\int_{\mathbf{R}^n}(1+|x|)^m\left[\int_{\mathbf{R}^n}\tfrac{1}{s^n}\big|\Omega\big(\tfrac{x-y}{s}\big)\big|\,|\Psi(y)|\,dy\right]dx\\
&\qquad\le C_m\,\chi_{[\frac{1}{2},1]}(s)\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(1+|y+sx|)^m|\Omega(x)|\,|\Psi(y)|\,dy\,dx\\
&\qquad\le C_m\,\chi_{[\frac{1}{2},1]}(s)\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(1+|sx|)^m|\Omega(x)|\,(1+|y|)^m|\Psi(y)|\,dy\,dx
\end{aligned}$$

$$\leq C_m \chi_{[\frac{1}{2},1]}(s)\Big(\int_{\mathbf{R}^n}(1+|x|)^m|\Omega(x)|\,dx\Big)\Big(\int_{\mathbf{R}^n}(1+|y|)^m|\Psi(y)|\,dy\Big)$$
$$\leq C_0'(\Phi,m)\,s^m\,\mathfrak{N}_m(\Psi),$$

since $\chi_{[\frac{1}{2},1]}(s)\leq 2^m s^m$. To obtain a similar estimate for the first term on the right in (2.1.19), we argue as follows:

$$\begin{aligned}
&\int_{\mathbf{R}^n}(1+|x|)^m|\zeta(s)|\left|\frac{d^{m+1}(\Omega_s*\Psi)}{ds^{m+1}}(x)\right|dx\\
&=\int_{\mathbf{R}^n}(1+|x|)^m\,|\zeta(s)|\left|\frac{d^{m+1}}{ds^{m+1}}\int_{\mathbf{R}^n}\frac{1}{s^n}\Omega\Big(\frac{x-y}{s}\Big)\Psi(y)\,dy\right|dx\\
&=\int_{\mathbf{R}^n}(1+|x|)^m\,|\zeta(s)|\left|\int_{\mathbf{R}^n}\Omega(y)\frac{d^{m+1}\Psi(x-sy)}{ds^{m+1}}\,dy\right|dx\\
&\leq C_m'\int_{\mathbf{R}^n}(1+|x|)^m\,|\zeta(s)|\int_{\mathbf{R}^n}|\Omega(y)|\Big[\sum_{|\alpha|\leq m+1}|\partial^\alpha\Psi(x-sy)|\,|y|^{|\alpha|}\Big]\,dy\,dx\\
&\leq C_m'|\zeta(s)|\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(1+|x+sy|)^m|\Omega(y)|\sum_{|\alpha|\leq m+1}|\partial^\alpha\Psi(x)|\,(1+|y|)^{m+1}\,dy\,dx\\
&\leq C_m'|\zeta(s)|\int_{\mathbf{R}^n}(1+|y|)^{m+1}\,|\Omega(y)|\,(1+|y|)^m\,dy\int_{\mathbf{R}^n}(1+|x|)^m\sum_{|\alpha|\leq m+1}|\partial^\alpha\Psi(x)|\,dx\\
&\leq C_0''(\Phi,m)\,s^m\,\mathfrak{N}_m(\Psi).
\end{aligned}$$

We now set $C_0(\Phi,m)=C_0'(\Phi,m)+C_0''(\Phi,m)$ to conclude the proof of (2.1.18). □

Next, we discuss the proof of Theorem 2.1.4.

Proof. (a) We pick a continuous and integrable function $\psi(s)$ on the interval $[1,\infty)$ that decays faster than any negative power of s (i.e., $|\psi(s)|\leq C_N s^{-N}$ for all $N>0$) and such that

$$\int_1^\infty s^k\,\psi(s)\,ds=\begin{cases}1 & \text{if } k=0,\\ 0 & \text{if } k=1,2,3,\dots.\end{cases} \tag{2.1.20}$$

Such a function exists; see Exercise 2.1.3. In fact, we may take

$$\psi(s)=\frac{e}{\pi}\,\frac{1}{s}\,e^{-\frac{\sqrt{2}}{2}(s-1)^{\frac{1}{4}}}\sin\Big(\frac{\sqrt{2}}{2}(s-1)^{\frac{1}{4}}\Big). \tag{2.1.21}$$

We now define the function

$$\Phi^o(x)=\int_1^\infty\psi(s)P_s(x)\,ds, \tag{2.1.22}$$

where P_s is the Poisson kernel. Note that the double integral

$$\int_{\mathbf{R}^n}\int_1^\infty\frac{s}{(s^2+|x|^2)^{\frac{n+1}{2}}}\,s^{-N}\,ds\,dx$$

converges and so it follows from (2.1.20) and (2.1.22) via Fubini's theorem that

$$\int_{\mathbf{R}^n} \Phi^o(x)\,dx = 1\,.$$

Moreover, another application of Fubini's theorem yields that

$$\widehat{\Phi^o}(\xi) = \int_1^\infty \psi(s)\widehat{P_s}(\xi)\,ds = \int_1^\infty \psi(s)e^{-2\pi s|\xi|}\,ds$$

using that $\widehat{P_s}(\xi) = e^{-2\pi s|\xi|}$ (cf. Exercise 2.2.11 in [156]). This function is rapidly decreasing as $|\xi| \to \infty$ and the same is true for all the derivatives

$$\partial^\alpha \widehat{\Phi^o}(\xi) = \int_1^\infty \psi(s)\partial_\xi^\alpha\big(e^{-2\pi s|\xi|}\big)\,ds\,. \tag{2.1.23}$$

Moreover, the function $\widehat{\Phi^o}$ is clearly smooth on $\mathbf{R}^n \setminus \{0\}$ and we will show that it is also smooth at the origin. Notice that for all multi-indices α we have

$$\partial_\xi^\alpha(e^{-2\pi s|\xi|}) = s^{|\alpha|}p_\alpha(\xi)|\xi|^{-m_\alpha}e^{-2\pi s|\xi|}$$

for some $m_\alpha \in \mathbf{Z}^+$ and some polynomial $p_\alpha(\xi)$. By Taylor's theorem, for some function $v(s,|\xi|)$ with $0 \le v(s,|\xi|) \le 2\pi s|\xi|$, we have

$$e^{-2\pi s|\xi|} = \sum_{k=0}^{L}(-2\pi)^k\frac{|\xi|^k}{k!}s^k + \frac{(-2\pi s|\xi|)^{L+1}}{(L+1)!}e^{-v(s,|\xi|)}\,.$$

Choosing $L > m_\alpha$ gives

$$\partial_\xi^\alpha(e^{-2\pi s|\xi|}) = \sum_{k=0}^{L}(-2\pi)^k\frac{|\xi|^k}{k!}s^{k+|\alpha|}\frac{p_\alpha(\xi)}{|\xi|^{m_\alpha}} + s^{|\alpha|}\frac{p_\alpha(\xi)}{|\xi|^{m_\alpha}}\frac{(-2\pi s|\xi|)^{L+1}}{(L+1)!}e^{-v(s,|\xi|)},$$

which, inserted in (2.1.23) and in view of (2.1.20), yields that when $|\alpha| > 0$, the derivative $\partial^\alpha\widehat{\Phi^o}(\xi)$ tends to zero as $\xi \to 0$ and when $\alpha = 0$, $\widehat{\Phi^o}(\xi) \to 1$ as $\xi \to 0$. We conclude that $\widehat{\Phi^o}$ is continuously differentiable and hence smooth at the origin (cf. Exercise 1.1.1); hence it lies in the Schwartz class, and thus so does Φ^o.

Finally, we have the estimate

$$\begin{aligned} M(f;\Phi^o)(x) &= \sup_{t>0}|(\Phi_t^o * f)(x)| \\ &= \sup_{t>0}\left|\int_1^\infty \psi(s)(f * P_{ts})(x)\,ds\right| \\ &\le \int_1^\infty |\psi(s)|\,ds\,M(f;P)(x)\,, \end{aligned}$$

and the required conclusion follows since $\int_1^\infty |\psi(s)|\,ds \le 500$. Note that we actually obtained the stronger pointwise estimate

$$M(f;\Phi^o) \le 500\,M(f;P)$$

rather than (2.1.12).

(b) The control of the nontagential maximal function $M_a^*(\cdot\,;\Phi)$ in terms of the vertical maximal function $M(\cdot\,;\Phi)$ is the hardest and most technical part of the proof. For matters of exposition, we present the proof only in the case that $a=1$ and we note that the case of general $a>0$ presents only notational differences. We derive (2.1.13) as a consequence of the estimate

$$\big\|M_1^*(f;\Phi)\big\|_{L^p}^p \le C_2''(n,p,\Phi)^p\big\|M(f;\Phi)\big\|_{L^p}^p + \frac{1}{2}\big\|M_1^*(f;\Phi)\big\|_{L^p}^p\,, \tag{2.1.24}$$

which is useful only if we know that $\|M_1^*(f;\Phi)\|_{L^p}<\infty$. This presents a significant hurdle that needs to be overcome by an approximation. For this reason we introduce a family of maximal functions $M_1^*(f;\Phi)^{\varepsilon,N}$ for $0\le\varepsilon,N<\infty$ such that $\|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p}<\infty$ and such that $M_1^*(f;\Phi)^{\varepsilon,N}\uparrow M_1^*(f;\Phi)$ as $\varepsilon\downarrow 0$ and we prove (2.1.24) with $M_1^*(f;\Phi)^{\varepsilon,N}$ in place of $M_1^*(f;\Phi)^{\varepsilon,N}$. In other words we prove

$$\big\|\,M_1^*(f;\Phi)^{\varepsilon,N}\big\|_{L^p}^p \le C_2'(n,p,\Phi,N)^p\big\|M(f;\Phi)\big\|_{L^p}^p + \frac{1}{2}\big\|M_1^*(f;\Phi)^{\varepsilon,N}\big\|_{L^p}^p\,, \tag{2.1.25}$$

where there is an additional dependence on N in the constant $C_2'(n,p,\Phi,N)$, but there is no dependence on ε. The $M_1^*(f;\Phi)^{\varepsilon,N}$ are defined as follows: for a bounded distribution f in $\mathscr{S}'(\mathbf{R}^n)$ such that $M(f;\Phi)\in L^p$ we define

$$M_1^*(f;\Phi)^{\varepsilon,N}(x) = \sup_{0<t<\frac{1}{\varepsilon}}\ \sup_{|y-x|\le t}\big|(\Phi_t*f)(y)\big|\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y|)^N}\,.$$

We first show that $M_1^*(f;\Phi)^{\varepsilon,N}$ lies in $L^p(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ if N is large enough depending on f. Indeed, using that $(\Phi_t*f)(x)=\langle f,\Phi_t(x-\cdot)\rangle$ and the fact that f is in $\mathscr{S}'(\mathbf{R}^n)$, we obtain constants C_f and $m=m_f$ such that:

$$\begin{aligned}
|(\Phi_t*f)(y)| &\le C_f\sum_{|\gamma|\le m,|\beta|\le m}\ \sup_{w\in\mathbf{R}^n}|w^\gamma(\partial^\beta\Phi_t)(y-w)|\\
&\le C_f\sum_{|\beta|\le m}\sup_{z\in\mathbf{R}^n}(1+|y|^m+|z|^m)|(\partial^\beta\Phi_t)(z)|\\
&\le C_f\,(1+|y|^m)\sum_{|\beta|\le m}\sup_{z\in\mathbf{R}^n}(1+|z|^m)|(\partial^\beta\Phi_t)(z)|\\
&\le C_f\,\frac{(1+|y|^m)}{\min(t^n,t^{n+m})}\sum_{|\beta|\le m}\sup_{z\in\mathbf{R}^n}(1+|z|^m)|(\partial^\beta\Phi)(z/t)|
\end{aligned}$$

$$\leq C_f \frac{(1+|y|)^m}{\min(t^n,t^{n+m})}(1+t^m)\sum_{|\beta|\leq m}\sup_{z\in\mathbf{R}^n}(1+|z/t|^m)|(\partial^\beta \Phi)(z/t)|$$
$$\leq C_{f,\Phi}(1+\varepsilon|y|)^m\varepsilon^{-m}(1+t^m)(t^{-n}+t^{-n-m}).$$

Multiplying by $(\frac{t}{t+\varepsilon})^N(1+\varepsilon|y|)^{-N}$ for some $0<t<\frac{1}{\varepsilon}$ and $|y-x|<t$ yields

$$\big|(\Phi_t * f)(y)\big|\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y|)^N}\leq C_{f,\Phi}\frac{\varepsilon^{-m}(1+\varepsilon^{-m})(\varepsilon^{n-N}+\varepsilon^{n+m-N})}{(1+\varepsilon|y|)^{N-m}},$$

and using that $1+\varepsilon|y|\geq\frac{1}{2}(1+\varepsilon|x|)$, we obtain for some $C''(f,\Phi,\varepsilon,n,m,N)<\infty$,

$$M_1^*(f;\Phi)^{\varepsilon,N}(x)\leq\frac{C''(f,\Phi,\varepsilon,n,m,N)}{(1+\varepsilon|x|)^{N-m}}.$$

Taking $N>m+n/p$, we have that $M_1^*(f;\Phi)^{\varepsilon,N}$ lies in $L^p(\mathbf{R}^n)$. This choice of N depends on m and hence on the distribution f.

We now introduce functions

$$U(f;\Phi)^{\varepsilon,N}(x)=\sup_{0<t<\frac{1}{\varepsilon}}\sup_{|y-x|<t} t\big|\nabla(\Phi_t * f)(y)\big|\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y|)^N}$$

and

$$V(f;\Phi)^{\varepsilon,N}(x)=\sup_{0<t<\frac{1}{\varepsilon}}\sup_{y\in\mathbf{R}^n}\big|(\Phi_t * f)(y)\big|\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y|)^N}\Big(\frac{t}{t+|x-y|}\Big)^{[\frac{2n}{p}]+1}.$$

Let $C(n)=\|M\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)}$, where M is the Hardy–Littlewood maximal operator. We need the norm estimate

$$\big\|V(f;\Phi)^{\varepsilon,N}\big\|_{L^p}\leq C(n)^{\frac{2}{p}}\big\|M_1^*(f;\Phi)^{\varepsilon,N}\big\|_{L^p} \tag{2.1.26}$$

and the pointwise estimate

$$U(f;\Phi)^{\varepsilon,N}\leq A(n,p,\Phi,N)\,V(f;\Phi)^{\varepsilon,N}, \tag{2.1.27}$$

where

$$A(\Phi,N,n,p)=2^{[\frac{2n}{p}]+1}C_0(\partial_j\Phi,N+[\tfrac{2n}{p}]+1)\,\mathfrak{N}_{N+[\frac{2n}{p}]+1}(\partial_j\Phi).$$

To prove (2.1.26) we observe that when $z\in B(y,t)\subseteq B(x,|x-y|+t)$ we have

$$\big|(\Phi_t * f)(y)\big|\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y|)^N}\leq M_1^*(f;\Phi)^{\varepsilon,N}(z),$$

from which it follows that for any $y \in \mathbf{R}^n$

$$\begin{aligned}
&\big|(\Phi_t * f)(y)\big|\Big(\frac{t}{t+\varepsilon}\Big)^N \frac{1}{(1+\varepsilon|y|)^N} \\
&\qquad \le \left(\frac{1}{|B(y,t)|}\int_{B(y,t)} \big[M_1^*(f;\Phi)^{\varepsilon,N}(z)\big]^{\frac{p}{2}}\,dz\right)^{\frac{2}{p}} \\
&\qquad \le \left(\frac{|x-y|+t}{t}\right)^{\frac{2n}{p}}\left(\frac{1}{|B(x,|x-y|+t)|}\int_{B(x,|x-y|+t)} \big[M_1^*(f;\Phi)^{\varepsilon,N}(z)\big]^{\frac{p}{2}}\,dz\right)^{\frac{2}{p}} \\
&\qquad \le \left(\frac{|x-y|+t}{t}\right)^{[\frac{2n}{p}]+1} M\Big(\big[M_1^*(f;\Phi)^{\varepsilon,N}\big]^{\frac{p}{2}}\Big)^{\frac{2}{p}}(x)\,.
\end{aligned}$$

We now use the boundedness of the Hardy–Littlewood maximal operator M on L^2 to obtain (2.1.26).

In proving (2.1.27), we may assume that Φ has integral 1; otherwise we can multiply Φ by a suitable constant to arrange for this to happen. We note that

$$t\big|\nabla(\Phi_t * f)\big| = \big|(\nabla\Phi)_t * f\big| \le \sqrt{n}\sum_{j=1}^n |(\partial_j\Phi)_t * f|\,,$$

and it suffices to work with each partial derivative $\partial_j\Phi$ of Φ. Using Lemma 2.1.5 we write

$$\partial_j\Phi = \int_0^1 \Theta^{(s)} * \Phi_s\,ds$$

for suitable Schwartz functions $\Theta^{(s)}$. Fix $x \in \mathbf{R}^n$, $t > 0$, and y with $|y-x| < t < 1/\varepsilon$. Then we have

$$\begin{aligned}
&\big|\big((\partial_j\Phi)_t * f\big)(y)\big|\Big(\frac{t}{t+\varepsilon}\Big)^N \frac{1}{(1+\varepsilon|y|)^N} \\
&\qquad = \Big(\frac{t}{t+\varepsilon}\Big)^N \frac{1}{(1+\varepsilon|y|)^N}\left|\int_0^1 \big((\Theta^{(s)})_t * \Phi_{st} * f\big)(y)\,ds\right| \qquad (2.1.28)\\
&\qquad \le \Big(\frac{t}{t+\varepsilon}\Big)^N \int_0^1\int_{\mathbf{R}^n} t^{-n}\big|\Theta^{(s)}(t^{-1}z)\big|\,\frac{\big|\big(\Phi_{st} * f\big)(y-z)\big|}{(1+\varepsilon|y|)^N}\,dz\,ds\,.
\end{aligned}$$

Inserting the factor 1 written as

$$\left(\frac{ts}{ts+|x-(y-z)|}\right)^{[\frac{2n}{p}]+1}\left(\frac{ts}{ts+\varepsilon}\right)^N\left(\frac{ts+|x-(y-z)|}{ts}\right)^{[\frac{2n}{p}]+1}\left(\frac{ts+\varepsilon}{ts}\right)^N$$

in the preceding z-integral and using that

$$\frac{1}{(1+\varepsilon|y|)^N} \le \frac{(1+\varepsilon|z|)^N}{(1+\varepsilon|y-z|)^N}$$

and the fact that $|x-y|<t<1/\varepsilon$, we obtain the estimate

$$\begin{aligned}
&\Big(\frac{t}{t+\varepsilon}\Big)^N\int_0^1\int_{\mathbf{R}^n} t^{-n}\big|\Theta^{(s)}(t^{-1}z)\big|\,\frac{\big|\big(\Phi_{st}*f\big)(y-z)\big|}{(1+\varepsilon|y|)^N}\,dz\,ds\\
&\quad\le V(f;\Phi)^{\varepsilon,N}(x)\int_0^1\int_{\mathbf{R}^n}(1+\varepsilon|z|)^N\Big(\frac{ts+|x-(y-z)|}{ts}\Big)^{[\frac{2n}{p}]+1}t^{-n}\big|\Theta^{(s)}(t^{-1}z)\big|\,dz\,\frac{ds}{s^N}\\
&\quad\le V(f;\Phi)^{\varepsilon,N}(x)\int_0^1\int_{\mathbf{R}^n}s^{-[\frac{2n}{p}]-1-N}(1+\varepsilon t|z|)^N(s+1+|z|)^{[\frac{2n}{p}]+1}\big|\Theta^{(s)}(z)\big|\,dz\,ds\\
&\quad\le 2^{[\frac{2n}{p}]+1}C_0(\partial_j\Phi,N+[\tfrac{2n}{p}]+1)\,\mathfrak{N}_{N+[\frac{2n}{p}]+1}(\partial_j\Phi)\,V(f;\Phi)^{\varepsilon,N}(x)
\end{aligned}$$

in view of conclusion (2.1.18) of Lemma 2.1.5. Combining this estimate with (2.1.28), we deduce (2.1.27). Estimates (2.1.26) and (2.1.27) together yield

$$\big\|U(f;\Phi)^{\varepsilon,N}\big\|_{L^p}\le C(n)A(n,p,\Phi,N)\big\|M_1^*(f;\Phi)^{\varepsilon,N}\big\|_{L^p}. \tag{2.1.29}$$

We now set

$$E_\varepsilon=\big\{x\in\mathbf{R}^n:\ U(f;\Phi)^{\varepsilon,N}(x)\le K\,M_1^*(f;\Phi)^{\varepsilon,N}(x)\big\}$$

for some constant K to be determined shortly. With $A=A(n,p,\Phi,N)$, we have

$$\begin{aligned}
\int_{(E_\varepsilon)^c}\big[M_1^*(f;\Phi)^{\varepsilon,N}(x)\big]^p\,dx &\le \frac{1}{K^p}\int_{(E_\varepsilon)^c}\big[U(f;\Phi)^{\varepsilon,N}(x)\big]^p\,dx\\
&\le \frac{1}{K^p}\int_{\mathbf{R}^n}\big[U(f;\Phi)^{\varepsilon,N}(x)\big]^p\,dx\\
&\le \frac{C(n)^pA^p}{K^p}\int_{\mathbf{R}^n}\big[M_1^*(f;\Phi)^{\varepsilon,N}(x)\big]^p\,dx\\
&\le \frac{1}{2}\int_{\mathbf{R}^n}\big[M_1^*(f;\Phi)^{\varepsilon,N}(x)\big]^p\,dx,
\end{aligned} \tag{2.1.30}$$

provided we choose K such that $K^p=2C(n)^pA(n,p,\Phi,N)^p$. Obviously K is a function of n,p,Φ,N and in particular depends on N.

It remains to estimate the contribution of the integral of $\big[M_1^*(f;\Phi)^{\varepsilon,N}(x)\big]^p$ over the set E_ε. We claim that the following pointwise estimate is valid:

$$M_1^*(f;\Phi)^{\varepsilon,N}(x)\le 4C'(n,N,K)^{\frac{1}{q}}\Big[M\big(M(f;\Phi)^q\big)(x)\Big]^{\frac{1}{q}} \tag{2.1.31}$$

for any $x\in E_\varepsilon$ and $0<q<\infty$ and some constant $C'(n,N,K)$, where M is the Hardy–Littlewood maximal operator. To prove (2.1.31) we fix $x\in E_\varepsilon$ and we also fix y such that $|y-x|<t$.

By the definition of $M_1^*(f;\Phi)^{\varepsilon,N}(x)$ there exists a point $(y_0,t)\in\mathbf{R}^{n+1}_+$ such that $|x-y_0|<t<\frac{1}{\varepsilon}$ and

$$\big|(\Phi_t*f)(y_0)\big|\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y_0|)^N}\ge\frac{1}{2}M_1^*(f;\Phi)^{\varepsilon,N}(x). \tag{2.1.32}$$

Also by the definitions of E_ε and $U(f;\Phi)^{\varepsilon,N}$, for any $x\in E_\varepsilon$ we have

$$t\big|\nabla(\Phi_t*f)(\xi)\big|\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|\xi|)^N}\le KM_1^*(f;\Phi)^{\varepsilon,N}(x) \tag{2.1.33}$$

for all ξ satisfying $|\xi-x|<t<\frac{1}{\varepsilon}$. It follows from (2.1.32) and (2.1.33) that

$$t\big|\nabla(\Phi_t*f)(\xi)\big|\le 2K\big|(\Phi_t*f)(y_0)\big|\left(\frac{1+\varepsilon|\xi|}{1+\varepsilon|y_0|}\right)^N \tag{2.1.34}$$

for all ξ satisfying $|\xi-x|<t<\frac{1}{\varepsilon}$. We let z be such that $|z-x|<t$. Applying the mean value theorem and using (2.1.34), we obtain, for some ξ between y_0 and z,

$$\begin{aligned}\big|(\Phi_t*f)(z)-(\Phi_t*f)(y_0)\big| &= \big|\nabla(\Phi_t*f)(\xi)\big|\,|z-y_0|\\ &\le \frac{2K}{t}\big|(\Phi_t*f)(\xi)\big|\left(\frac{1+\varepsilon|\xi|}{1+\varepsilon|y_0|}\right)^N|z-y_0|\\ &\le \frac{2^{N+1}K}{t}\big|(\Phi_t*f)(y_0)\big|\,|z-y_0|\\ &\le \frac{1}{2}\big|(\Phi_t*f)(y_0)\big|,\end{aligned}$$

provided z also satisfies $|z-y_0|<2^{-N-2}K^{-1}t$ in addition to $|z-x|<t$. Therefore, for z satisfying $|z-y_0|<2^{-N-2}K^{-1}t$ and $|z-x|<t$ we have

$$\big|(\Phi_t*f)(z)\big|\ge\frac{1}{2}\big|(\Phi_t*f)(y_0)\big|\ge\frac{1}{4}M_1^*(f;\Phi)^{\varepsilon,N}(x),$$

where the last inequality uses (2.1.32). Thus we have

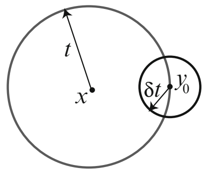
$$\begin{aligned}M\big(M(f;\Phi)^q\big)(x) &\ge \frac{1}{|B(x,t)|}\int_{B(x,t)}\big[M(f;\Phi)(w)\big]^q\,dw\\ &\ge \frac{1}{|B(x,t)|}\int_{B(x,t)\cap B(y_0,2^{-N-2}K^{-1}t)}\big[M(f;\Phi)(w)\big]^q\,dw\\ &\ge \frac{1}{|B(x,t)|}\int_{B(x,t)\cap B(y_0,2^{-N-2}K^{-1}t)}\frac{1}{4^q}\big[M_1^*(f;\Phi)^{\varepsilon,N}(x)\big]^q\,dw\\ &\ge \frac{|B(x,t)\cap B(y_0,2^{-N-2}K^{-1}t)|}{|B(x,t)|}\frac{1}{4^q}\big[M_1^*(f;\Phi)^{\varepsilon,N}(x)\big]^q\\ &\ge C'(n,N,K)^{-1}4^{-q}\big[M_1^*(f;\Phi)^{\varepsilon,N}(x)\big]^q,\end{aligned}$$

where we used the simple geometric fact that if $|x-y_0|\le t$ and $\delta>0$, then

$$\frac{|B(x,t)\cap B(y_0,\delta t)|}{|B(x,t)|}\ge c_{n,\delta}>0,$$

the minimum of this constant being obtained when $|x-y_0|=t$. See Figure 2.1.

Fig. 2.1 The ball $B(y_0,\delta t)$ captures at least a fixed proportion of the ball $B(x,t)$.



This proves (2.1.31). Taking $q=p/2$ and applying the boundedness of the Hardy–Littlewood maximal operator on L^2 yields

$$\int_{E_\varepsilon}\left[M_1^*(f;\Phi)^{\varepsilon,N}(x)\right]^p dx \le C_2'(n,p,\Phi,N)\int_{\mathbf{R}^n} M(f;\Phi)(x)^p\,dx. \tag{2.1.35}$$

Combining this estimate with (2.1.30), we finally prove (2.1.25).

Recalling the fact (obtained earlier) that $\|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p}<\infty$, we deduce from (2.1.25) that

$$\big\|M_1^*(f;\Phi)^{\varepsilon,N}\big\|_{L^p} \le 2^{\frac{1}{p}}C_2'(n,p,\Phi,N)\big\|M(f;\Phi)\big\|_{L^p}. \tag{2.1.36}$$

The previous constant depends on f but is independent of ε. Notice that

$$M_1^*(f;\Phi)^{\varepsilon,N}(x) \ge \frac{2^{-N}}{(1+\varepsilon|x|)^N}\sup_{0<t<1/\varepsilon}\left(\frac{t}{t+\varepsilon}\right)^N \sup_{|y-x|<t}\big|(\Phi_t*f)(y)\big|$$

and that the preceding expression on the right increases to

$$2^{-N}M_1^*(f;\Phi)(x)$$

as $\varepsilon\downarrow 0$. Since the constant in (2.1.36) does not depend on ε, an application of the Lebesgue monotone convergence theorem yields

$$\|M_1^*(f;\Phi)\|_{L^p} \le 2^{N+\frac{1}{p}}C_2'(n,p,\Phi,N)\|M(f;\Phi)\|_{L^p}. \tag{2.1.37}$$

The problem with this estimate is that the finite constant $2^N C_2'(n,p,\Phi,N)$ depends on N and thus on f. However, we have managed to show that under the assumption $\|M(f;\Phi)\|_{L^p}<\infty$, one must necessarily have $\|M_1^*(f;\Phi)\|_{L^p}<\infty$.

Keeping this significant observation in mind, we repeat the preceding argument from the point where the functions $U(f;\phi)^{\varepsilon,N}$ and $V(f;\phi)^{\varepsilon,N}$ are introduced, setting $\varepsilon=N=0$. Then we arrive at (2.1.24) with a constant $C_2''(n,p,\Phi)=C_2'(n,p,\Phi,0)$ which is independent of N and thus of f. We conclude the validity of (2.1.13) with $C_2(n,p,1,\Phi)=2^{1/p}C_2''(n,p,\Phi)$ when $a=1$. A similar constant (depending on a) is obtained for different values of $a>0$.

(c) As usual, $B(x,R)$ denotes a ball centered at x with radius R. Recall that

$$M_b^{**}(f;\Phi)(x)=\sup_{t>0}\sup_{y\in\mathbf{R}^n}\frac{|(\Phi_t*f)(x-y)|}{\left(\frac{|y|}{t}+1\right)^b}.$$

It follows from the definition of $M_a^*(f;\Phi)(z) = \sup_{t>0}\sup_{|w-z|<at}|(\Phi_t * f)(w)|$ that

$$|(\Phi_t * f)(x-y)| \le M_a^*(f;\Phi)(z) \qquad \text{if } z \in B(x-y,at)\,.$$

But the ball $B(x-y,at)$ is contained in the ball $B(x,|y|+at)$; hence it follows that

$$\begin{aligned}
|(\Phi_t * f)(x-y)|^{\frac{n}{b}} &\le \frac{1}{|B(y,at)|}\int_{B(y,at)} M_a^*(f;\Phi)(z)^{\frac{n}{b}}\,dz \\
&\le \frac{1}{|B(y,at)|}\int_{B(x,|y|+at)} M_a^*(f;\Phi)(z)^{\frac{n}{b}}\,dz \\
&\le \left(\frac{|y|+at}{at}\right)^n M\big(M_a^*(f;\Phi)^{\frac{n}{b}}\big)(x) \\
&\le \max(1,a^{-n})\left(\frac{|y|}{t}+1\right)^n M\big(M_a^*(f;\Phi)^{\frac{n}{b}}\big)(x)\,,
\end{aligned}$$

from which we conclude that for all $x \in \mathbf{R}^n$ we have

$$M_b^{**}(f;\Phi)(x) \le \max(1,a^{-b})\left\{M\big(M_a^*(f;\Phi)^{\frac{n}{b}}\big)(x)\right\}^{\frac{b}{n}}.$$

Raising to the power p and using the fact that $p > n/b$ and the boundedness of the Hardy–Littlewood maximal operator M on $L^{pb/n}$, we obtain the required conclusion (2.1.14).

(d) In proving (d) we may replace b by the integer $b_0 = [b]+1$. Let Φ be a Schwartz function with integral equal to 1. Applying Lemma 2.1.5 with $m = b_0$, we write any function φ in $\mathscr{F}_N$ as

$$\varphi(y) = \int_0^1 (\Theta^{(s)} * \Phi_s)(y)\,ds$$

for some choice of Schwartz functions $\Theta^{(s)}$. Then we have

$$\varphi_t(y) = \int_0^1 ((\Theta^{(s)})_t * \Phi_{ts})(y)\,ds$$

for all $t > 0$. Fix $x \in \mathbf{R}^n$. Then for y in $B(x,t)$ we have

$$\begin{aligned}
|(\varphi_t * f)(y)| &\le \int_0^1\int_{\mathbf{R}^n} |(\Theta^{(s)})_t(z)|\,|(\Phi_{ts} * f)(y-z)|\,dz\,ds \\
&\le \int_0^1\int_{\mathbf{R}^n} |(\Theta^{(s)})_t(z)|\,M_{b_0}^{**}(f;\Phi)(x)\left(\frac{|x-(y-z)|}{st}+1\right)^{b_0} dz\,ds \\
&\le \int_0^1 s^{-b_0}\int_{\mathbf{R}^n} |(\Theta^{(s)})_t(z)|\,M_{b_0}^{**}(f;\Phi)(x)\left(\frac{|x-y|}{t}+\frac{|z|}{t}+1\right)^{b_0} dz\,ds \\
&\le 2^{b_0}M_{b_0}^{**}(f;\Phi)(x)\int_0^1 s^{-b_0}\int_{\mathbf{R}^n} |\Theta^{(s)}(w)|\,\big(|w|+1\big)^{b_0}\,dw\,ds \\
&\le 2^{b_0}M_{b_0}^{**}(f;\Phi)(x)\int_0^1 s^{-b_0}C_0(\Phi,b_0)\,s^{b_0}\,\mathfrak{N}_{b_0}(\varphi)\,ds\,,
\end{aligned}$$

where we applied conclusion (2.1.18) of Lemma 2.1.5. Setting $N = b_0 = [b]+1$, we obtain for y in $B(x,t)$ and $\varphi \in \mathscr{F}_N$,

$$|(\varphi_t * f)(y)| \le 2^{b_0} C_0(\Phi, b_0)\, M_{b_0}^{**}(f;\Phi)(x)\,.$$

Taking the supremum over all y in $B(x,t)$, over all $t>0$, and over all φ in $\mathscr{F}_N$, we obtain the pointwise estimate

$$\mathscr{M}_N(f)(x) \le 2^{b_0} C_0(\Phi, b_0)\, M_{b_0}^{**}(f;\Phi)(x)\,, \qquad x \in \mathbf{R}^n,$$

where $N = b_0+1$. This clearly yields (2.1.15) if we set $C_4 = 2^{b_0} C_0(\Phi, b_0)$.

(e) We fix an $f \in \mathscr{S}'(\mathbf{R}^n)$ that satisfies $\|\mathscr{M}_N(f)\|_{L^p} < \infty$ for some fixed positive integer N. To show that f is a bounded distribution, we fix a Schwartz function φ and we observe that for some positive constant $c = c_\varphi$, we have that $c\,\varphi$ is an element of $\mathscr{F}_N$ and thus $M_1^*(f;c\,\varphi) \le \mathscr{M}_N(f)$. Then

$$\begin{aligned}
c^p\,|(\varphi * f)(x)|^p &\le \inf_{|y-x|\le 1} \sup_{|z-y|\le 1} |(c\,\varphi * f)(z)|^p \\
&\le \inf_{|y-x|\le 1} M_1^*(f;c\,\varphi)(y)^p \\
&\le \frac{1}{v_n}\int_{|y-x|\le 1} M_1^*(f;c\,\varphi)(y)^p\,dy \\
&\le \frac{1}{v_n}\int_{\mathbf{R}^n} M_1^*(f;c\,\varphi)(y)^p\,dy \\
&\le \frac{1}{v_n}\int_{\mathbf{R}^n} \mathscr{M}_N(f)(y)^p\,dy < \infty,
\end{aligned}$$

which implies that $\varphi * f$ is a bounded function. We conclude that f is a bounded distribution. We now proceed to show that f is an element of H^p. We fix a smooth radial nonnegative compactly supported function θ such that

$$\theta(x) = \begin{cases} 1 & \text{if} \quad |x|<1, \\ 0 & \text{if} \quad |x|>2. \end{cases}$$

We observe that the identity

$$\begin{aligned}
P(x) &= P(x)\theta(x) + \sum_{k=1}^{\infty} \big(\theta(2^{-k}x)P(x) - \theta(2^{-(k-1)}x)P(x)\big) \\
&= P(x)\theta(x) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} \bigg(\frac{\theta(\cdot)-\theta(2(\cdot))}{(2^{-2k}+|\cdot|^2)^{\frac{n+1}{2}}}\bigg)_{2^k}(x)
\end{aligned}$$

is valid for all $x \in \mathbf{R}^n$. We set

$$\Phi^{(k)}(x) = \big(\theta(x)-\theta(2x)\big)\frac{1}{(2^{-2k}+|x|^2)^{\frac{n+1}{2}}}\,,$$

and we claim that for all bounded tempered distributions f and for all $t>0$ we have

$$P_t * f = (\theta P)_t * f + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} (\Phi^{(k)})_{2^k t} * f\,, \tag{2.1.38}$$

where the series converges in $\mathscr{S}'(\mathbf{R}^n)$; see Exercise 2.1.5.

Assuming (2.1.38), we claim that for some fixed constant $c_0 = c_0(n,N)$, the functions $c_0\,\theta\,P$ and $c_0\Phi^{(k)}$ lie in $\mathscr{F}_N$ uniformly in $k=1,2,3,\dots$.

To verify this assertion for $|\alpha| \le N+1$, we apply Leibniz's rule to write

$$\begin{aligned} \left|\partial^\alpha\left[\frac{\theta(x)-\theta(2x)}{(2^{-2k}+|x|^2)^{\frac{n+1}{2}}}\right]\right| &= \left|\sum_{\beta\le\alpha} c_{\alpha,\beta}\partial_x^{\alpha-\beta}(\theta(x)-\theta(2x))\partial_x^\beta\left(\frac{1}{(2^{-2k}+|x|^2)^{\frac{n+1}{2}}}\right)\right| \\ &\le \sum_{\beta\le\alpha}|c'_{\alpha,\beta}|\,\chi_{\frac12\le|x|\le2}\frac{K_\beta}{(2^{-2k}+|x|^2)^{\frac{n+1}{2}-|\beta|}}\,, \end{aligned}$$

where

$$K_\beta = \sup_{\substack{m,\gamma\\ m+|\gamma|=|\beta|}} \; \sup_{\substack{t,x\\ t^2+|x|^2}} \left|\frac{\partial^m}{\partial t^m}\frac{\partial^\gamma}{\partial x^\gamma}\frac{1}{(t^2+|x|^2)^{\frac{n+1}{2}}}\right|,$$

and this estimate follows from the fact that the function $(t^2+|x|^2)^{-\frac{n+1}{2}}$ is homogeneous of degree $-n-1$ on $\mathbf{R}^{n+1}$ and smooth on the sphere $\mathbf{S}^n$. These estimates are uniform in $k=0,1,2,\dots$ and thus $\mathfrak{N}_N(\theta P)+\mathfrak{N}_N(\Phi^{(k)}) \le 1/c_0(n,N)$ for all some constant $c_0=c_0(n,N)$ for all $k=0,1,2,\dots$.

Then we obtain

$$\begin{aligned} \sup_{t>0}|P_t*f| &\le \sup_{t>0}|(\theta P)_t * f| + \frac{1}{c_0}\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\sum_{k=1}^{\infty}2^{-k}\sup_{t>0}\left|(c_0\Phi^{(k)})_{2^k t}*f\right| \\ &\le C_5(n,N)\mathscr{M}_N(f)\,, \end{aligned}$$

which proves the required conclusion (2.1.16).

We observe that the last estimate also yields the stronger estimate

$$M_1^*(f;P)(x) = \sup_{t>0}\sup_{\substack{y\in\mathbf{R}^n\\ |y-x|\le t}}|(P_t*f)(y)| \le C_5(n,N)\mathscr{M}_N(f)(x)\,. \tag{2.1.39}$$

It follows that the quasi-norm $\|M_1^*(f;P)\|_{L^p(\mathbf{R}^n)}$ is also equivalent to $\|f\|_{H^p}$. □

Remark 2.1.6. To simplify the understanding of the equivalences just proved, a first-time reader may wish to define the H^p quasi-norm of a distribution f as

$$\big\|f\big\|_{H^p} = \big\|M_1^*(f;P)\big\|_{L^p}$$

and then study only the implications (a) $\Longrightarrow$ (c), (c) $\Longrightarrow$ (d), (d) $\Longrightarrow$ (e), and (e) $\Longrightarrow$ (a) in the proof of Theorem 2.1.4. In this way one avoids passing through the statement in part (b). For many applications, the identification of $\|f\|_{H^p}$ with $\|M_1^*(f;\Phi)\|_{L^p}$ for some Schwartz function Φ (with nonvanishing integral) suffices.

We also remark that the proof of Theorem 2.1.4 yields

$$\big\|f\big\|_{H^p(\mathbf{R}^n)} \approx \big\|\mathscr{M}_N(f)\big\|_{L^p(\mathbf{R}^n)},$$

where $N = [\frac{n}{p}] + 1$.

2.1.3 Consequences of the Characterizations of Hardy Spaces

In this subsection we look at a few consequences of Theorem 2.1.4. In many applications we need to be working with dense subspaces of H^p. It turns out that both $H^p \cap L^2$ and $H^p \cap L^1$ are dense in H^p.

Proposition 2.1.7. *Let $0 < p \le 1$ and let r satisfy $p \le r \le \infty$. Then $L^r \cap H^p$ is dense in H^p. Hence, $H^p \cap L^2$ and $H^p \cap L^1$ are dense in H^p.*

Proof. Let f be a distribution in $H^p(\mathbf{R}^n)$. Recall the Poisson kernel $P(x)$ and set $N = [\frac{n}{p}] + 1$. For any fixed $x \in \mathbf{R}^n$ and $t > 0$ we have

$$|(P_t * f)(x)| \le M_1^*(f;P)(y) \le C\mathscr{M}_N(f)(y) \tag{2.1.40}$$

for any $|y-x| \le t$. Indeed, the first estimate in (2.1.40) follows from the definition of $M_1^*(f;P)$, and the second estimate by (2.1.39). Raising (2.1.40) to the power p and averaging over the ball $B(x,t)$, we obtain

$$|(P_t * f)(x)|^p \le \frac{C^p}{v_n t^n} \int_{B(x,t)} \mathscr{M}_N(f)(y)^p\,dy \le \frac{C_1^p}{t^n}\big\|f\big\|_{H^p}^p. \tag{2.1.41}$$

It follows that the function $P_t * f$ is in $L^\infty(\mathbf{R}^n)$ with norm at most a constant multiple of $t^{-n/p}\big\|f\big\|_{H^p}$. Moreover, this function is also in $L^p(\mathbf{R}^n)$, since it is controlled by $M(f;P)$. Therefore, the functions $P_t * f$ lie in $L^r(\mathbf{R}^n)$ for all r with $p \le r \le \infty$. It remains to show that $P_t * f$ also lie in H^p and that $P_t * f \to f$ in H^p as $t \to 0$.

To see that $P_t * f$ lies in H^p, we use the semigroup formula $P_t * P_s = P_{t+s}$ for the Poisson kernel, which is a consequence of the fact that $\widehat{P_t}(\xi) = e^{-2\pi t|\xi|}$ by applying the Fourier transform. Therefore, for any $t > 0$ we have

$$\sup_{s>0} |P_s * P_t * f| = \sup_{s>0} |P_{s+t} * f| \le \sup_{s>0} |P_s * f|,$$

which implies that

$$\big\|P_t * f\big\|_{H^p} \le \big\|f\big\|_{H^p}$$

for all $t > 0$. We now need to show that $P_t * f \to f$ in H^p as $t \to 0$. This will be a consequence of the Lebesgue dominated convergence theorem once we know that

$$\sup_{s>0} |P_s * P_t * f - P_s * f| \le 2\sup_{s>0} |P_s * f| \in L^p(\mathbf{R}^n) \tag{2.1.42}$$

and also that

$$\sup_{s>0} |(P_s * P_t * f - P_s * f)(x)| \to 0 \qquad \text{as} \quad t \to 0 \tag{2.1.43}$$

pointwise for all $x \in \mathbf{R}^n$. Statement (2.1.42) is a trivial consequence of the semigroup formula for the Poisson kernel.

The proof of (2.1.43) requires considerable more work. In proving (2.1.43), by a translation, we may assume that $x = 0$. Let us fix $\varepsilon > 0$. In view of (2.1.41), we have

$$\sup_{s \ge M} |(P_s * P_t * f - P_s * f)(0)| \le C' M^{-n/p}$$

and we pick M such that $C' M^{-n/p} < \varepsilon$. It will suffice to show that

$$\sup_{0<s<M} |P_t * P_s * f(0) - P_s * f(0)| < 3\varepsilon \tag{2.1.44}$$

for t sufficiently close to zero. Let η_0 be a Schwartz function whose Fourier transform $\widehat{\eta_0}$ is equal to 1 on the ball $B(0,1)$ and vanishes outside $B(0,2)$. We write $1 = \widehat{\eta_0} + \widehat{\eta_\infty}$. Then $\eta_\infty = \delta_0 - \eta_0$,

$$P_s * f = P_s * \eta_0 * f + P_s * \eta_\infty * f\,,$$

and we will show that

$$\sup_{0<s<M} |P_t * P_s * \eta_0 * f(0) - P_s * \eta_0 * f(0)| < 2\varepsilon \tag{2.1.45}$$

and

$$\sup_{0<s<M} |P_t * P_s * \eta_\infty * f(0) - P_s * \eta_\infty * f(0)| < \varepsilon \tag{2.1.46}$$

for t sufficiently small. In order to prove (2.1.45), we write

$$\begin{aligned}
&P_t * P_s * \eta_0 * f(0) - P_s * \eta_0 * f(0) \\
&= \int_{\mathbf{R}^n} P_s(y) (P_t * \eta_0 * f(y) - \eta_0 * f(y)) dy \\
&= \int_{\mathbf{R}^n} P_s(y) \bigg(\int_{\mathbf{R}^n} P_t(z) \big(\eta_0 * f(y-z) - \eta_0 * f(y) \big)\, dz \bigg) dy\,.
\end{aligned}$$

Note that $\eta_0 * f$ and $P_t * \eta_0 * f$ are in $L^\infty \cap \mathscr{C}^\infty$, since f is a bounded distribution. There is an $A > 0$ such that $\int_{|y| \ge A/M} P(y)\, dy < \varepsilon$ and so

$$\left| \int_{|y| \ge A} P_s(y) \big(P_t * \eta_0 * f(y) - \eta_0 * f(y) \big) dy \right| \le \|\eta_0 * f\|_{L^\infty} \int_{|y| \ge A} P_s(y) dy < \varepsilon$$

for all $s \le M$. For $|y| \le A$, $\eta_0 * f$ is uniformly continuous in this region, so

$$\sup_{0<s<M} \int_{|y|\le A} P_s(y) \big\| P_t * \eta_0 * f - \eta_0 * f \big\|_{L^\infty} dy < \varepsilon \|P_s\|_{L^1} = \varepsilon$$

for t sufficiently small, since $\{P_t\}_{t>0}$ is an approximate identity; see Theorem 1.2.19 (2) in [156]. Therefore (2.1.45) holds.

Next, we write (2.1.46) as

$$\sup_{0<s<M} |\langle f, P_s * \eta_\infty * P_t - P_s * \eta_\infty \rangle|,$$

and since $f \in \mathscr{S}'(\mathbf{R}^n)$, this is controlled by a finite sum of expressions of the form:

$$\begin{aligned}
&\sup_{0<s<M} \sup_{x\in\mathbf{R}^n} |x^\alpha \partial_x^\beta (P_s * \eta_\infty * P_t - P_s * \eta_\infty)(x)| \\
&= \sup_{0<s<M} \sup_{x\in\mathbf{R}^n} \left| x^\alpha \int_{\mathbf{R}^n} (\partial_x^\beta (P_s * \eta_\infty * P_t - P_s * \eta_\infty))\widehat{\ }(\xi) e^{2\pi i \xi\cdot x} d\xi \right| \\
&=(2\pi)^{|\beta|} \sup_{0<s<M} \sup_{x\in\mathbf{R}^n} \left| x^\alpha \int_{\mathbf{R}^n} \xi^\beta \widehat{\eta_\infty}(\xi) (e^{-2\pi t|\xi|} - 1) e^{-2\pi s|\xi|} e^{2\pi i \xi\cdot x} d\xi \right|. \qquad (2.1.47)
\end{aligned}$$

For a fixed x, find a j such that $|x_j| = \sup_{1\le k\le n} |x_k|$. Set $N = |\alpha| + |\beta| + n + 2$. Integrate (2.1.47) by parts to rewrite it as

$$(2\pi)^{|\beta|} \sup_{0<s<M} \sup_{x\in\mathbf{R}^n} \left| \frac{x^\alpha}{(2\pi i x_j)^N} \int_{\mathbf{R}^n} \partial_j^N \left(\xi^\beta \widehat{\eta_\infty}(\xi) (e^{-2\pi t|\xi|} - 1) e^{-2\pi s|\xi|} \right) e^{2\pi i \xi\cdot x} d\xi \right|.$$

Note that the choice of N yields $\sup_{x\in\mathbf{R}^n} \frac{|x|^{|\alpha|}}{|x_j|^N} < \infty$. To compute the ∂_j^N derivative, we need the estimate for $0 \le m \le N$:

$$|\partial_j^m \xi^\beta| \le C|\xi|^{|\beta|-m}$$

and the following estimates for $1 \le m \le N$ (c.f. Exercise 1.1.6(b)):

$$\begin{aligned}
|\partial_j^m \widehat{\eta_\infty}(\xi)| &\le C\chi_{[1,2]}(\xi) \\
|\partial_j^m e^{-s|\xi|}| &\le \frac{C}{|\xi|^m} \frac{s|\xi| + \cdots + (s|\xi|)^m}{e^{s|\xi|}} \\
|\partial_j^m (e^{-t|\xi|} - 1)| &\le \frac{C}{|\xi|^m} \frac{t|\xi| + \cdots + (t|\xi|)^m}{e^{t|\xi|}}.
\end{aligned}$$

Let $N = a_1 + a_2 + a_3 + a_4$, where $a_1, a_2, a_3, a_4 \in \{0, 1, \ldots, N\}$. Then

$$\begin{aligned}
&\partial_j^N \left(\xi^\beta \widehat{\eta_\infty}(\xi) (e^{-2\pi t|\xi|} - 1) e^{-2\pi s|\xi|} \right) \\
&= \sum_{a_1,a_2,a_3,a_4} c(a_1,a_2,a_3,a_4) (\partial_j^{a_1} \xi^\beta)(\partial_j^{a_2} \widehat{\eta_\infty})(\xi)(\partial_j^{a_3} (e^{-2\pi t|\xi|} - 1))(\partial_j^{a_4} e^{-2\pi s|\xi|}),
\end{aligned}$$

for some suitable constants $c(a_1,a_2,a_3,a_4)$, in view of Leibniz's rule. We claim that for all $a_1,a_2,a_3,a_4 \in \{0,1,\dots,N\}$ we have

$$\int_{\mathbf{R}^n} \big|(\partial_j^{a_1}\xi^\beta)(\partial_j^{a_2}\widehat{\eta_\infty}(\xi))(\partial_j^{a_3}(e^{-2\pi t|\xi|}-1))(\partial_j^{a_4}e^{-2\pi s|\xi|})\big|\,d\xi \le C'\,t \tag{2.1.48}$$

for all $0<s<M$. Obviously $C'\,t$ multiplied by $(2\pi)^{|\beta|}\sup_{x\in\mathbf{R}^n}\frac{|x|^{|\alpha|}}{|x_j|^N}<\infty$ can be made smaller than the given ε if t is sufficiently close to zero.

Let us now prove (2.1.48). If $a_2>0$, then the integral is over the annulus $1\le|\xi|\le 2$ and we can easily derive (2.1.48), since for ξ in this range we have $|\partial_j^{a_3}(e^{-t|\xi|}-1)|\le C''t$. If $a_2=0$, $a_3>0$, $a_4>0$ then the integral is over the region $|\xi|\ge 1$, and using the preceding estimates we write

$$\begin{aligned}
&\int_{|\xi|\ge 1} |(\partial_j^{a_1}\xi^\beta)\widehat{\eta_\infty}(\xi)(\partial_j^{a_3}(e^{-2\pi t|\xi|}-1))(\partial_j^{a_4}e^{-2\pi s|\xi|})|\,d\xi\\
=&\,t\int_{|\xi|\ge 1} C|\xi|^{|\beta|-a_1-a_3-a_4+1}\,\frac{1+(t|\xi|)^2+\cdots+(t|\xi|)^{a_3-1}}{e^{t|\xi|}}\,\frac{s|\xi|+\cdots+(s|\xi|)^{a_4}}{e^{s|\xi|}}\,d\xi\\
\le&\, Ct\int_{|\xi|\ge 1}|\xi|^{|\beta|-N+1}C_1C_2\,d\xi\\
\le&\, C't,
\end{aligned}$$

by the choice of N. In the case $a_2=a_3=0$, $a_4>0$ we use the inequality $|e^t-1|\le Ct$ and argue in a similar fashion to prove (2.1.48). The same argument is valid in the last case $a_2=a_3=a_4=0$. □

Next we observe the following consequence of Theorem 2.1.4.

Corollary 2.1.8. *For any two Schwartz functions Φ and Θ with nonvanishing integral we have*

$$\big\|\sup_{t>0}|\Theta_t * f|\big\|_{L^p}\approx\big\|\sup_{t>0}|\Phi_t * f|\big\|_{L^p}\approx\big\|f\big\|_{H^p}$$

for all $f\in\mathscr{S}'(\mathbf{R}^n)$, with constants depending only on n,p,Φ, and Θ.

Proof. See the discussion after Theorem 2.1.4. □

Next we define a *norm* on Schwartz functions relevant in the theory of Hardy spaces:

$$\mathfrak{N}_N(\varphi;x_0,R)=\int_{\mathbf{R}^n}\Big(1+\Big|\frac{x-x_0}{R}\Big|\Big)^N\sum_{|\alpha|\le N+1}R^{|\alpha|}|\partial^\alpha\varphi(x)|\,dx\,.$$

Note that $\mathfrak{N}_N(\varphi;0,1)=\mathfrak{N}_N(\varphi)$.

Corollary 2.1.9. *(a) For any $0<p\le 1$, every $f\in H^p(\mathbf{R}^n)$, and any $\varphi\in\mathscr{S}(\mathbf{R}^n)$, we have*

$$\big|\langle f,\varphi\rangle\big|\le\mathfrak{N}_N(\varphi)\inf_{|z|\le 1}\mathscr{M}_N(f)(z)\,, \tag{2.1.49}$$

where $N = [\frac{n}{p}] + 1$, *and consequently there is a constant* $C_{n,p}$ *such that*

$$\big|\big\langle f,\varphi\big\rangle\big| \le \mathfrak{N}_N(\varphi)\, C_{n,p}\, \big\|f\big\|_{H^p}\,. \tag{2.1.50}$$

(b) Let $0 < p \le 1$, $N = [n/p]+1$, *and* $p \le r \le \infty$. *Then there is a constant* $C(p,n,r)$ *such that for any* $f \in H^p$ *and* $\varphi \in \mathscr{S}(\mathbf{R}^n)$ *we have*

$$\big\|\varphi * f\big\|_{L^r} \le C(p,n,r)\mathfrak{N}_N(\varphi)\big\|f\big\|_{H^p}\,. \tag{2.1.51}$$

(c) For any $x_0 \in \mathbf{R}^n$, *for all* $R > 0$, *and any* $\psi \in \mathscr{S}(\mathbf{R}^n)$, *we have*

$$\big|\big\langle f,\psi\big\rangle\big| \le \mathfrak{N}_N(\psi; x_0, R) \inf_{|z-x_0|\le R} \mathscr{M}_N(f)(z)\,. \tag{2.1.52}$$

Proof. (a) We use that $\langle f,\varphi\rangle = (\widetilde{\varphi} * f)(0)$, where $\widetilde{\varphi}(x) = \varphi(-x)$ and we observe that $\mathfrak{N}_N(\varphi) = \mathfrak{N}_N(\widetilde{\varphi})$. Then (2.1.49) follows from the inequality

$$|(\widetilde{\varphi} * f)(0)| \le \mathfrak{N}_N(\varphi) M_1^*\Big(f;\frac{\widetilde{\varphi}}{\mathfrak{N}_N(\varphi)}\Big)(z) \le \mathfrak{N}_N(\varphi)\mathscr{M}_N(f)(z)$$

for all $|z| < 1$, which is valid, since $\widetilde{\varphi}/\mathfrak{N}_N(\varphi)$ lies in $\mathscr{F}_N$. We deduce (2.1.50) as follows:

$$\begin{aligned}
\big|\big\langle f,\varphi\big\rangle\big|^p &\le \mathfrak{N}_N(\varphi)^p \inf_{|z|\le 1} \mathscr{M}_N(f)(z)^p \\
&\le \mathfrak{N}_N(\varphi)^p \frac{1}{|B(0,1)|}\int_{|z|\le 1} \mathscr{M}_N(f)^p\,dz \\
&\le \mathfrak{N}_N(\varphi)^p\, C_{n,p}^p\, \big\|f\big\|_{H^p}^p\,.
\end{aligned}$$

(b) For any fixed $x \in \mathbf{R}^n$ and $t > 0$ we have

$$|(\varphi_t * f)(x)| \le \mathfrak{N}_N(\varphi) M_1^*\Big(f;\frac{\varphi}{\mathfrak{N}_N(\varphi)}\Big)(y) \le \mathfrak{N}_N(\varphi)\mathscr{M}_N(f)(y) \tag{2.1.53}$$

for all y satisfying $|y-x| \le t$. Restricting to $t = 1$ yields

$$|(\varphi * f)(x)|^p \le \frac{\mathfrak{N}_N(\varphi)^p}{|B(x,1)|}\int_{B(x,1)} \mathscr{M}_N(f)^p(y)\,dy \le \mathfrak{N}_N(\varphi)^p C_{p,n}^p \big\|f\big\|_{H^p}^p\,.$$

This implies that $\|\varphi * f\|_{L^\infty} \le C_{p,n}\mathfrak{N}_N(\varphi)\|f\|_{H^p}$. Choosing $y = x$ and $t = 1$ in (2.1.53) and then taking L^p quasi-norms yields a similar estimate for $\|\varphi * f\|_{L^p}$. By interpolation we deduce $\|\varphi * f\|_{L^r} \le C(p,n,r)\mathfrak{N}_N(\varphi)\|f\|_{H^p}$, when $r \le p \le \infty$.

(c) To prove (2.1.52), given a Schwartz function ψ and $R > 0$, define $\varphi(y) = \psi(-Ry + x_0)$ so that $\psi(x) = \varphi(\frac{x_0-x}{R}) = R^n\varphi_R(x_0-x)$. In view of (2.1.53) we have

$$\big|\big\langle f,\psi\big\rangle\big| = R^n\big|(\varphi_R * f)(x_0)\big| \le R^n\,\mathfrak{N}_N(\varphi) \inf_{|z-x_0|\le R} \mathscr{M}_N(f)(z)\,.$$

But a simple change of variables shows that $R^n \mathfrak{N}(\varphi) = \mathfrak{N}(\psi; x_0, R)$ and this combined with the preceding inequality yields (2.1.52).

□

Proposition 2.1.10. *Let $0 < p \le 1$. Then the following statements are valid:*
(a) Convergence in H^p implies convergence in $\mathscr{S}'$.
(b) If $f_k \in H^p$ satisfy $\sup_{k\in\mathbf{Z}^+} \|f_k\|_{H^p} \le C < \infty$ and $f_k \to f$ in $\mathscr{S}'(\mathbf{R}^n)$ as $k \to \infty$, then $f \in H^p$.
(c) H^p is a complete quasi-normed metrizable space.

Proof. (a) Let f_j, f in $H^p(\mathbf{R}^n)$ and suppose that $f_j \to f$ in $H^p(\mathbf{R}^n)$. Applying (2.1.50) we obtain that for any $\varphi \in \mathscr{S}(\mathbf{R}^n)$ we have $\langle f_j - f, \varphi\rangle \to 0$; hence $f_j \to f$ in $\mathscr{S}'(\mathbf{R}^n)$.

(b) For any $\Phi \in \mathscr{S}(\mathbf{R}^n)$ with integral one and $t > 0$ we have $\Phi_t * f_k \to \Phi_t * f$ as $k \to \infty$, since $f_k \to f$ in $\mathscr{S}'(\mathbf{R}^n)$. Thus

$$|\Phi_t * f| = \liminf_{k\to\infty} |\Phi_t * f_k| \le \liminf_{k\to\infty} \sup_{t>0} |\Phi_t * f_k| .$$

Taking the supremum over t, we obtain $\sup_{t>0} |\Phi_t * f| \le \liminf_{k\to\infty} \sup_{t>0} |\Phi_t * f_k|$. Then we apply L^p quasi-norms and Fatou's lemma to deduce that $\|M(f;\Phi)\|_{L^p}$ is bounded by a multiple of C; thus, $f \in H^p$.

(c) Suppose $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $H^p(\mathbf{R}^n)$. Then there is a constant C_0 such that $\sup_{j\ge1} \|f_j\|_{H^p} \le C_0$. Using (2.1.50) (with $f_j - f_k$ in place of f) we obtain that for every φ in $\mathscr{S}'(\mathbf{R}^n)$ the sequence $\{\langle f_j, \varphi\rangle\}_{j=1}^\infty$ is Cauchy in $\mathbf{C}$ and thus it converges to a complex number $f(\varphi)$. We claim that the mapping $\varphi \mapsto f(\varphi)$ is a tempered distribution. We clearly have

$$|f(\varphi)| = \lim_{k\to\infty} \big|\langle f_k, \varphi\rangle\big| \le C_{n,p} \mathfrak{N}_N(\varphi) C_0 .$$

But an easy calculation shows that $\mathfrak{N}_N(\varphi)$ is controlled by the finite sum of seminorms $\rho_{\alpha,\beta}(\varphi)$ with $|\alpha|, |\beta| \le N + n + 1$. This yields that f lies in $\mathscr{S}'(\mathbf{R}^n)$, in particular f is a bounded distribution, and obviously $f_j \to f$ in $\mathscr{S}'(\mathbf{R}^n)$. Part (b) implies that f is an element of $H^p(\mathbf{R}^n)$.

Next we show that $f_k \to f$ in H^p. Given $\Phi \in \mathscr{S}(\mathbf{R}^n)$ with integral 1, we have for any $t > 0$ and any $k \ge 1$

$$|(f_k - f) * \Phi_t| = \liminf_{\ell\to\infty} |(f_k - f_\ell) * \Phi_t| \le \liminf_{\ell\to\infty} \sup_{t>0} |(f_k - f_\ell) * \Phi_t| .$$

Taking the supremum over $t > 0$ on the left and then the L^p quasi-norm and applying Fatou's lemma we deduce that

$$\big\|M(f_k - f;\Phi)\big\|_{L^p} \le \liminf_{\ell\to\infty} \big\|M(f_k - f_\ell;\Phi)\big\|_{L^p} .$$

Letting $k \to \infty$ we obtain that

$$\limsup_{k\to\infty} \big\|M(f_k - f;\Phi)\big\|_{L^p} \le \limsup_{k,\ell\to\infty} \big\|M(f_k - f_\ell;\Phi)\big\|_{L^p} = 0;$$

thus $\|f_k - f\|_{H^p} \to 0$ as $k \to \infty$. Therefore H^p is complete. Finally we observe that the map $(f,g) \mapsto \|f-g\|_{H^p}^p$ is a metric on H^p that generates the same topology as the quasi-norm $f \mapsto \|f\|_{H^p}$; hence H^p is metrizable. $\square$

2.1.4 Vector-Valued H^p and Its Characterizations

We now obtain a vector-valued analogue of Theorem 2.1.4 crucial in the characterization of Hardy spaces using Littlewood–Paley theory. To state this analogue we need to extend the definitions of the maximal operators to finite sequences of distributions. Let $a,b>0$ and let Φ be a Schwartz function on $\mathbf{R}^n$. In accordance with Definition 2.1.3, we give the following definitions.

Definition 2.1.11. Let $L \in \mathbf{Z}^+$. We denote by ℓ^2_L the space of all complex-valued sequences $\vec{a} = (a_1, \dots, a_L)$ of length L with norm $\|\vec{a}\|_{\ell^2_L} = (|a_1|^2 + \cdots + |a_L|^2)^{1/2}$. For a sequence $\vec{f} = \{f_j\}_{j=1}^L$ of tempered distributions on $\mathbf{R}^n$ we define the *smooth maximal function of $\vec{f}$ with respect to Φ* as

$$M(\vec{f};\Phi)(x) = \sup_{t>0} \big\|\{(\Phi_t * f_j)(x)\}_j\big\|_{\ell^2_L}.$$

We define the *nontangential maximal function (with aperture a) of f with respect to Φ* as

$$M_a^*(\vec{f};\Phi)(x) = \sup_{t>0} \sup_{\substack{y\in\mathbf{R}^n \\ |y-x|\le at}} \big\|\{(\Phi_t * f_j)(y)\}_j\big\|_{\ell^2_L}.$$

We also define the *auxiliary maximal function*

$$M_b^{**}(\vec{f};\Phi)(x) = \sup_{t>0} \sup_{y\in\mathbf{R}^n} \frac{\big\|\{(\Phi_t * f_j)(x-y)\}_j\big\|_{\ell^2_L}}{(1+t^{-1}|y|)^b}.$$

We note that if the function Φ is not assumed to be Schwartz but merely integrable, for example, if Φ is the Poisson kernel, the maximal functions $M(\vec{f};\Phi)$, $M_a^*(\vec{f};\Phi)$, and $M_b^{**}(\vec{f};\Phi)$ are well defined for sequences $\vec{f} = \{f_j\}_{j=1}^L$ whose terms are bounded tempered distributions on $\mathbf{R}^n$.

For a fixed positive integer N we define the *grand maximal function of $\vec{f}$ (with respect to N)* as

$$\mathscr{M}_N(\vec{f}) = \sup_{\varphi\in\mathscr{F}_N} M_1^*(\vec{f};\varphi), \tag{2.1.54}$$

where

$$\mathscr{F}_N = \Big\{\varphi \in \mathscr{S}(\mathbf{R}^n) :\ \mathfrak{N}_N(\varphi) \le 1\Big\}$$

is as defined in (2.1.11) and

$$\mathfrak{N}_N(\varphi) = \int_{\mathbf{R}^n} (1+|x|)^N \sum_{|\alpha|\le N+1} |\partial^\alpha \varphi(x)|\, dx\,.$$

We note that as in the scalar case, we have the sequence of simple inequalities

$$M(\vec{f};\Phi) \le M_a^*(\vec{f};\Phi) \le (1+a)^b M_b^{**}(\vec{f};\Phi)\,. \tag{2.1.55}$$

We now define the vector-valued Hardy space $H^p(\mathbf{R}^n,\ell^2_L)$.

Definition 2.1.12. Let $\vec{f}=\{f_j\}_{j=1}^L$ be a finite sequence of bounded tempered distributions on $\mathbf{R}^n$ and let $0<p<\infty$. We say that $\vec{f}$ lies in the vector-valued Hardy space $H^p(\mathbf{R}^n,\ell^2_L)$ if the *Poisson maximal function*

$$M(\vec{f};P)(x) = \sup_{t>0} \big\| \{(P_t * f_j)(x)\}_j \big\|_{\ell^2_L}$$

lies in $L^p(\mathbf{R}^n)$. If this is the case, we set

$$\big\|\vec{f}\,\big\|_{H^p(\mathbf{R}^n,\ell^2_L)} = \big\|M(\vec{f};P)\big\|_{L^p(\mathbf{R}^n)} = \bigg\| \sup_{\varepsilon>0} \Big(\sum_{j=1}^L |f_j * P_\varepsilon|^2 \Big)^{\frac{1}{2}} \bigg\|_{L^p(\mathbf{R}^n)}.$$

The next theorem provides a vector-valued analogue of Theorem 2.1.4.

Theorem 2.1.13. *Let $0<p<\infty$, $L\in\mathbf{Z}^+$. Then the following statements are valid:*
(a) There exists a Schwartz function Φ^o with $\int_{\mathbf{R}^n}\Phi^o(x)\,dx=1$ and a constant C_1 ($C_1=500$ works) such that

$$\big\|M(\vec{f};\Phi^o)\big\|_{L^p(\mathbf{R}^n)} \le C_1 \big\|\vec{f}\,\big\|_{H^p(\mathbf{R}^n,\ell^2_L)} \tag{2.1.56}$$

for every sequence $\vec{f}=\{f_j\}_{j=1}^L$ of bounded tempered distributions.
(b) For every $a>0$ and Φ in $\mathscr{S}(\mathbf{R}^n)$ there exists a constant $C_2(n,p,a,\Phi)$ such that

$$\big\|M_a^*(\vec{f};\Phi)\big\|_{L^p(\mathbf{R}^n)} \le C_2(n,p,a,\Phi)\big\|M(\vec{f};\Phi)\big\|_{L^p(\mathbf{R}^n,\ell^2_L)} \tag{2.1.57}$$

for every sequence $\vec{f}=\{f_j\}_{j=1}^L$ of tempered distributions.
(c) For every $a>0$, $b>n/p$, and Φ in $\mathscr{S}(\mathbf{R}^n)$ there exists a constant $C_3(n,p,a,b,\Phi)$ such that

$$\big\|M_b^{**}(\vec{f};\Phi)\big\|_{L^p(\mathbf{R}^n)} \le C_3(n,p,a,b,\Phi)\big\|M_a^*(\vec{f};\Phi)\big\|_{L^p(\mathbf{R}^n,\ell^2_L)} \tag{2.1.58}$$

for every sequence $\vec{f}=\{f_j\}_{j=1}^L$ of tempered distributions.
(d) For every $b>0$ and Φ in $\mathscr{S}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n}\Phi(x)\,dx\neq 0$ there exists a constant $C_4(b,\Phi)$ such that if $N=[\frac{n}{p}]+1$ we have

$$\big\|\mathscr{M}_N(\vec{f})\big\|_{L^p(\mathbf{R}^n)} \le C_4(b,\Phi)\big\|M_b^{**}(\vec{f};\Phi)\big\|_{L^p(\mathbf{R}^n,\ell^2_L)} \tag{2.1.59}$$

for every sequence $\vec{f}=\{f_j\}_{j=1}^L$ of tempered distributions.

(e) For every positive integer N there exists a constant $C_5(n,N)$ such that for all $f_j \in \mathscr{S}'(\mathbf{R}^n)$, $j = 1,\dots,L$ with $\|\mathscr{M}_N(\vec{f})\|_{L^p(\mathbf{R}^n,\ell^2_L)} < \infty$ (where $\vec{f} = \{f_j\}_{j=1}^L$) we must have that f_j are bounded distributions and satisfy

$$\big\|\vec{f}\,\big\|_{H^p(\mathbf{R}^n)} \le C_5(n,N)\big\|\mathscr{M}_N(\vec{f}\,)\big\|_{L^p(\mathbf{R}^n,\ell^2_L)}, \tag{2.1.60}$$

that is, $\vec{f}$ lies in the Hardy space $H^p(\mathbf{R}^n,\ell^2_L)$.

Proof. The proof of this theorem is obtained via a step-by-step repetition of the proof of Theorem 2.1.4 in which the scalar absolute values of complex numbers are replaced by ℓ^2_L norms. The verification of the details of this extension is omitted. The crucial observation in the adaptation of the proof of Theorem 2.1.4 is that the constants that appear in all inequalities do not depend on L. $\square$

We end this subsection by observing the validity of the following vector-valued analogue of (2.1.52):

$$\Big(\sum_{j=1}^{L}\big|\langle f_j,\varphi\rangle\big|^2\Big)^{\frac{1}{2}} \le \mathfrak{N}_N(\varphi;x_0,R)\inf_{|z-x_0|\le R}\mathscr{M}_N(\vec{f}\,)(z). \tag{2.1.61}$$

The proof of (2.1.61) is identical to the corresponding estimate for scalar-valued functions. Set $\psi(x) = \varphi(-Rx+x_0)$. It follows directly from Definition 2.1.11 that for any fixed z with $|z-x_0| \le R$ we have

$$\begin{aligned}
\Big(\sum_{j=1}^{L}\big|\langle f_j,\varphi\rangle\big|^2\Big)^{\frac{1}{2}} &= R^n\big\|\{(f_j*\psi_R)(x_0)\}_j\big\|_{\ell^2_L}\\
&\le \sup_{y:\,|y-z|\le R} R^n\big\|\{(f_j*\psi_R)(y)\}_j\big\|_{\ell^2_L}\\
&\le R^n\,\mathfrak{N}_N(\psi)\,\mathscr{M}_N(\vec{f}\,)(z),
\end{aligned}$$

which, combined with the observation

$$R^n\,\mathfrak{N}_N(\psi) = \mathfrak{N}_N(\varphi;x_0,R),$$

yields (2.1.61) when we take the infimum over all z with $|z-x_0| \le R$.

2.1.5 Singular Integrals on vector-valued Hardy Spaces

To obtain the Littlewood–Paley characterization of Hardy spaces, we need a multiplier theorem for vector-valued Hardy spaces.

Fix $L \in \mathbf{Z}^+$. Suppose that $\{K_j(x)\}_{j=1}^L$ is a family of functions defined on $\mathbf{R}^n \setminus \{0\}$ with the following properties: There exist constants $A, B < \infty$ and an integer N such that for all multi-indices α with $|\alpha| \le N$ and $x \neq 0$ we have

$$\sum_{j=1}^{L} \left|\partial^\alpha K_j(x)\right| \le A\,|x|^{-n-|\alpha|} < \infty \tag{2.1.62}$$

and also

$$\sup_{\xi \in \mathbf{R}^n} \sum_{j=1}^{L} \left|\widehat{K_j}(\xi)\right| \le B < \infty . \tag{2.1.63}$$

Note that for $h \in L^1(\mathbf{R}^n)$, $K_j * h$ is a well-defined function in $L^{1,\infty}(\mathbf{R}^n)$.

An example of such a sequence of kernels is given by $K_j(x) = \Psi_{2^{-j}}(x)$, where Ψ is a fixed Schwartz function on $\mathbf{R}^n$ whose Fourier transform is supported in a compact annulus that does not contain the origin.

Theorem 2.1.14. *Suppose that a finite sequence of kernels $\{K_j\}_{j=1}^L$ satisfies (2.1.62) and (2.1.63) with $N = [\frac{n}{p}] + 1$, for some $0 < p \le 1$. Then there exists a constant $C_{n,p}$ that depends only on the dimension n and on p such that for all sequences of integrable functions $\{f_j\}_{j=1}^L$ we have the estimate*

$$\Big\| \sum_{j=1}^{L} K_j * f_j \Big\|_{H^p(\mathbf{R}^n)} \le C_{n,p}(A+B) \big\| \{f_j\}_j \big\|_{H^p(\mathbf{R}^n, \ell^2_L)} .$$

Moreover, the space $L^1(\mathbf{R}^n, \ell^2_L)$ is dense in $H^p(\mathbf{R}^n, \ell^2_L)$ and thus there is a unique bounded extension of the operator

$$\{f_j\}_{j=1}^L \mapsto \sum_{j=1}^{L} K_j * f_j \tag{2.1.64}$$

from $H^p(\mathbf{R}^n, \ell^2_L)$ to $H^p(\mathbf{R}^n)$.

Proof. We fix a smooth positive function Φ supported in the unit ball $B(0,1)$ with $\int_{\mathbf{R}^n} \Phi(x)\,dx = 1$ and we consider the maximal function

$$M\Big(\sum_{j=1}^{L} K_j * f_j; \Phi\Big) = \sup_{\varepsilon > 0} \Big|\Phi_\varepsilon * \sum_{j=1}^{L} K_j * f_j\Big| ,$$

defined for $f_j \in L^1(\mathbf{R}^n)$.We will show that this maximal function lies in $L^p(\mathbf{R}^n)$.

We now fix a $\lambda > 0$ and we set $N = [\frac{n}{p}] + 1$. We also fix $\gamma > 0$ to be chosen later and we define the set

$$\Omega_\lambda = \{x \in \mathbf{R}^n : \ \mathscr{M}_N(\vec{f})(x) > \gamma\lambda\} .$$

The set Ω_λ is open, and we may use the Whitney decomposition (Appendix J in [156]) to write it as a union of cubes Q_k such that

(a) $\bigcup_k Q_k = \Omega_\lambda$ and the Q_k's have disjoint interiors;

(b) $\sqrt{n}\,\ell(Q_k) \le \operatorname{dist}(Q_k, (\Omega_\lambda)^c) \le 4\sqrt{n}\,\ell(Q_k)$.

We denote by $c(Q_k)$ the center of the cube Q_k. For each k we set

$$d_k = \operatorname{dist}(Q_k, (\Omega_\lambda)^c) + 2\sqrt{n}\,\ell(Q_k) \approx \ell(Q_k)\,,$$

so that

$$B(c(Q_k), d_k) \cap (\Omega_\lambda)^c \neq \emptyset\,.$$

We now introduce a partition of unity $\{\varphi_k\}_k$ adapted to the sequence of cubes $\{Q_k\}_k$ such that

(c) $\chi_{\Omega_\lambda} = \sum_k \varphi_k$ and each φ_k satisfies $0 \le \varphi_k \le 1$;

(d) each φ_k is supported in $\frac{6}{5}\,Q_k$ and satisfies $I_k = \int_{\mathbf{R}^n} \varphi_k\,dx \approx d_k^n$;

(e) $\|\partial^\alpha \varphi_k\|_{L^\infty} \le C_\alpha d_k^{-|\alpha|}$ for all multi-indices α and some constants C_α independent of k.

We fix a sequence of integrable functions f_j and we decompose each function as

$$f_j = g_j + \sum_k b_{j,k}\,,$$

where g_j is the *good function* of the decomposition given by

$$g_j = f_j \chi_{\mathbf{R}^n \setminus \Omega_\lambda} + \sum_k \frac{\int_{\mathbf{R}^n} f_j \varphi_k\,dx}{I_k}\,\varphi_k$$

and $b_j = \sum_k b_{j,k}$ is the *bad function* of the decomposition given by

$$b_{j,k} = \left(f_j - \frac{\int_{\mathbf{R}^n} f_j \varphi_k\,dx}{I_k} \right) \varphi_k\,.$$

We note that each $b_{j,k}$ has integral zero. We define $\vec{g} = \{g_j\}_{j=1}^L$ and $\vec{b} = \{b_j\}_{j=1}^L$. At this point we appeal to (2.1.61) and to properties (d) and (e) to obtain

$$\left(\sum_{j=1}^L \left| \frac{\int_{\mathbf{R}^n} f_j \varphi_k\,dx}{I_k} \right|^2 \right)^{\frac{1}{2}} \le \frac{\mathfrak{N}_N\big(\varphi_k; c(Q_k), d_k\big)}{I_k} \inf_{|z - c(Q_k)| \le d_k} \mathscr{M}_N(\vec{f})(z)\,. \qquad (2.1.65)$$

But since

$$\frac{\mathfrak{N}_N\big(\varphi_k; c(Q_k), d_k\big)}{I_k} \le \left[\int_{\frac{6}{5}Q_k} \left(1 + \frac{|x - c(Q_k)|}{d_k} \right)^N \sum_{|\alpha| \le N+1} \frac{d_k^{|\alpha|} C_\alpha d_k^{-|\alpha|}}{I_k}\,dx \right] \le C_{N,n}\,,$$

it follows that (2.1.65) is at most a constant multiple of λ, since the ball $B(c(Q_k),d_k)$ meets the complement of Ω_λ. We conclude that

$$\big\|\vec{g}\big\|_{L^\infty(\Omega_\lambda,\ell^2_L)} \le C_{N,n}\,\gamma\lambda\,. \tag{2.1.66}$$

We now estimate $M(\sum_{j=1}^{L} K_j * b_{j,k};\Phi)$. For fixed k and $\varepsilon>0$ we have

$$\begin{aligned}
&\Big(\Phi_\varepsilon * \sum_{j=1}^{L} K_j * b_{j,k}\Big)(x)\\
&\quad= \int_{\mathbf{R}^n}\Big(\Phi_\varepsilon * \sum_{j=1}^{L} K_j\Big)(x-y)\left[f_j(y)\varphi_k(y) - \frac{\int_{\mathbf{R}^n} f_j\varphi_k\,dx}{I_k}\,\varphi_k(y)\right]dy\\
&\quad= \int_{\mathbf{R}^n}\sum_{j=1}^{L}\left\{\big(\Phi_\varepsilon * K_j\big)(x-z) - \int_{\mathbf{R}^n}\big(\Phi_\varepsilon * K_j\big)(x-y)\frac{\varphi_k(y)}{I_k}\,dy\right\}\varphi_k(z)f_j(z)\,dz\\
&\quad= \int_{\mathbf{R}^n}\sum_{j=1}^{L} R_{j,k}(x,z)\varphi_k(z)f_j(z)\,dz\,,
\end{aligned}$$

where we set $R^{\varepsilon}_{j,k}(x,z)$ for the expression inside the curly brackets. Using (2.1.52), we obtain

$$\begin{aligned}
&\left|\int_{\mathbf{R}^n}\sum_{j=1}^{L} R^{\varepsilon}_{j,k}(x,z)\varphi_k(z)f_j(z)\,dz\right|\\
&\qquad\le \sum_{j=1}^{L}\mathfrak{N}_N\big(R^{\varepsilon}_{j,k}\big(x,\cdot\,\big)\varphi_k;c(Q_k),d_k\big)\inf_{|z-c(Q_k)|\le d_k}\mathscr{M}_N(f_j)(z)\\
&\qquad\le \sum_{j=1}^{L}\mathfrak{N}_N\big(R^{\varepsilon}_{j,k}\big(x,\cdot\,\big)\varphi_k;c(Q_k),d_k\big)\inf_{|z-c(Q_k)|\le d_k}\mathscr{M}_N(\vec{f}\,)(z)\,.
\end{aligned}\tag{2.1.67}$$

Since $\varphi_k(z)$ is supported in $\frac{6}{5}Q_k$, the term $(1+\frac{|z-c(Q_k)|}{d_k})^N$ contributes only a constant factor in the integral defining $\mathfrak{N}_N\big(R^{\varepsilon}_{j,k}\big(x,\cdot\,\big)\varphi_k;c(Q_k),d_k\big)$, and we obtain

$$\begin{aligned}
&\mathfrak{N}_N\big(R^{\varepsilon}_{j,k}\big(x,\cdot\,\big)\varphi_k;c(Q_k),d_k\big)\\
&\qquad\le C_{N,n}\int_{\frac{6}{5}Q_k}\sum_{|\alpha|\le N+1} d_k^{|\alpha|}\left|\frac{\partial^\alpha}{\partial z^\alpha}\big(R^{\varepsilon}_{j,k}(x,z)\varphi_k(z)\big)\right|dz\,.
\end{aligned}\tag{2.1.68}$$

For notational convenience we set $K^{\varepsilon}_j = \Phi_\varepsilon * K_j$. We observe that the family $\{K^{\varepsilon}_j\}_j$ satisfies (2.1.62) and (2.1.63) with constants A' and B' that are only multiples of A and B, respectively, uniformly in ε; see Exercise 2.1.13. We now obtain a pointwise estimate for $\mathfrak{N}_N\big(R^{\varepsilon}_{j,k}\big(x,\cdot\,\big)\varphi_k;c(Q_k),d_k\big)$ when $x\in\mathbf{R}^n\setminus\Omega_\lambda$. For fixed $x\in\mathbf{R}^n\setminus\Omega_\lambda$ we have

$$R^{\varepsilon}_{j,k}(x,z)\varphi_k(z) = \int_{\mathbf{R}^n}\varphi_k(z)\Big\{K^{\varepsilon}_j(x-z) - K^{\varepsilon}_j(x-y)\Big\}\frac{\varphi_k(y)\,dy}{I_k}\,,$$

from which it follows that

$$\left|\frac{\partial^\alpha}{\partial z^\alpha} R^\varepsilon_{j,k}(x,z)\varphi_k(z)\right| \le \int_{\mathbf{R}^n} \left|\frac{\partial^\alpha}{\partial z^\alpha}\left\{\varphi_k(z)\left[K^\varepsilon_j(x-z) - K^\varepsilon_j(x-y)\right]\right\}\right| \frac{\varphi_k(y)\,dy}{I_k}.$$

Using hypothesis (2.1.62), we can obtain the estimate

$$\sum_{j=1}^L \left|\frac{\partial^\alpha}{\partial z^\alpha}\left\{\varphi_k(z)\left\{K^\varepsilon_j(x-z) - K^\varepsilon_j(x-y)\right\}\right\}\right| \le C_{N,n} A \frac{d_k d_k^{-|\alpha|}}{|x-c(Q_k)|^{n+1}} \tag{2.1.69}$$

for all $|\alpha| \le N$, for all $y,z \in Q_k$ and all $x \in \mathbf{R}^n \setminus \Omega_\lambda$. Indeed, by Leibniz's rule, the left-hand side of (2.1.69) is controlled by

$$\begin{aligned} C'_\alpha \sum_{|\beta|\le|\alpha|} \quad d_k^{-|\alpha|+|\beta|} \sum_{j=1}^L \left|\frac{\partial^\beta}{\partial z^\beta}\left\{K^\varepsilon_j(x-z) - K^\varepsilon_j(x-y)\right\}\right| & \\ \le C''_\alpha \left[\sum_{\substack{|\beta|\le|\alpha| \\ \beta\neq 0}} \frac{A d_k^{-|\alpha|+|\beta|}}{|x-z|^{n+|\beta|}} + \frac{d_k^{-|\alpha|} A\, d_k}{|x-z|^{n+1}}\right] & \\ = C''_\alpha A \left[\sum_{\substack{|\beta|\le|\alpha| \\ \beta\neq 0}} \frac{d_k^{-|\alpha|+1}}{|x-z|^{n+1}} \left(\frac{d_k}{|x-z|}\right)^{|\beta|-1} + \frac{d_k^{-|\alpha|} d_k}{|x-z|^{n+1}}\right] & \\ \le C_{N,n} \frac{d_k^{-|\alpha|} A d_k}{|x-c(Q_k)|^{n+1}} & \end{aligned}$$

since $|x-z| \ge c d_k$ and $|x-z| \approx |x-c(Q_k)|$. This proves (2.1.69).

It follows from (2.1.69) that

$$d_k^{|\alpha|} \sum_{j=1}^L \left|\frac{\partial^\alpha}{\partial z^\alpha}\left\{R^\varepsilon_{j,k}(x,z)\varphi_k(z)\right\}\right| \le C_{N,n} A \frac{d_k}{|x-c(Q_k)|^{n+1}}.$$

Inserting this estimate in (2.1.68) and summing over all j yields for $x \in \mathbf{R}^n \setminus \Omega_\lambda$

$$\sum_{j=1}^L \mathfrak{N}_N(R^\varepsilon_{j,k}(x,\cdot)\varphi_k; c(Q_k), d_k) \le C_{N,n} A \frac{d_k^{n+1}}{|x-c(Q_k)|^{n+1}}. \tag{2.1.70}$$

Combining (2.1.70) with (2.1.67) gives for $x \in \mathbf{R}^n \setminus \Omega_\lambda$,

$$\sum_{j=1}^L \left|\int_{\mathbf{R}^n} R^\varepsilon_{j,k}(x,z)\varphi_k(z) f_j(z)\,dz\right| \le \frac{C_{N,n} A\, d_k^{n+1}}{|x-c(Q_k)|^{n+1}} \inf_{|z-c(Q_k)|\le d_k} \mathscr{M}_N(\vec{f}\,)(z).$$

This provides the estimate

$$\sup_{\varepsilon>0}\Big|\sum_{j=1}^{L}(K_j^{\varepsilon}*b_{j,k})(x)\Big|\le \frac{C_{N,n}A\,d_k^{n+1}}{|x-c(Q_k)|^{n+1}}\,\gamma\lambda$$

for all $x\in\mathbf{R}^n\setminus\Omega_\lambda$, since the ball $B(c(Q_k),d_k)$ intersects $(\Omega_\lambda)^c$. Summing over k and using the sublinearity of $M(\cdot;\Phi)$ results in

$$M\Big(\sum_{j=1}^{L}K_j*b_j;\Phi\Big)(x)\le\sum_k\frac{C_{N,n}A\,\gamma\lambda\,d_k^{n+1}}{|x-c(Q_k)|^{n+1}}\le\sum_k\frac{C'_{N,n}A\,\gamma\lambda\,d_k^{n+1}}{(d_k+|x-c(Q_k)|)^{n+1}}$$

for all $x\in(\Omega_\lambda)^c$. It is a simple fact that the *Marcinkiewicz function* below satisfies

$$\int_{\mathbf{R}^n}\sum_k\frac{d_k^{n+1}}{(d_k+|x-c(Q_k)|)^{n+1}}\,dx\le C_n\sum_k|Q_k|=C_n\,|\Omega_\lambda|;$$

see Exercise 5.6.6 in [156]. We have therefore shown that

$$\begin{aligned}\tfrac{\lambda}{2}\big|(\Omega_\lambda)^c\cap\big\{M(\vec{K}*\vec{b};\Phi)>\tfrac{\lambda}{2}\big\}\big|&\le\int_{(\Omega_\lambda)^c}M(\vec{K}*\vec{b};\Phi)(x)\,dx\\&\le C_{N,n}A\gamma\lambda\,|\Omega_\lambda|,\end{aligned}\tag{2.1.71}$$

where we used the notation $\vec{K}*\vec{b}=\sum_{j=1}^{L}K_j*b_j$. Also define $\vec{K}*\vec{g}=\sum_{j=1}^{L}K_j*g_j$.

We now combine the information we have acquired so far. First we have

$$\big|\{M(\vec{K}*\vec{f};\Phi)>\lambda\}\big|\le\big|\{M(\vec{K}*\vec{g};\Phi)>\tfrac{\lambda}{2}\}\big|+\big|\{M(\vec{K}*\vec{b};\Phi)>\tfrac{\lambda}{2}\}\big|.$$

For the good function $\vec{g}$ we have the estimate

$$\begin{aligned}\big|\{M(\vec{K}*\vec{g};\Phi)>\tfrac{\lambda}{2}\}\big|&\le\frac{4}{\lambda^2}\int_{\mathbf{R}^n}M(\vec{K}*\vec{g};\Phi)(x)^2\,dx\\&\le\frac{4}{\lambda^2}\int_{\mathbf{R}^n}M(\vec{K}*\vec{g})(x)^2\,dx\\&\le\frac{C_n}{\lambda^2}\int_{\mathbf{R}^n}|(\vec{K}*\vec{g})(x)|^2\,dx\\&=\frac{C_n}{\lambda^2}\int_{\mathbf{R}^n}\Big|\sum_{j=1}^{L}\widehat{K_j}(\xi)\widehat{g_j}(\xi)\Big|^2d\xi\\&\le\frac{C_n}{\lambda^2}\int_{\mathbf{R}^n}\Big(\sum_{j=1}^{L}|\widehat{K_j}(\xi)|\Big)^2\Big(\sum_{j=1}^{L}|\widehat{g_j}(\xi)|^2\Big)d\xi\\&\le\frac{C_nB^2}{\lambda^2}\int_{\mathbf{R}^n}\sum_{j=1}^{L}|g_j(x)|^2\,dx\end{aligned}$$

$$\leq \frac{C_n B^2}{\lambda^2} \int_{\Omega_\lambda} \sum_{j=1}^{L} |g_j(x)|^2 \, dx + \frac{C_n B^2}{\lambda^2} \int_{(\Omega_\lambda)^c} \sum_{j=1}^{L} |f_j(x)|^2 \, dx$$

$$\leq B^2 C_{N,n} \gamma^2 \, |\Omega_\lambda| + \frac{C_n B^2}{\lambda^2} \int_{(\Omega_\lambda)^c} \mathscr{M}_N(\vec{f})(x)^2 \, dx,$$

where we used Corollary 2.1.12 in [156], the L^2 boundedness of the Hardy–Littlewood maximal operator, hypothesis (2.1.63), the fact that $f_j = g_j$ on $(\Omega_\lambda)^c$, estimate (2.1.66), and the fact that $\|\vec{f}\|_{\ell^2_L} \leq \mathscr{M}_N(\vec{f})$ in the preceding sequence of estimates.

On the other hand, estimate (2.1.71) gives

$$\big|\{M(\vec{K} * \vec{b}; \Phi) > \tfrac{\lambda}{2}\}\big| \leq |\Omega_\lambda| + 2C_{N,n} A\, \gamma |\Omega_\lambda|,$$

which, combined with the previously obtained estimate for $\vec{g}$, gives

$$\big|\big\{M(\vec{K} * \vec{f}; \Phi) > \lambda\big\}\big| \leq 2C_{N,n}(1 + A\gamma + B^2\gamma^2)\, |\Omega_\lambda| + \frac{C_n B^2}{\lambda^2} \int_{(\Omega_\lambda)^c} \mathscr{M}_N(\vec{f})(x)^2 \, dx.$$

Multiplying this estimate by $p\lambda^{p-1}$, recalling that $\Omega_\lambda = \{\mathscr{M}_N(\vec{f}) > \gamma\lambda\}$, and integrating in λ from 0 to ∞, we can easily obtain

$$\big\|M(\vec{K} * \vec{f}; \Phi)\big\|^p_{L^p(\mathbf{R}^n)} \leq 2C_{N,n}(1 + A\gamma + B^2\gamma^2)\gamma^{-p} \big\|\mathscr{M}_N(\vec{f})\big\|^p_{L^p(\mathbf{R}^n, \ell^2_L)}. \tag{2.1.72}$$

Choosing $\gamma = (A+B)^{-1}$ and recalling that $N = [\frac{n}{p}] + 1$ gives the required conclusion for some constant $C_{n,p}$ that depends only on n and p.

Finally, we discuss the extension of the operator (2.1.64) to the entire $H^p(\mathbf{R}^n, \ell^2_L)$. In view of Proposition 2.1.7, $L^1(\mathbf{R}^n) \cap H^p(\mathbf{R}^n)$ is dense in $H^p(\mathbf{R}^n)$. It follows that $L^1(\mathbf{R}^n, \ell^2_L) \cap H^p(\mathbf{R}^n, \ell^2_L)$ is dense in $H^p(\mathbf{R}^n, \ell^2_L)$. Indeed, given $\vec{f} = (f_1, \dots, f_L)$ in $H^p(\mathbf{R}^n, \ell^2_L)$, find sequences $h_j^{(k)}$ in $L^1(\mathbf{R}^n)$ such that $h_j^{(k)} \to f_j$ in $H^p(\mathbf{R}^n)$ as $k \to \infty$. Set $\vec{h}^{(k)} = (h_1^{(k)}, \dots, h_L^{(k)})$. Then for any $\Phi \in \mathscr{S}(\mathbf{R}^n)$ with integral one we have

$$M(\vec{f} - \vec{h}^{(k)}; \Phi) \leq M(f_1 - \vec{h}_1^{(k)}; \Phi) + \cdots + M(f_L - \vec{h}_L^{(k)}; \Phi).$$

Apply the L^p quasi-norm on both sides of the preceding expression and then let $k \to \infty$ to obtain the density of $L^1(\mathbf{R}^n, \ell^2_L) \cap H^p(\mathbf{R}^n, \ell^2_L)$ in $H^p(\mathbf{R}^n, \ell^2_L)$. In view of this, the operator in (2.1.64) admits a unique bounded extension from $H^p(\mathbf{R}^n, \ell^2_L)$ to $H^p(\mathbf{R}^n)$. □

Exercises

2.1.1. Prove that if v is a bounded tempered distribution and h_1, h_2 are in $\mathscr{S}(\mathbf{R}^n)$, then

$$(h_1 * h_2) * v = h_1 * (h_2 * v).$$

2.1.2. (a) Show that the H^1 norm remains invariant under the L^1 dilation $f_t(x) = t^{-n}f(t^{-1}x)$.
(b) Show that the H^p norm remains invariant under the L^p dilation $t^{n-n/p}f_t(x)$ interpreted in the sense of distributions.

2.1.3. Show that the continuous function $\psi(s) = \frac{e}{\pi}\frac{1}{s}e^{-\frac{\sqrt{2}}{2}(s-1)^{\frac{1}{4}}}\sin\big(\frac{\sqrt{2}}{2}(s-1)^{\frac{1}{4}}\big)$ defined on $[1,\infty)$ satisfies

$$\int_1^\infty s^k\,\psi(s)\,ds = \begin{cases} 1 & \text{if } k=0,\\ 0 & \text{if } k=1,2,3,\dots. \end{cases}$$

$\big[$*Hint:* Consider the analytic function $F(z) = \frac{4e}{\pi}z^3(z^4+1)^{k-1}e^{-\frac{\sqrt{2}}{2}z}e^{i\frac{\sqrt{2}}{2}z}$ integrated over the boundary of the domain formed by the disc of radius R intersected with the quadrant $\operatorname{Re} z \ge 0, \operatorname{Im} z \ge 0$. Apply Cauchy's residue theorem to F over this contour, noting that a pole appears only when $k=0$. In this case a simple pole at the point $\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$ produces a nonzero residue.$\big]$

2.1.4. Let P_t be the Poisson kernel. Show that for any bounded tempered distribution f we have

$$P_t * f \to f \qquad \text{in } \mathscr{S}'(\mathbf{R}^n) \text{ as } t \to 0.$$

$\big[$*Hint:* Fix a smooth function ϕ whose Fourier transform is equal to 1 in a neighborhood of zero. Show that $P_t*(\phi*f)\to\phi*f$ in $\mathscr{S}'(\mathbf{R}^n)$ and that $\widehat{P_t}(1-\widehat{\phi})\widehat{f} \to (1-\widehat{\phi})\widehat{f}$ in $\mathscr{S}'(\mathbf{R}^n)$ as $t\to 0$. $\big]$

2.1.5. Fix a smooth radial nonnegative compactly supported function θ on $\mathbf{R}^n$ such that $\theta = 1$ on the unit ball and vanishing outside the ball of radius 2. Set $\Phi^{(k)}(x) = \big(\theta(x)-\theta(2x)\big)(2^{-2k}+|x|^2)^{-\frac{n+1}{2}}$ for $k \ge 1$. Prove that for all bounded tempered distributions f and for all $t>0$ we have

$$P_t * f = (\theta P)_t * f + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\sum_{k=1}^\infty 2^{-k}(\Phi^{(k)})_{2^k t} * f,$$

where the series converges in $\mathscr{S}'(\mathbf{R}^n)$. Here $P(x) = \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}(1+|x|^2)^{\frac{n+1}{2}}$ is the Poisson kernel.
$\big[$*Hint:* Fix a function $\phi \in \mathscr{S}(\mathbf{R}^n)$ whose Fourier transform is equal to 1 in a neighborhood of zero and prove the required conclusion for $\phi * f$ and for $(\delta_0-\phi)*f$. In the first case use the Lebesgue dominated convergence theorem and in the second case the Fourier transform.$\big]$

2.1.6. Let $0<p<\infty$ be fixed. Show that a bounded tempered distribution f lies in H^p if and only if the nontangential Poisson maximal function

$$M_1^*(f;P)(x) = \sup_{t>0}\sup_{\substack{y\in\mathbf{R}^n\\ |y-x|\le t}} |(P_t*f)(y)|$$

lies in L^p, and in this case we have $\|f\|_{H^p} \approx \|M_1^*(f;P)\|_{L^p}$.

$\big[$*Hint:* Observe that $M(f;P)$ can be replaced with $M_1^*(f;P)$ in the proof of part (e) of Theorem 2.1.4.$\big]$

2.1.7. (a) Let $1 < q \le \infty$ and let g in $L^q(\mathbf{R}^n)$ be a compactly supported function with integral zero. Show that g lies in the Hardy space $H^1(\mathbf{R}^n)$.
(b) Prove the same conclusion when L^q is replaced by $L\log^+ L$.
$\big[$*Hint:* Part (a): Pick a $\mathscr{C}_0^\infty$ function Φ supported in the unit ball with nonvanishing integral and suppose that the support of g is contained in the ball $B(0,R)$. For $|x| \le 2R$ we have that $M(f;\Phi)(x) \le C_\Phi M(g)(x)$, and since $M(g)$ lies in L^q, it also lies in $L^1(B(0,2R))$. For $|x| > 2R$, write $(\Phi_t * g)(x) = \int_{\mathbf{R}^n} \big(\Phi_t(x-y) - \Phi_t(x)\big) g(y)\,dy$ and use the mean value theorem to estimate this expression by $t^{-n-1}\|\nabla\Phi\|_{L^\infty}\|g\|_{L^1} \le |x|^{-n-1}C_\Phi\|g\|_{L^q}$, since $t \ge |x-y| \ge |x| - |y| \ge |x|/2$ whenever $|x| \ge 2R$ and $|y| \le R$. Thus $M(f;\Phi)$ lies in $L^1(\mathbf{R}^n)$. Part (b): You may use Exercise 2.1.4(b) in [156] to deduce that $M(g)$ is integrable over $B(0,2R)$.$\big]$

2.1.8. Show that for every integrable function g with mean value zero and support inside a ball B, we have $M(g;\Phi) \in L^p((3B)^c)$ for $p > n/(n+1)$. Here Φ is in $\mathscr{S}$.

2.1.9. Show that the space of all Schwartz functions whose Fourier transform is supported away from a neighborhood of the origin is dense in H^p.
$\big[$*Hint:* Use the square function characterization of H^p.$\big]$

2.1.10. (a) Suppose that $f \in H^p(\mathbf{R}^n)$ for some $0 < p \le 1$ and Φ in $\mathscr{S}(\mathbf{R}^n)$. Then show that for all $t > 0$ the function $\Phi_t * f$ belongs to $L^r(\mathbf{R}^n)$ for all $p \le r \le \infty$. Find an estimate for the L^r norm of $\Phi_t * f$ in terms of $\|f\|_{H^p}$ and $t > 0$.
(b) Let $0 < p \le 1$. Show that for all f in $H^p(\mathbf{R}^n)$, $\widehat{f}$ is a continuous function and prove that there exists a constant $C_{n,p}$ such that for all $\xi \neq 0$

$$|\widehat{f}(\xi)| \le C_{n,p}|\xi|^{\frac{n}{p}-n}\|f\|_{H^p}.$$

$\big[$*Hint:* Part (a): Use Proposition 2.1.7. Part (b): Use part (a) with $r = 1$.$\big]$

2.1.11. Show that $H^p(\mathbf{R}^n,\ell^2) = L^p(\mathbf{R}^n,\ell^2)$ whenever $1<p<\infty$ and that $H^1(\mathbf{R}^n,\ell^2)$ is contained in $L^1(\mathbf{R}^n,\ell^2)$.
$\big[$*Hint:* Prove these assertions for ℓ^2_L first for some $L \in \mathbf{Z}^+$.$\big]$

2.1.12. For a sequence of tempered distributions $\vec{f} = \{f_j\}_j$, define the following variant of the grand maximal function:

$$\widetilde{\mathscr{M}}_N(\vec{f})(x) = \sup_{\{\varphi_j\}_j \in \widetilde{\mathscr{F}}_N} \sup_{\varepsilon>0} \sup_{\substack{y\in\mathbf{R}^n \\ |y-x|<\varepsilon}} \Big(\sum_j \big|((\varphi_j)_\varepsilon * f_j)(y)\big|^2\Big)^{\frac{1}{2}},$$

where $N \ge [\frac{n}{p}] + 1$ and

$$\widetilde{\mathscr{F}}_N = \Big\{\{\varphi_j\}_j \in \mathscr{S}(\mathbf{R}^n) : \Big(\sum_j \mathfrak{N}_N(\varphi_j)^2\Big)^{1/2} \le 1\Big\}.$$

Show that for all sequences of tempered distributions $\vec{f} = \{f_j\}_j$ we have

$$\|\widetilde{\mathscr{M}}_N(\vec{f})\|_{L^p(\mathbf{R}^n,\ell^2)} \approx \|\mathscr{M}_N(\vec{f})\|_{L^p(\mathbf{R}^n,\ell^2)}$$

with constants depending only on n and p.
[*Hint:* Fix Φ in $\mathscr{S}(\mathbf{R}^n)$ with integral 1. Using Lemma 2.1.5, write

$$(\varphi_j)_t(y) = \int_0^1 ((\Theta_j^{(s)})_t * \Phi_{ts})(y)\,ds$$

and adapt the proof of part (d) of Theorem 2.1.4 to obtain the pointwise estimate

$$\widetilde{\mathscr{M}}_N(\vec{f}) \le C_{n,p} M_m^{**}(\vec{f};\Phi),$$

where $m > n/p$.]

2.1.13. Suppose that the family $\{K_j\}_{j=1}^L$ satisfies (2.1.62) and (2.1.63) and let Φ be a smooth function supported in the unit ball $B(0,1)$. If $\Phi_\varepsilon(x) = \varepsilon^{-n}\Phi(x/\varepsilon)$, then the family $\{\Phi_\varepsilon * K_j\}_{j=1}^L$ also satisfies (2.1.62) and (2.1.63) with constants A' and B' proportional to $A+B$ and B, respectively.
[*Hint:* In the proof of (2.1.62) consider the cases $|x| \ge 2\varepsilon$ and $|x| \le 2\varepsilon$. In the second case write

$$\begin{aligned}\text{p.v.}\int_{\mathbf{R}^n} \Phi_\varepsilon(x-y)K_j(y)\,dy = &\int_{\mathbf{R}^n} \big(\Phi_\varepsilon(x-y)-\Phi_\varepsilon(x)\big)K_j(y)\Phi_0(y/\varepsilon)\,dy \\ &+\left(\text{p.v.}\int_{\mathbf{R}^n} K_j(y)\Phi_0(y/\varepsilon)\,dy\right)\Phi_\varepsilon(x),\end{aligned}$$

where $\Phi_0(y)$ is a smooth function which is equal to 1 on the ball $|y| \le 3$ and vanishes outside the ball $|y| \le 4$.]

2.2 Function Spaces and the Square Function Characterization of Hardy Spaces

In Sections 1.2 and 1.3 we obtained a remarkable characterization of Sobolev and Lipschitz using the Littlewood–Paley operators Δ_j. In this section we achieve a similar characterization for the Hardy spaces. These characterizations motivate the introduction of classes of spaces defined in terms of mixed (discrete and continuous) quasi-norms of the sequences $\Delta_j^\Psi(f)$, for a suitable $\Psi \in \mathscr{S}(\mathbf{R}^n)$. Within the general framework of these classes, one can launch a study of function spaces from a unified perspective.

We have encountered two expressions involving the operators Δ_j^Ψ in the characterizations of Sobolev and Lipschitz spaces. Sobolev spaces were characterized by an L^p norm of the Littlewood–Paley square function

$$\Big(\sum_j |2^{j\alpha}\Delta_j^\Psi(f)|^2\Big)^{\frac{1}{2}},$$

but Lipschitz spaces were characterized by an ℓ^q norm of the sequence of quantities $\big\|2^{j\alpha}\Delta_j^\Psi(f)\big\|_{L^p}$. These examples motivate the introduction of two fundamental scales of function spaces, called the Triebel–Lizorkin and Besov–Lipschitz spaces, respectively.

2.2.1 Introduction to Function Spaces

Before we give the pertinent definitions, we recall the setup that we developed in Section 1.3 and used in Section 1.4. Throughout this section we fix a radial Schwartz function Ψ on $\mathbf{R}^n$ whose Fourier transform is nonnegative, is supported in the annulus $1-\frac{1}{7}\le|\xi|\le 2$, is equal to one on the smaller annulus $1\le|\xi|\le 2-\frac{2}{7}$, and satisfies

$$\sum_{j\in\mathbf{Z}}\widehat{\Psi}(2^{-j}\xi)=1\,, \qquad\qquad \xi\neq 0\,. \tag{2.2.1}$$

Associated with this bump, we define the Littlewood–Paley operators Δ_j^Ψ given by multiplication on the Fourier transform side by the function $\widehat{\Psi}(2^{-j}\xi)$. We also define a Schwartz function Φ such that

$$\widehat{\Phi}(\xi)=\begin{cases}\sum_{j\le 0}\widehat{\Psi}(2^{-j}\xi) & \text{when } \xi\neq 0,\\ 1 & \text{when } \xi=0.\end{cases} \tag{2.2.2}$$

Note that $\widehat{\Phi}(\xi)$ is equal to 1 for $|\xi|\le 2-\frac{2}{7}$ and vanishes when $|\xi|\ge 2$. It follows from these definitions that

$$S_0+\sum_{j=1}^{\infty}\Delta_j^\Psi=I\,, \tag{2.2.3}$$

where S_0 is the operator given by convolution with the bump Φ and the convergence of the series in (2.2.3) is in $\mathscr{S}'(\mathbf{R}^n)$. Moreover, we also have the identity

$$\sum_{j\in\mathbf{Z}}\Delta_j^\Psi=I\,, \tag{2.2.4}$$

where the convergence of the series in (2.2.4) is in the sense of $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}$.

Definition 2.2.1. Let $\alpha \in \mathbf{R}$ and $0 < p,q \le \infty$. For $f \in \mathscr{S}'(\mathbf{R}^n)$ we set

$$\|f\|_{B_p^{\alpha,q}} = \|S_0(f)\|_{L^p} + \Big(\sum_{j=1}^{\infty} \big(2^{j\alpha}\|\Delta_j^{\Psi}(f)\|_{L^p}\big)^q\Big)^{\frac{1}{q}}$$

with the obvious modification when $p,q=\infty$. When $p,q<\infty$ we also define

$$\|f\|_{F_p^{\alpha,q}} = \|S_0(f)\|_{L^p} + \Big\|\Big(\sum_{j=1}^{\infty} \big(2^{j\alpha}|\Delta_j^{\Psi}(f)|\big)^q\Big)^{\frac{1}{q}}\Big\|_{L^p}.$$

The space of all tempered distributions f for which the quantity $\|f\|_{B_p^{\alpha,q}}$ is finite is called the (inhomogeneous) *Besov–Lipschitz* space with indices α,p,q and is denoted by $B_p^{\alpha,q}$. The space of all tempered distributions f for which the quantity $\|f\|_{F_p^{\alpha,q}}$ is finite is called the (inhomogeneous) *Triebel–Lizorkin* space with indices α,p,q and is denoted by $F_p^{\alpha,q}$.

We now define the corresponding homogeneous versions of these spaces. For an element f of $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}$ we let

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \Big(\sum_{j\in\mathbf{Z}} \big(2^{j\alpha}\|\Delta_j^{\Psi}(f)\|_{L^p}\big)^q\Big)^{\frac{1}{q}}$$

and

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \Big\|\Big(\sum_{j\in\mathbf{Z}} \big(2^{j\alpha}|\Delta_j^{\Psi}(f)|\big)^q\Big)^{\frac{1}{q}}\Big\|_{L^p}.$$

The space of all f in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}$ for which the quantity $\|f\|_{\dot{B}_p^{\alpha,q}}$ is finite is called the (homogeneous) *Besov–Lipschitz* space with indices α,p,q and is denoted by $\dot{B}_p^{\alpha,q}$. The space of f in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}$ such that $\|f\|_{\dot{F}_p^{\alpha,q}} < \infty$ is called the (homogeneous) *Triebel–Lizorkin* space with indices α,p,q and is denoted by $\dot{F}_p^{\alpha,q}$.

We now make several observations related to these definitions. First we note that the expressions $\|\cdot\|_{\dot{F}_p^{\alpha,q}}$, $\|\cdot\|_{F_p^{\alpha,q}}$, $\|\cdot\|_{\dot{B}_p^{\alpha,q}}$, and $\|\cdot\|_{B_p^{\alpha,q}}$ are built in terms of L^p quasi-norms of ℓ^q quasi-norms of $2^{j\alpha}\Delta_j$ or ℓ^q quasi-norms of L^p quasi-norms of the same expressions. As a result, we can see that these quantities satisfy the triangle inequality with a constant (which may be taken to be 1 when $1 \le p,q < \infty$). To determine whether these quantities are indeed quasi-norms, we need to check whether the following property holds:

$$\|f\|_X = 0 \implies f = 0, \tag{2.2.5}$$

where X is one of the $\dot{F}_p^{\alpha,q}$, $F_p^{\alpha,q}$, $\dot{B}_p^{\alpha,q}$, and $B_p^{\alpha,q}$. Since these are spaces of distributions, the identity $f=0$ in (2.2.5) should be interpreted in the sense of distributions. If $\|f\|_X = 0$ for some inhomogeneous space X, then $S_0(f)=0$ and $\Delta_j^{\Psi}(f)=0$ for all $j \ge 1$. Using (2.2.3), we conclude that $f=0$; thus the quantities $\|\cdot\|_{F_p^{\alpha,q}}$ and $\|\cdot\|_{B_p^{\alpha,q}}$ are indeed quasi-norms. Let us investigate what happens when $\|f\|_X = 0$ for some homogeneous space X. In this case we must have $\Delta_j(f)=0$, and using (2.2.4) we conclude that $\widehat{f}$ must be supported at the origin. Proposition 2.4.1 in [156] yields that f must be a polynomial and thus f must be zero (since distributions whose difference is a polynomial are identified in homogeneous spaces).

Remark 2.2.2. We interpret the previous definition in certain cases. According to what we have seen so far, we have

$$\begin{aligned}
\dot{F}_p^{0,2} &\approx F_p^{0,2} \approx L^p, && 1<p<\infty,\\
F_p^{s,2} &\approx L_s^p, && 1<p<\infty,\\
\dot{F}_p^{s,2} &\approx \dot{L}_s^p, && 1<p<\infty,\\
B_\infty^{\gamma,\infty} &\approx \Lambda_\gamma, && \gamma>0,\\
\dot{B}_\infty^{\gamma,\infty} &\approx \dot{\Lambda}_\gamma, && \gamma>0,
\end{aligned}$$

where $\approx$ indicates that the corresponding norms are equivalent. Moreover, later in this section we will see that

$$\dot{F}_p^{0,2} \approx H^p \qquad 0<p\le 1.$$

Although in this text we restrict attention to the case $p<\infty$, it is worth noting that when $p=\infty$, $\dot{F}_\infty^{\alpha,q}$ can be defined as the space of all $f\in\mathscr{S}'/\mathscr{P}$ that satisfy

$$\big\|f\big\|_{\dot{F}_\infty^{\alpha,q}} = \sup_{Q \text{ dyadic cube}} \int_Q \frac{1}{|Q|}\bigg(\sum_{j=-\log_2 \ell(Q)}^{\infty} (2^{j\alpha}|\Delta_j^\Psi(f)|)^q\bigg)^{\frac{1}{q}} < \infty .$$

In the particular case $q=2$ and $\alpha=0$, the space obtained in this way is called BMO and coincides with the space introduced and studied in Chapter 3; this space serves as a substitute for L^∞ and plays a fundamental role in analysis. It should now be clear that several important spaces in analysis can be thought of as elements of the scale of Triebel–Lizorkin spaces.

It would have been more natural to denote Besov–Lipschitz and Triebel–Lizorkin spaces by $B^p_{\alpha,q}$ and $F^p_{\alpha,q}$ to maintain the upper and lower placements of the corresponding indices analogous to those in the previously defined Lebesgue, Sobolev, Lipschitz, and Hardy spaces. However, the notation in Definition 2.2.1 is more or less prevalent in the field of function spaces, and we adhere to it.

2.2.2 Properties of Functions with Compactly Supported Fourier Transforms

The definitions of the quasi-norms of the spaces $B_p^{\alpha,q}$, $F_p^{\alpha,q}$, $\dot{B}_p^{\alpha,q}$, and $\dot{F}_p^{\alpha,q}$ depend on the function Ψ (and Φ which is defined in terms of Ψ). It is not clear from Definition 2.2.1 whether a different choice of bump Ψ produces equivalent quasi-norms for these spaces. In this subsection we show that if Ω is another function that satisfies (2.2.1) and Θ is defined in terms of Ω in the same way that Φ is defined in terms of Ψ, [i.e., via (2.2.2)], then the norms defined in Definition 2.2.1 with respect to the pairs (Φ,Ψ) and (Θ,Ω) are comparable. To prove this assertion we need the following lemma.

Lemma 2.2.3. *Let $0<r<\infty$. Then there exist constants C_1 and C_2 such that for all $t>0$ and for all $\mathscr{C}^1$ functions u on $\mathbf{R}^n$ whose distributional Fourier transform is supported in the ball $|\xi|\le t$ we have*

$$\sup_{z\in\mathbf{R}^n}\frac{1}{t}\frac{|\nabla u(x-z)|}{(1+t|z|)^{\frac{n}{r}}}\le C_1\sup_{z\in\mathbf{R}^n}\frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}}\,, \tag{2.2.6}$$

$$\sup_{z\in\mathbf{R}^n}\frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}}\le C_2 M(|u|^r)(x)^{\frac{1}{r}}\,, \tag{2.2.7}$$

where M denotes the Hardy–Littlewood maximal operator. The constants C_1 and C_2 depend only on the dimension n and r; in particular they are independent of t.

Proof. Select a Schwartz function Φ whose Fourier transform is supported in the ball $|\xi|\le 2$ and is equal to 1 on the unit ball $|\xi|\le 1$. Then $\widehat{\Phi}(\frac{\xi}{t})$ is equal to 1 on the support of $\widehat{u}$ and we can write

$$u(x-z)=(\Phi*u)(x-z)=\int_{\mathbf{R}^n}t^n\Phi(t(x-z-y))u(y)\,dy\,.$$

Taking partial derivatives and using that Φ is a Schwartz function, we obtain

$$|\nabla u(x-z)|\le C_N\int_{\mathbf{R}^n}t^{n+1}(1+t|x-z-y|)^{-N}|u(y)|\,dy\,,$$

where N is arbitrarily large. Using that for all $x,y,z\in\mathbf{R}^n$ we have

$$1\le(1+t|x-z-y|)^{\frac{n}{r}}\frac{(1+t|z|)^{\frac{n}{r}}}{(1+t|x-y|)^{\frac{n}{r}}}\,,$$

we obtain

$$\frac{1}{t}\frac{|\nabla u(x-z)|}{(1+t|z|)^{\frac{n}{r}}}\le C_N\int_{\mathbf{R}^n}t^n(1+t|x-z-y|)^{\frac{n}{r}-N}\frac{|u(y)|}{(1+t|x-y|)^{\frac{n}{r}}}\,dy\,,$$

from which (2.2.6) follows easily by choosing $N=n+1+n/r$.

We now turn to the proof of (2.2.7). We first prove this estimate under the additional assumption that u is a bounded function. Let $|y|\le\delta$ for some $\delta>0$ to be chosen later. We now apply the mean value theorem to write

$$u(x-z)=(\nabla u)(x-z-\xi_y)\cdot y+u(x-z-y)$$

for some ξ_y satisfying $|\xi_y| \le |y| \le \delta$. This implies that

$$|u(x-z)| \le \sup_{|w|\le|z|+\delta} |(\nabla u)(x-w)|\,\delta + |u(x-z-y)|.$$

Raising the preceding inequality to the power r, averaging over the ball $|y|\le\delta$, and then raising to the power $\frac{1}{r}$ yields

$$|u(x-z)| \le c_r\left[\sup_{|w|\le|z|+\delta} |(\nabla u)(x-w)|\,\delta + \left(\frac{1}{v_n\delta^n}\int_{|y|\le\delta}|u(x-z-y)|^r\,dy\right)^{\frac{1}{r}}\right]$$

with $c_r = \max(2^{1/r}, 2^r)$. Here v_n is the volume of the unit ball in $\mathbf{R}^n$. Then

$$\frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} \le c_r\left[\sup_{|w|\le|z|+\delta} \frac{|(\nabla u)(x-w)|}{(1+t|z|)^{\frac{n}{r}}}\,\delta + \frac{\left(\frac{1}{v_n\delta^n}\int_{|y|\le\delta+|z|}|u(x-y)|^r\,dy\right)^{\frac{1}{r}}}{(1+t|z|)^{\frac{n}{r}}}\right].$$

We now set $\delta = \varepsilon/t$ for some $\varepsilon\le 1$. Then we have

$$|w| \le |z| + \frac{\varepsilon}{t} \implies \frac{1}{1+t|z|} \le \frac{2}{1+t|w|},$$

and we can use this to obtain the estimate

$$\frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} \le c_{r,n}\left[\sup_{w\in\mathbf{R}^n} \frac{1}{t}\frac{|(\nabla u)(x-w)|}{(1+t|w|)^{\frac{n}{r}}}\,\varepsilon + \frac{\left(\frac{t^n}{v_n\varepsilon^n}\int_{|y|\le\frac{1}{t}+|z|}|u(x-y)|^r\,dy\right)^{\frac{1}{r}}}{(1+t|z|)^{\frac{n}{r}}}\right]$$

with $c_{r,n} = \max(2^{1/r}, 2^r)2^{n/r}$. It follows that

$$\sup_{z\in\mathbf{R}^n}\frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} \le c_{r,n}\left[\sup_{w\in\mathbf{R}^n}\frac{1}{t}\frac{|(\nabla u)(x-w)|}{(1+t|w|)^{\frac{n}{r}}}\,\varepsilon + \varepsilon^{-\frac{n}{r}}M(|u|^r)(x)^{\frac{1}{r}}\right].$$

We apply inequality (2.2.6) and we select $\varepsilon = \frac{1}{2}(c_{r,n}C_1)^{-1}$, where C_1 is the constant in the inequality in (2.2.6). We obtain

$$\sup_{z\in\mathbf{R}^n}\frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} \le \frac{1}{2}\sup_{z\in\mathbf{R}^n}\frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} + c_{r,n}\varepsilon^{-\frac{n}{r}}M(|u|^r)(x)^{\frac{1}{r}}.$$

Using that

$$\sup_{z\in\mathbf{R}^n}\frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} \le \|u\|_{L^\infty} < \infty,$$

we deduce (2.2.7) with constant $C_2 = 2\,c_{r,n}\,\varepsilon^{-n/r}$, where $\varepsilon = \frac{1}{2}(c_{r,n}C_1)^{-1}$.

We now discuss inequality (2.2.7) when u is not a bounded function. Since

$$u = (\widehat{u})^{\vee} = \big(\widehat{u}\,\widehat{\Phi}(\tfrac{\cdot}{t})\big)^{\vee} = u * t^n \Phi(t(\cdot))$$

and $t^n\Phi(tx)$ is a Schwartz function, we have that $|u(x)| \le C'(t,u)(1+|x|)^{Q_u}$ for some constant $C'(t,u)$ and some $Q_u \in \mathbf{Z}^+$; see Theorem 2.3.20 in [156]. We pick a function ϕ in $\mathscr{S}(\mathbf{R}^n)$ whose Fourier transform is nonnegative, is supported in the unit ball, and has integral one. For $\delta \le \min(1,t)$ consider the $\mathscr{C}^1$ function $x \mapsto \phi(\delta x)u(x)$ whose Fourier transform is supported in $B(0,\delta)+B(0,t)$, which is contained in the ball $B(0,2t)$. Certainly ϕ is a Schwartz function, and so for every $N>0$ there is a constant $C_0(N)$ such that $|\phi(y)| \le C_0(N)(1+|y|)^{-N}$ for all $y \in \mathbf{R}^n$. For $N = Q_u$ and $y = \delta x$, $\delta \le 1$, we have

$$|\phi(\delta x)u(x)| \le C_0(Q_u)\frac{C'(t,u)(1+|x|)^{Q_u}}{(1+|\delta x|)^{Q_u}} \le C_0(Q_u)C'(t,u)\frac{1}{\delta^{Q_u}}\frac{(1+|x|)^{Q_u}}{(1+|x|)^{Q_u}}$$

and this is a bounded function with L^∞ norm $C_0(Q_u)C'(t,u)\delta^{-Q_u}$. By the preceding case, we have

$$\frac{\phi(\delta(x-z))|u(x-z)|}{(1+2t|z|)^{\frac{n}{r}}} \le C_2 M(|u|^r)(x)^{\frac{1}{r}}\|\phi\|_{L^\infty}$$

for every $x,z \in \mathbf{R}^n$. Letting $\delta \to 0$ and using that $\phi(0)=1$ we deduce (2.2.7) with the constant $2^{n/r}C_2\|\phi\|_{L^\infty}$ in place of C_2. $\square$

Corollary 2.2.4. *Let $0<p\le\infty$ and α a multi-index. Then there are constants $C = C(\alpha,n,p)$ and $C' = C(\alpha,n,p)$ such that for all Schwartz functions u on $\mathbf{R}^n$ whose Fourier transform is supported in the ball $B(0,t)$, for some $t>0$, we have*

$$\big\|\partial^\alpha u\big\|_{L^p(\mathbf{R}^n)} \le C t^{|\alpha|}\|u\|_{L^p(\mathbf{R}^n)} \tag{2.2.8}$$

and

$$\big\|\partial^\alpha u\big\|_{L^\infty(\mathbf{R}^n)} \le C' t^{|\alpha|+\frac{n}{p}}\|u\|_{L^p(\mathbf{R}^n)}\,. \tag{2.2.9}$$

Proof. Given $0<p\le\infty$, pick $0<r<p$. Then (2.2.6) and (2.2.7) imply that

$$\frac{1}{t}\,|\nabla u(x)| \le \sup_{z\in\mathbf{R}^n}\frac{1}{t}\frac{|\nabla u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} \le C_1C_2 M(|u|^r)(x)^{\frac{1}{r}}\,, \tag{2.2.10}$$

where M is the Hardy–Littlewood maximal operator and C_1 and C_2 depend only on n and r. Taking L^p quasi-norms and using the boundedness of M on $L^{p/r}$ we obtain (2.2.8) when $|\alpha|=1$. Since every derivative of u also has Fourier transform supported in $B(0,t)$, we obtain (2.2.8) for $|\alpha|\ge 2$ by iteration.

Select a Schwartz function Φ whose Fourier transform is supported in the ball $|\xi|\le 2$ and is equal to 1 on the unit ball $|\xi|\le 1$. Then $\widehat{\Phi}(\frac{\xi}{t})$ is equal to 1 on the support of $\widehat{u}$ and we can write $u = u * t^n\Phi(t(\cdot))$, hence

$$\partial^\alpha u(x) = \int_{\mathbf{R}^n} t^{n+|\alpha|}(\partial^\alpha\Phi)(t(x-y))u(y)\,dy\,. \tag{2.2.11}$$

If $1 \le p \le \infty$, Hölder's inequality gives that

$$\|\partial^\alpha u\|_{L^\infty} \le t^{|\alpha|+\frac{n}{p}} \|u\|_{L^p} \|\partial^\alpha \Phi\|_{L^{p'}} .$$

When $0 < p < 1$ we obtain from (2.2.11) that

$$|\partial^\alpha u(x)| \le t^{n+|\alpha|} \|\partial^\alpha \Phi\|_{L^\infty} \|u\|_{L^\infty}^{1-p} \int_{\mathbf{R}^n} |u(y)|^p \, dy , \tag{2.2.12}$$

which certainly implies (2.2.9) when $t = 1$, by taking the supremum over all x in $\mathbf{R}^n$. If $\widehat{u}$ is supported in $B(0,t)$ for some $t \ne 1$, we apply (2.2.9) when $t = 1$ to the Schwartz function $u_t(x) = t^{-n} u(t^{-1}x)$ whose Fourier transform is supported in $B(0,1)$. The inequality

$$\left\|\partial^\alpha u_t\right\|_{L^\infty(\mathbf{R}^n)} \le C' \left\|u_t\right\|_{L^p(\mathbf{R}^n)} .$$

transforms into (2.2.9) by changing variables. We note that if $p < 1$ and $t < 1$, then (2.2.12) implies the estimate $\|\partial^\alpha u\|_{L^\infty(\mathbf{R}^n)} \le C' t^{\frac{|\alpha|}{p}+\frac{n}{p}} \|u\|_{L^p(\mathbf{R}^n)}$, which is stronger than (2.2.9). $\square$

2.2.3 *Equivalence of Function Space Norms*

We now derive other consequences of Lemma 2.2.3 that will allow us to prove that different norms in Triebel–Lizorkin spaces are equivalent.

Corollary 2.2.5. *Let $\Phi, \Omega, \Psi \in \mathscr{S}(\mathbf{R}^n)$. Suppose that the Fourier transforms of Ω, Ψ are supported in the annulus $1 - \frac{1}{7} \le |\xi| \le 2$. Let $0 < r < \infty$. Then for all f in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ and for all $x \in \mathbf{R}^n$ and $t > 0$ we have*

$$|\Phi_t * \Delta_j^\Psi(f)(x)| \le C_{\Phi,n,r} (M(|\Delta_j^\Psi(f)|^r)(x))^{\frac{1}{r}} . \tag{2.2.13}$$

In particular, for any $k, j \in \mathbf{Z}$ and $x \in \mathbf{R}^n$ we have

$$\left|\Delta_k^\Omega \Delta_j^\Psi(f)(x)\right| \le C_{\Omega,n,r} (M(|\Delta_j^\Psi(f)|^r)(x))^{\frac{1}{r}} . \tag{2.2.14}$$

Proof. Given r pick $N = \frac{n}{r} + n + 1$. Then we have

$$\begin{aligned} \left|(\Phi_t * \Delta_j^\Psi(f))(x)\right| &\le C_{\Phi,N} \int_{\mathbf{R}^n} \frac{\left|\Delta_j^\Psi(f)(x-z)\right|}{(1+t^{-1}|z|)^{\frac{n}{r}}} \frac{t^{-n} dz}{(1+t^{-1}|z|)^{N-\frac{n}{r}}} \\ &\le C'_{\Phi,n,r} \sup_{z \in \mathbf{R}^n} \frac{\left|\Delta_j^\Psi(f)(x-z)\right|}{(1+t^{-1}|z|)^{\frac{n}{r}}} \int_{\mathbf{R}^n} \frac{t^{-n} dz}{(1+t^{-1}|z|)^{N-\frac{n}{r}}} \\ &\le C_{\Phi,n,r} (M(|\Delta_j^\Psi(f)|^r)(x))^{\frac{1}{r}} , \end{aligned} \tag{2.2.15}$$

in view of Lemma 2.2.3, since $\Delta_j^{\Psi}(f)$ is a $\mathscr{C}^1$ function whose Fourier transform is supported in the ball $B(0,2^{j+1})$. This proves (2.2.13), which implies (2.2.14). □

We now return to a point alluded to earlier, that replacing Ψ by another function Ω with similar properties yields equivalent quasi-norms for the function spaces in Definition 2.2.1.

Corollary 2.2.6. *Let Ψ, Ω be Schwartz functions whose Fourier transforms are supported in the annulus $1-\frac{1}{7} \le |\xi| \le 2$ and satisfy (2.2.1). Let Φ be as in (2.2.2) and let*

$$\widehat{\Theta}(\xi) = \begin{cases} \sum_{j\le 0} \widehat{\Omega}(2^{-j}\xi) & \text{when } \xi \neq 0, \\ 1 & \text{when } \xi = 0. \end{cases}$$

Then the homogeneous Triebel–Lizorkin and Besov–Lipschitz quasi-norms defined with respect to Ψ and Ω are equivalent. Likewise, the inhomogeneous Triebel–Lizorkin and Besov–Lipschitz quasi-norms defined with respect to the pairs (Ψ,Φ) and (Ω,Θ) are also equivalent.

Proof. The support properties of Ψ and Ω imply the identity

$$\Delta_j^{\Omega} = \Delta_j^{\Omega}(\Delta_{j-1}^{\Psi} + \Delta_j^{\Psi} + \Delta_{j+1}^{\Psi}). \tag{2.2.16}$$

Thus for any $f \in \mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$, the L^p quasi-norm of $\Delta_j^{\Omega}(f)$ is controlled by the finite sum of the L^p quasi-norms of $\Delta_j^{\Omega}\Delta_{j+i}^{\Psi}(f)$ over $i \in \{-1,0,1\}$. Using (2.2.14) with $r<p$ and applying the boundedness of the Hardy–Littlewood maximal operator on $L^{p/r}(\mathbf{R}^n)$, we deduce that any homogeneous Besov–Lipschitz quasi-norm defined in terms of Ω is controlled by the corresponding norm defined in terms of Ψ.

The corresponding result for Triebel–Lizorkin quasi-norms is as follows:

$$\begin{aligned} \Big\| \Big(\sum_{j\in\mathbf{Z}} |2^{j\alpha}\Delta_j^{\Omega}(f)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p} &\le C_{p,q} \sum_{i\in\{-1,0,1\}} \Big\| \Big(\sum_{j\in\mathbf{Z}} |2^{j\alpha}\Delta_j^{\Omega}\Delta_{j+i}^{\Psi}(f)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p} \\ &\le C_{p,q,n,r,\Omega} \Big\| \Big(\sum_{j\in\mathbf{Z}} \big|M(|2^{j\alpha}\Delta_j^{\Psi}(f)|^r)\big|^{\frac{q}{r}} \Big)^{\frac{1}{q}} \Big\|_{L^p} \\ &= C_{p,q,n,r,\Omega} \Big\| \Big(\sum_{j\in\mathbf{Z}} \big|M(|2^{j\alpha}\Delta_j^{\Psi}(f)|^r)\big|^{\frac{q}{r}} \Big)^{\frac{r}{q}} \Big\|_{L^{p/r}}^{\frac{1}{r}} \end{aligned}$$

for all $f \in \mathscr{S}'(\mathbf{R}^n)$. Picking $r<p,q$, we use the $L^{p/r}(\mathbf{R}^n,\ell^{q/r})$ to $L^{p/r}(\mathbf{R}^n,\ell^{q/r})$ boundedness of the Hardy–Littlewood maximal operator (Theorem 5.6.6 in [156]) to complete the proof of the equivalence of the Triebel–Lizorkin quasi-norms in the homogeneous case.

In the case of the inhomogeneous spaces, we let S_0^{Φ} and S_0^{Θ} be the operators given by convolution with the bumps Φ and Θ, respectively. Then for $f \in \mathscr{S}'(\mathbf{R}^n)$ we have

$$\Theta * f = \Theta * (\Phi * f) + \Theta * (\Psi_{2^{-1}} * f), \tag{2.2.17}$$

since the Fourier transform of the function $\Phi + \Psi_{2^{-1}}$ is equal to 1 on the support of $\widehat{\Theta}$. Applying Corollary 2.2.5 (with $t = 1$), we obtain that

$$|\Theta * (\Phi * f)| \le C_r M(|\Phi * f|^r)^{\frac{1}{r}}$$

and also

$$|\Theta * (\Psi_{2^{-1}} * f)| \le C_r M(|\Psi_{2^{-1}} * f|^r)^{\frac{1}{r}}$$

for any $0 < r < \infty$. Picking $r < p$, we obtain that

$$\big\|\Theta * (\Phi * f)\big\|_{L^p} \le C \big\|S_0^{\Phi}(f)\big\|_{L^p}$$

and also

$$\big\|\Theta * (\Psi_{2^{-1}} * f)\big\|_{L^p} \le C \big\|\Delta_1^{\Psi}(f)\big\|_{L^p} .$$

Inserting the last two estimates in (2.2.17), we obtain that $\|S_0^{\Theta}(f)\|_{L^p}$ is controlled by a multiple of

$$\|S_0^{\Phi}(f)\|_{L^p} + \|\Delta_1^{\Psi}(f)\|_{L^p}$$

which is in turn bounded by a multiple of the $F_p^{\alpha,q}$ quasi-norm of f defined in terms of the pair (Ψ, Φ). This gives the equivalence of quasi-norms in the inhomogeneous case. $\square$

The idea behind the proof of the equivalence of function space quasi-norms defined in terms of different bumps is quite useful. In the rest of this subsection, we take this idea a bit further.

Definition 2.2.7. Let $\Psi \in \mathscr{S}(\mathbf{R}^n)$. For $b > 0$, $j \in \mathbf{R}$, and $f \in \mathscr{S}'(\mathbf{R}^n)$, we introduce the notation

$$M_{b,j}^{**}(f;\Psi)(x) = \sup_{y \in \mathbf{R}^n} \frac{|(\Psi_{2^{-j}} * f)(x-y)|}{(1+2^j|y|)^b} .$$

Note that

$$\sup_{j>0} M_{b,j}^{**}(f;\Psi) \le M_b^{**}(f;\Psi),$$

where M_b^{**} was introduced in (2.1.8). The operator $M_{b,j}^{**}(f;\Psi)$ is called the *Peetre maximal function of f (with respect to Ψ).*

We clearly have

$$|\Delta_j^{\Psi}(f)| \le M_{b,j}^{**}(f;\Psi),$$

but the next result shows that a certain converse of this inequality is also valid.

Theorem 2.2.8. *Let $\alpha \in \mathbf{R}$, $b > n(\min(p,q))^{-1}$, and $0 < p,q < \infty$. Let Ψ be a Schwartz function whose Fourier transform is supported in the annulus $1 - \frac{1}{7} \le |\xi| \le 2$, is equal to 1 on the annulus $1 \le |\xi| \le 2 - \frac{2}{7}$, and satisfies (2.2.1). Let Ω be another Schwartz function which has vanishing moments of all order, i.e., $\int \Omega(y) y^\gamma\, dy = 0$ for all multi-indices γ. Then there is a constant $C = C_{\alpha,p,q,n,b,\Psi,\Omega}$, such that*

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\big|2^{j\alpha}M^{**}_{b,j}(f;\Omega)\big|^q\Big)^{\frac{1}{q}}\Big\|_{L^p}\le C\Big\|\Big(\sum_{j\in\mathbf{Z}}\big|2^{j\alpha}\Delta^{\Psi}_j(f)\big|^q\Big)^{\frac{1}{q}}\Big\|_{L^p} \tag{2.2.18}$$

for all $f\in\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$.

Proof. We start with a Schwartz function Θ whose Fourier transform is nonnegative, supported in the annulus $1-\frac{2}{7}\le|\xi|\le 2$, and satisfies

$$\sum_{j\in\mathbf{Z}}\widehat{\Theta}(2^{-j}\xi)^2=1,\qquad \xi\in\mathbf{R}^n\setminus\{0\}. \tag{2.2.19}$$

Using (2.2.19), we have

$$\Omega_{2^{-k}}*f=\sum_{j\in\mathbf{Z}}(\Omega_{2^{-k}}*\Theta_{2^{-j}})*(\Theta_{2^{-j}}*f).$$

It follows that

$$\begin{aligned}
&2^{k\alpha}\frac{|(\Omega_{2^{-k}}*f)(x-z)|}{(1+2^k|z|)^b}\\
&\le\sum_{j\in\mathbf{Z}}2^{k\alpha}\int_{\mathbf{R}^n}|(\Omega_{2^{-k}}*\Theta_{2^{-j}})(y)|\,\frac{|(\Theta_{2^{-j}}*f)(x-z-y)|}{(1+2^k|z|)^b}\,dy\\
&=\sum_{j\in\mathbf{Z}}2^{k\alpha}\int_{\mathbf{R}^n}2^{kn}|(\Omega*\Theta_{2^{-(j-k)}})(2^ky)|\,\frac{(1+2^j|y+z|)^b}{(1+2^k|z|)^b}\,\frac{|(\Theta_{2^{-j}}*f)(x-z-y)|}{(1+2^j|y+z|)^b}\,dy\\
&\le\sum_{j\in\mathbf{Z}}2^{k\alpha}\int_{\mathbf{R}^n}|(\Omega*\Theta_{2^{-(j-k)}})(y)|\,\frac{(1+2^j|2^{-k}y+z|)^b}{(1+2^k|z|)^b}\,\frac{|(\Theta_{2^{-j}}*f)(x-z-2^{-k}y)|}{(1+2^j|2^{-k}y+z|)^b}\,dy\\
&\le\sum_{j\in\mathbf{Z}}2^{(k-j)\alpha}\int_{\mathbf{R}^n}|(\Omega*\Theta_{2^{-(j-k)}})(y)|\,\frac{(1+2^{j-k}|y|+2^j|z|)^b}{(1+2^k|z|)^b}\,dy\,2^{j\alpha}M^{**}_{b,j}(f;\Theta)(x)\\
&\le\sum_{j\in\mathbf{Z}}2^{(k-j)\alpha}\int_{\mathbf{R}^n}|(\Omega*\Theta_{2^{-(j-k)}})(y)|\,(1+2^{j-k})^b(1+2^{j-k}|y|)^b dy\,2^{j\alpha}M^{**}_{b,j}(f;\Theta)(x).
\end{aligned}$$

We conclude that

$$2^{k\alpha}M^{**}_{b,k}(f;\Omega)(x)\le\sum_{j\in\mathbf{Z}}V_{j-k}\,2^{j\alpha}M^{**}_{b,j}(f;\Theta)(x), \tag{2.2.20}$$

where

$$V_j=2^{-j\alpha}(1+2^j)^b\int_{\mathbf{R}^n}|(\Omega*\Theta_{2^{-j}})(y)|\,(1+2^j|y|)^b\,dy.$$

We now use the facts that both Ω and Θ have vanishing moments of all orders and the result in Appendix B.4 to obtain

$$|(\Omega*\Theta_{2^{-j}})(y)|\le C_{L,N,n,\Theta,\Omega}\,\frac{2^{jn}2^{-|j|L}}{(1+2^{\min(0,j)}|y|)^N}$$

for all $L,N>0$. We deduce the estimate

$$|V_j| \le C_{L,M,n,\Theta,\Omega} 2^{-|j|M}$$

for all M sufficiently large, which, in turn, yields the estimate

$$\sum_{j\in\mathbf{Z}} |V_j|^{\min(1,q)} < \infty .$$

We deduce from (2.2.20) that for all $x\in\mathbf{R}^n$ we have

$$\big\|\{2^{k\alpha}M_{b,k}^{**}(f;\Omega)(x)\}_k\big\|_{\ell^q} \le C_{\alpha,p,q,n,\Theta,\Omega}\big\|\{2^{k\alpha}M_{b,k}^{**}(f;\Theta)(x)\}_k\big\|_{\ell^q} . \tag{2.2.21}$$

Lemma 2.2.3 gives

$$2^{k\alpha}M_{b,k}^{**}(f;\Theta) \le C_2 2^{k\alpha} M(|\Delta_k^{\Theta}(f)|^r)^{\frac{1}{r}} = C_2\, M(|2^{k\alpha}\Delta_k^{\Theta}(f)|^r)^{\frac{1}{r}} . \tag{2.2.22}$$

In view of (2.2.1) we have the identity

$$\Delta_k^{\Theta} = \Delta_k^{\Theta}\big(\Delta_{k-1}^{\Psi}+\Delta_k^{\Psi}+\Delta_{k+1}^{\Psi}\big),$$

and applying (2.2.14) to each term of the preceding sum yields

$$M(|2^{k\alpha}\Delta_k^{\Theta}(f)|^r)^{\frac{1}{r}} \le C'\Big(MM(|2^{k\alpha}\Delta_k^{\Psi}(f)|^r)\Big)^{\frac{1}{r}} . \tag{2.2.23}$$

We now choose $r<\min(p,q)$, we combine (2.2.21), (2.2.22), (2.2.23), and we use twice the $L^{p/r}(\mathbf{R}^n,\ell^{q/r})$ to $L^{p/r}(\mathbf{R}^n,\ell^{q/r})$ boundedness of the Hardy–Littlewood maximal operator (Theorem 5.6.6 in [156]) to complete the proof. □

2.2.4 The Littlewood–Paley Characterization of Hardy Spaces

We discuss an important characterization of Hardy spaces in terms of Littlewood–Paley square functions. The vector-valued Hardy spaces and the action of singular integrals on them are crucial tools in obtaining this characterization.

We have the following.

Theorem 2.2.9. *Let Ψ be a Schwartz function on $\mathbf{R}^n$ whose Fourier transform is nonnegative, supported in $\frac{6}{7}\le|\xi|\le 2$, equal to 1 on $1\le|\xi|\le\frac{12}{7}$, and satisfies for all $\xi\neq 0$*

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1 . \tag{2.2.24}$$

Let Δ_j^{Ψ} be the Littlewood–Paley operators associated with Ψ and let $0<p\le 1$. Then there exists a constant $C=C_{n,p,\Psi}$ such that for all $f\in H^p(\mathbf{R}^n)$ we have

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}|\Delta_j^{\Psi}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \le C\|f\|_{H^p} . \tag{2.2.25}$$

Conversely, suppose that a tempered distribution f satisfies

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}|\Delta_j^{\Psi}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}<\infty. \tag{2.2.26}$$

Then there exists a unique polynomial $Q(x)$ such that $f-Q$ lies in the Hardy space H^p and satisfies for some constant $C=C_{n,p,\Psi}$

$$\frac{1}{C}\big\|f-Q\big\|_{H^p}\le\Big\|\Big(\sum_{j\in\mathbf{Z}}|\Delta_j^{\Psi}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}. \tag{2.2.27}$$

Proof. We fix $\Phi\in\mathscr{S}(\mathbf{R}^n)$ with integral equal to 1 and we take $f\in H^p\cap L^1$ and M in $\mathbf{Z}^+$. Let r_j be the Rademacher functions, defined in Appendix C.1 in [156], reindexed so that their index set is the set of all integers (not the set of nonnegative integers). We begin with the estimate

$$\Big|\sum_{j=-M}^{M}r_j(\omega)\Delta_j^{\Psi}(f)\Big|\le\sup_{\varepsilon>0}\Big|\Phi_\varepsilon*\sum_{j=-M}^{M}r_j(\omega)\Delta_j^{\Psi}(f)\Big|,$$

which holds since $\{\Phi_\varepsilon\}_{\varepsilon>0}$ is an approximate identity. We raise this inequality to the power p, we integrate over $x\in\mathbf{R}^n$ and $\omega\in[0,1]$, and we use the maximal function characterization of H^p [Theorem 2.1.4(a)] to obtain

$$\int_0^1\int_{\mathbf{R}^n}\Big|\sum_{j=-M}^{M}r_j(\omega)\Delta_j^{\Psi}(f)(x)\Big|^p\,dx\,d\omega\le C_{p,n}^p\int_0^1\Big\|\sum_{j=-M}^{M}r_j(\omega)\Delta_j^{\Psi}(f)\Big\|_{H^p}^p d\omega.$$

Applying Fubini's theorem and the lower inequality for the Rademacher functions in Appendix C.2 in [156], yields

$$\int_{\mathbf{R}^n}\Big(\sum_{j=-M}^{M}|\Delta_j^{\Psi}(f)(x)|^2\Big)^{\frac{p}{2}}dx\le C_p^pC_{p,n}^p\int_0^1\Big\|\sum_{j=-M}^{M}r_j(\omega)\Delta_j^{\Psi}(f)\Big\|_{H^p}^p d\omega. \tag{2.2.28}$$

Next, for any fixed $M\in\mathbf{Z}^+$ and $\omega\in[0,1]$, we consider the mapping

$$f\mapsto\sum_{j=-M}^{M}r_j(\omega)\Delta_j^{\Psi}(f)$$

whose kernel

$$\sum_{k=-M}^{M}r_k(\omega)\Psi_{2^{-k}}(x)$$

satisfies (2.1.62) and (2.1.63) with constants A and B depending only on n and Ψ (thus, independent of ω and M). Applying Theorem 2.1.14 (the scalar version, i.e., the case where $L=1$) we obtain

$$\Big\|\sum_{j=-M}^{M}r_j(\omega)\Delta_j^{\Psi}(f)\Big\|_{H^p}^p\le C(n,p,\Psi)\big\|f\big\|_{H^p}^p.$$

Using this fact and (2.2.28), we conclude that

$$\Big\|\Big(\sum_{j=-M}^{M}|\Delta_j^{\Psi}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\le C_{n,p,\Psi}\big\|f\big\|_{H^p},$$

from which (2.2.25) follows directly by letting $M\to\infty$. We have now established (2.2.25) for $f\in H^p\cap L^1$. Using density, we can extend this estimate to all $f\in H^p$.

We now turn to the converse statement of the theorem. Assume that (2.2.26) holds for some tempered distribution f.

Set $\widehat{\eta}(\xi)=\widehat{\Psi}(\frac{1}{2}\xi)+\widehat{\Psi}(\xi)+\widehat{\Psi}(2\xi)$. Then $\widehat{\eta}$ is supported in an annulus and is equal to 1 on the support of $\widehat{\Psi}$. Using Theorem 2.1.14 we obtain that for any $L\in\mathbf{Z}^+$ and $L'\in\mathbf{Z}^+\cup\{0\}$ with $L'<L$ the mapping

$$\{f_j\}_{L'\le|j|<L}\mapsto\sum_{L'\le|j|<L}\Delta_j^{\eta}(f_j)$$

maps $H^p(\mathbf{R}^n,\ell^2_{2L-2L'})$ to $H^p(\mathbf{R}^n)$; note that if $L'=0$, then $\ell^2_{2L-2L'}$ should be ℓ^2_{2L-1}. Indeed, Theorem 2.1.14 can be applied, since the family of kernels $\{\eta_{2^{-j}}\}_{L'\le|j|<L}$ satisfies $\sum_{L'\le|j|<L}|\partial_x^\alpha(\eta_{2^{-j}})(x)|\le C_\alpha|x|^{-n-|\alpha|}$, $x\neq0$, for all multilindices α and $\sum_{L'\le|j|<L}|\widehat{\eta_{2^{-j}}}|\le c'$ with constants independent of L,L'. Thus we have

$$\Big\|\sum_{L'\le|j|<L}\Delta_j^{\eta}(f_j)\Big\|_{H^p}\le C_{p,n,\Phi}\Big\|\sup_{t>0}\Big(\sum_{L'\le|j|<L}|\Phi_t*f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}$$

for any Φ Schwartz function with nonvanishing integral and any $f_j\in H^p$. Taking[1] $f_j=\Delta_j^{\Psi}(f)$ and using that $\Delta_j^{\eta}\Delta_j^{\Psi}=\Delta_j^{\Psi}$, we deduce that for all $L\in\mathbf{Z}^+$ we have

$$\Big\|\sum_{L'\le|j|<L}\Delta_j^{\Psi}(f)\Big\|_{H^p}\le C_{p,n,\Phi}\Big\|\sup_{t>0}\Big(\sum_{L'\le|j|<L}|\Phi_t*\Delta_j^{\Psi}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}.$$

Applying Corollary 2.2.5 for some $r<p$ we arrive at the estimate

$$\begin{aligned}\Big\|\sum_{L'\le|j|<L}\Delta_j^{\Psi}(f)\Big\|_{H^p}&\le C_{p,n}\Big\|\Big(\sum_{L'\le|j|<L}|M(|\Delta_j^{\Psi}(f)|^r)|^{\frac{2}{r}}\Big)^{\frac{1}{2}}\Big\|_{L^p}\\&=C_{p,n}\Big\|\Big(\sum_{L'\le|j|<L}|M(|\Delta_j^{\Psi}(f)|^r)|^{\frac{2}{r}}\Big)^{\frac{r}{2}}\Big\|_{L^{\frac{p}{r}}}^{\frac{1}{r}}.\end{aligned}$$

Since $r<\min(2,p)$, we use the $L^{p/r}(\mathbf{R}^n,\ell^{2/r}_{2L-2L'})$ to $L^{p/r}(\mathbf{R}^n,\ell^{2/r}_{2L-2L'})$ boundedness of the Hardy–Littlewood maximal operator (Theorem 5.6.6 in [156]) to obtain the inequality

[1] $f_j\in H^p$ since $\sup_{t>0}|\Phi_t*\Delta_j^{\Psi}(f)|\le C'M(|\Delta_j^{\Psi}(f)|^r)^{1/r}\in L^p$ for $r<p$ in view of (2.2.26).

$$\sup_{L\in\mathbf{Z}^+}\sup_{0\le L'<L}\Big\|\sum_{L'\le|j|<L}\Delta_j^{\Psi}(f)\Big\|_{H^p}\le C'_{p,n}\Big\|\Big(\sum_{L'\le|j|<L}|\Delta_j^{\Psi}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}. \tag{2.2.29}$$

Thus the sequence $S^L(f)=\sum_{|j|<L}\Delta_j^{\Psi}(f)$, $L=1,2,\dots$ is Cauchy in H^p and by the completeness of H^p [Proposition 2.1.10(c)] it converges to an element $u_f\in H^p$. Obviously (2.2.29) has as a consequence that

$$\|u_f\|_{H^p}\le C'_{p,n}\Big\|\Big(\sum_{j\in\mathbf{Z}}|\Delta_j^{\Psi}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}. \tag{2.2.30}$$

It remains to relate u_f and f. In view of (1.1.6) we know that $S^L(f)\to f$ in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$; thus for any Schwartz function ψ whose support is disjoint from $\{0\}$ we have $\langle S^L(f)\widehat{\ },\psi\rangle\to\langle\widehat{f},\psi\rangle$. Thus $\langle\widehat{u_f},\psi\rangle=\langle\widehat{f},\psi\rangle$ and this implies that the support of $\widehat{u_f}-\widehat{f}$ is $\{0\}$. Proposition 2.4.1 in [156] gives the existence of a unique polynomial Q such that $u_f=f-Q$. Then clearly (2.2.30) implies (2.2.27). $\square$

The preceding proof can be modified to provide the following extension.

Corollary 2.2.10. *Fix Ψ in $\mathscr{S}(\mathbf{R}^n)$ with Fourier transform supported in $\frac{6}{7}\le|\xi|\le 2$, equal 1 on the annulus $1\le|\xi|\le\frac{12}{7}$, and satisfying $\sum_{j\in\mathbf{Z}}\widehat{\Psi}(2^{-j}\xi)=1$ for $\xi\neq 0$. Fix b_1,b_2 with $b_1<b_2$ and define a Schwartz function Ω via*

$$\widehat{\Omega}(\xi)=\sum_{j=b_1}^{b_2}\widehat{\Psi}(2^{-j}\xi).$$

Define $\Delta_k^{\Omega}(g)\widehat{\ }(\xi)=\widehat{g}(\xi)\widehat{\Omega}(2^{-k}\xi)$, $k\in\mathbf{Z}$. Let $q=b_2-b_1+1$, $0<p\le 1$, and fix $r\in\{0,1,\dots,q-1\}$. Then there exists a constant $C=C_{n,p,b_1,b_2,\Psi}$ such that for all $f\in H^p(\mathbf{R}^n)$ we have

$$\Big\|\Big(\sum_{j=r\mod q}|\Delta_j^{\Omega}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\le C\|f\|_{H^p}. \tag{2.2.31}$$

Conversely, suppose that a tempered distribution f satisfies

$$\Big\|\Big(\sum_{j=r\mod q}|\Delta_j^{\Omega}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}<\infty. \tag{2.2.32}$$

Then there exists a unique polynomial $Q(x)$ such that $f-Q$ lies in the Hardy space H^p and satisfies for some constant $C=C_{n,p,b_1,b_2,\Psi}$

$$\frac{1}{C}\|f-Q\|_{H^p}\le\Big\|\Big(\sum_{j=r\mod q}|\Delta_j^{\Omega}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}. \tag{2.2.33}$$

Proof. Inequality (2.2.31) is a direct consequence of (2.2.25) since Δ_k^{Ω} can be written as a finite sum of Δ_j^{Ω}'s. Conversely, we introduce a Schwartz function η whose Fourier transform $\widehat{\eta}$ is supported in an annulus of the form $0 < c_1 \le |\xi| \le c_2 < \infty$ and is equal to 1 on the support of Ω. Then (2.2.30) with Ω in place of Ψ follows as in the preceding proof. Since $\sum_{j=r \bmod q} \Delta_j^{\Omega}(f) = f$ in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$, which is a consequence of the fact that $\sum_{j=r \bmod q} \widehat{\Omega}(2^{-j}\xi) = 1$ for all $\xi \neq 0$, we conclude that there is a unique polynomial such that $f - Q$ lies in H^p and satisfies (2.2.33).

Exercises

2.2.1. Let $0 < q_0 \le q_1 < \infty$, $0 < p < \infty$, $\varepsilon > 0$, and $\alpha \in \mathbf{R}$. Prove the embeddings

$$\begin{aligned} B_p^{\alpha,q_0} &\subseteqq B_p^{\alpha,q_1}\,, \\ F_p^{\alpha,q_0} &\subseteqq F_p^{\alpha,q_1}\,, \\ B_p^{\alpha+\varepsilon,q_0} &\subseteqq B_p^{\alpha,q_1}\,, \\ F_p^{\alpha+\varepsilon,q_0} &\subseteqq F_p^{\alpha,q_1}\,, \end{aligned}$$

where p and q_1 are allowed to be infinite in the case of Besov spaces.

2.2.2. Let $0 < q < \infty$, $0 < p < \infty$, and $\alpha \in \mathbf{R}$. Prove that the embeddings

$$B_p^{\alpha,\min(p,q)} \subseteqq F_p^{\alpha,q} \subseteqq B_p^{\alpha,\max(p,q)}$$

hold with norm one, if the norms in the spaces are defined with respect to the same Schwartz function Ψ.
$\big[$*Hint:* When $p \ge q$ use Minkowski's inequality for $L^{p/q}$ for one embedding and the embedding $\ell^q \subseteqq \ell^p$ for the other. When $p < q$ use the reverse Minkowski inequality for $L^{p/q}$ for one embedding and the fact $(\sum_k |a_k|)^{p/q} \le \sum_k |a_k|^{p/q}$ for the other.$\big]$

2.2.3. Let $-\infty < \alpha < \infty$ and $0 < p, \beta < \infty$. Let $1' = \infty$ and $p' = p/(p-1)$ for $p \neq 1$.
(a) Suppose that the Fourier transform of function g is $\mathscr{C}^\infty$ and is equal to $|\xi|^{-\alpha}$ for $|\xi| \ge 10$. Show that g lies in $B_p^{\gamma,q}(\mathbf{R}^n)$ if and only if $0 < q < \infty$ and $\gamma < \alpha - \frac{n}{p'}$ or $q = \infty$ and $\gamma \le \alpha - \frac{n}{p'}$.
(b) If the Fourier transform of function g is $\mathscr{C}^\infty$ and is equal to $|\xi|^{-\alpha}(\log|\xi|)^{-\beta}$ for $|\xi| \ge 10$, then show that g lies in $B_p^{\alpha-\frac{n}{p'},q}(\mathbf{R}^n)$ if and only if $q > 1/\beta$.

2.2.4. Let $0 < p, q < \infty$ and $\alpha \in \mathbf{R}$. Show that the space of Schwartz functions is dense in all the spaces $B_p^{\alpha,q}(\mathbf{R}^n)$ and $F_p^{\alpha,q}(\mathbf{R}^n)$.

[*Hint:* Fix a function $\varphi \in \mathscr{S}(\mathbf{R}^n)$ whose Fourier transform has integral one and is supported in a ball of radius 1 centered at zero. Given $f \in F_p^{\alpha,q}(\mathbf{R}^n)$ consider the family of Schwartz functions

$$f_{N,\delta}(x) = S_0^{\Phi}(f)(x)\varphi(\delta x) + \sum_{j=1}^{N} \Delta_j^{\Psi}(f)(x)\varphi(\delta x)$$

for $0 < \delta < 1/10$.]

2.2.5. Let $\alpha \in \mathbf{R}$, let $0 < p,q < \infty$, and let $N = [\frac{n}{2} + \frac{n}{\min(p,q)}] + 1$. Assume that m is a $\mathscr{C}^N$ function on $\mathbf{R}^n \setminus \{0\}$ that satisfies

$$|\partial^{\gamma} m(\xi)| \le C_{\gamma} |\xi|^{-|\gamma|}$$

for all $|\gamma| \le N$. Show that there exists a constant C such that for all $f \in \mathscr{S}'/\mathscr{P}'$ we have

$$\big\|(m\widehat{f}\,)^{\vee}\big\|_{\dot{B}_p^{\alpha,q}} \le C\big\|f\big\|_{\dot{B}_p^{\alpha,q}}.$$

[*Hint:* Pick $r < \min(p,q)$ such that $N > \frac{n}{2} + \frac{n}{r}$. Write $m_j(\xi) = m(\xi)(\widehat{\Psi}(2^{-j+1}\xi) + \widehat{\Psi}(2^{-j}\xi) + \widehat{\Psi}(2^{-j-1}\xi))$. Then $\Delta_j^{\Psi}((m\widehat{f}\,)^{\vee}) = m_j^{\vee} * \Delta_j^{\Psi}(f)$. Obtain the estimate

$$\begin{aligned}\big|\big(m_j^{\vee} * \Delta_j^{\Psi}(f)\big)(x)\big| &\le C \sup_{y \in \mathbf{R}^n} \frac{\big|\Delta_j^{\Psi}(f)(x-y)\big|}{(1+2^j|y|)^{\frac{n}{r}}} \int_{\mathbf{R}^n} |m_j^{\vee}(y)|(1+2^j|y|)^{\frac{n}{r}}\, dy \\ &\le C' M(|\Delta_j^{\Psi}(f)|^r)^{\frac{1}{r}}(x) \bigg(\int_{\mathbf{R}^n} |m_j(2^j(\cdot))^{\vee}(y)|^2 (1+|y|)^{2N}\, dy \bigg)^{\frac{1}{2}}.\end{aligned}$$

The hypothesis on m implies that the preceding integral is bounded by a constant.]

2.2.6. ([293]) Let m be as in Exercise 2.2.5. Show that there exists a constant C such that for all $f \in \mathscr{S}'(\mathbf{R}^n)/\mathscr{P}'(\mathbf{R}^n)$ we have

$$\big\|(m\widehat{f}\,)^{\vee}\big\|_{\dot{F}_p^{\alpha,q}} \le C\big\|f\big\|_{\dot{F}_p^{\alpha,q}}.$$

[*Hint:* Use the hint of Exercise 2.2.5 and Theorem 5.6.6 in [156].]

2.2.7. (a) Suppose that $B_{p_0}^{\alpha_0,q_0} = B_{p_1}^{\alpha_1,q_1}$ with equivalent norms. Prove that $\alpha_0 = \alpha_1$ and $p_0 = p_1$. Prove the same result for the scale of Triebel–Lizorkin spaces.
(b) Suppose that $B_{p_0}^{\alpha_0,q_0} = B_{p_1}^{\alpha_1,q_1}$ with equivalent norms. Prove that $q_0 = q_1$. Argue similarly with the scale of Triebel–Lizorkin spaces.
[*Hint:* Part (a): Test the corresponding norms on the function $\eta(2^j x)$, where η is chosen so that its Fourier transform is supported in $1 \le |\xi| \le \frac{12}{7}$. Part (b): Try a function f of the form $\widehat{f}(\xi) = \sum_{j=1}^{N} a_j \widehat{\varphi}(\xi_1 - 2^j, \xi_2, \dots, \xi_n)$, where φ is a Schwartz function whose Fourier transform is supported in a small neighborhood of the origin.]

2.3 Atomic Decomposition of Homogeneous Triebel–Lizorkin Spaces

In this section we focus attention on the homogeneous Triebel–Lizorkin spaces $\dot{F}_p^{\alpha,q}$, which include the Hardy spaces discussed in Section 2.1. Most results discussed in this section are also valid for the inhomogeneous Triebel–Lizorkin spaces and for the Besov–Lipschitz spaces via a similar or simpler analysis.

2.3.1 Embeddings and Completeness of Triebel–Lizorkin Spaces

Proposition 2.3.1. *Let $0 < p,q < \infty$, and $\alpha \in \mathbf{R}$. The homogeneous Triebel–Lizorkin space $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ is continuously embedded in the Besov space $\dot{B}_p^{\alpha,\infty}(\mathbf{R}^n)$ which is in turn continuously embedded in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$. Moreover, the space $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ is complete.*

Proof. Given $f \in \mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ we have the sequence of inequalities

$$\sup_{j\in\mathbf{Z}} 2^{j\alpha}\|\Delta_j^\Psi(f)\|_{L^p} \le \Big\|\sup_{j\in\mathbf{Z}} |2^{j\alpha}\Delta_j^\Psi(f)|\Big\|_{L^p} \le \Big\|\Big(\sum_{j\in\mathbf{Z}} |2^{j\alpha}\Delta_j^\Psi(f)|^q\Big)^{\frac{1}{q}}\Big\|_{L^p}, \quad (2.3.1)$$

which shows that $\|f\|_{\dot{B}_p^{\alpha,\infty}} \le \|f\|_{\dot{F}_p^{\alpha,q}}$. Thus we proved the embedding $\dot{F}_p^{\alpha,q} \subseteqq \dot{B}_p^{\alpha,\infty}$.

Next we prove that $\dot{B}_p^{\alpha,\infty}(\mathbf{R}^n)$ continuously embeds in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$. Let ψ be in $\mathscr{S}_0(\mathbf{R}^n)$. Then given Ψ as in (2.2.1), let $\widehat{\Omega}(\xi) = \widehat{\Psi}(\frac{1}{2}\xi) + \widehat{\Psi}(\xi) + \widehat{\Psi}(2\xi)$. Given $f \in \mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ we have

$$\langle f,\psi\rangle = \sum_{j\in\mathbf{Z}} \langle \Delta_j^\Psi(f),\psi\rangle = \sum_{j\in\mathbf{Z}} \langle \Delta_j^\Psi(f),\Delta_j^\Omega(\psi)\rangle,$$

where the first identity is due to the fact that the series $\sum_{j\in\mathbf{Z}} \Delta_j^\Psi$ converges in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ and the second identity to the fact that $\widehat{\Omega}$ is equal to one on the support of $\widehat{\Psi}$. It follows that

$$\begin{aligned}
\big|\langle f,\psi\rangle\big| &\le \sum_{j\in\mathbf{Z}} \big\|\Delta_j^\Psi(f)\big\|_{L^\infty}\big\|\Delta_j^\Omega(\psi)\big\|_{L^1} \\
&\le C\sum_{j\in\mathbf{Z}} 2^{\frac{jn}{p}-j\alpha}\big\|2^{j\alpha}\Delta_j^\Psi(f)\big\|_{L^p}\big\|\Delta_j^\Omega(\psi)\big\|_{L^1} \\
&\le \|f\|_{\dot{F}_p^{\alpha,q}} C\sum_{j\in\mathbf{Z}} 2^{\frac{jn}{p}-j\alpha}\big\|\Delta_j^\Omega(\psi)\big\|_{L^1} \\
&= C\|f\|_{\dot{F}_p^{\alpha,q}}\|\psi\|_{\dot{B}_1^{\frac{n}{p}-\alpha,1}},
\end{aligned}$$

where we used Corollary 2.2.4 in the second inequality and (2.3.1) in the last inequality.

Next, we show that $\|\psi\|_{\dot{B}_1^{\frac{n}{p}-\alpha,1}}$ is controlled by a finite sum of Schwartz seminorms of $\psi \in \mathscr{S}_0(\mathbf{R}^n)$. Using the result in Appendix B.4, we obtain the following estimate for all $L \in \mathbf{Z}^+$ and $N > 0$ satisfying $N < N' - (L+1+n)$

$$\left|\Delta_j^{\Omega}(\psi)(x)\right| \le C''_{N,N',L,n}\left[\sup_{|\gamma|\le L}\sup_{x\in\mathbf{R}^n}|\partial^\gamma\psi(x)|(1+|x|)^{N'}\right]\frac{2^{\min(j,0)n-|j|(L+1)}}{(1+2^{\min(j,0)}|x|)^N},$$

where the constant $C''_{N,N',L,n}$ also depends on Ω. Consequently we obtain that

$$\left\|\Delta_j^{\Omega}(\psi)\right\|_{L^p} \le C'''_{N,N',L,n}\left[\sup_{|\gamma|\le L}\sup_{x\in\mathbf{R}^n}|\partial^\gamma\psi(x)|(1+|x|)^{N'}\right]2^{\min(j,0)\frac{n}{p'}-|j|(L+1)}$$

if $N > n/p$. Choosing $L > n+|\alpha|$, it follows that

$$\left\|\psi\right\|_{\dot{B}_1^{\frac{n}{p}-\alpha,1}} = \sum_{j\in\mathbf{Z}} 2^{j(\frac{n}{p}-\alpha)}\left\|\Delta_j^{\Omega}(\psi)\right\|_{L^p}$$

is bounded by a constant multiple of the expression

$$\sup_{|\gamma|\le L}\sup_{x\in\mathbf{R}^n}|\partial^\gamma\psi(x)|(1+|x|)^{N'}$$

which is controlled by a finite sum of seminorms $\rho_{\alpha,\beta}(\psi)$. This proves that $\dot{B}_p^{\alpha,\infty}(\mathbf{R}^n)$ is continuously embedded in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$.

Finally, we turn to the last assertion that the space $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ is complete. Since $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ is continuously embedded in $\mathscr{S}'/\mathscr{P}$, every Cauchy sequence $\{u_M\}_{M=0}^\infty$ in $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ is Cauchy in $\mathscr{S}'/\mathscr{P}$ and thus it converges to an element $u \in \mathscr{S}'/\mathscr{P}$, defined by $\langle u,\psi\rangle = \lim_{M\to\infty}\langle u_M,\psi\rangle$ for all $\psi\in\mathscr{S}_0(\mathbf{R}^n)$.

Since $u_M \to u$ in $\mathscr{S}'/\mathscr{P}$, it follows that for every $j \in \mathbf{Z}$

$$\Delta_j^{\Psi}(u_M - u_{M'}) \to \Delta_j^{\Psi}(u-u_{M'})$$

as $M\to\infty$. Thus for any $J\in\mathbf{Z}^+$ we have

$$\begin{aligned}\Big(\sum_{|j|\le J}(2^{j\alpha}|\Delta_j^{\Psi}(u-u_{M'})|)^q\Big)^{\frac{1}{q}} &= \liminf_{M\to\infty}\Big(\sum_{|j|\le J}(2^{j\alpha}|\Delta_j^{\Psi}(u_M-u_{M'})|)^q\Big)^{\frac{1}{q}}\\ &\le \liminf_{M\to\infty}\Big(\sum_{j\in\mathbf{Z}}(2^{j\alpha}|\Delta_j^{\Psi}(u_M-u_{M'})|)^q\Big)^{\frac{1}{q}}.\end{aligned}$$

First we let $J\to\infty$, then we take L^p quasi-norms and we apply Fatou's lemma. We obtain

$$\|u-u_{M'}\|_{\dot{F}_p^{\alpha,q}} \le \liminf_{M\to\infty}\|u_M-u_{M'}\|_{\dot{F}_p^{\alpha,q}}.$$

If we replace $u_{M'}$ by 0, this implies that u lies in $\dot{F}_p^{\alpha,q}$ since $\sup_{M\geq 0}\|u_M\|_{\dot{F}_p^{\alpha,q}} < \infty$. Then we have

$$\limsup_{M'\to\infty}\|u-u_{M'}\|_{\dot{F}_p^{\alpha,q}} \leq \limsup_{M'\to\infty}\limsup_{M\to\infty}\|u_M-u_{M'}\|_{\dot{F}_p^{\alpha,q}},$$

but the expression on the right is zero since the sequence $\{u_M\}_{M=0}^{\infty}$ is Cauchy in $\dot{F}_p^{\alpha,q}$. It follows that $u_M \to u$ in $\dot{F}_p^{\alpha,q}$ as $M\to\infty$; thus $\dot{F}_p^{\alpha,q}$ is complete. □

2.3.2 The Space of Triebel–Lizorkin Sequences

To provide further intuition into the understanding of the homogeneous Triebel–Lizorkin spaces we introduce a related space consisting of sequences of scalars. This space is denoted by $\dot{f}_p^{\alpha,q}$ and is related to $\dot{F}_p^{\alpha,q}$ in a way similar to that in which $\ell^2(\mathbf{Z})$ is related to $L^2([0,1])$.

Definition 2.3.2. Let $0<q\leq\infty$ and $\alpha\in\mathbf{R}$. Let $\mathscr{D}$ be the set of all dyadic cubes in $\mathbf{R}^n$. We consider the set of all sequences $\{s_Q\}_{Q\in\mathscr{D}}$ such that the function

$$g^{\alpha,q}(\{s_Q\}_Q) = \Big(\sum_{Q\in\mathscr{D}}(|Q|^{-\frac{\alpha}{n}-\frac{1}{2}}|s_Q|\chi_Q)^q\Big)^{\frac{1}{q}} \tag{2.3.2}$$

is in $L^p(\mathbf{R}^n)$. For such sequences $s=\{s_Q\}_Q$ we set

$$\|s\|_{\dot{f}_p^{\alpha,q}} = \big\|g^{\alpha,q}(s)\big\|_{L^p(\mathbf{R}^n)}.$$

2.3.3 The Smooth Atomic Decomposition of Homogeneous Triebel–Lizorkin Spaces

We discuss the smooth atomic decomposition of homogeneous Triebel–Lizorkin spaces. We denote by $\mathscr{D}$ the space of all dyadic cubes on $\mathbf{R}^n$. For any fixed $j\in\mathbf{Z}$ we let $\mathscr{D}_j=\{Q\in\mathscr{D}:\ \ell(Q)=2^{-j}\}$. We begin with the definition of smooth atoms on $\mathbf{R}^n$.

Definition 2.3.3. Let Q be a dyadic cube and let L be a nonnegative integer. A $\mathscr{C}^\infty$ function a_Q on $\mathbf{R}^n$ is called a *smooth L-atom for Q* if it satisfies the following properties:

(a) a_Q is supported in $3Q$ (the cube concentric with Q having three times its side length);

(b) $\displaystyle\int_{\mathbf{R}^n} x^\gamma a_Q(x)\,dx = 0$ for all multi-indices γ with $|\gamma|\leq L$;

(c) $|\partial^\gamma a_Q| \leq |Q|^{-\frac{|\gamma|}{n}-\frac{1}{2}}$ for all multi-indices γ satisfying $|\gamma|\leq L+1$.

In view of properties (a) and (c) of Definition 1.3.2, for every $M>0$ there is a constant $C(n,M,L)$ such that every smooth L-atom a_Q supported in Q with center c_Q and side length $\ell(Q)$ satisfies

$$|\partial^\gamma a_Q(x)| \le C(n,M,L)\ell(Q)^{\frac{n}{2}} \frac{\ell(Q)^{-n-|\gamma|}}{\left(1+\frac{|x-c_Q|}{\ell(Q)}\right)^M} \tag{2.3.3}$$

for all $x\in\mathbf{R}^n$ and for all multi-indices γ with $|\gamma|\le L+1$.

We now prove a theorem stating that elements of $\dot F_p^{\alpha,q}$ can be decomposed as sums of smooth atoms.

Theorem 2.3.4. *Let* $0<p,q<\infty$*,* $\alpha\in\mathbf{R}$*, and let*

$$L=\left[\max\left(n\max(1,\tfrac{1}{p},\tfrac{1}{q})-n-\alpha,\alpha\right)\right].$$

Then there is a constant $C_{n,p,q,\alpha}$ *such that for every sequence of smooth L-atoms* $\{a_Q\}_{Q\in\mathscr{D}}$ *and every sequence of complex scalars* $\{s_Q\}_{Q\in\mathscr{D}}$ *in* $\dot f_p^{\alpha,q}$ *we have that the series* $\sum_{\mu\in\mathbf{Z}}\left(\sum_{Q\in\mathscr{D}_\mu} s_Q a_Q\right)$ *converges in* $\dot F_p^{\alpha,q}(\mathbf{R}^n)$ *to an element* f *of* $\dot F_p^{\alpha,q}(\mathbf{R}^n)$ *with quasi-norm*

$$\big\|f\big\|_{\dot F_p^{\alpha,q}} \le C_{n,p,q,\alpha}\big\|\{s_Q\}_Q\big\|_{\dot f_p^{\alpha,q}}. \tag{2.3.4}$$

Conversely, there is a constant $C'_{n,p,q,\alpha}$ *such that given any distribution* f *in* $\dot F_p^{\alpha,q}$ *and any* $L\in\mathbf{Z}^+$*, there exist a sequence of smooth L-atoms* $\{a_Q\}_{Q\in\mathscr{D}}$ *and a sequence of complex scalars* $\{s_Q\}_{Q\in\mathscr{D}}$ *such that the series* $\sum_{\mu\in\mathbf{Z}}\left(\sum_{Q\in\mathscr{D}_\mu} s_Q a_Q\right)$ *converges to* f *in* $\dot F_p^{\alpha,q}(\mathbf{R}^n)$ *and*

$$\big\|\{s_Q\}_Q\big\|_{\dot f_p^{\alpha,q}} \le C'_{n,p,q,\alpha}\big\|f\big\|_{\dot F_p^{\alpha,q}}. \tag{2.3.5}$$

We observe that for any given x the expression $\sum_{Q\in\mathscr{D}_\mu} s_Q a_Q(x)$ is a finite sum with at most 3^n summands, so the convergence concerns the series in μ.

Proof. We prove the first assertion of the theorem. We let Δ_j^Ψ be the Littlewood–Paley operator associated with a Schwartz function Ψ whose Fourier transform is compactly supported away from the origin in $\mathbf{R}^n$. Let a_Q be a smooth L-atom supported in a cube $3Q$ with center c_Q and let the side length of Q be $\ell(Q)=2^{-\mu}$. It follows from (2.3.3) that a_Q satisfies

$$|\partial_y^\gamma a_Q(y)| \le C_{N',n}2^{-\frac{\mu n}{2}}\frac{2^{\mu|\gamma|+\mu n}}{(1+2^\mu|y-c_Q|)^{N'}} \tag{2.3.6}$$

for all $N'>0$ and for all multi-indices γ satisfying $|\gamma|\le L+1$. Moreover, the function $y\mapsto \Psi_{2^{-j}}(y-x)$ satisfies

$$|\partial_y^\beta \Psi_{2^{-j}}(y-x)| \le C_{N',n,\beta}\frac{2^{j|\beta|+jn}}{(1+2^j|y-x|)^{N'}} \tag{2.3.7}$$

for all $N'>0$ and for all multi-indices β.

The function a_Q has vanishing moments of all orders up to and including $L = (L+1)-1$ and satisfies (2.3.6) for all multi-indices γ with $|\gamma| \le L+1$. The function $y \mapsto \Psi_{2^{-j}}(y-x)$ has vanishing moments of all orders and satisfies (2.3.7) for all multi-indices β. Using the result in Appendix B.4, we deduce the following estimate for all $N>0$ satisfying $N < N' - (L+1+n)$

$$\big|\Delta_j^{\Psi}(a_Q)(x)\big| \le C_{N,n,L}\, 2^{-\frac{\mu n}{2}} \frac{2^{\min(j,\mu)n-|\mu-j|(L+1)}}{(1+2^{\min(j,\mu)}|x-c_Q|)^N}\,. \tag{2.3.8}$$

Now fix $0<b<\min(1,p,q)$ so that

$$L+1 > \frac{n}{b} - n - \alpha\,. \tag{2.3.9}$$

This can be achieved by taking b close enough to $\min(1,p,q)$, since our assumption $L = \big[\max\big(n\max\big(1,\frac{1}{p},\frac{1}{q}\big)-n-\alpha,\alpha\big)\big]$ yields that $L+1 > n\max\big(1,\frac{1}{p},\frac{1}{q}\big)-n-\alpha$ and also that $L+1>\alpha$. These two conditions imply that the function $d(k)$ defined for $k\in\mathbf{Z}$ by

$$d(k) = 2^{\min(k,0)(n-\frac{n}{b})+k\alpha-|k|(L+1)}$$

satisfies for some $\delta>0$

$$d(k) \le C\,2^{-|k|\delta} \tag{2.3.10}$$

for all $k\in\mathbf{Z}$. Using Exercise 2.3.6, we obtain

$$\sum_{Q\in\mathscr{D}_\mu} \frac{|s_Q|}{(1+2^{\min(j,\mu)}|x-c_Q|)^N} \le c\,2^{\max(\mu-j,0)\frac{n}{b}} \bigg\{ M\Big(\sum_{Q\in\mathscr{D}_\mu} |s_Q|^b\chi_Q\Big)(x)\bigg\}^{\frac{1}{b}}$$

whenever $N>n/b$, where M is the Hardy–Littlewood maximal operator. It follows from the preceding estimate and (2.3.8) that

$$2^{j\alpha}\sum_{\mu\in\mathbf{Z}}\sum_{Q\in\mathscr{D}_\mu}|s_Q|\big|\Delta_j^{\Psi}(a_Q)\big| \le C_0 \sum_{\mu\in\mathbf{Z}} d(j-\mu)\bigg\{M\Big(\sum_{Q\in\mathscr{D}_\mu}\big(|s_Q|\,|Q|^{-\frac{1}{2}-\frac{\alpha}{n}}\big)^b\chi_Q\Big)\bigg\}^{\frac{1}{b}},$$

where $C_0 = c\,C_{N,n,L}$. In particular this estimate is valid for any finite subset $\mathbf{Z}'$ of $\mathbf{Z}$. For such a subset we have

$$2^{j\alpha}\Delta_j^{\Psi}\Big(\sum_{\mu\in\mathbf{Z}'}\sum_{Q\in\mathscr{D}_\mu} s_Q a_Q\Big) = 2^{j\alpha}\sum_{\mu\in\mathbf{Z}'}\sum_{Q\in\mathscr{D}_\mu} s_Q\Delta_j^{\Psi}(a_Q)\,. \tag{2.3.11}$$

Raise the last displayed inequality to the power q and sum over $j\in\mathbf{Z}$; then raise to the power $1/q$ and take $\|\cdot\|_{L^p}$ quasi-norms. We obtain

$$\begin{aligned}&\Big\|\sum_{\mu\in\mathbf{Z}'}\sum_{Q\in\mathscr{D}_\mu} s_Q a_Q\Big\|_{\dot F_p^{\alpha,q}}\\ &\le C_0\bigg\|\bigg\{\sum_{j\in\mathbf{Z}}\bigg[\sum_{\mu\in\mathbf{Z}'} d(j-\mu)\Big\{M\Big(\sum_{Q\in\mathscr{D}_\mu}\big(|s_Q|\,|Q|^{-\frac{1}{2}-\frac{\alpha}{n}}\big)^b\chi_Q\Big)\Big\}^{\frac{1}{b}}\bigg]^q\bigg\}^{\frac{1}{q}}\bigg\|_{L^p}.\end{aligned}$$

We now estimate the expression inside the preceding L^p norm by

$$\Big\{\sum_{j\in\mathbf{Z}} d(j)^{\min(1,q)}\Big\}^{\frac{1}{\min(1,q)}}\Big\{\sum_{\mu\in\mathbf{Z}'}\Big\{M\Big(\sum_{Q\in\mathscr{D}_\mu}\big(|s_Q|\,|Q|^{-\frac12-\frac{\alpha}{n}}\big)^b\chi_Q\Big)\Big\}^{\frac{q}{b}}\Big\}^{\frac1q},$$

and we note that the first term is a constant in view of (2.3.10). We conclude that

$$\begin{aligned}\Big\|\sum_{\mu\in\mathbf{Z}'}\sum_{Q\in\mathscr{D}_\mu}s_Q\Big\|_{\dot F_p^{\alpha,q}} &\le C_0C\Big\|\Big\{\sum_{\mu\in\mathbf{Z}'}\Big\{M\Big(\sum_{Q\in\mathscr{D}_\mu}\big(|s_Q|\,|Q|^{-\frac12-\frac{\alpha}{n}}\big)^b\chi_Q\Big)\Big\}^{\frac{q}{b}}\Big\}^{\frac1q}\Big\|_{L^p}\\ &= C_0C\Big\|\Big\{\sum_{\mu\in\mathbf{Z}'}\Big\{M\Big(\sum_{Q\in\mathscr{D}_\mu}\big(|s_Q|\,|Q|^{-\frac12-\frac{\alpha}{n}}\big)^b\chi_Q\Big)\Big\}^{\frac{q}{b}}\Big\}^{\frac{b}{q}}\Big\|_{L^{\frac{p}{b}}}^{\frac1b}\\ &\le C_0C'\Big\|\Big\{\sum_{\mu\in\mathbf{Z}'}\Big\{\sum_{Q\in\mathscr{D}_\mu}\big(|s_Q|\,|Q|^{-\frac12-\frac{\alpha}{n}}\big)^b\chi_Q\Big\}^{\frac{q}{b}}\Big\}^{\frac{b}{q}}\Big\|_{L^{\frac{p}{b}}}^{\frac1b}\\ &= C_0C'\Big\|\Big\{\sum_{\mu\in\mathbf{Z}'}\sum_{Q\in\mathscr{D}_\mu}\big(|s_Q|\,|Q|^{-\frac12-\frac{\alpha}{n}}\big)^q\chi_Q\Big\}^{\frac1q}\Big\|_{L^p},\end{aligned}\tag{2.3.12}$$

where in the last inequality we used Theorem 5.6.6 in [156], which is valid under the assumption $1<\frac{p}{b},\frac{q}{b}<\infty$. We now take $\mathbf{Z}'=\{\mu\in\mathbf{Z}:\ M'<|\mu|\le M\}$, for some integers $M'<M$, and we use the following consequence of the Lebesgue dominated convergence theorem

$$\lim_{M',M\to\infty}\Big\|\Big\{\sum_{M'<|\mu|\le M}\sum_{Q\in\mathscr{D}_\mu}\big(|s_Q|\,|Q|^{-\frac12-\frac{\alpha}{n}}\big)^q\chi_Q\Big\}^{\frac1q}\Big\|_{L^p}=0,$$

since $s=\{s_Q\}_{Q\in\mathscr{D}}\in\dot f_p^{\alpha,q}$. We obtain that the sequence

$$F_M=\sum_{|\mu|\le M}\sum_{Q\in\mathscr{D}_\mu}s_Qa_Q$$

is Cauchy in $\dot F_p^{\alpha,q}$. Proposition 2.3.1 yields that it converges to an element f in $\dot F_p^{\alpha,q}$.

We now repeat the preceding argument replacing $\mathbf{Z}'$ by $\mathbf{Z}$ and $\sum_{\mu\in\mathbf{Z}'}\sum_{Q\in\mathscr{D}_\mu}s_Q$ by f noting that (2.3.11) holds for $\mathbf{Z}$ in place of $\mathbf{Z}'$ since we can interchange Δ_j^{Ψ} with the infinite sum over μ (and certainly with the finite sum in $Q\in\mathscr{D}_\mu$) in view of the convergence of the sequence $\sum_{|\mu|\le M}\big(\sum_{Q\in\mathscr{D}_\mu}s_Qa_Q\big)$ to f in $\dot F_p^{\alpha,q}$ (and thus in $\mathscr{S}'/\mathscr{P}$). This proves (2.3.4) since (2.3.12) is controlled by $\|s\|_{\dot f_p^{\alpha,q}}$.

We now turn to the converse statement of the theorem. It is not difficult to see that given $L\in\mathbf{Z}^+$ there exist Schwartz functions Θ and Ψ (unrelated to the previous one) such that $\widehat{\Psi}$ is supported in the annulus $\frac12\le|\xi|\le 2$ and Θ is supported in the ball $|x|\le 1$ and satisfies $\int_{\mathbf{R}^n}x^\gamma\Theta(x)\,dx=0$ for all $|\gamma|\le L$, such that the identity

$$\sum_{j\in\mathbf{Z}}\widehat{\Psi}(2^{-j}\xi)\widehat{\Theta}(2^{-j}\xi)=1\tag{2.3.13}$$

holds for all $\xi\in\mathbf{R}^n\setminus\{0\}$; see Exercise 2.3.1.

Given a distribution $f \in \dot{F}_p^{\alpha,q}$, using identity (2.3.13), we write

$$f = \sum_{j\in\mathbf{Z}} \Psi_{2^{-j}} * \Theta_{2^{-j}} * f\,,$$

where the convergence is in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ in view of Corollary 1.1.7.

For each Q in $\mathscr{D}_j$ define a constant

$$s_Q = |Q|^{\frac{1}{2}} \sup_{y\in Q} |(\Psi_{\ell(Q)} * f)(y)| \sup_{|\gamma|\le L+1} \big\|\partial^\gamma \Theta\big\|_{L^1}$$

and a function

$$a_Q(x) = \frac{1}{s_Q} \int_Q \Theta_{\ell(Q)}(x-y)(\Psi_{\ell(Q)} * f)(y)\,dy\,. \tag{2.3.14}$$

It is straightforward to verify that a_Q is supported in $3Q$ and that it has vanishing moments up to and including order L, since θ does so. Moreover, using (2.3.14) we obtain for all $|\gamma| \le L+1$

$$|\partial^\gamma a_Q| \le \frac{1}{s_Q} \big\|\partial^\gamma \Theta\big\|_{L^1} \ell(Q)^{-|\gamma|} \sup_Q |\Psi_{\ell(Q)} * f| \le |Q|^{-\frac{1}{2}-\frac{|\gamma|}{n}}\,,$$

which makes the function a_Q a smooth L-atom.

Using this notation, we write

$$f = \sum_{j\in\mathbf{Z}} \sum_{Q\in\mathscr{D}_j} \int_Q \Theta_{2^{-j}}(x-y)(\Psi_{2^{-j}} * f)(y)\,dy = \sum_{j\in\mathbf{Z}} \Big(\sum_{Q\in\mathscr{D}_j} s_Q a_Q \Big)\,,$$

where the series in j converges in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$.

Let b be as in (2.3.9). Now note that

$$\begin{aligned}
&\sum_{\ell(Q)=2^{-j}} \big(|Q|^{-\frac{\alpha}{n}-\frac{1}{2}} s_Q \chi_Q(x)\big)^q \\
&\qquad = C \sum_{\ell(Q)=2^{-j}} \big(2^{j\alpha} \sup_{y\in Q} |(\Psi_{2^{-j}} * f)(y)| \chi_Q(x)\big)^q \\
&\qquad \le C \sup_{|z|\le\sqrt{n}2^{-j}} \big(2^{j\alpha}(1+2^j|z|)^{-b} |(\Psi_{2^{-j}} * f)(x-z)|\big)^q (1+2^j|z|)^{bq} \\
&\qquad \le C\big(2^{j\alpha} M_{b,j}^{**}(f,\Psi)(x)\big)^q\,,
\end{aligned}$$

where we used the fact that in the first inequality there is only one nonzero term in the sum because of the appearance of the characteristic function. Summing over all $j \in \mathbf{Z}^n$, raising to the power $1/q$, and taking L^p norms yields the estimate

$$\big\|\{s_Q\}_Q\big\|_{\dot{f}_p^{\alpha,q}} \le C \Big\| \Big(\sum_{j\in\mathbf{Z}} |2^{j\alpha} M_{b,j}^{**}(f;\Psi)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p} \le C \|f\|_{\dot{F}_p^{\alpha,q}}\,,$$

where the last inequality follows from Theorem 2.2.8. This proves (2.3.5). It follows from (2.3.5) that $\{s_Q\}_{Q\in\mathscr{D}}$ lies in $\dot{f}_p^{\alpha,q}$ and thus by the first assertion of the theorem we have that the series

$$\sum_{\mu}\Big(\sum_{Q\in\mathscr{D}_\mu} s_Q a_Q\Big)$$

converges to some element in $\dot{F}_p^{\alpha,q}$. Since it converges to f in $\mathscr{S}'/\mathscr{P}$, it follows that $\sum_\mu\big(\sum_{Q\in\mathscr{D}_\mu} s_Q a_Q\big)$ converges to f in $\dot{F}_p^{\alpha,q}$, and this completes the proof. □

2.3.4 The Nonsmooth Atomic Decomposition of Homogeneous Triebel–Lizorkin Spaces

We now discuss the main theorem of this section, the nonsmooth atomic decomposition of the homogeneous Triebel–Lizorkin spaces $\dot{F}_p^{\alpha,q}$, which in particular includes that of the Hardy spaces H^p. We begin with a definition.

Definition 2.3.5. Let $0 < p,q < \infty$. A sequence of complex numbers $r=\{r_Q\}_{Q\in\mathscr{D}}$ is called an ∞-atom for $\dot{f}_p^{\alpha,q}$ if there exists a dyadic cube Q_0 such that

(a) $r_Q = 0$ if $Q \nsubseteq Q_0$;

(b) $\big\|g^{\alpha,q}(r)\big\|_{L^\infty} \le |Q_0|^{-\frac{1}{p}}$,

where, recalling from (2.3.2),

$$g^{\alpha,q}(\{r_Q\}_Q) = \Big(\sum_{Q\in\mathscr{D}} (|Q|^{-\frac{\alpha}{n}-\frac{1}{2}}|r_Q|\chi_Q)^q\Big)^{\frac{1}{q}}.$$

We observe that every ∞-atom $r=\{r_Q\}$ for $\dot{f}_p^{\alpha,q}$ satisfies $\|r\|_{\dot{f}_p^{\alpha,q}} \le 1$. Indeed,

$$\|r\|_{\dot{f}_p^{\alpha,q}}^p = \int_{Q_0} |g^{\alpha,q}(r)|^p\,dx \le |Q_0|^{-1}|Q_0| = 1.$$

The following theorem concerns the atomic decomposition of the spaces $\dot{f}_p^{\alpha,q}$.

Theorem 2.3.6. *Let $\alpha\in\mathbf{R}$, $0<p,q<\infty$, and $s=\{s_Q\}_{Q\in\mathscr{D}}$ be in $\dot{f}_p^{\alpha,q}$. Then there exist $C_{n,p,q}>0$, a sequence of scalars λ_j, and a sequence of ∞-atoms $r_j=\{r_{j,Q}\}_{Q\in\mathscr{D}}$ for $\dot{f}_p^{\alpha,q}$ such that for each $Q\in\mathscr{D}$ the series $\sum_{j=1}^\infty \lambda_j r_{j,Q}$ is absolutely convergent and equal to s_Q, i.e.,*

$$s=\{s_Q\}_{Q\in\mathscr{D}} = \sum_{j=1}^\infty \lambda_j\{r_{j,Q}\}_{Q\in\mathscr{D}} = \sum_{j=1}^\infty \lambda_j r_j,$$

and such that

$$\Big(\sum_{j=1}^\infty |\lambda_j|^p\Big)^{\frac{1}{p}} \le C_{n,p,q}\|s\|_{\dot{f}_p^{\alpha,q}}. \tag{2.3.15}$$

Proof. We fix α, p, q, and a sequence $s = \{s_Q\}_{Q\in\mathscr{D}}$ in $\dot{f}_p^{\alpha,q}$. For a dyadic cube R in $\mathscr{D}$ we define the function

$$g_R^{\alpha,q}(s)(x) = \Big(\sum_{\substack{Q\in\mathscr{D}\\ R\subseteq Q}} \big(|Q|^{\frac{\alpha}{n}-\frac{1}{2}}|s_Q|\,\chi_Q(x)\big)^q \Big)^{\frac{1}{q}}$$

and we observe that this function is constant on R. We also note that for dyadic cubes R_1 and R_2 with $R_1 \subseteq R_2$ we have

$$g_{R_2}^{\alpha,q}(s) \le g_{R_1}^{\alpha,q}(s)\,.$$

Finally, we observe that

$$\lim_{\substack{\ell(R)\to\infty\\ x\in R}} g_R^{\alpha,q}(s)(x) = 0$$

and

$$\lim_{\substack{\ell(R)\to 0\\ x\in R}} g_R^{\alpha,q}(s)(x) = g^{\alpha,q}(s)(x)\,,$$

where $g^{\alpha,q}(s)$ is the function defined in (2.3.2).

For $k \in \mathbf{Z}$ we set

$$\mathscr{A}_k = \big\{R \in \mathscr{D}:\ g_R^{\alpha,q}(s)(x) > 2^k \quad \text{for all } x \in R\big\}\,.$$

We note that $\mathscr{A}_{k+1} \subseteq \mathscr{A}_k$ for all k in $\mathbf{Z}$ and that

$$\{x \in \mathbf{R}^n:\ g^{\alpha,q}(s)(x) > 2^k\} = \bigcup_{R\in\mathscr{A}_k} R\,. \tag{2.3.16}$$

Moreover, we have for all $k \in \mathbf{Z}$,

$$\Big(\sum_{Q\in\mathscr{D}\setminus\mathscr{A}_k} \big(|Q|^{-\frac{\alpha}{n}-\frac{1}{2}}|s_Q|\,\chi_Q(x)\big)^q \Big)^{\frac{1}{q}} \le 2^k\,, \tag{2.3.17}$$

for all $x \in \mathbf{R}^n$.

To prove (2.3.17) we assume that $g^{\alpha,q}(s)(x) > 2^k$; otherwise, the conclusion is trivial. Then there exists a maximal dyadic cube $R_{\max}$ in $\mathscr{A}_k$ such that $x \in R_{\max}$. Letting R_0 be the unique dyadic cube that contains $R_{\max}$ and has twice its side length, we have that the left-hand side of (2.3.17) is equal to $g_{R_0}^{\alpha,q}(s)(x)$, which is at most 2^k, since R_0 is not contained in $\mathscr{A}_k$.

Since $g^{\alpha,q}(s) \in L^p(\mathbf{R}^n)$, by our assumption, and $g^{\alpha,q}(s) > 2^k$ for all $x \in Q$ if $Q \in \mathscr{A}_k$, the cubes in $\mathscr{A}_k$ must have size bounded above by some constant. We set

$$\mathscr{B}_k = \big\{J \in \mathscr{D}: \quad J \text{ is a maximal dyadic cube in } \mathscr{A}_k \setminus \mathscr{A}_{k+1}\big\}\,.$$

For J in $\mathscr{B}_k$ we define a sequence $t(k,J) = \{t(k,J)_Q\}_{Q\in\mathscr{D}}$ by setting

$$t(k,J)_Q = \begin{cases} s_Q & \text{if } Q \subseteq J \text{ and } Q \in \mathscr{A}_k \setminus \mathscr{A}_{k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$\text{if} \quad Q \notin \bigcup_{k\in\mathbf{Z}} \mathscr{A}_k\,, \qquad \text{then} \qquad s_Q = 0\,.$$

Moreover, the identity

$$s = \sum_{k\in\mathbf{Z}} \sum_{J\in\mathscr{B}_k} t(k,J) \tag{2.3.18}$$

is valid and it is worth noticing that for each $Q \in \mathscr{D}$, there is at most one $k \in \mathbf{Z}$ and at most one $J \in \mathscr{B}_k$ such that $t(k,J)_Q$ is nonzero, i.e., the sum in (2.3.18) evaluated at Q has at most one nonzero term.

For all $x \in \mathbf{R}^n$ we have

$$\begin{aligned} g^{\alpha,q}(t(k,J))(x) &= \Big(\sum_{\substack{Q\subseteq J \\ Q\in\mathscr{A}_k\setminus\mathscr{A}_{k+1}}} \big(|Q|^{-\frac{\alpha}{n}-\frac{1}{2}}|s_Q|\,\chi_Q(x)\big)^q \Big)^{\frac{1}{q}} \\ &\le \Big(\sum_{\substack{Q\subseteq J \\ Q\in\mathscr{D}\setminus\mathscr{A}_{k+1}}} \big(|Q|^{-\frac{\alpha}{n}-\frac{1}{2}}|s_Q|\,\chi_Q(x)\big)^q \Big)^{\frac{1}{q}} \\ &\le 2^{k+1}, \end{aligned} \tag{2.3.19}$$

where we used (2.3.17) in the last estimate. We define atoms $r(k,J) = \{r(k,J)_Q\}_{Q\in\mathscr{D}}$ by setting

$$r(k,J)_Q = 2^{-k-1}|J|^{-\frac{1}{p}}\,t(k,J)_Q\,, \tag{2.3.20}$$

and we also define scalars

$$\lambda_{k,J} = 2^{k+1}|J|^{\frac{1}{p}}\,.$$

To see that each $r(k,J)$ is an ∞-atom for $\dot{f}_p^{\alpha,q}$, we observe that $r(k,J)_Q = 0$ if $Q \nsubseteq J$ and that

$$g^{\alpha,q}(r(k,J))(x) \le |J|^{-\frac{1}{p}}\,, \qquad \text{for all } x \in \mathbf{R}^n,$$

in view of (2.3.19) and (2.3.20). Also using (2.3.18) and (2.3.20), we obtain that

$$s = \sum_{k\in\mathbf{Z}} \sum_{J\in\mathscr{B}_k} \lambda_{k,J}\, r(k,J)\,, \tag{2.3.21}$$

which says that s can be written as a countably infinite sum of atoms. We now reindex the countable set $\mathscr{U} = \{(k,J) :\ k \in \mathbf{Z}\,,\ J \in \mathscr{B}_k\}$ by $\mathbf{Z}^+$ and write

$$s = \sum_{j=1}^{\infty} \lambda_j r_j\,, \tag{2.3.22}$$

where $\{\lambda_1, \lambda_2, \dots\} = \{\lambda_{k,J} :\ (k,J) \in \mathscr{U}\}$ and $\{r_1, r_2, \dots\} = \{r(k,J) :\ (k,J) \in \mathscr{U}\}$. As observed the sum in (2.3.21) has the property that for each $Q \in \mathscr{D}$, there is at most one $k \in \mathbf{Z}$ and at most one $J \in \mathscr{B}_k$ such that $\lambda_{k,J}\, r(k,J)_Q = t(k,J)_Q$ is nonzero. Thus for each $Q \in \mathscr{D}$, at most one term in the sum $\sum_{j=1}^{\infty} \lambda_j r_{j,Q}$ is nonzero; in particular, this series is absolutely convergent.

Finally, we estimate the sum of the pth power of the coefficients $\lambda_{k,J}$. We have

$$
\begin{aligned}
\sum_{j=1}^{\infty} |\lambda_j|^p &= \sum_{k\in\mathbf{Z}} \sum_{J\in\mathscr{B}_k} \lambda_{k,J}^p \\
&= \sum_{k\in\mathbf{Z}} 2^{(k+1)p} \sum_{J\in\mathscr{B}_k} |J| \\
&\le 2^p \sum_{k\in\mathbf{Z}} 2^{kp} \Big| \bigcup_{Q\in\mathscr{A}_k} Q \Big| \\
&= 2^p \sum_{k\in\mathbf{Z}} 2^{k(p-1)} 2^k |\{x \in \mathbf{R}^n :\ g^{\alpha,q}(s)(x) > 2^k\}| \\
&\le 2^p \sum_{k\in\mathbf{Z}} \int_{2^k}^{2^{k+1}} 2^{k(p-1)} |\{x \in \mathbf{R}^n :\ g^{\alpha,q}(s)(x) > \tfrac{\lambda}{2}\}|\, d\lambda \\
&\le 2^p \sum_{k\in\mathbf{Z}} \int_{2^k}^{2^{k+1}} \lambda^{p-1} |\{x \in \mathbf{R}^n :\ g^{\alpha,q}(s)(x) > \tfrac{\lambda}{2}\}|\, d\lambda \\
&= \frac{2^{2p}}{p} \big\| g^{\alpha,q}(s) \big\|_{L^p}^p \\
&= \frac{2^{2p}}{p} \|s\|_{\dot{f}_p^{\alpha,q}}^p .
\end{aligned}
$$

Taking the pth root yields (2.3.15). The proof of the theorem is now complete. □

We now deduce a corollary concerning a new characterization of the space $\dot{f}_p^{\alpha,q}$.

Corollary 2.3.7. *Suppose $\alpha \in \mathbf{R}$, $0 < p \le 1$, and $p \le q < \infty$. Then for a given sequence $s \in \dot{f}_p^{\alpha,q}$ we have the following equivalence:*

$$
\|s\|_{\dot{f}_p^{\alpha,q}} \approx \inf \bigg\{ \Big(\sum_{j=1}^{\infty} |\lambda_j|^p \Big)^{\frac{1}{p}} :\ \lim_{N\to\infty} \Big\| s - \sum_{j=1}^{N} \lambda_j r_j \Big\|_{\dot{f}_p^{\alpha,q}} = 0,\ r_j \textit{ are } \infty\textit{-atoms for } \dot{f}_p^{\alpha,q} \bigg\}.
$$

Remark 2.3.8. Notice that $\dot{f}_p^{\alpha,q}$ is complete (Exercise 2.3.6(b)), so if r_j are ∞-atoms for $\dot{f}_p^{\alpha,q}$, if $(\sum_{j=1}^{\infty} |\lambda_j|^p)^{\frac{1}{p}} < \infty$ and if

$$
\Big\| s - \sum_{j=1}^{N} \lambda_j r_j \Big\|_{\dot{f}_p^{\alpha,q}} \to 0
$$

as $N \to \infty$, then s must be an element of $\dot{f}_p^{\alpha,q}$.

Proof. Given $s \in \dot{f}_p^{\alpha,q}$, let r_j be ∞-atoms for $\dot{f}_p^{\alpha,q}$ and λ_j be as in Theorem 2.3.6. Then $\sum_{j=1}^\infty |\lambda_j|^p \le C_{n,p,q}^p \|s\|_{\dot{f}_p^{\alpha,q}}^p < \infty$ and $s_Q = \sum_{j=1}^\infty \lambda_j r_{j,Q}$, where the series converges absolutely. Then as $N \to \infty$ we have

$$\Big\| s - \sum_{j=1}^N \lambda_j r_j \Big\|_{\dot{f}_p^{\alpha,q}}^p = \Big\| \sum_{j=1}^\infty \lambda_j r_j - \sum_{j=1}^N \lambda_j r_j \Big\|_{\dot{f}_p^{\alpha,q}}^p \le \Big\| \sum_{j=N+1}^\infty |\lambda_j r_j| \Big\|_{\dot{f}_p^{\alpha,q}}^p \le \sum_{j=N+1}^\infty |\lambda_j|^p \to 0,$$

where we used Exercise 2.3.6(a) in the last inequality together with the observation made after Definition 2.3.5 that every ∞-atom r for $\dot{f}_p^{\alpha,q}$ satisfies $\|r\|_{\dot{f}_p^{\alpha,q}} \le 1$. (Here $|r_j| = \{|r_{j,Q}|\}_{Q\in\mathscr{D}}$.) Then λ_j and s are as in the statement of the corollary and (2.3.15) implies the $\ge$ inequality in the claimed equivalence.

The converse inequality ($\le$) is easier since for any r_j ∞-atoms for $\dot{f}_p^{\alpha,q}$ we have

$$\|s\|_{\dot{f}_p^{\alpha,q}}^p \le \Big\| s - \sum_{j=1}^N \lambda_j r_j \Big\|_{\dot{f}_p^{\alpha,q}}^p + \Big\| \sum_{j=1}^N \lambda_j r_j \Big\|_{\dot{f}_p^{\alpha,q}}^p \le \Big\| s - \sum_{j=1}^N \lambda_j r_j \Big\|_{\dot{f}_p^{\alpha,q}}^p + \sum_{j=1}^\infty |\lambda_j|^p ;$$

thus letting $N \to \infty$ and taking the infimum over all $(\sum_{j=1}^\infty |\lambda_j|^p)^{1/p}$ yields the desired inequality. □

Theorem 2.3.6 allows us to obtain an atomic decomposition for the space $\dot{F}_p^{\alpha,q}$ as well. Indeed, we have the following result.

Corollary 2.3.9. *Let $\alpha \in \mathbf{R}$, $0 < p \le 1$, $L \ge [\max(\frac{n}{p} - n - \alpha, \alpha)]$, and let q satisfy $p \le q < \infty$. Then for a given $f \in \dot{F}_p^{\alpha,q}$ we have the following equivalence:*

$$\|f\|_{\dot{F}_p^{\alpha,q}} \approx \inf\Big\{ \Big(\sum_{j=1}^\infty |\lambda_j|^p \Big)^{\frac{1}{p}} : \lim_{N\to\infty} \|f - \sum_{j=1}^N \lambda_j A_j\|_{\dot{F}_p^{\alpha,q}} = 0,$$
$$\text{where } A_j = \sum_{\mu\in\mathbf{Z}} \Big(\sum_{Q\in\mathscr{D}_\mu} r_{j,Q} a_Q \Big) \textit{ converging in } \dot{F}_p^{\alpha,q},$$
$$a_Q \textit{ are smooth L-atoms, and } r_j = \{r_{j,Q}\}_{Q\in\mathscr{D}} \textit{ are } \infty\textit{-atoms for } \dot{f}_p^{\alpha,q} \Big\}.$$

Proof. Let λ_j, A_j be as above such that

$$\lim_{N\to\infty} \|f - \sum_{j=1}^N \lambda_j A_j\|_{\dot{F}_p^{\alpha,q}} = 0 .$$

In view of the subadditivity of the expression $h \mapsto \|f\|_{\dot{F}_p^{\alpha,q}}^p$ (Exercise 2.3.2) we have that

$$\|f\|_{\dot{F}_p^{\alpha,q}}^p \le \sum_{j=1}^N \|\lambda_j A_j\|_{\dot{F}_p^{\alpha,q}}^p + \|f - \sum_{j=1}^N \lambda_j A_j\|_{\dot{F}_p^{\alpha,q}}^p .$$

It follows from this that

$$\|f\|_{\dot F_p^{\alpha,q}}^p \le \limsup_{N\to\infty} \sum_{j=1}^N |\lambda_j|^p \|A_j\|_{\dot F_p^{\alpha,q}}^p \le \sum_{j=1}^\infty |\lambda_j|^p \|A_j\|_{\dot F_p^{\alpha,q}}^p \le c_{n,p} \sum_{j=1}^\infty |\lambda_j|^p \|r_j\|_{\dot f_p^{\alpha,q}}^p ,$$

where the last inequality is due to Theorem 2.3.4. Using the fact that every ∞-atom $r=\{r_Q\}$ for $\dot f_p^{\alpha,q}$ satisfies $\|r\|_{\dot f_p^{\alpha,q}} \le 1$, we take the infimum over all representations of f as above to deduce the $\le$ part of the claimed equivalence.

Conversely, given f in $\dot F_p^{\alpha,q}$, we use Theorem 2.3.4 to write

$$f = \sum_{\mu\in\mathbf{Z}} \Big(\sum_{Q\in\mathscr{D}_\mu} s_Q a_Q \Big)$$

where $s=\{s_Q\}_{Q\in\mathscr{D}}$ lies in $\dot f_p^{\alpha,q}$, each a_Q is a smooth L-atom for the cube Q and the series converges in $\dot F_p^{\alpha,q}$. Now Theorem 2.3.6 gives that $s=\{s_Q\}_Q$ can be written as a sum of r_j, ∞-atoms for $\dot f_p^{\alpha,q}$, that is,

$$s = \sum_{j=1}^\infty \lambda_j r_j ,$$

where

$$\Big(\sum_{j=1}^\infty |\lambda_j|^p\Big)^{\frac1p} \le c\|s\|_{\dot f_p^{\alpha,q}} .$$

Since $\|s\|_{\dot f_p^{\alpha,q}} \le C_{p,q,n,\alpha}\big\|f\big\|_{\dot F_p^{\alpha,q}}$, Theorem 2.3.4 implies that

$$\Big(\sum_{j=1}^\infty |\lambda_j|^p\Big)^{\frac1p} \le c'\big\|f\big\|_{\dot F_p^{\alpha,q}} . \tag{2.3.23}$$

For $j=1,2,\dots$ set

$$A_j = \sum_{\mu\in\mathbf{Z}} \Big(\sum_{Q\in\mathscr{D}_\mu} r_{j,Q} a_Q \Big) \tag{2.3.24}$$

and note that Theorem 2.3.4 gives that the series in μ in (2.3.24) converges in $\dot F_p^{\alpha,q}$ and the $\dot F_p^{\alpha,q}$ quasi-norm of A_j is bounded by a constant in view of (2.3.4), since $\|r_j\|_{\dot f_p^{\alpha,q}} \le 1$. Appealing again to (2.3.4) in Theorem 2.3.4 we obtain

$$\Big\|\sum_{j=1}^N \lambda_j A_j - f\Big\|_{\dot F_p^{\alpha,q}}^p \le C_{n,p,q,\alpha}^p \Big\|\sum_{j=1}^N \lambda_j r_j - s\Big\|_{\dot f_p^{\alpha,q}}^p \le C_{n,p,q,\alpha}^p \sum_{j=N+1}^\infty |\lambda_j|^p \to 0$$

as $N\to\infty$. The $\ge$ part of the claimed equivalence is a consequence of (2.3.23). $\square$

2.3.5 Atomic Decomposition of Hardy Spaces

We now pass to one of the main theorems of this chapter, the atomic decomposition of $H^p(\mathbf{R}^n)$ for $0 < p \le 1$. We begin by defining atoms for H^p.

Definition 2.3.10. Let $1 < q \le \infty$. A function A is called *an L^q-atom for $H^p(\mathbf{R}^n)$* if there exists a cube Q such that

(a) A is supported in Q;

(b) $\big\|A\big\|_{L^q} \le |Q|^{\frac{1}{q}-\frac{1}{p}}$;

(c) $\displaystyle\int x^\gamma A(x)\,dx = 0$ for all multi-indices γ with $|\gamma| \le [\frac{n}{p} - n]$.

Notice that any L^r-atom for H^p is also an L^q-atom for H^p whenever $0 < p \le 1$ and $1 < q < r \le \infty$. It is also simple to verify that an L^q-atom A for H^p is in fact in H^p. We prove this result in the next theorem for $q = 2$, and we refer to Exercise 2.3.4 for a general q.

Theorem 2.3.11. *Let $0 < p \le 1$. There is a constant $C_{n,p} < \infty$ such that every L^2-atom A for $H^p(\mathbf{R}^n)$ satisfies*

$$\|A\|_{H^p} \le C_{n,p}\,.$$

Proof. We could prove this theorem either by showing that the smooth maximal function $M(A;\Phi)$ is in L^p or by showing that the square function $\big(\sum_{j\in\mathbf{Z}} |\Delta_j^\Psi(A)|^2\big)^{1/2}$ is in L^p. Both proofs are similar and we choose to present the second.

Let $A(x)$ be an atom supported in a cube Q centered at the origin; otherwise apply the argument to the atom $A(x - c_Q)$, where c_Q is the center of Q. We control the L^p quasi-norm of $\big(\sum_{j\in\mathbf{Z}} |\Delta_j^\Psi(A)|^2\big)^{1/2}$ by estimating it over the cube Q^* and over $(Q^*)^c$, where $Q^* = 2\sqrt{n}\,Q$. We have

$$\bigg(\int_{Q^*}\Big(\sum_{j\in\mathbf{Z}} |\Delta_j^\Psi(A)|^2\Big)^{\frac{p}{2}}dx\bigg)^{\frac{1}{p}} \le \bigg(\int_{Q^*}\sum_{j\in\mathbf{Z}} |\Delta_j^\Psi(A)|^2\,dx\bigg)^{\frac{1}{2}} |Q^*|^{\frac{1}{p(2/p)'}}\,.$$

Using that the square function $f \mapsto \big(\sum_{j\in\mathbf{Z}} |\Delta_j^\Psi(f)|^2\big)^{\frac{1}{2}}$ is L^2 bounded, we obtain

$$\begin{aligned}\bigg(\int_{Q^*}\Big(\sum_{j\in\mathbf{Z}} |\Delta_j^\Psi(A)|^2\Big)^{\frac{p}{2}}dx\bigg)^{\frac{1}{p}} &\le C_n\|A\|_{L^2}|Q^*|^{\frac{1}{p(2/p)'}} \\ &\le C_n(2\sqrt{n})^{\frac{n}{p}-\frac{n}{2}}|Q|^{\frac{1}{2}-\frac{1}{p}}|Q|^{\frac{1}{p}-\frac{1}{2}} \\ &= C_n'\,.\end{aligned} \tag{2.3.25}$$

To estimate the contribution of the square function outside Q^*, we use the cancellation of the atoms. Let $k = [\frac{n}{p} - n] + 1$. We have

$$\begin{aligned}
\Delta_j^{\Psi}(A)(x) &= \int_Q A(y)\Psi_{2^{-j}}(x-y)\,dy \\
&= 2^{jn}\int_Q A(y)\Big[\Psi(2^j x - 2^j y) - \sum_{|\beta|\le k-1} (\partial^\beta \Psi)(2^j x)\frac{(-2^j y)^\beta}{\beta!}\Big]dy \\
&= 2^{jn}\int_Q A(y)\Big[\sum_{|\beta|=k} (\partial^\beta \Psi)(2^j x - 2^j\theta y)\frac{(-2^j y)^\beta}{\beta!}\Big]dy\,,
\end{aligned}$$

where $0 \le \theta \le 1$. Taking absolute values, using the fact that $\partial^\beta \Psi$ are Schwartz functions, and that $|x-\theta y| \ge |x| - |y| \ge \frac{1}{2}|x|$ whenever $y \in Q$ and $x \notin Q^*$, we obtain the estimate

$$\begin{aligned}
|\Delta_j^{\Psi}(A)(x)| &\le 2^{jn}\int_Q |A(y)| \sum_{|\beta|=k} \frac{C_N}{(1+2^j\frac{1}{2}|x|)^N}\frac{|2^j y|^k}{\beta!}dy \\
&\le \frac{C_{N,p,n}2^{j(k+n)}}{(1+2^j|x|)^N}\Big(\int_Q |A(y)|^2\,dy\Big)^{\frac{1}{2}}\Big(\int_Q |y|^{2k}\,dy\Big)^{\frac{1}{2}} \\
&\le \frac{C'_{N,p,n}2^{j(k+n)}}{(1+2^j|x|)^N}|Q|^{\frac{1}{2}-\frac{1}{p}}|Q|^{\frac{k}{n}+\frac{1}{2}} \\
&= \frac{C_{N,p,n}2^{j(k+n)}}{(1+2^j|x|)^N}|Q|^{1+\frac{k}{n}-\frac{1}{p}}
\end{aligned}$$

for $j \in \mathbf{Z}$ and $x \in (Q^*)^c$. For such x we now have

$$\Big(\sum_{j\in\mathbf{Z}} |\Delta_j^{\Psi}(A)(x)|^2\Big)^{\frac{1}{2}} \le C_{N,p,n}|Q|^{1+\frac{k}{n}-\frac{1}{p}}\Big(\sum_{j\in\mathbf{Z}} \frac{2^{2j(k+n)}}{(1+2^j|x|)^{2N}}\Big)^{\frac{1}{2}}. \tag{2.3.26}$$

It is a simple fact that the series in (2.3.26) converges. Indeed, considering the cases $2^j \le 1/|x|$ and $2^j > 1/|x|$ we see that the series on the right in (2.3.26) contributes at most a fixed multiple of $|x|^{-2k-2n}$. It remains to estimate the L^p quasi-norm of the expression on the right in (2.3.26) over $(Q^*)^c$. This is bounded by a constant multiple of

$$\Big(\int_{(Q^*)^c} \frac{(|Q|^{1+\frac{k}{n}-\frac{1}{p}})^p}{|x|^{p(k+n)}}\,dx\Big)^{\frac{1}{p}} \le C_{n,p}|Q|^{1+\frac{k}{n}-\frac{1}{p}}\Big(\int_{c|Q|^{\frac{1}{n}}}^{\infty} r^{-p(k+n)+n-1}\,dr\Big)^{\frac{1}{p}},$$

for some constant c, and the latter is easily seen to be bounded above by an absolute constant. Here we used the fact that $p(k+n) > n$ or, equivalently, $k > \frac{n}{p} - n$, which is certainly true, since k was chosen to be $[\frac{n}{p} - n] + 1$. Combining this estimate with that in (2.3.25), we conclude the proof of the theorem. □

We have now proved that L^q-atoms for H^p are indeed elements of H^p. We now obtain the converse statement, i.e., every element of H^p can be decomposed as a sum of L^2-atoms for H^p. Applying the same idea as in Corollary 2.3.9 to H^p, we obtain the following result.

Theorem 2.3.12. *Let $0 < p \le 1$. Given a distribution $f \in H^p(\mathbf{R}^n)$, there exists a sequence of L^2-atoms for H^p, $\{A_j\}_{j=1}^\infty$, and a sequence of scalars $\{\lambda_j\}_{j=1}^\infty$ such that*

$$\sum_{j=1}^{N} \lambda_j A_j \to f \qquad \text{in} \;\; H^p. \tag{2.3.27}$$

Thus the space of all finite linear combinations of L^2-atoms for H^p is dense in H^p. Moreover, we have

$$\begin{aligned} \|f\|_{H^p} \approx \inf\Big\{ \Big(\sum_{j=1}^{\infty} |\lambda_j|^p \Big)^{\frac{1}{p}} :\; \lim_{N\to\infty} \Big\| \sum_{j=1}^{N} \lambda_j A_j - f \Big\|_{H^p} = 0 \\ \text{where } A_j \text{ are } L^2\text{-atoms for } H^p. \Big\}. \end{aligned} \tag{2.3.28}$$

Proof. Fix $f \in H^p(\mathbf{R}^n)$. Let A_j be L^2-atoms for H^p and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$ such that (2.3.27) holds. It follows from Theorem 2.3.11 and the sublinearity of the expression $g \mapsto \|g\|_{H^p}^p$ that

$$\Big\| \sum_{j=1}^{N} \lambda_j A_j \Big\|_{H^p}^p \le C_{n,p}^p \sum_{j=1}^{N} |\lambda_j|^p .$$

Thus if the sequence $\sum_{j=1}^N \lambda_j A_j$ converges to f in H^p, then

$$\|f\|_{H^p}^p \le \Big\| f - \sum_{j=1}^{N} \lambda_j A_j \Big\|_{H^p}^p + C_{n,p}^p \sum_{j=1}^{N} |\lambda_j|^p ,$$

and letting $N \to \infty$ proves the direction $\le$ in (2.3.28).

We now focus on the converse statement, which is similar to the analogous statement in Corollary 2.3.9. Let $L = [\frac{n}{p} - n]$. Given f in $\dot F_p^{0,2} = H^p$, via Theorem 2.3.4 we write

$$f = \sum_{\mu \in \mathbf{Z}} \Big(\sum_{Q \in \mathscr{D}_\mu} s_Q a_Q \Big)$$

where $s = \{s_Q\}_{Q \in \mathscr{D}}$ lies in $\dot f_p^{0,2}$, each a_Q is a smooth L-atom for the cube Q and the series converges in H^p. Theorem 2.3.6 gives that $s = \{s_Q\}_Q$ can be written as a sum of r_j, ∞-atoms for $\dot f_p^{0,2}$, that is, $s = \sum_{j=1}^\infty \lambda_j r_j$, where

$$\Big(\sum_{j=1}^{\infty} |\lambda_j|^p \Big)^{\frac{1}{p}} \le c \|s\|_{\dot f_p^{0,2}} .$$

Since $\|s\|_{\dot{f}_p^{0,2}} \le C_{n,p}\big\|f\big\|_{H^p}$, Theorem 2.3.4 implies that

$$\Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{\frac{1}{p}} \le c'\big\|f\big\|_{H^p}. \tag{2.3.29}$$

For $j = 1,2,\dots$ set

$$A_j = \sum_{\mu\in\mathbf{Z}}\Big(\sum_{Q\in\mathscr{D}_\mu} r_{j,Q}a_Q\Big)$$

where the series converges in H^p (see Theorem 2.3.4) and the H^p quasi-norm of A_j is bounded by a constant in view of (2.3.4), since $\|r_j\|_{\dot{f}_p^{0,2}} \le 1$. Using again (2.3.4) in Theorem 2.3.4 we obtain

$$\begin{aligned}\Big\|\sum_{j=1}^{N}\lambda_jA_j - f\Big\|_{H^p}^p &\le C_{n,p}^p\Big\|\sum_{j=1}^{N}\lambda_jr_j - s\Big\|_{\dot{f}_p^{0,2}}^p\\ &\le C_{n,p}^p\Big\|\sum_{j=N+1}^{\infty}|\lambda_j|\,|r_j|\Big\|_{\dot{f}_p^{0,2}}^p\\ &\le C_{n,p}^p\sum_{j=N+1}^{\infty}|\lambda_j|^p \to 0,\end{aligned}$$

as $N\to\infty$, where the last inequality follows from Exercise 2.3.6(a).

Next we show that each A_j is a fixed multiple of an L^2-atom for H^p. Let us fix an index j. By the definition of the ∞-atom for $\dot{f}_p^{0,2}$, there exists a dyadic cube Q_0^j such that $r_{j,Q} = 0$ for all dyadic cubes Q not contained in Q_0^j. Then the support of each a_Q is contained in $3Q$, hence in $3Q_0^j$. This implies that the function A_j is supported in $3Q_0^j$. The same is true for the function $g^{0,2}(r_j)$ defined in (2.3.2). Using this fact, we have

$$\begin{aligned}\big\|A_j\big\|_{L^2} &\approx \big\|A_j\big\|_{\dot{F}_2^{0,2}}\\ &\le c\,\big\|r_j\big\|_{\dot{f}_2^{0,2}}\\ &= c\,\big\|g^{0,2}(r_j)\big\|_{L^2}\\ &\le c\,\big\|g^{0,2}(r_j)\big\|_{L^\infty}|3Q_0^j|^{\frac{1}{2}}\\ &\le c\,|3Q_0^j|^{-\frac{1}{p}+\frac{1}{2}}.\end{aligned}$$

Since

$$g^{0,2}(r_j) = \Big(\sum_{Q\in\mathscr{D}}|Q|^{-1}|r_{j,Q}|^2\chi_Q\Big)^{\frac{1}{2}}$$

the estimate $\|g^{0,2}(r_j)\|_{L^2} \le |3Q_0^j|^{-\frac{1}{p}+\frac{1}{2}}$ we proved implies that

$$\sum_{Q\in\mathscr{D}} |r_{j,Q}|^2 < \infty.$$

Let $M' < M$ be positive integers. Then

$$\begin{aligned}\Big\|\sum_{M'<|\mu|\le M}\sum_{Q\in\mathscr{D}_\mu} r_{j,Q}a_Q\Big\|_{L^1} &\le |3Q_0^j|^{\frac{1}{2}}\Big\|\sum_{M'<|\mu|\le M}\sum_{Q\in\mathscr{D}_\mu} r_{j,Q}a_Q\Big\|_{L^2}\\ &= |3Q_0^j|^{\frac{1}{2}}\Big(\sum_{M'<|\mu|\le M}\sum_{Q\in\mathscr{D}_\mu} |r_{j,Q}|^2\Big)^{\frac{1}{2}} \to 0\end{aligned}$$

as $M',M\to\infty$. Therefore the sequence $\sum_{|\mu|\le M}\sum_{Q\in\mathscr{D}_\mu} r_{j,Q}a_Q$ is Cauchy in L^1 and hence it converges in L^1. But this sequence converges in H^p to A_j by Theorem 2.3.4, so finally it converges to A_j in L^1.

The fact that $A_j = \sum_{\mu\in\mathbf{Z}}\sum_{Q\in\mathscr{D}_\mu} r_{j,Q}a_Q$ with convergence in L^1 allows us to deduce that vanishing moments of a_Q pass on to A_j. We conclude that each A_j is a fixed multiple of an L^2-atom for H^p. The $\ge$ direction in (2.3.28) now follows from (2.3.29), given that we have now established all the remaining properties. □

Remark 2.3.13. Property (c) in Definition 2.3.10 can be replaced by

$$\int x^\gamma A(x)\,dx = 0 \qquad \text{for all multi-indices } \gamma \text{ with } |\gamma|\le L,$$

for any $L\ge[\frac{n}{p}-n]$, and the atomic decomposition of H^p holds unchanged. In fact, in the proof of Theorem 2.3.12 we may take $L\ge[\frac{n}{p}-n]$ instead of $L=[\frac{n}{p}-n]$ and then apply Theorem 2.3.4 for this L. Note that Theorem 2.3.4 was valid for all $L\ge[\frac{n}{p}-n]$. This observation turns out to be quite useful in certain applications.

Exercises

2.3.1. (a) Given $N\in\mathbf{Z}^+$, prove that there exists a Schwartz function Θ supported in the unit ball $|x|\le 1$ such that $\int_{\mathbf{R}^n} x^\gamma\Theta(x)\,dx = 0$ for all multi-indices γ with $|\gamma|\le N$ and such that $|\widehat{\Theta}(\xi)|\ge\frac{1}{2}$ for all ξ in the annulus $\frac{1}{2}\le|\xi|\le 2$.
(b) Prove there exists a Schwartz function Ψ whose Fourier transform is supported in the annulus $\frac{1}{2}\le|\xi|\le 2$ and is at least $c>0$ in the smaller annulus $\frac{3}{5}\le|\xi|\le\frac{5}{3}$ and which satisfies for all $\xi\in\mathbf{R}^n\setminus\{0\}$

$$\sum_{j\in\mathbf{Z}}\widehat{\Psi}(2^{-j}\xi)\widehat{\Theta}(2^{-j}\xi) = 1\,.$$

$\big[$*Hint:* Part (a): Let θ be an even real-valued Schwartz function supported in the ball $|x|\le 1$ and such that $\widehat{\theta}(0)=1$. Then for some $\varepsilon\in(0,\frac{1}{2})$ we have $\widehat{\theta}(\xi)\ge\frac{1}{2}$ for all

ξ satisfying $|\xi| < 2\varepsilon$. Set $\Theta = (-\Delta)^N(\theta_\varepsilon)$. Part (b): Define the function $\widehat{\Psi}(\xi) = \widehat{\eta}(\xi)\big(\sum_{j\in\mathbf{Z}}\widehat{\eta}(2^{-j}\xi)\widehat{\Theta}(2^{-j}\xi)\big)^{-1}$, where $\widehat{\eta}(\xi)$ is a Schwartz function supported in the annulus $\frac{1}{2} \le |\xi| \le 2$ and equal to 1 on the smaller annulus $\frac{3}{5} \le |\xi| \le \frac{5}{3}$.]

2.3.2. Let $\alpha \in \mathbf{R}$, $0 < p \le 1$, $p \le q < +\infty$.
(a) For all f, g in $\dot{F}_p^{\alpha,q}$ show that

$$\big\|f+g\big\|_{\dot{F}_p^{\alpha,q}}^p \le \big\|f\big\|_{\dot{F}_p^{\alpha,q}}^p + \big\|g\big\|_{\dot{F}_p^{\alpha,q}}^p .$$

(b) For all sequences $\{s_Q\}_{Q\in\mathscr{D}}$ and $\{t_Q\}_{Q\in\mathscr{D}}$ show that

$$\big\|\{s_Q\}_Q + \{t_Q\}_Q\big\|_{\dot{f}_p^{\alpha,q}}^p \le \big\|\{s_Q\}_Q\big\|_{\dot{f}_p^{\alpha,q}}^p + \big\|\{t_Q\}_Q\big\|_{\dot{f}_p^{\alpha,q}}^p .$$

[*Hint:* Use $|a+b|^p \le |a|^p + |b|^p$ and apply Minkowski's inequality on $L^{q/p}$ (or on $\ell^{q/p}$).]

2.3.3. Let Φ be a smooth function supported in the unit ball of $\mathbf{R}^n$. Use the same idea as in Theorem 2.3.11 to show directly (without appealing to any other theorem) that the smooth maximal function $M(\,\cdot\,;\Phi)$ of an L^2-atom for H^p lies in L^p when $p < 1$. Recall that $M(f;\Phi) = \sup_{t>0}|\Phi_t * f|$.

2.3.4. Extend Theorem 2.3.11 to the case $1 < q \le \infty$. Precisely, prove that there is a constant $C_{n,p,q}$ such that every L^q-atom A for H^p satisfies

$$\|A\|_{H^p} \le C_{n,p,q} .$$

[*Hint:* If $1 < q < 2$ use the boundedness of the square function on L^q while for $2 \le q \le \infty$ use its boundedness on L^2.]

2.3.5. (a) Suppose that $s_Q^k \ge 0$ for all $Q \in \mathscr{D}$ and $k = 1, 2, \dots$. Prove that

$$\big\|\{\sum_{k=1}^{\infty} s_Q^k\}_Q\big\|_{\dot{f}_p^{\alpha,q}}^p \le \sum_{k=1}^{\infty} \big\|\{s_Q^k\}_Q\big\|_{\dot{f}_p^{\alpha,q}}^p .$$

(b) Prove the completeness of the spaces $\dot{f}_p^{\alpha,q}$ when $\alpha \in \mathbf{R}$, $0 < p \le 1$, $p \le q < \infty$.
[*Hint:* Part (b): You may want to use part (a) together with the fact that if a quasi-normed space $(X, ||\cdot||)$ has the property $||x+y||^p \le ||x||^p + ||y||^p$ for all $x, y \in X$, then $(X, ||\cdot||)$ is complete if and only if for every sequence $x_k \in X$ with the property $\sum_{k=1}^{\infty} ||x_k||^p < \infty$ there is an x_* such that $||\sum_{k=1}^{N} x_k - x_*|| \to 0$ as $N \to \infty$.]

2.3.6. Show that for all $\mu, j \in \mathbf{Z}$, all $N, b > 0$ satisfying $N > n/b$ and $b < 1$, all scalars s_Q (indexed by dyadic cubes Q with centers c_Q), and all $x \in \mathbf{R}^n$ we have

$$\sum_{Q\in\mathscr{D}_\mu} \frac{|s_Q|}{(1+2^{\min(j,\mu)}|x-c_Q|)^N} \le c(n,N,b)\, 2^{\max(\mu-j,0)\frac{n}{b}} \left\{ M\Big(\sum_{Q\in\mathscr{D}_\mu} |s_Q|^b \chi_Q \Big)(x) \right\}^{\frac{1}{b}},$$

where M is the Hardy–Littlewood maximal operator and $c(n,N,b)$ is a constant. $\big[$*Hint:* Fix $x\in\mathbf{R}^n$ and define $\mathscr{F}_0 = \big\{Q\in\mathscr{D}_\mu : |c_Q-x|\,2^{\min(j,\mu)}\le 1\big\}$ and for $k\ge 1$ $\mathscr{F}_k = \big\{Q\in\mathscr{D}_\mu : 2^{k-1} < |c_Q-x|\,2^{\min(j,\mu)} \le 2^k\big\}$. Break up the sum on the left as a sum over the families $\mathscr{F}_k$ and use that $\sum_{Q\in\mathscr{F}_k}|s_Q| \le \big(\sum_{Q\in\mathscr{F}_k}|s_Q|^b\big)^{1/b}$ and the fact that $\big|\bigcup_{Q\in\mathscr{F}_k} Q\big| \le c_n 2^{-\min(j,\mu)n+kn}$.$\big]$

2.3.7. Let A be an L^2-atom for $H^p(\mathbf{R}^n)$ for some $0<p<1$. Show that there is a constant C such that for all multi-indices α with $|\alpha|\le k = [\frac{n}{p}-n]$ we have

$$\sup_{\xi\in\mathbf{R}^n} |\xi|^{|\alpha|-k-1}\big|(\partial^\alpha\widehat{A})(\xi)\big| \le C\,\big\|A\big\|_{L^2(\mathbf{R}^n)}^{-\frac{2p}{2-p}(\frac{k+1}{n}+\frac{1}{2})+1}.$$

[*Hint:* Subtract the Taylor polynomial of degree $k-|\alpha|$ at 0 of the function $x\mapsto e^{-2\pi i x\cdot\xi}$.$\big]$

2.3.8. Let A be an L^2-atom for $H^p(\mathbf{R}^n)$ for some $0<p<1$. Show that for all multi-indices α and all $1\le r\le\infty$ there is a constant C such that

$$\big\|\,|\partial^\alpha\widehat{A}|^2\big\|_{L^{r'}(\mathbf{R}^n)} \le C\,\big\|A\big\|_{L^2(\mathbf{R}^n)}^{-\frac{2p}{2-p}(\frac{2|\alpha|}{n}+\frac{1}{r})+2}.$$

$\big[$*Hint:* In the case $r=1$ use the $L^1\to L^\infty$ boundedness of the Fourier transform and in the case $r=\infty$ use Plancherel's theorem. For general r use interpolation.$\big]$

2.3.9. Let f be in $H^p(\mathbf{R}^n)$ for some $0<p\le 1$. Then the Fourier transform of f, originally defined as a tempered distribution, is a continuous function that satisfies

$$|\widehat{f}(\xi)| \le C_{n,p}\big\|f\big\|_{H^p(\mathbf{R}^n)}|\xi|^{\frac{n}{p}-n}$$

for some constant $C_{n,p}$ independent of f.
$\big[$*Hint:* If f is an L^2-atom for H^p, combine the estimates of Exercises 2.3.7 and 2.3.8 with $\alpha=0$ (and $r=1$). In general, apply Theorem 2.3.12.$\big]$

2.3.10. Let A be an L^∞-atom for $H^p(\mathbf{R}^n)$ for some $0<p<1$ and let $\alpha=\frac{n}{p}-n$. Show that there is a constant $C_{n,p}$ such that for all g in $\dot{\Lambda}_\alpha(\mathbf{R}^n)$ we have

$$\left|\int_{\mathbf{R}^n} A(x)g(x)\,dx\right| \le C_{n,p}\,\big\|g\big\|_{\dot{\Lambda}_\alpha(\mathbf{R}^n)}.$$

$\big[$*Hint:* Suppose that A is supported in a cube Q of side length $2^{-\nu}$ and center c_Q. Write the previous integrand as $\sum_{j\in\mathbf{Z}}\Delta_j^{\Omega}(A)\Delta_j^{\Psi}(g)$ for a Littlewood–Paley operator

Δ_j^{Ψ} associated with a function $\Psi \in \mathscr{S}$ whose Fourier transform is nonnegative, supported in $\frac{6}{7} < |\xi| < 2$, and satisfies $\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$, while $\widehat{\Omega}(\xi)$ is $\mathscr{C}^\infty$, supported in $\frac{6}{14} < |\xi| < 4$, and is equal to one on the support of $\widehat{\Psi}(\xi)$. Then apply the result of Appendix B.4 to obtain the estimate

$$\left|\Delta_j^{\Omega}(\overline{A})(x)\right| \le C_N |Q|^{-\frac{1}{p}+1} \frac{2^{\min(j,\nu)n} 2^{-|j-\nu|D}}{\left(1+2^{\min(j,\nu)}|x-c_Q|\right)^N},$$

where $D = [\alpha]+1$ when $\nu \ge j$ and $D = 0$ when $\nu < j$. Use Theorem 1.4.6.$\big]$

2.3.11. Let $\varepsilon > 0$. Show that the function

$$h(x) = \frac{\chi_{|x|<1/2}}{x\left(\log \frac{1}{|x|}\right)^{1+\varepsilon}}$$

lies in the Hardy space $H^1(\mathbf{R})$ although

$$\int_{-1/2}^{1/2} |h(t)| \log |h(t)| \, dt = \infty.$$

$\big[$*Hint:* For $j = 1,2,\dots$ define atoms $a_j(x) = c\, j^{1+\varepsilon}\big(h\chi_{I_j} - \mathrm{Avg}_{I_j}(h\chi_{I_j})\big)$ supported in $I_j = (2^{-j}, 2^{-j+1})$ and $b_j(x) = c\, j^{1+\varepsilon}\big(h\chi_{L_j} - \mathrm{Avg}_{I_j}(h\chi_{L_j})\big)$ supported in $L_j = (-2^{-j+1}, -2^{-j})$ for a suitable $c > 0$. Then write $h = \sum_{j=1}^{\infty} \frac{1}{c\, j^{1+\varepsilon}}(a_j + b_j)$.$\big]$

2.4 Singular Integrals on Function Spaces

Our final task in this chapter is to investigate the action of singular integrals on function spaces. The emphasis of our study focuses on Hardy spaces, although with no additional effort the action of singular integrals on other function spaces can also be obtained.

2.4.1 Singular Integrals on the Hardy Space H^1

Before we discuss the main results in this topic, we review some background on singular integrals.

Let $K(x)$ be a function defined away from the origin on $\mathbf{R}^n$ that satisfies the size estimate

$$\sup_{0<R<\infty} \frac{1}{R} \int_{|x|\le R} |K(x)|\, |x| \, dx \le A_1, \tag{2.4.1}$$

the smoothness condition,

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}}\int_{|x|\ge 2|y|}|K(x-y)-K(x)|\,dx\le A_2\,, \tag{2.4.2}$$

and the cancellation condition

$$\sup_{0<R_1<R_2<\infty}\bigg|\int_{R_1<|x|<R_2}K(x)\,dx\bigg|\le A_3\,, \tag{2.4.3}$$

for some $A_1,A_2,A_3<\infty$. Condition (2.4.3) implies that there exists a sequence $\varepsilon_j\downarrow 0$ as $j\to\infty$ such that the following limit exists:

$$\lim_{j\to\infty}\int_{\varepsilon_j\le|x|\le 1}K(x)\,dx=L_0.$$

This gives that for a smooth and compactly supported function f on $\mathbf{R}^n$, the limit

$$\lim_{j\to\infty}\int_{|x-y|>\varepsilon_j}K(x-y)f(y)\,dy=T(f)(x) \tag{2.4.4}$$

exists and defines a linear operator T. This operator T is given by convolution with a tempered distribution W that coincides with the function K on $\mathbf{R}^n\setminus\{0\}$.

We know that such a T, initially defined on $\mathscr{C}_0^\infty(\mathbf{R}^n)$, admits an extension that is L^p bounded for all $1<p<\infty$ and is also of weak type $(1,1)$. All these norms are bounded above by dimensional constant multiples of the quantity $A_1+A_2+A_3$ (cf. Theorem 5.4.1 in [156]). Therefore, such a T is well defined on $L^1(\mathbf{R}^n)$ and in particular on $H^1(\mathbf{R}^n)$, which is contained in $L^1(\mathbf{R}^n)$. The following result concerns the H^1 to L^1 boundedness of T.

Theorem 2.4.1. *Let K satisfy (2.4.1), (2.4.2), and (2.4.3), and let T be defined as in (2.4.4). Then there is a constant C_n such that for all f in $H^1(\mathbf{R}^n)$ we have*

$$\big\|T(f)\big\|_{L^1}\le C_n(A_1+A_2+A_3)\big\|f\big\|_{H^1}\,. \tag{2.4.5}$$

Proof. To prove this theorem we have a powerful tool at our disposal, the atomic decomposition of $H^1(\mathbf{R}^n)$. It is therefore natural to start by checking the validity of (2.4.5) whenever f is an L^2-atom for H^1.

Since T is a convolution operator (i.e., it commutes with translations), it suffices to take the atom f supported in a cube Q centered at the origin. Let $f=a$ be such an atom, supported in Q, and let $Q^*=2\sqrt{n}\,Q$. We write

$$\int_{\mathbf{R}^n}|T(a)(x)|\,dx=\int_{Q^*}|T(a)(x)|\,dx+\int_{(Q^*)^c}|T(a)(x)|\,dx \tag{2.4.6}$$

and we estimate each term separately. We have

$$\begin{aligned}\int_{Q^*}|T(a)(x)|\,dx &\le |Q^*|^{\frac12}\bigg(\int_{Q^*}|T(a)(x)|^2\,dx\bigg)^{\frac12}\\ &\le C_n(A_1+A_2+A_3)|Q^*|^{\frac12}\bigg(\int_Q|a(x)|^2\,dx\bigg)^{\frac12}\\ &\le C_n(A_1+A_2+A_3)|Q^*|^{\frac12}|Q|^{\frac12-\frac11}\\ &= C_n'(A_1+A_2+A_3)\,,\end{aligned}$$

where we used the L^2 boundedness of T and property (b) of atoms in Definition 2.3.10. Now note that if $x\notin Q^*$ and $y\in Q$, then $|x|\ge 2|y|$ and $x-y$ stays away from zero; thus $K(x-y)$ is well defined. Moreover, in this case $T(a)(x)$ can be expressed as a convergent integral of $a(y)$ against $K(x-y)$. We have

$$\begin{aligned}\int_{(Q^*)^c}|T(a)(x)|\,dx &= \int_{(Q^*)^c}\bigg|\int_Q K(x-y)a(y)\,dy\bigg|\,dx\\ &= \int_{(Q^*)^c}\bigg|\int_Q \big(K(x-y)-K(x)\big)a(y)\,dy\bigg|\,dx\\ &\le \int_Q\int_{(Q^*)^c}\big|K(x-y)-K(x)\big|\,dx\,|a(y)|\,dy\\ &\le \int_Q\int_{|x|\ge 2|y|}\big|K(x-y)-K(x)\big|\,dx\,|a(y)|\,dy\\ &\le A_2\int_Q|a(x)|\,dx\\ &\le A_2|Q|^{\frac12}\bigg(\int_Q|a(x)|^2\,dx\bigg)^{\frac12}\\ &\le A_2|Q|^{\frac12}|Q|^{\frac12-\frac11}=A_2\,.\end{aligned}$$

Combining this calculation with the previous one and inserting the final conclusions in (2.4.6) we deduce that L^2-atoms a for H^1 satisfy

$$\big\|T(a)\big\|_{L^1}\le (C_n'+1)(A_1+A_2+A_3)\,. \tag{2.4.7}$$

We now pass to general functions in H^1. In view of Theorem 2.3.12 we can write an $f\in H^1$ as

$$f=\sum_{j=1}^\infty \lambda_j a_j\,,$$

where the series converges in H^1, the a_j are L^2-atoms for H^1, and

$$\big\|f\big\|_{H^1}\approx\sum_{j=1}^\infty|\lambda_j|<\infty\,. \tag{2.4.8}$$

Since T maps L^1 to $L^{1,\infty}$ (Theorem 5.3.3 in [156]), $T(f)$ is already a well-defined $L^{1,\infty}$ function.

We claim that

$$T(f)=\sum_{j=1}^{\infty}\lambda_j T(a_j) \qquad \text{a.e.} \tag{2.4.9}$$

noting that the series in (2.4.9) converges in L^1 and produces a well-defined integrable function. Once (2.4.9) is established, the required conclusion (2.4.5) follows easily by taking L^1 norms in (2.4.9) and using (2.4.7) and (2.4.8).

To prove (2.4.9), we make use of the fact that T is of weak type $(1,1)$. For a given $\delta>0$ we have

$$\begin{aligned}
&\big|\big\{\big|T(f)-\sum_{j=1}^{\infty}\lambda_j T(a_j)\big|>\delta\big\}\big| \\
&\le \big|\big\{\big|T(f)-\sum_{j=1}^{N}\lambda_j T(a_j)\big|>\delta/2\big\}\big|+\big|\big\{\big|\sum_{j=N+1}^{\infty}\lambda_j T(a_j)\big|>\delta/2\big\}\big| \\
&\le \frac{2}{\delta}\big\|T\big\|_{L^1\to L^{1,\infty}}\Big\|f-\sum_{j=1}^{N}\lambda_j a_j\Big\|_{L^1}+\frac{2}{\delta}\Big\|\sum_{j=N+1}^{\infty}\lambda_j T(a_j)\Big\|_{L^1} \\
&\le \frac{2}{\delta}\big\|T\big\|_{L^1\to L^{1,\infty}}\Big\|f-\sum_{j=1}^{N}\lambda_j a_j\Big\|_{H^1}+\frac{2}{\delta}(C_n'+1)(A_1+A_2+A_3)\sum_{j=N+1}^{\infty}|\lambda_j|\,.
\end{aligned}$$

Since $\sum_{j=1}^{N}\lambda_j a_j$ converges to f in H^1 and $\sum_{j=1}^{\infty}|\lambda_j|<\infty$, both terms in the sum converge to zero as $N\to\infty$. We conclude that

$$\big|\big\{x\in\mathbf{R}^n:\ \big|T(f)(x)-\sum_{j=1}^{\infty}\lambda_j T(a_j)(x)\big|>\delta\big\}\big|=0$$

for all $\delta>0$, which implies (2.4.9). □

2.4.2 Singular Integrals on Besov–Lipschitz Spaces

We continue with a corollary concerning Besov–Lipschitz spaces.

Corollary 2.4.2. *Let K satisfy (2.4.1), (2.4.2), and (2.4.3), and let T be defined as in (2.4.4). Let $1\le p\le\infty$, $0<q\le\infty$, and $\alpha\in\mathbf{R}$. Then there is a constant $C_{n,p,q,\alpha}$ such that for all f in $\mathscr{S}(\mathbf{R}^n)$ we have*

$$\big\|T(f)\big\|_{\dot B_p^{\alpha,q}}\le C_n(A_1+A_2+A_3)\big\|f\big\|_{\dot B_p^{\alpha,q}}\,. \tag{2.4.10}$$

Therefore, T admits a bounded extension on all homogeneous Besov–Lipschitz spaces $\dot B_p^{\alpha,q}$ with $p\ge1$, in particular, on all homogeneous Lipschitz spaces.

Proof. Let Ψ be a Schwartz function whose Fourier transform is supported in the annulus $1-\frac{1}{7} \le |\xi| \le 2$ and that satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1, \qquad \xi \neq 0.$$

Pick a Schwartz function ζ whose Fourier transform $\widehat{\zeta}$ is supported in the annulus $\frac{1}{4} < |\xi| < 8$ and that is equal to one on the support of $\widehat{\Psi}$. Let W be the tempered distribution that coincides with K on $\mathbf{R}^n \setminus \{0\}$ so that $T(f) = f * W$. Then we have $\zeta_{2^{-j}} * \Psi_{2^{-j}} = \Psi_{2^{-j}}$ for all j and hence

$$\begin{aligned} \big\|\Delta_j(T(f))\big\|_{L^p} &= \big\|\zeta_{2^{-j}} * \Psi_{2^{-j}} * W * f\big\|_{L^p} \\ &\le \big\|\zeta_{2^{-j}} * W\big\|_{L^1} \big\|\Delta_j(f)\big\|_{L^p}, \end{aligned} \tag{2.4.11}$$

since $1 \le p \le \infty$. It is not hard to check that the function $\zeta_{2^{-j}}$ is in H^1 with norm independent of j. Therefore, $\zeta_{2^{-j}}$ is in H^1. Using Theorem 2.4.1, we conclude that

$$\big\|T(\zeta_{2^{-j}})\big\|_{L^1} = \big\|\zeta_{2^{-j}} * W\big\|_{L^1} \le C\big\|\zeta_{2^{-j}}\big\|_{H^1} = C'.$$

Inserting this in (2.4.11), multiplying by $2^{j\alpha}$, and taking ℓ^q quasi-norms, we obtain the required conclusion. $\square$

2.4.3 Singular Integrals on $H^p(\mathbf{R}^n)$

It is possible to extend Theorem 2.4.1 to $H^p(\mathbf{R}^n)$ for $p < 1$, provided the kernel K has additional smoothness.

For the purposes of this subsection, we fix a $\mathscr{C}^\infty$ function $K(x)$ on $\mathbf{R}^n \setminus \{0\}$. We suppose that there is a positive integer N (to be specified later) such that

$$|\partial^\beta K(x)| \le A\,|x|^{-n-|\beta|} \qquad \text{for all } |\beta| \le N \tag{2.4.12}$$

and that

$$\sup_{0<R_1<R_2<\infty} \left| \int_{R_1<|x|<R_2} K(x)\,dx \right| \le A, \tag{2.4.13}$$

for some $A < \infty$.

We fix a nonnegative smooth function η on $\mathbf{R}^n$ that vanishes in the unit ball of $\mathbf{R}^n$ and is equal to 1 outside the ball $B(0,2)$ and for $0 < \varepsilon < 1/2$ we define the smoothly truncated family of kernels

$$K^{(\varepsilon)}(x) = K(x)\eta(x/\varepsilon)$$

and the doubly smoothly truncated family of kernels

$$K_{(\varepsilon)}(x) = K^{(\varepsilon)}(x) - K^{(1/\varepsilon)}(x).$$

Condition (2.4.12) with $\beta = 0$ and (2.4.13) imply that

$$\left| \int_{|x|\le 1} K(x)\eta(x/\varepsilon)\,dx \right| \le (1+\omega_{n-1}\log 2)A$$

for all $\varepsilon < 1/2$; hence there exists a sequence $\varepsilon_j < 1/2$ with $\varepsilon_j \downarrow 0$ as $j \to \infty$ such that the following limit exists:

$$\lim_{j\to\infty} \int_{|x|\le 1} K(x)\eta(x/\varepsilon_j)\,dx = L_0.$$

We now define W in $\mathscr{S}'(\mathbf{R}^n)$ by setting

$$\begin{aligned} \langle W, \varphi \rangle &= \lim_{j\to\infty} \int_{\mathbf{R}^n} K_{(\varepsilon_j)}(x)\varphi(x)\,dx \qquad (2.4.14)\\ &= L_0\varphi(0) + \int_{|x|\le 1} K(x)(\varphi(x)-\varphi(0))\,dx + \int_{|x|\ge 1} K(x)\varphi(x)\,dx \end{aligned}$$

for φ in $\mathscr{S}$. It is quite easy to verify that the preceding expression is bounded by a constant multiple of a finite sum of Schwartz seminorms of φ. Note that this distribution[2] depends on the number L_0 and hence on the bump η.

We define the associated doubly smoothly truncated singular integral by setting

$$T_{(\varepsilon)}(\varphi)(x) = \int_{\mathbf{R}^n} K_{(\varepsilon)}(y)\varphi(x-y)\,dy \qquad (2.4.15)$$

for Schwartz functions φ on $\mathbf{R}^n$.

We also define an operator T given by convolution with W by setting

$$T(\varphi) = \lim_{j\to\infty} T_{(\varepsilon_j)}(\varphi) = \varphi * W \qquad (2.4.16)$$

for $\varphi \in \mathscr{S}(\mathbf{R}^n)$. Observe that W coincides with K on $\mathbf{R}^n \setminus \{0\}$, since if φ is supported in $|x| \ge t_0 > 0$, (2.4.14) implies that the action of W on $\varphi \in \mathscr{S}$ coincides with that of $K^{(\varepsilon_j)}$ on φ when $\varepsilon_j < t_0/2$. Condition (2.4.12) with $|\beta| = 1$ implies

$$\sup_{y\ne 0} \int_{|x|\ge 2|y|} |K(x-y) - K(x)|\,dx \le cA; \qquad (2.4.17)$$

hence Theorem 5.4.1 in [156] yields the L^2 boundedness of T. Note that (2.4.17) also holds for $K_{(\varepsilon)}$ in place of K uniformly in ε; thus again by Theorem 5.4.1 in [156] the operators $T_{(\varepsilon)}$ are uniformly bounded on $L^2(\mathbf{R}^n)$.

---

[2] Alternatively, we could have defined W as an element of $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ acting on functions $\varphi \in \mathscr{S}_0$; in this case W would have been independent of L_0 and η.

We summarize these and other observations about $K_{(\varepsilon)}$, $T_{(\varepsilon)}$, and T.

(i) The kernels $K_{(\varepsilon)}$ satisfy the same estimates as K uniformly in ε with constant A' in place of A, where A' is comparable to A.
(ii) $T_{(\varepsilon)}$ are uniformly bounded on L^2.
(iii) $T_{(\varepsilon_j)}(g)$ tends to $T(g)$ in L^2 for any $g \in L^2(\mathbf{R}^n)$.
(iv) T is L^2 bounded with norm $\|\widehat{W}\|_{L^\infty} \le CA$.
(v) For any $f \in H^p$, $T(f)$ is a well-defined element of $\mathscr{S}'$.

We have already explained assertions (i) and (ii) and (iv).

We explain (iii). Theorem 5.3.4 in [156] gives that for all $g \in L^2$ we have

$$\sup_{\varepsilon>0} |T_{(\varepsilon)}(g)| \le M(T(f)) + C_n A\, M(g);$$

hence the maximal operator $T^{(**)}(g) = \sup_{\varepsilon>0} |T_{(\varepsilon)}(g)|$ is L^2 bounded. Moreover, as an easy consequence of (2.4.14), for each $\varphi \in \mathscr{S}$ we have $T_{(\varepsilon_j)}(\varphi) \to T(\varphi)$ pointwise everywhere. In view of Theorem 2.1.14 in [156], for every $g \in L^2(\mathbf{R}^n)$ we have $T_{(\varepsilon_j)}(g) - T(g) \to 0$ a.e. as $j \to \infty$. Since

$$|T_{(\varepsilon_j)}(g) - T(g)| \le 2\,T^{(**)}(g) \in L^2,$$

the Lebesgue dominated convergence theorem yields that $T_{(\varepsilon_j)}(g) - T(g) \to 0$ in L^2.

To verify the validity of (v) we write $W = W_0 + K_\infty$, where $W_0 = \Phi W$ and $K_\infty = (1-\Phi)K$, where Φ is a smooth function equal to one on the ball $B(0,1)$ and vanishing outside the ball $B(0,2)$. Then for f in $H^p(\mathbf{R}^n)$, $0 < p \le 1$, we define a tempered distribution $T(f) = W * f$ by setting

$$\big\langle T(f), \phi \big\rangle = \big\langle f, \phi * \widetilde{W_0} \big\rangle + \big\langle \widetilde{\phi} * f, \widetilde{K_\infty} \big\rangle \tag{2.4.18}$$

for ϕ in $\mathscr{S}(\mathbf{R}^n)$. (Here $\widetilde{\varphi}(x) = \varphi(-x)$ for functions and analogously for distributions.) The function $\phi * \widetilde{W_0}$ is in $\mathscr{S}$, so the action of f on it is well defined. Also $\widetilde{\phi} * f$ is in L^1 (see Proposition 2.1.9), while $\widetilde{K_\infty}$ is in L^∞; hence the second term on the right in (2.4.18) represents an absolutely convergent integral. Moreover, in view of Theorem 2.3.20 in [156] and Corollary 2.1.9, both terms on the right in (2.4.18) are controlled by a finite sum of seminorms $\rho_{\alpha,\beta}(\phi)$ (cf. Definition 2.2.1 in [156]). This defines $T(f)$ as a tempered distribution for every $f \in H^p$.

The following is an extension of Theorem 2.4.1 for $p < 1$.

Theorem 2.4.3. *Let $0 < p < 1$ and $N = [\frac{n}{p} - n] + 1$. Let K be a $\mathscr{C}^\infty$ function on $\mathbf{R}^n \setminus \{0\}$ that satisfies (2.4.13) and (2.4.12) for some $A < \infty$ for all multi-indices $|\beta| \le N$ and all $x \ne 0$. Let W be a tempered distribution that coincides with K on $\mathbf{R}^n \setminus \{0\}$, as defined in (2.4.14). Then there is a constant $C_{n,p}$ such that for all $f \in H^p$ the distribution $T(f)$ defined in (2.4.18) coincides with an L^p function that satisfies*

$$\big\|T(f)\big\|_{L^p} \le C_{n,p} A \big\|f\big\|_{H^p}.$$

Proof. The proof of this theorem is based on the atomic decomposition of H^p.

We first take $f = a$ to be an L^2-atom for H^p, and without loss of generality we may assume that a is supported in a cube Q centered at the origin. We let Q^* be the cube with side length $2\sqrt{n}\,\ell(Q)$, where $\ell(Q)$ is the side length of Q. We have

$$\begin{aligned}\left(\int_{Q^*}|T(a)(x)|^p\,dx\right)^{\frac{1}{p}} &\le C|Q^*|^{\frac{1}{p}-\frac{1}{2}}\left(\int_{Q^*}|T(a)(x)|^2\,dx\right)^{\frac{1}{2}}\\ &\le C''A|Q|^{\frac{1}{p}-\frac{1}{2}}\left(\int_{Q}|a(x)|^2\,dx\right)^{\frac{1}{2}}\\ &\le C_nA\,|Q|^{\frac{1}{p}-\frac{1}{2}}|Q|^{\frac{1}{2}-\frac{1}{p}}\\ &= C_nA\,,\end{aligned}$$

where we used that T is L^2 bounded with norm at most a constant multiple of A.

For $x \notin Q^*$ and $y \in Q$, we have $|x| \ge 2|y|$, and thus $x-y$ stays away from zero and $K(x-y)$ is well defined. We have

$$T(a)(x) = \int_Q K(x-y)\,a(y)\,dy\,.$$

Recall that $N = [\frac{n}{p} - n] + 1$. Using the cancellation of atoms in H^p, we deduce

$$\begin{aligned}T(a)(x) &= \int_Q a(y)K(x-y)\,dy\\ &= \int_Q a(y)\left[K(x-y) - \sum_{|\beta|\le N-1}(\partial^\beta K)(x)\frac{y^\beta}{\beta!}\right]dy\\ &= \int_Q a(y)\left[\sum_{|\beta|=N}(\partial^\beta K)(x-\theta_y y)\frac{y^\beta}{\beta!}\right]dy\,,\end{aligned}$$

for some $0 \le \theta_y \le 1$. The fact that $|x| \ge 2|y|$ implies that $|x-\theta_y y| \ge \frac{1}{2}|x|$ and using (2.4.12) we obtain the estimate

$$|T(a)(x)| \le c_{n,N}\frac{A}{|x|^{N+n}}\int_Q |a(y)|\,|y|^N\,dy\,,$$

from which it follows that for $x \notin Q^*$ we have

$$|T(a)(x)| \le c_{n,p}\frac{A}{|x|^{N+n}}|Q|^{1+\frac{N}{n}-\frac{1}{p}}$$

via a calculation using Hölder's inequality and the fact that $\|a\|_{L^q} \le |Q|^{\frac{1}{q}-\frac{1}{p}}$. Integrating over $(Q^*)^c$, we obtain that

$$\left(\int_{(Q^*)^c}|T(a)(x)|^p dx\right)^{\frac{1}{p}} \le c_{n,p}A\,|Q|^{1+\frac{N}{n}-\frac{1}{p}}\left(\int_{(Q^*)^c}\frac{1}{|x|^{p(N+n)}}\,dx\right)^{\frac{1}{p}} \le c'_{n,p}A\,.$$

We have now shown that there exists a constant $C_{n,p}$ such that

$$\big\|T(a)\big\|_{L^p} \le C_{n,p}A \tag{2.4.19}$$

whenever a is an L^2-atom for H^p.

To replace a by general $f \in H^p$ we need Lemma 2.4.4 that follows immediately, which we apply to the family Schwartz functions $\zeta_{\varepsilon_j}(x) = K(x)\big(\eta(x/\varepsilon_j) - \eta(\varepsilon_j x)\big)$ and the associated operators $T_{(\varepsilon_j)}$. Fix $f \in H^p \cap L^2$, with atomic decomposition $f = \sum_{j=1}^\infty \lambda_j a_j$ where a_j are L^2-atoms for H^p and $\sum_{j=1}^\infty |\lambda_j|^p \le 2^p \|f\|_{H^p}^p$. Then Lemma 2.4.4 yields

$$T(f) = \sum_{j=1}^\infty \lambda_j T(a_j) \qquad \text{a.e.}, \tag{2.4.20}$$

where the series converges in L^p. We apply L^p quasi-norms on both sides of (2.4.20), we use the sublinearity of $h \mapsto \|h\|_{L^p}^p$, and (2.4.19) to deduce that

$$\big\|T(f)\big\|_{L^p}^p \le C_{n,p}^p A^p \sum_{j=1}^\infty |\lambda_j|^p \le 2^p C_{n,p}^p A^p \|f\|_{H^p}^p \tag{2.4.21}$$

for all $f \in L^2 \cap H^p$. Recall that $T(f)$ is well defined for all $f \in H^p$, as observed in item (v) in the introductory comments of this subsection. Then by the density of $L^2 \cap H^p$ in H^p, the estimate

$$\big\|T(f)\big\|_{L^p}^p \le 2^p C_{n,p}^p A^p \|f\|_{H^p}^p$$

obtained in (2.4.21) for $f \in L^2 \cap H^p$ extends to any $f \in H^p$. □

Lemma 2.4.4. *Let $\{\zeta_\varepsilon\}_{\varepsilon>0}$ be a family of Schwartz functions and for each $\varepsilon > 0$ let T_ε be the operator given by convolution with ζ_ε. Suppose that the T_ε's are uniformly (in $\varepsilon > 0$) bounded on $L^2(\mathbf{R}^n)$ and that there is an $L^2(\mathbf{R}^n)$-bounded operator T such that for each $g \in L^2(\mathbf{R}^n)$, we have*

$$\big\|T_\varepsilon(g) - T(g)\big\|_{L^2} \to 0 \quad \text{ as } \varepsilon \to 0. \tag{2.4.22}$$

Suppose moreover that for a given $0 < p \le 1$ there is a constant C_0 such that for all a that are L^2-atoms for H^p we have

$$\sup_{\varepsilon>0} \|T_\varepsilon(a)\|_{L^p} \le C_0\,. \tag{2.4.23}$$

Then for every $f \in L^2 \cap H^p$ with atomic decomposition $f = \sum_{j=1}^\infty \lambda_j a_j$, where a_j are L^2-atoms for H^p and $\sum_{j=1}^\infty |\lambda_j|^p \le 2^p \|f\|_{H^p}^p$, the sequence $\sum_{j=1}^N \lambda_j T(a_j)$ is Cauchy in L^p and converges in L^p to a well-defined L^p function $\sum_{j=1}^\infty \lambda_j T(a_j)$ which is equal almost everywhere to $T(f)$, i.e., we have

$$T(f) = \sum_{j=1}^\infty \lambda_j T(a_j) \qquad \text{a.e.} \tag{2.4.24}$$

Proof. We begin the proof by observing that as a consequence of (2.4.23) we have

$$\|T(a)\|_{L^p} \le C_0 \tag{2.4.25}$$

for all a that are L^2-atoms for H^p. Indeed, (2.4.22) implies that for a given L^2 atom a for H^p, there is sequence $\varepsilon_k \downarrow 0$ such that

$$T(a) = \lim_{k\to\infty} T_{\varepsilon_k}(a) = \liminf_{k\to\infty} T_{\varepsilon_k}(a) \qquad \text{a.e.}$$

Then Fatou's lemma on L^p together with (2.4.23) imply (2.4.25).

Given $f \in H^p \cap L^2$, we write $f = \sum_{j=1}^\infty \lambda_j a_j$ in an atomic decomposition, where a_j are L^2-atoms for H^p, the series converges to f in H^p, and $\sum_{j=1}^\infty |\lambda_j|^p \le 2^p \|f\|_{H^p}^p$.

We observe that the sequence $\big\{\sum_{j=1}^N \lambda_j T(a_j)\big\}_{N=1}^\infty$ is Cauchy in L^p since

$$\Big\| \sum_{j=N'}^{N} \lambda_j T(a_j)\Big\|_{L^p}^p \le \sum_{j=N'}^{N} |\lambda_j|^p C_0^p ,$$

which tends to zero as $N', N \to \infty$. Thus the sequence $\sum_{j=1}^N \lambda_j T(a_j)(x)$ converges in L^p to a well-defined L^p function. We set

$$\sum_{j=1}^{\infty} \lambda_j T(a_j) = L^p \text{ limit of } \sum_{j=1}^{N} \lambda_j T(a_j) .$$

To prove (2.4.24), we first prove an analogous result about T_ε, namely,

$$T_\varepsilon(f) = \sum_{j=1}^{\infty} \lambda_j T_\varepsilon(a_j) \qquad \text{a.e.} \tag{2.4.26}$$

where $\sum_{j=1}^\infty \lambda_j T_\varepsilon(a_j)$ denotes the L^p limit of the Cauchy sequence $\sum_{j=1}^N \lambda_j T_\varepsilon(a_j)$. We fix $\varepsilon, \delta > 0$. Then by the linearity of T_ε for each $L \in \mathbf{Z}^+$ we have

$$\begin{aligned}
&\Big|\Big\{x \in \mathbf{R}^n :\ |T_\varepsilon(f)(x) - \sum_{j=1}^{\infty} \lambda_j T_\varepsilon(a_j)(x)| > \delta\Big\}\Big| \\
&\le \Big|\Big\{x \in \mathbf{R}^n :\ |T_\varepsilon\Big(\sum_{j=L+1}^{\infty} \lambda_j a_j\Big)(x) - \sum_{j=L+1}^{\infty} \lambda_j T_\varepsilon(a_j)(x)| > \delta\Big\}\Big| \\
&\le \Big|\Big\{x \in \mathbf{R}^n : |T_\varepsilon\Big(\sum_{j=L}^{\infty} \lambda_j a_j\Big)(x)| > \frac{\delta}{2}\Big\}\Big| + \Big|\Big\{x \in \mathbf{R}^n : |\sum_{j=L+1}^{\infty} \lambda_j T_\varepsilon(a_j)(x)| > \frac{\delta}{2}\Big\}\Big| \\
&\le \frac{2^p}{\delta^p}\Big\|T_\varepsilon\Big(\sum_{j=L+1}^{\infty} \lambda_j a_j\Big)\Big\|_{L^p}^p + \frac{2^p}{\delta^p} \sum_{j=L+1}^{\infty} |\lambda_j|^p \big\|T_\varepsilon(a_j)\big\|_{L^p}^p .
\end{aligned} \tag{2.4.27}$$

By assumption (2.4.23) the second term in the sum in (2.4.27) is controlled by $C_0^p(\frac{2}{\delta})^p\sum_{j=L+1}^{\infty}|\lambda_j|^p$ which tends to zero as $L\to\infty$.

To show the same conclusion for the first sum in (2.4.27) we recall the grand maximal function

$$\mathscr{M}_N(f)(x) = \sup_{\varphi\in\mathscr{F}_N}\sup_{t>0}\sup_{\substack{y\in\mathbf{R}^n\\|y-x|\le t}}|(\varphi_t*f)(y)|$$

where

$$\mathscr{F}_N = \Big\{\varphi\in\mathscr{S}(\mathbf{R}^n):\ \int_{\mathbf{R}^n}(1+|x|)^N\sum_{|\alpha|\le N+1}|\partial^\alpha\varphi(x)|\,dx\le 1\Big\}.$$

The function ζ_ε lies in $\mathscr{S}(\mathbf{R}^n)$; thus there is a constant $c_{\varepsilon,N}$ such that $c_{\varepsilon,N}\zeta_\varepsilon$ lies in $\mathscr{F}_N$. Then we have

$$|T_\varepsilon\big(\sum_{j=L+1}^{\infty}\lambda_j a_j\big)|\le\frac{1}{c_{\varepsilon,N}}\mathscr{M}_N\big(f-\sum_{j=1}^{L}\lambda_j a_j\big).$$

Taking L^p quasi-norms we obtain

$$\Big\|T_\varepsilon\big(\sum_{j=L+1}^{\infty}\lambda_j a_j\big)\Big\|_{L^p}^p\le\frac{1}{c_{\varepsilon,N}^p}\Big\|\mathscr{M}_N\big(f-\sum_{j=1}^{L}\lambda_j a_j\big)\Big\|_{L^p}^p\le\frac{C_{n,p}^p}{c_{\varepsilon,N}^p}\Big\|f-\sum_{j=1}^{L}\lambda_j a_j\Big\|_{L^p}^p,$$

and since $\sum_{j=1}^{L}\lambda_j a_j\to f$ in H^p as $L\to\infty$, we deduce that the first sum in (2.4.27) tends to zero as $L\to\infty$. This proves that for any $\varepsilon,\delta>0$ we have

$$\Big|\Big\{x\in\mathbf{R}^n:\ |T_\varepsilon(f)(x)-\sum_{j=1}^{\infty}\lambda_j T_\varepsilon(a_j)(x)|>\delta\Big\}\Big|=0;$$

hence (2.4.26) holds.

Next, we claim that $\sum_{j=1}^{\infty}\lambda_j T_\varepsilon(a_j)\to\sum_{j=1}^{\infty}\lambda_j T(a_j)$ in measure as $\varepsilon\to 0$. Indeed, given $\delta>0$, write

$$\begin{aligned}
&\Big|\Big\{\sum_{j=1}^{\infty}\lambda_j T_\varepsilon(a_j)-\sum_{j=1}^{\infty}\lambda_j T(a_j)|>\delta\Big\}\Big|\\
&\le\Big|\Big\{\Big|\sum_{j=1}^{L}\lambda_j\big(T_\varepsilon(a_j)-T(a_j)\big)\Big|>\frac{\delta}{2}\Big\}\Big|+\Big|\Big\{\Big|\sum_{j=L+1}^{\infty}\lambda_j T_\varepsilon(a_j)-\sum_{j=L+1}^{\infty}\lambda_j T(a_j)\Big|>\frac{\delta}{2}\Big\}\Big|\\
&\le\frac{2^2}{\delta^2}\Big\|\sum_{j=1}^{L}\lambda_j\big(T_\varepsilon(a_j)-T(a_j)\big)\Big\|_{L^2}^2+\frac{2^p}{\delta^p}\sum_{j=L+1}^{\infty}|\lambda_j|^p\Big[\big\|T_\varepsilon(a_j)\big\|_{L^p}^p+\big\|T(a_j)\big\|_{L^p}^p\Big]\\
&\le\frac{2^2}{\delta^2}\Big\|T_\varepsilon\big(\sum_{j=1}^{L}\lambda_j a_j\big)-T\big(\sum_{j=1}^{L}\lambda_j a_j\big)\Big\|_{L^2}^2+\frac{2^{p+1}C_0^p}{\delta^p}\sum_{j=L+1}^{\infty}|\lambda_j|^p,
\end{aligned}\tag{2.4.28}$$

where we made use of (2.4.23) and (2.4.25) in the last estimate above. The second term in (2.4.28) can be made less than any given number $\tau > 0$ if L is chosen to be large enough. Once we fix L, then there is a $\varepsilon_0 > 0$ such that for $\varepsilon_0 < \varepsilon$ the first term in (2.4.28) is controlled by τ too, since $T_\varepsilon(\sum_{j=1}^{L} \lambda_j a_j)$ converges to $T(\sum_{j=1}^{L} \lambda_j a_j)$ in L^2 in view of (2.4.22). Therefore (2.4.28) can be made arbitrarily small for ε sufficiently small, and the claimed convergence in measure is valid. By Theorem 1.1.11 in [156] there is sequence ε_i (subsequence of $\varepsilon > 0$) such that

$$\sum_{k=1}^{\infty} \lambda_k T_{\varepsilon_i}(a_k)(x) \to \sum_{k=1}^{\infty} \lambda_k T(a_k)(x) \qquad \text{a.e. as } i \to \infty. \tag{2.4.29}$$

Since $T_{\varepsilon_i}(f)$ tends to $T(f)$ in L^2, we can find a subsequence $\{\varepsilon_{i_\ell}\}$ of the subsequence $\{\varepsilon_i\}$ such that

$$T_{\varepsilon_{i_\ell}}(f)(x) \to T(f)(x) \qquad \text{a.e. as } \ell \to \infty. \tag{2.4.30}$$

Using identity (2.4.26) with ε_{i_ℓ} in place of ε, together with (2.4.29) with i_ℓ in place of i, along with (2.4.30), letting $\ell \to \infty$, we deduce (2.4.24). □

We discuss a version of the Theorem 2.4.3 in which the target space is H^p.

Theorem 2.4.5. *Under the hypotheses of Theorem 2.4.3, there is a constant $C_{n,p}$ such that for all $f \in H^p$,*

$$\big\|T(f)\big\|_{H^p} \le C_{n,p} A \big\|f\big\|_{H^p}. \tag{2.4.31}$$

Proof. We fix a smooth function Φ supported in the unit ball $B(0,1)$ in $\mathbf{R}^n$ whose mean value is not equal to zero. For $t > 0$ we define the smooth functions

$$K^{(t)} = \Phi_t * W$$

and for $f \in H^p$, we define an operator

$$T^{(t)}(f) = \Phi_t * T(f)$$

noting that the convolution is well defined since $T(f)$ lies in $\mathscr{S}'$ and Φ_t is in $\mathscr{C}_0^\infty$.

We observe that the family of kernels $K^{(t)}$ satisfies

$$\sup_{t>0} \big|\widehat{K^{(t)}}(\xi)\big| \le \big\|\widehat{\Phi}\big\|_{L^\infty} \big\|\widehat{W}\big\|_{L^\infty} \le CA \big\|\widehat{\Phi}\big\|_{L^\infty} \tag{2.4.32}$$

and that

$$\sup_{t>0} |\partial^\beta K^{(t)}(x)| \le C_\Phi A\, |x|^{-n-|\beta|} \tag{2.4.33}$$

for all $|\beta| \le N$, where

$$C_\Phi = \sup_{|\gamma| \le N} \int_{\mathbf{R}^n} |\xi|^{|\gamma|} |\widehat{\Phi}(\xi)|\, d\xi\,.$$

Indeed, assertion (2.4.32) is easily verified. When $|x| \le 2t$ assertion (2.4.33) follows from the identity

$$K^{(t)}(x) = \big((\Phi_t * W)^{\widehat{}}\big)^{\vee}(x) = \int_{\mathbf{R}^n} e^{2\pi i x\cdot\xi}\, \widehat{W}(\xi)\, \widehat{\Phi}(t\xi)\, d\xi\,,$$

while whenever $|x| \ge 2t$, (2.4.33) follows from (2.4.12) and from the integral representation

$$\partial^\beta K^{(t)}(x) = \int_{|y|\le t} \partial^\beta K(x-y)\, \Phi_t(y)\, dy\,.$$

We now take $f = a$ to be an L^2-atom for H^p, and without loss of generality we may assume that a is supported in a cube Q centered at the origin. We let Q^* be the cube with side length $2\sqrt{n}\,\ell(Q)$, where $\ell(Q)$ is the side length of Q. Recall the smooth maximal function $M(f;\Phi)$ from Section 2.1. Then $M(T(a);\Phi)$ is pointwise controlled by the Hardy–Littlewood maximal function of $T(a)$. Using an argument similar to that in Theorem 2.4.1, we have

$$\begin{aligned}
\bigg(\int_{Q^*} |M(T(a);\Phi)(x)|^p\, dx\bigg)^{\frac{1}{p}} &\le \big\|\Phi\big\|_{L^1}\bigg(\int_{Q^*} |M(T(a))(x)|^p\, dx\bigg)^{\frac{1}{p}}\\
&\le C|Q^*|^{\frac{1}{p}-\frac{1}{2}}\bigg(\int_{Q^*} |M(T(a))(x)|^2\, dx\bigg)^{\frac{1}{2}}\\
&\le C'|Q|^{\frac{1}{p}-\frac{1}{2}}\bigg(\int_{\mathbf{R}^n} |T(a)(x)|^2\, dx\bigg)^{\frac{1}{2}}\\
&\le C''A|Q|^{\frac{1}{p}-\frac{1}{2}}\bigg(\int_{Q} |a(x)|^2\, dx\bigg)^{\frac{1}{2}}\\
&\le C_n A\,|Q|^{\frac{1}{p}-\frac{1}{2}}|Q|^{\frac{1}{2}-\frac{1}{p}}\\
&= C_n A\,.
\end{aligned}$$

It therefore remains to estimate the contribution of $M(T(a);\Phi)$ on the complement of Q^*.

For $x \notin Q^*$ we write

$$T^{(t)}(a)(x) = (a * K^{(t)})(x) = \int_Q K^{(t)}(x-y)\, a(y)\, dy\,.$$

Recall that $N = [\frac{n}{p} - n] + 1$. Using the cancellation of L^2 atoms for H^p we deduce

$$\begin{aligned}
T^{(t)}(a)(x) &= \int_Q a(y)\bigg[K^{(t)}(x-y) - \sum_{|\beta|\le N-1} (\partial^\beta K^{(t)})(x)\frac{y^\beta}{\beta!}\bigg] dy\\
&= \int_Q a(y)\bigg[\sum_{|\beta|=N} (\partial^\beta K^{(t)})(x-\theta_y y)\frac{y^\beta}{\beta!}\bigg] dy
\end{aligned}$$

for some $0 \le \theta_y \le 1$. Since $x \notin Q^*$ and $y \in Q$ we have $|x-\theta_y y| \ge |x| - |y| \ge \frac{1}{2}|x|$; thus using (2.4.33) we obtain the estimate

$$|T^{(t)}(a)(x)| \le c_{n,N} \frac{A}{|x|^{N+n}} \int_Q |a(y)|\,|y|^N\,dy,$$

from which it follows that for $x \notin Q^*$ we have

$$|T^{(t)}(a)(x)| \le c_{n,p} \frac{A}{|x|^{N+n}} |Q|^{1+\frac{N}{n}-\frac{1}{p}}$$

via a calculation using properties of atoms (see the proof of Theorem 2.3.11). Taking the supremum over all $t>0$ and integrating over $(Q^*)^c$, we obtain that

$$\left(\int_{(Q^*)^c} \sup_{t>0} |(T(a)*\Phi_t)(x)|^p dx \right)^{\frac{1}{p}} \le c_{n,p} A\, |Q|^{1+\frac{N}{n}-\frac{1}{p}} \left(\int_{(Q^*)^c} \frac{1}{|x|^{p(N+n)}}\,dx \right)^{\frac{1}{p}},$$

and the latter is easily seen to be finite and controlled by a constant multiple of A. Combining this estimate with the previously obtained estimate for the integral of $M(T(a);\Phi) = \sup_{t>0} |T^{(t)}(a)|$ over Q^* yields the conclusion of the theorem when $f=a$ is an atom.

We have now shown that there exists a constant $C_{n,p}$ such that

$$\big\|T^{(t)}(a)\big\|_{L^p} \le \big\|T(a)\big\|_{H^p} \le C_{n,p} A \tag{2.4.34}$$

whenever a is an L^2-atom for H^p. We now extend this estimate to arbitrary f in $L^2 \cap H^p$. To achieve this, we verify that the assumptions of Lemma 2.4.4 are valid for the family of Schwartz functions $\zeta_\varepsilon = \Phi_t * K_{(\varepsilon)}$ and the family of operators $T_\varepsilon^{(t)}(g) = \Phi_t * K_{(\varepsilon)} * g$, which are uniformly bounded on L^2 and converge in L^2 to $T^{(t)}(g) = \Phi_t * g$ for any $g \in L^2(\mathbf{R}^n)$.

Fix $f \in L^2 \cap H^p$, with atomic decomposition $f = \sum_{j=1}^\infty \lambda_j a_j$, where $\sum_{j=1}^\infty |\lambda_j|^p \le 2^p \|f\|_{H^p}^p$. Observe that the sequences $\sum_{j=1}^N |\lambda_j T^{(t)}(a_j)|$ and $\sum_{j=1}^N \lambda_j T^{(t)}(a_j)$ are Cauchy in L^p and thus they converge in L^p. We set

$$\sum_{j=1}^\infty \lambda_j T^{(t)}(a_j) = L^p \text{ limit of } \sum_{j=1}^N \lambda_j T^{(t)}(a_j) \text{ as } N \to \infty$$

$$\sum_{j=1}^\infty |\lambda_j T^{(t)}(a_j)| = L^p \text{ limit of } \sum_{j=1}^N |\lambda_j T^{(t)}(a_j)| \text{ as } N \to \infty,$$

and extracting subsequences that converge almost everywhere, we notice that

$$\Big|\sum_{j=1}^\infty \lambda_j T^{(t)}(a_j)\Big| \le \sum_{j=1}^\infty |\lambda_j T^{(t)}(a_j)| \qquad \text{a.e.} \tag{2.4.35}$$

To apply Lemma 2.4.4, we consider the family of Schwartz functions $\zeta_{\varepsilon_j}(x) = \Phi_t * K_{(\varepsilon_j)}$ and the associated operators $T^{(t)}_{(\varepsilon_j)} = \Phi_t * T_{(\varepsilon_j)}$, which are L^2-bounded uniformly in ε_j and $t > 0$. It is easy to see that for $g \in L^2$, $\Phi_t * T_{(\varepsilon_j)}(g) \to T^{(t)}(g)$ in L^2 as $j \to \infty$, and furthermore (2.4.34) also holds if $T^{(t)}$ is replaced by $T^{(t)}_{(\varepsilon_j)}$ uniformly in ε_j via a similar argument; hence the hypotheses of Lemma 2.4.4 are valid. Using the conclusion of Lemma 2.4.4 we write

$$T^{(t)}(f) = \sum_{j=1}^{\infty} \lambda_j T^{(t)}(a_j) \qquad \text{a.e.} \tag{2.4.36}$$

It follows from this fact and (2.4.35) that

$$|T^{(t)}(f)| \le \sum_{j=1}^{\infty} |\lambda_j T^{(t)}(a_j)| \le \sum_{j=1}^{\infty} |\lambda_j| M(T(a_j);\Phi) \qquad \text{a.e.} \tag{2.4.37}$$

Taking the supremum over $t > 0$ in (2.4.37) we deduce

$$M(T(f);\Phi) \le \sum_{j=1}^{\infty} |\lambda_j| M(T(a_j);\Phi) \qquad \text{a.e.} \tag{2.4.38}$$

and applying L^p quasi-norms on both sides and using (2.4.34) yields the desired conclusion (2.4.31) for $f \in L^2 \cap H^p$. The extension to general $f \in H^p$ follows by density and the fact that $T(f)$ is well defined for all $f \in H^p$, as observed in item (v) in the introduction of this subsection. □

2.4.4 A Singular Integral Characterization of $H^1(\mathbf{R}^n)$

We showed in Section 2.4.1 that singular integrals map H^1 to L^1. In particular, the Riesz transforms have this property. In this subsection we obtain a converse to this statement. We show that if $R_j(f)$ are integrable functions for some $f \in L^1$ and all $j = 1, \dots, n$, then f must be an element of the Hardy space H^1. This provides a characterization of $H^1(\mathbf{R}^n)$ in terms of the Riesz transforms.

Theorem 2.4.6. *For $n \ge 2$, there exists a constant C_n such that for f in $L^1(\mathbf{R}^n)$, we have*

$$C_n \|f\|_{H^1} \le \|f\|_{L^1} + \sum_{k=1}^{n} \|R_k(f)\|_{L^1}. \tag{2.4.39}$$

When $n = 1$ the corresponding statement is

$$C_1 \|f\|_{H^1} \le \|f\|_{L^1} + \|H(f)\|_{L^1} \tag{2.4.40}$$

for all $f \in L^1(\mathbf{R})$. Naturally, these statements are interesting when the expressions on the right in (2.4.39) and (2.4.40) are finite.

Before we prove this theorem we discuss a couple of corollaries.

Corollary 2.4.7. *An integrable function on the line lies in the Hardy space $H^1(\mathbf{R})$ if and only if its Hilbert transform is integrable. For $n \geq 2$, an integrable function on $\mathbf{R}^n$ lies in the Hardy space $H^1(\mathbf{R}^n)$ if and only its Riesz transforms are also in $L^1(\mathbf{R}^n)$.*

Proof. The corollary follows by combining Theorems 2.4.1 and 2.4.6. □

Corollary 2.4.8. *Functions in $H^1(\mathbf{R}^n)$, $n \geq 1$, have integral zero.*

Proof. Indeed, if $f \in H^1(\mathbf{R}^n)$, we must have $R_1(f) \in L^1(\mathbf{R}^n)$ by Theorem 2.4.1; thus $\widehat{R_1(f)}$ is uniformly continuous. But since

$$\widehat{R_1(f)}(\xi) = -i\widehat{f}(\xi)\frac{\xi_1}{|\xi|},$$

it follows that $\widehat{R_1(f)}$ is continuous at zero if and only if $\widehat{f}(0) = 0$. But this happens exactly when f has integral zero. □

We now discuss the proof of Theorem 2.4.6.

Proof. We consider the case $n \geq 2$, although the argument below also works in the case $n = 1$ with a suitable change of notation. Let P_t be the Poisson kernel. In the proof we may assume that f is real-valued, since it can be written as $f = f_1 + if_2$, where f_k are real-valued and $R_j(f_k)$ are also integrable. Given a real-valued function $f \in L^1(\mathbf{R}^n)$ such that $R_j(f)$ are integrable over $\mathbf{R}^n$ for every $j = 1,\dots,n$, we associate with it the $n+1$ functions

$$\begin{aligned} u_1(x,t) &= (P_t * R_1(f))(x), \\ \dots &= \dots \\ u_n(x,t) &= (P_t * R_n(f))(x), \\ u_{n+1}(x,t) &= (P_t * f)(x), \end{aligned}$$

which are harmonic on the space $\mathbf{R}^{n+1}_+$ (see Example 2.1.13 in [156]). It is convenient to denote the last variable t by x_{n+1}. One may check using the Fourier transform that these harmonic functions satisfy the following system:

$$\begin{aligned} \sum_{j=1}^{n+1} \frac{\partial u_j}{\partial x_j} &= 0, \\ \frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j} &= 0, \qquad k, j \in \{1,\dots,n+1\}, \quad k \neq j. \end{aligned} \tag{2.4.41}$$

This system of equations may also be expressed as $\operatorname{div} F = 0$ and $\operatorname{curl} F = \vec{0}$, where $F = (u_1,\dots,u_{n+1})$ is a vector field in $\mathbf{R}^{n+1}_+$. Note that when $n = 1$, the equations in (2.4.41) are the usual Cauchy–Riemann equations, which assert that the function

$F = (u_1, u_2) = u_1 + iu_2$ is holomorphic in the upper half-space. For this reason, when $n \geq 2$, the equations in (2.4.41) are often referred to as the *system of generalized Cauchy–Riemann equations*.

The function $|F|$ enjoys a crucial property in the study of this problem.

Lemma 2.4.9. *Let u_j be real-valued harmonic functions on $\mathbf{R}^{n+1}$ satisfying the system of equations* (2.4.41) *and let $F = (u_1, \dots, u_{n+1})$. Then the function*

$$|F|^q = \Big(\sum_{j=1}^{n+1} |u_j|^2\Big)^{q/2}$$

is subharmonic when $q \geq (n-1)/n$, i.e., it satisfies $\Delta(|F|^q) \geq 0$, on $\mathbf{R}^{n+1}_+$.

Lemma 2.4.10. *Let $0 < q < p < \infty$. Suppose that the function $|F(x,t)|^q$ defined on $\mathbf{R}^{n+1}_+$ is subharmonic and satisfies*

$$\sup_{t>0} \bigg(\int_{\mathbf{R}^n} |F(x,t)|^p \, dx \bigg)^{1/p} \leq A < \infty . \tag{2.4.42}$$

Then there is a constant $C_{n,p,q} < \infty$ such that the nontangential maximal function $|F|^*(x) = \sup_{t>0} \sup_{|y-x|<t} |F(y,t)|$, $x \in \mathbf{R}^n$, (cf. Definition 3.3.1) satisfies

$$\big\| |F|^* \big\|_{L^p(\mathbf{R}^n)} \leq C_{n,p,q} A .$$

Assuming these lemmas, whose proofs are postponed until the end of this section, we return to the proof of the theorem.

Without loss of generality, we may assume that the given integrable function f is real-valued, so that $R_j(f)$ are also real-valued and we are able to apply Lemma 2.4.9. Since the Poisson kernel is an approximate identity, the function $x \mapsto u_{n+1}(x,t)$ converges to $f(x)$ in L^1 as $t \to 0$. To show that $f \in H^1(\mathbf{R}^n)$, it suffices to show that the Poisson maximal function

$$M(f;P)(x) = \sup_{t>0} |(P_t * f)(x)| = \sup_{t>0} |u_{n+1}(x,t)|$$

is integrable. But this maximal function is pointwise controlled by

$$\sup_{t>0} |F(x,t)| \leq \sup_{t>0} \Big[|(P_t * f)(x)| + \sum_{j=1}^{n} |(P_t * R_j(f))(x)| \Big] ,$$

and certainly it satisfies

$$\sup_{t>0} \int_{\mathbf{R}^n} |F(x,t)| \, dx \leq A_f , \tag{2.4.43}$$

where

$$A_f = \big\| f \big\|_{L^1} + \sum_{k=1}^{n} \big\| R_k(f) \big\|_{L^1} .$$

We now have

$$M(f;P)(x) \le \sup_{t>0} |u_{n+1}(x,t)| \le \sup_{t>0} |F(x,t)| \le |F|^*(x)\,, \tag{2.4.44}$$

and using Lemma 2.4.9 with $q = \frac{n-1}{n}$ and Lemma 2.4.10 with $p = 1$ we obtain that

$$\big\| |F|^* \big\|_{L^1(\mathbf{R}^n)} \le C_n A_f\,. \tag{2.4.45}$$

Combining (2.4.43), (2.4.44), and (2.4.45), we deduce that

$$\big\| M(f;P)(x) \big\|_{L^1(\mathbf{R}^n)} \le C_n \bigg(\big\| f \big\|_{L^1} + \sum_{k=1}^{n} \big\| R_k(f) \big\|_{L^1} \bigg)\,,$$

from which (2.4.39) follows. This proof is also valid when $n = 1$, provided one replaces the Riesz transforms with the Hilbert transform; hence the proof of (2.4.40) is subsumed in that of (2.4.39). □

We now give a proof of Lemma 2.4.9

Proof. Denoting the variable t by x_{n+1}, we have

$$\frac{\partial}{\partial x_j} |F|^q = q |F|^{q-2} \Big(F \cdot \frac{\partial F}{\partial x_j} \Big)$$

and also

$$\frac{\partial^2}{\partial x_j^2} |F|^q = q\,|F|^{q-2} \bigg[F \cdot \frac{\partial^2 F}{\partial x_j^2} + \frac{\partial F}{\partial x_j} \cdot \frac{\partial F}{\partial x_j} \bigg] + q(q-2)|F|^{q-4} \Big(F \cdot \frac{\partial F}{\partial x_j} \Big)^2$$

for all $j = 1, 2, \dots, n+1$. Summing over all these j's, we obtain

$$\Delta(|F|^q) = q\,|F|^{q-4} \bigg[|F|^2 \sum_{j=1}^{n+1} \Big| \frac{\partial F}{\partial x_j} \Big|^2 + (q-2) \sum_{j=1}^{n+1} \Big| F \cdot \frac{\partial F}{\partial x_j} \Big|^2 \bigg]\,, \tag{2.4.46}$$

since the term containing $F \cdot \Delta(F) = \sum_{j=1}^{n+1} u_j \Delta(u_j)$ vanishes because each u_j is harmonic. The only term that could be negative in (2.4.46) is that containing the factor $q-2$ and naturally, if $q \ge 2$, the conclusion is obvious. Let us assume that $\frac{n-1}{n} \le q < 2$. Since $q \ge \frac{n-1}{n}$, we must have that $2 - q \le \frac{n+1}{n}$. Thus (2.4.46) is nonnegative if

$$\sum_{j=1}^{n+1} \Big| F \cdot \frac{\partial F}{\partial x_j} \Big|^2 \le \frac{n}{n+1} |F|^2 \sum_{j=1}^{n+1} \Big| \frac{\partial F}{\partial x_j} \Big|^2 . \tag{2.4.47}$$

This is certainly valid for points (x,t) such that $F(x,t) = 0$. To prove (2.4.47) for points (x,t) with $F(x,t) \ne 0$, it suffices to show that for every vector $v \in \mathbf{R}^{n+1}$ with Euclidean norm $|v| = 1$, we have

$$\sum_{j=1}^{n+1} \Big| v \cdot \frac{\partial F}{\partial x_j} \Big|^2 \le \frac{n}{n+1} \sum_{j=1}^{n+1} \Big| \frac{\partial F}{\partial x_j} \Big|^2 . \tag{2.4.48}$$

Denoting by A the $(n+1)\times(n+1)$ matrix whose entries are $a_{j,k} = \partial u_k/\partial x_j$, we rewrite (2.4.48) as

$$\big|Av\big|^2 \le \frac{n}{n+1} \big\|A\big\|^2 , \tag{2.4.49}$$

where

$$\big\|A\big\|^2 = \sum_{j=1}^{n+1}\sum_{k=1}^{n+1} |a_{j,k}|^2 .$$

By assumption, the functions u_j are real-valued and thus the numbers $a_{j,k}$ are real. In view of identities (2.4.41), the matrix A is real symmetric and has zero trace (i.e., $\sum_{j=1}^{n+1} a_{j,j} = 0$). A real symmetric matrix A can be written as $A = PDP^t$, where P is an orthogonal matrix and D is a real diagonal matrix. Since orthogonal matrices preserve the Euclidean distance, estimate (2.4.49) follows from the corresponding one for a diagonal matrix D. If $A = PDP^t$, then the traces of A and D are equal; hence $\sum_{j=1}^{n+1} \lambda_j = 0$, where λ_j are entries on the diagonal of D. Notice that estimate (2.4.49) with the matrix D in the place of A is equivalent to

$$\sum_{j=1}^{n+1} |\lambda_j|^2 |v_j|^2 \le \frac{n}{n+1} \Big(\sum_{j=1}^{n+1} |\lambda_j|^2 \Big) , \tag{2.4.50}$$

where we set $v = (v_1, \dots, v_{n+1})$ and we are assuming that $|v|^2 = \sum_{j=1}^{n+1} |v_j|^2 = 1$. Estimate (2.4.50) is certainly a consequence of

$$\sup_{1\le j\le n+1} |\lambda_j|^2 \le \frac{n}{n+1} \Big(\sum_{j=1}^{n+1} |\lambda_j|^2 \Big) . \tag{2.4.51}$$

But this is easy to prove. Let $|\lambda_{j_0}| = \max_{1\le j\le n+1} |\lambda_j|$. Then

$$|\lambda_{j_0}|^2 = \Big| - \sum_{j\neq j_0} \lambda_j \Big|^2 \le \Big(\sum_{j\neq j_0} |\lambda_j| \Big)^2 \le n \sum_{j\neq j_0} |\lambda_j|^2 . \tag{2.4.52}$$

Adding $n|\lambda_{j_0}|^2$ to both sides of (2.4.52), we deduce (2.4.51) and thus (2.4.47). □

We now give the proof of Lemma 2.4.10.

Proof. A consequence of the subharmonicity of $|F|^q$ is that

$$|F(x,t+\varepsilon)|^q \le (|F(\cdot,\varepsilon)|^q * P_t)(x) \tag{2.4.53}$$

for all $x \in \mathbf{R}^n$ and $t, \varepsilon > 0$. To prove (2.4.53), fix $\varepsilon > 0$ and consider the functions

$$U(x,t) = |F(x,t+\varepsilon)|^q , \qquad V(x,t) = (|F(\cdot,\varepsilon)|^q * P_t)(x) .$$

Given $\eta > 0$, we find a half-ball

$$B_{R_0} = \{(x,t) \in \mathbf{R}^{n+1}_+ : \ |x|^2 + t^2 < R_0^2\}$$

such that for $(x,t) \in \mathbf{R}^{n+1}_+ \setminus B_{R_0}$ we have

$$U(x,t) - V(x,t) \le \eta \,. \tag{2.4.54}$$

Suppose that this is possible. Since $U(x,0) = V(x,0)$, then (2.4.54) actually holds on the entire boundary of B_{R_0}. The function V is harmonic and U is subharmonic; thus $U - V$ is subharmonic. The maximum principle for subharmonic functions implies that (2.4.54) holds in the interior of B_{R_0}, and since it also holds on the exterior, it must be valid for all (x,t) with $x \in \mathbf{R}^n$ and $t \ge 0$. Since η was arbitrary, letting $\eta \to 0+$ implies (2.4.53).

We now prove that R_0 exists such that (2.4.54) is possible for $(x,t) \in \mathbf{R}^{n+1}_+ \setminus B_{R_0}$. Let $B((x,t),t/2)$ be the $(n+1)$-dimensional ball of radius $t/2$ centered at (x,t). The subharmonicity of $|F|^q$ is reflected in the inequality

$$|F(x,t)|^q \le \frac{1}{|B((x,t),\,t/2)|} \int_{B((x,t),t/2)} |F(y,s)|^q \, dy ds \,,$$

which by Hölder's inequality and the fact $p > q$ gives

$$|F(x,t)|^q \le \left(\frac{1}{|B((x,t),\,t/2)|} \int_{B((x,t),t/2)} |F(y,s)|^p \, dy ds \right)^{\frac{q}{p}} .$$

From this we deduce that

$$|F(x,t+\varepsilon)|^q \le \left[\frac{2^{n+1}/v_{n+1}}{(t+\varepsilon)^{n+1}} \int_{\frac{1}{2}(t+\varepsilon)}^{\frac{3}{2}(t+\varepsilon)} \int_{|y| \ge |x| - \frac{1}{2}(t+\varepsilon)} |F(y,s)|^p \, dy ds \right]^{\frac{q}{p}} . \tag{2.4.55}$$

If $t + \varepsilon \ge |x|$, using (2.4.42), we see that the expression on the right in (2.4.55) is bounded by $c' A^q (t+\varepsilon)^{-(n+1)q/p}$, and thus it can be made smaller than $\eta/2$ by taking $t \ge R_1 = \max\big(\varepsilon, (\eta/2c'A^q)^{-p/q(n+1)}\big)$. Since $R_1 \ge \varepsilon$, we must have $2t \ge t + \varepsilon \ge |x|$, which implies that $t \ge |x|/2$, and thus with $R_0' = \sqrt{5} R_1$, if $|(x,t)| > R_0'$ then $t \ge R_1$. Hence, the expression in (2.4.55) can be made smaller than $\eta/2$ for $|(x,t)| > R_0'$.

If $t + \varepsilon < |x|$ we estimate the expression on the right in (2.4.55) by

$$\left(\frac{2^{n+1}}{v_{n+1}} \frac{1}{(t+\varepsilon)^{n+1}} \int_{\frac{1}{2}(t+\varepsilon)}^{\frac{3}{2}(t+\varepsilon)} \left[\int_{|y| \ge \frac{1}{2}|x|} |F(y,s)|^p \, dy \right] ds \right)^{\frac{q}{p}} ,$$

and we notice that the preceding expression is bounded by

$$\left(\frac{3^{n+1}}{v_{n+1}} \int_{\frac{1}{2}\varepsilon}^{\infty} \left[\int_{|y| \ge \frac{1}{2}|x|} |F(y,s)|^p \, dy \right] \frac{ds}{s^{n+1}} \right)^{\frac{q}{p}} . \tag{2.4.56}$$

Let $G_{|x|}(s)$ be the function inside the square brackets in (2.4.56). Then $G_{|x|}(s) \to 0$ as $|x| \to \infty$ for all s. The hypothesis (2.4.42) implies that $G_{|x|}$ is bounded by a constant and it is therefore integrable over the interval $\big[\frac{1}{2}\varepsilon,\infty\big)$ with respect to the measure $s^{-n-1}ds$. By the Lebesgue dominated convergence theorem we deduce that the expression in (2.4.56) converges to zero as $|x| \to \infty$ and thus it can be made smaller that $\eta/2$ for $|x| \ge R_2$, for some constant R_2. Then with $R_0'' = \sqrt{2}R_2$ we have that if $|(x,t)| \ge R_0''$ then (2.4.56) is at most $\eta/2$. Since $U - V \le U$, we deduce the validity of (2.4.54) for $|(x,t)| > R_0 = \max(R_0', R_0'')$.

Let $r = p/q > 1$. Assumption (2.4.42) implies that the functions $x \mapsto |F(x,t)|^q$ are in L^r uniformly in t. Since any closed ball of L^r is weak* compact, there is a sequence $\varepsilon_k \to 0$ such that $|F(x,\varepsilon_k)|^q \to h$ weakly in L^r as $k \to \infty$ to some function $h \in L^r$. Since $P_t \in L^{r'}$, this implies that

$$(|F(\cdot,\varepsilon_k)|^q * P_t)(x) \to (h * P_t)(x)$$

for all $x \in \mathbf{R}^n$. Using (2.4.53) we obtain

$$|F(x,t)|^q = \limsup_{k\to\infty} |F(x,t+\varepsilon_k)|^p \le \limsup_{k\to\infty} \big(|F(x,\varepsilon_k)|^q * P_t\big)(x) = (h * P_t)(x)\,,$$

which gives for all $x \in \mathbf{R}^n$,

$$|F|^*(x) \le \Big[\sup_{t>0}\sup_{|y-x|<t} (|h| * P_t)(x)\Big]^{1/q} \le C_n' M(h)(x)^{1/q}\,. \tag{2.4.57}$$

Let $g \in L^{r'}(\mathbf{R}^n)$ with $L^{r'}$ norm at most one. The weak convergence yields

$$\int_{\mathbf{R}^n} |F(x,\varepsilon_k)|^q g(x)\,dx \to \int_{\mathbf{R}^n} h(x)\,g(x)\,dx$$

as $k \to \infty$, and consequently we have

$$\left|\int_{\mathbf{R}^n} h(x)\,g(x)\,dx\right| \le \sup_k \int_{\mathbf{R}^n} |F(x,\varepsilon_k)|^q |g(x)|\,dx \le \big\|g\big\|_{L^{r'}} \sup_{t>0}\left(\int_{\mathbf{R}^n} |F(x,t)|^p\,dx\right)^{\frac{1}{r}}.$$

Since g is arbitrary with $L^{r'}$ norm at most one, this implies that

$$\big\|h\big\|_{L^r} \le \sup_{t>0}\left(\int_{\mathbf{R}^n} |F(x,t)|^p\,dx\right)^{\frac{1}{r}}. \tag{2.4.58}$$

Putting things together, we have

$$\begin{aligned}
\big\||F|^*\big\|_{L^p} &\le C_n' \big\|M(h)^{1/q}\big\|_{L^p} \\
&= C_n' \big\|M(h)\big\|_{L^r}^{1/q} \\
&\le C_{n,p,q} \big\|h\big\|_{L^r}^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&= C_{n,p,q} \sup_{t>0} \left(\int_{\mathbf{R}^n} |F(x,t)|^p \, dx \right)^{1/qr} \\
&\le C_{n,p,q} A \,,
\end{aligned}
$$

where we have used (2.4.57) and (2.4.58) in the last two displayed inequalities. □

Exercises

2.4.1. Let f be an integrable function on the line whose Fourier transform vanishes on $(-\infty,0)$. Show that f lies in $H^1(\mathbf{R})$.

2.4.2. (a) Let h be a function on $\mathbf{R}$ such that $h(x)$ and $xh(x)$ are in $L^2(\mathbf{R})$. Show that h is integrable over $\mathbf{R}$ and satisfies

$$
\|h\|_{L^1}^2 \le 8 \, \|h\|_{L^2} \big\|xh(x)\big\|_{L^2} \,.
$$

(b) Suppose that g is an integrable function on $\mathbf{R}$ with vanishing integral and $g(x)$ and $xg(x)$ are in $L^2(\mathbf{R})$. Show that g lies in $H^1(\mathbf{R})$ and that for some constant C we have

$$
\|g\|_{H^1}^2 \le C \, \|g\|_{L^2} \big\|xg(x)\big\|_{L^2} \,.
$$

$\big[$*Hint:* Part (a): Split the integral of $|h(x)|$ over the regions $|x| \le R$ and $|x| > R$ and pick a suitable R. Part (b): Show that both $H(g)$ and $H(yg(y))$ lie in L^2. But since g has vanishing integral, we have $xH(g)(x) = H(yg(y))(x)$.$\big]$

2.4.3. (a) Let H be the Hilbert transform on the real line. Prove the identity

$$
H(fg - H(f)H(g)) = fH(g) + gH(f)
$$

for all f,g real-valued Schwartz functions. (b) Show that the bilinear operators

$$
\begin{aligned}
(f,g) &\mapsto f\,H(g) + H(f)\,g \,, \\
(f,g) &\mapsto f\,g - H(f)\,H(g) \,,
\end{aligned}
$$

map $L^p(\mathbf{R}) \times L^{p'}(\mathbf{R}) \to H^1(\mathbf{R})$ whenever $1 < p < \infty$.
$\big[$*Hint:* Part (a): Consider product $U_f(z)U_g(z)$, where $U_f(z) = \frac{i}{\pi} \int_{\mathbf{R}} \frac{f(t)}{z-t} \, dt$ is holomorphic on the upper half space and has boundary values $f + iH(f)$. Part (b): Use part (a) and Theorem 2.4.6.$\big]$

2.4.4. Follow the steps given to prove the following interpolation result. Let $1 < p_1 \le \infty$ and let T be a subadditive operator that maps $H^1(\mathbf{R}^n) + L^{p_1}(\mathbf{R}^n)$ into measurable functions on $\mathbf{R}^n$. Suppose that there is $A_0 < \infty$ such that for all $f \in H^1(\mathbf{R}^n)$ we have

$$
\sup_{\lambda>0} \lambda \, \big| \big\{ x \in \mathbf{R}^n : \, |T(f)(x)| > \lambda \big\} \big| \le A_0 \big\|f\big\|_{H^1}
$$

and that it also maps $L^{p_1}(\mathbf{R}^n)$ to $L^{p_1,\infty}(\mathbf{R}^n)$ with norm at most A_1. Show that for any $1<p<p_1$, T maps $L^p(\mathbf{R}^n)$ to itself with norm at most

$$CA_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{1-\frac{1}{p_1}}}A_1^{\frac{1-\frac{1}{p}}{1-\frac{1}{p_1}}},$$

where $C=C(n,p,p_1)$.

(a) Fix $1<q<p<p_1<\infty$ and f and let Q_j be the family of all maximal dyadic cubes such that $\lambda^q<|Q_j|^{-1}\int_{Q_j}|f|^q\,dx$. Write $E_\lambda=\bigcup Q_j$ and note that $E_\lambda\subseteqq\big\{M(|f|^q)^{\frac{1}{q}}>\lambda\big\}$ and that $|f|\le\lambda$ a.e. on $(E_\lambda)^c$. Write f as the sum of the *good function*

$$g_\lambda=f\chi_{(E_\lambda)^c}+\sum_j(\operatorname*{Avg}_{Q_j}f)\,\chi_{Q_j}$$

and the *bad function*

$$b_\lambda=\sum_j b_\lambda^j\,,\qquad\text{where}\qquad b_\lambda^j=\big(f-\operatorname*{Avg}_{Q_j}f\big)\,\chi_{Q_j}\,.$$

(b) Show that g_λ lies in $L^{p_1}(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$, $\|g_\lambda\|_{L^\infty}\le 2^{\frac{n}{q}}\lambda$, and that

$$\|g_\lambda\|_{L^{p_1}}^{p_1}\le\int_{|f|\le\lambda}|f(x)|^{p_1}\,dx+2^{\frac{np_1}{q}}\lambda^{p_1}|E_\lambda|<\infty.$$

(c) Show that for $c=2^{\frac{n}{q}+1}$, each $c^{-1}\lambda^{-1}|Q_j|^{-1}b_\lambda^j$ is an L^q-atom for H^1. Conclude that b_λ lies in $H^1(\mathbf{R}^n)$ and satisfies

$$\big\|b_\lambda\big\|_{H^1}\le c\lambda\sum_j|Q_j|\le c\lambda\,|E_\lambda|<\infty.$$

(d) Start with

$$\begin{aligned}\big\|T(f)\big\|_{L^p}^p\le p\gamma^p&\int_0^\infty\lambda^{p-1}\big|\big\{T(g_\lambda)|>\tfrac{1}{2}\gamma\lambda\big\}\big|\,d\lambda\\&+p\gamma^p\int_0^\infty\lambda^{p-1}\big|\big\{T(b_\lambda)|>\tfrac{1}{2}\gamma\lambda\big\}\big|\,d\lambda\end{aligned}$$

and use the results in parts (b) and (c) to obtain that the preceding expression is at most $C(n,p,q,p_1)\max(A_1\gamma^{p-p_1},\gamma^{p-1}A_0)$. Select $\gamma=A_1^{\frac{p_1}{p_1-1}}A_0^{-\frac{1}{p_1-1}}$ to obtain the required conclusion.

(e) In the case $p_1=\infty$ we have $|T(g_\lambda)|\le A_1 2^{\frac{n}{q}}\lambda$ and pick $\gamma>2A_1 2^{\frac{n}{q}}$ to make the integral involving g_λ vanishing.

2.4.5. Let P_t be the Poisson kernel and K_j be the kernel of the Riesz transform R_j. Let $\widehat{\varphi}\in\mathscr{S}$ be equal to 1 in a neighborhood of the origin. Then $\delta_0=\varphi+(\delta_0-\varphi)$ and for a bounded distribution f (cf. Section 2.1.1) and $t>0$ write

$$(P_t*K_j)*f=(P_t*K_j)*(\varphi*f)+(P_t*K_j)*(\delta_0-\varphi)*f\,.$$

Since P_t lies in L^1 and $\varphi * f$ in L^∞, $(K_j * P_t) * (\varphi * f) = R_j(P_t * \varphi * f)$ is a *BMO* function. The Fourier transform of $(P_t * K_j) * (\delta_0 - \varphi)$ is $-i\frac{\xi_j}{|\xi|}e^{-2\pi t|\xi|}(1-\widehat{\varphi}(\xi))$, which is a Schwartz function. Thus $(P_t * K_j) * (\delta_0 - \varphi)$ is also a Schwartz function and $(P_t * K_j) * (\delta_0 - \varphi) * f$ is a smooth function. Hence $(P_t * K_j) * f = P_t * R_j(f)$ is a well-defined function for all $t > 0$ and $j = 1, \dots, n$. Let $\frac{n-1}{n} < p < 1$.
(a) Show that there are constants C_n, C_1 such that for any $f \in H^p(\mathbf{R}^n)$ we have

$$\sup_{\delta>0}\Big[\big\|P_\delta * f\big\|_{L^p} + \sum_{k=1}^{n}\big\|P_\delta * R_k(f)\big\|_{L^p}\Big] \le C_n\big\|f\big\|_{H^p}$$

when $n \ge 2$ and

$$\sup_{\delta>0}\Big[\big\|P_\delta * f\big\|_{L^p} + \big\|P_\delta * H(f)\big\|_{L^p}\Big] \le C_1\big\|f\big\|_{H^p}$$

when $n = 1$.
(b) Prove that there are constants C_1, C_n such that for any bounded tempered distribution f on $\mathbf{R}^n$ we have

$$c_n\big\|f\big\|_{H^p} \le \sup_{\delta>0}\Big[\big\|P_\delta * f\big\|_{L^p} + \sum_{k=1}^{n}\big\|P_\delta * R_k(f)\big\|_{L^p}\Big]$$

when $n \ge 2$ and

$$c_1\big\|f\big\|_{H^p} \le \sup_{\delta>0}\Big[\big\|P_\delta * f\big\|_{L^p} + \big\|P_\delta * H(f)\big\|_{L^p}\Big]$$

when $n = 1$.
[*Hint:* Part (a): This is a consequence of Theorem 2.4.5. Part (b): Define $F_\delta = (P_\delta * u_1, \dots, P_\delta * u_n, P_\delta * u_{n+1})$, where $u_j(x,t) = (P_t * R_j(f))(x)$, $j = 1, \dots, n$, and $u_{n+1}(x,t) = (P_t * f)(x)$. Each $P_\delta * u_j$ is a harmonic function on $\mathbf{R}^{n+1}_+$ and continuous up to the boundary. The subharmonicity of $|F_\delta(x,t)|^p$ has as a consequence that $|F_\delta(x,t+\varepsilon)|^p \le (|F_\delta(\cdot,\varepsilon)|^p * P_t)(x)$ in view of (2.4.53). Letting $\varepsilon \to 0$ implies that $|F_\delta(x,t)|^p \le (|F_\delta(\cdot,0)|^p * P_t)(x)$, by the continuity of F_δ up to the boundary. Since $F_\delta(x,0) = (P_\delta * R_1(f), \dots, P_\delta * R_n(f), P_\delta * f)$, the hypothesis that $P_\delta * f, P_\delta * R_j(f)$ are in L^p uniformly in $\delta > 0$ yields $\sup_{t,\delta>0}\int_{\mathbf{R}^n}|F_\delta(x,t)|^p\,dx < \infty$. Fatou's lemma implies (2.4.42) for $F(x,t) = (u_1, \dots, u_{n+1})$ and then Lemma 2.4.10 yields the claim.]

HISTORICAL NOTES

The theory of Hardy spaces is vast and complicated. In classical complex analysis, the Hardy spaces H^p were spaces of analytic functions and were introduced to characterize boundary values of analytic functions on the unit disk.

Hardy [180] proved that the mean value of the pth power of the modulus of an analytic function on the unit disc is an increasing function of the radius and its logarithm is a convex function of the logarithm of the radius. The first systematic study of the class $H_p(\mathbb{D})$ of all analytic functions F on the unit disk $\mathbb{D}$ with the property that $\sup_{0<r<1}\int_0^1 |F(re^{2\pi i\theta})|^p\,d\theta<\infty$, $0<p<\infty$, can be traced to F. Riesz's article [303]. In this article Riesz proved the factorization theorem, the existence of boundary values, and other basic properties of such functions and adopted the symbol H_p, honoring Hardy for the fact that the aforementioned mean values are increasing as a function of the radius r. When $1<p<\infty$, the space $H_p(\mathbb{D})$ coincides with the space of analytic functions whose real parts are Poisson integrals of functions in $L^p(\mathbf{T}^1)$. But for $0<p\le 1$ this characterization fails and for several years a satisfactory characterization was missing. For a systematic treatment of these spaces we refer to the books of Duren [127] and Koosis [226].

With the illuminating work of Stein and Weiss [327] on systems of conjugate harmonic functions the road opened to higher-dimensional extensions of Hardy spaces. Burkholder, Gundy, and Silverstein [52] proved the fundamental theorem that an analytic function F lies in $H^p(\mathbf{R}^2_+)$ [i.e., $\sup_{y>0}\int_{\mathbf{R}}|F(x+iy)|^p\,dx<\infty$] if and only if the nontangential maximal function of its real part lies in $L^p(\mathbf{R})$. This result was proved using Brownian motion, but later Koosis [225] obtained another proof using complex analysis. This theorem spurred the development of the modern theory of Hardy spaces by providing the first characterization without the notion of conjugacy and indicating that Hardy spaces are intrinsically defined. The pioneering article of Fefferman and Stein [139] furnished three new characterizations of Hardy spaces: using a maximal function associated with a general approximate identity, using the grand maximal function, and using the area function of Luzin. From this point on, the role of the Poisson kernel faded into the background, when it turned out that it was not essential in the study of Hardy spaces. A previous characterization of Hardy spaces using the g-function, a radial analogue of the Luzin area function, was obtained by Calderón [54]. Two alternative characterizations of Hardy spaces were obtained by Uchiyama in terms of the generalized Littlewood–Paley g-function [356] and in terms of Fourier multipliers [357]. A characterization of $H^1(\mathbf{R})$ in terms of the variation of the function $m_f(y)=\int_{\mathbf{R}}f(x)\ln|x-y|^{-1}\,dx$ was obtained by Stefanov [321]. An extension of this result in higher dimensions was provided by Wang [364]. Necessary and sufficient conditions for systems of singular integral operators to characterize $H^1(\mathbf{R}^n)$ were also obtained by Uchiyama [355]. The characterization of H^p using Littlewood–Paley theory was observed by Peetre [292]. The case $p=1$ was later independently obtained by Rubio de Francia, Ruiz, and Torrea [308].

The one-dimensional atomic decomposition of Hardy spaces is due to Coifman [86] and its higher-dimensional extension to Latter [239]. A simplification of some of the technical details in Latter's proof was subsequently obtained by Latter and Uchiyama [240]. Using the atomic decomposition Coifman and Weiss [97] extended the definition of Hardy spaces to more general structures. The idea of obtaining the atomic decomposition from the reproducing formula (2.3.13) goes back to Calderón [56]. Another simple proof of the L^2-atomic decomposition for H^p (starting from the nontangential Poisson maximal function) was obtained by Wilson [370]. With only a little work, one can show that L^q-atoms for H^p can be written as sums of L^∞-atoms for H^p. We refer to the book of García-Cuerva and Rubio de Francia [150] for a proof of this fact. Although finite sums of atoms are dense in H^1, an example due to Y. Meyer (contained in [265]) shows that the H^1 norm of a function may not be comparable to $\inf\sum_{j=1}^N|\lambda_j|$, where the infimum is taken over all representations of the function as finite linear combinations $\sum_{j=1}^N\lambda_j a_j$ with the a_j being L^∞-atoms for H^1. Based on this idea, Bownik [48] constructed an example of a linear functional on a dense subspace of H^1 that is uniformly bounded on L^∞-atoms for H^1 but does not extend to a bounded linear functional on the whole H^1. However, if a Banach-valued linear operator is bounded uniformly on all L^q-atoms for H^p with $1<q<\infty$ and $0<p\le 1$, then it is bounded on the entire H^p as shown by Meda, Sjögren, and Vallarino [261]. This fact is also valid for quasi-Banach-valued linear operators, and when $q=2$ it was obtained independently by Yang and Zhou [374]. A related general result says that a sublinear operator maps the Triebel–Lizorkin space $\dot F^s_{p,q}(\mathbf{R}^n)$ to a quasi-Banach space if and only if it is uniformly bounded on certain infinitely differentiable atoms of the space; see Liu and Yang [250]. Atomic decompositions of general function spaces were obtained in the fundamental work of Frazier and Jawerth [143], [144]. The exposition in Section 2.3 is based on the article of Frazier and Jawerth [145]. The work of these authors provides a solid

manifestation that atomic decompositions are intrinsically related to Littlewood–Paley theory and not wedded to a particular space. Littlewood–Paley theory therefore provides a comprehensive and unifying perspective on function spaces.

Main references on H^p spaces and their properties are the books of Baernstein and Sawyer [14], Folland and Stein [142] in the context of homogeneous groups, Lu [252] (on which the proofs of Lemma 2.1.5 and Theorem 2.1.4 are based), Strömberg and Torchinsky [333] (on weighted Hardy spaces), and Uchiyama [358]. The articles of Calderón and Torchinsky [58], [59] develop and extend the theory of Hardy spaces to the nonisotropic setting. Hardy spaces can also be defined in terms of nonstandard convolutions, such as the "twisted convolution" on $\mathbf{R}^{2n}$. Characterizations of the space H^1 in this context have been obtained by Mauceri, Picardello, and Ricci [259].

The localized Hardy spaces h_p, $0 < p \le 1$, were introduced by Goldberg [155] as spaces of distributions for which the maximal operator $\sup_{0<t<1} |\Phi_t * f|$ lies in $L^p(\mathbf{R}^n)$ (here Φ is a Schwartz function with nonvanishing integral). These spaces can be characterized in ways analogous to those of the homogeneous Hardy spaces H^p; in particular, they admit an atomic decomposition. It was shown by Bui [50] that the space h^p coincides with the Triebel–Lizorkin space $F_p^{0,2}(\mathbf{R}^n)$; see also Meyer [263]. For the local theory of Hardy spaces one may consult the articles of Dafni [108] and Chang, Krantz, and Stein [73].

Interpolation of operators between Hardy spaces was originally based on complex function theory; see the articles of Calderón and Zygmund [57] and Weiss [365]. The real-interpolation approach discussed in Exercise 2.4.4 can be traced to the article of Igari [201]. Interpolation between Hardy spaces was further studied and extended by Riviere and Sagher [305]; Fefferman, Riviere, and Sagher [137]; and He [187].

The action of singular integrals on periodic spaces was studied by Calderón and Zygmund [61]. The preservation of Lipschitz spaces under singular integral operators is due to Taibleson [334]. The case $0 < \alpha < 1$ was earlier considered by Privalov [301] for the conjugate function on the circle. Fefferman and Stein [139] were the first to show that singular integrals map Hardy spaces to themselves. The boundedness of fractional integrals on H^p was obtained by Krantz [228]. The case $p = 1$ was earlier considered by Stein and Weiss [327]. An exposition on the subject of function spaces and the action of singular integrals on them was written by Frazier, Jawerth, and Weiss [146]. For a careful study of the action of singular integrals on function spaces, we refer to the book of Torres [352]. The study of anisotropic function spaces and the action of singular integrals on them has been studied by Bownik [47]. Weighted anisotropic Hardy spaces have been studied by Bownik, Li, Yang, and Zhou [49].

Besov spaces are named after Besov, who obtained a trace theorem and embeddings for them [34], [35]. The spaces $B_p^{\alpha,q}$, as defined in Section 2.2, were introduced by Peetre [290], although the case $p = q = 2$ was earlier considered by Hörmander [194]. The connection of Besov spaces with modern Littlewood–Paley theory was brought to the surface by Peetre [290]. The extension of the definition of Besov spaces to the case $p < 1$ is also due to Peetre [291], but there was a forerunner by Flett [140]. Peetre's monograph [294] contains an excellent exposition on the topic of Besov spaces. The spaces $F_p^{\alpha,q}$ with $1 < p, q < \infty$ were introduced by Triebel [353] and independently by Lizorkin [251]. The extension of the spaces $F_p^{\alpha,q}$ to the case $0 < p < \infty$ and $0 < q \le \infty$ first appeared in Peetre [293], who also obtained a maximal characterization for all of these spaces. Lemma 2.2.3 originated in Peetre [293]; the version given in the text is based on a refinement of Triebel [354]. The article of Lions, Lizorkin, and Nikol'skij [249] presents an account of the treatment of the spaces $F_p^{\alpha,q}$ introduced by Triebel and Lizorkin as well as the equivalent characterizations obtained by Lions, using interpolation between Banach spaces, and by Nikol'skij, using best approximation.

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