Corona Problem for H^∞ on Riemann Surfaces

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Abstract In this paper we survey some results and methods related to the famous corona problem for algebras H^∞ of bounded holomorphic functions on Caratheodory hyperbolic Riemann surfaces.

Keywords Corona theorem • Maximal ideal space • Bounded holomorphic function • Covering • Riemann surface of finite type

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1 Corona Problem

Let X be a complex manifold and $H^\infty(X)$ be the Banach algebra of bounded holomorphic functions on X equipped with the supremum norm. We assume that X is Caratheodory hyperbolic, that is, the functions in $H^\infty(X)$ separate the points of X. The maximal ideal space $\mathscr{M} = \mathscr{M}(H^\infty(X))$ is the set of all nonzero linear multiplicative functionals on $H^\infty(X)$. Since the norm of each $\phi \in \mathscr{M}$ is one, $\mathscr{M}$ is a subset of the closed unit ball of the dual space $(H^\infty(X))^*$. It is a compact Hausdorff space in the Gelfand topology, the weak* topology induced by $(H^\infty(X))^*$.

There is a continuous embedding $i : X \hookrightarrow \mathscr{M}$ taking $x \in X$ to the evaluation homomorphism $f \mapsto f(x)$, $f \in H^\infty(X)$. The complement to the closure of $i(X)$ in $\mathscr{M}$ is called the *corona*. The *corona problem* is: *given X to determine whether the corona is empty.* It was first posed in the case of the unit disc $\mathbb{D}$ in $\mathbb{C}$ by S. Kakutani in 1941. The complement of the closure of $i(\mathbb{D})$ in $\mathscr{M}$ was called the corona by D. Newman [N] (as in this case there would have been a set of maximal ideals

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suggestive of the sun's corona if the complement failed to be empty). Newman showed that the corona problem in the disk is equivalent to a certain interpolation problem [C1, p. 548]. The latter was solved by L. Carleson [C1] in 1962. The proof of the Carleson corona theorem was subsequently simplified by L. Hörmander [Ho] who used the Koszul complex technique to reduce it to a $\bar{\partial}$-problem on $\mathbb{D}$. Along these lines the simplest proof of the corona theorem was obtained by T. Wolff, see, e.g., [Ga].

Following the appearance of Carleson's proof, a number of authors have proved the corona theorem for finite bordered Riemann surfaces, e.g., [A1, A2, EM1, F, St1, St2, St3, HN, O2]. The corona problem was first solved for a class of infinitely connected domains (the, so-called, "roadrunner" domains) by M. Behrens [Be1, Be2] (his results were further extended in [D, DW, Nar]), and for a class of finitely sheeted covering Riemann surfaces by M. Nakai [Nak2]. Also, it was shown that there are non-planar Riemann surfaces for which the corona is non-trivial (see, e.g., [JM], [G1], [BD, L] and references therein). This is due to a structure that in a sense makes the surface seem higher dimensional. So there is a hope that the restriction to the Riemann sphere might prevent this obstacle. However, the general problem for planar domains is still open as is the problem in several variables for the ball and polydisk. (In fact, there are no known examples of domains in $\mathbb{C}^n$, $n \geq 2$, without corona.) In this direction, Gamelin [G2] has shown that the corona problem for planar domains is local in the sense that it depends only on the behavior of the domain locally about each boundary point.

At present, one of the strongest corona theorems for planar domains is due to Moore [M]. It states that the corona is empty for any domain with boundary contained in the graph of a $C^{1+\epsilon}$ function. This result is the extension of the earlier result of Jones and Garnett [GJ] for a Denjoy domain (i.e., a domain with boundary contained in $\mathbb{R}$). Among other results, it is worth mentioning recent results of Handy [Han] establishing the corona theorem for complements of certain square Cantor sets and of NewDelman [ND] who proves the corona theorem for the complement of a subset of a Lipschitz graph of homogeneous type (in the lines of Carleson's result [C2] on Denjoy domains).

The corona problem can be equivalently reformulated as follows, see, e.g., [Ga]:

A collection $f_1, \ldots, f_n$ of functions from $H^\infty(X)$ satisfies the *corona condition* if

$$1 \geq \max_{1 \leq j \leq n} |f_j(x)| \geq \delta > 0 \quad \text{for all} \quad x \in X. \tag{1}$$

The corona problem being solvable (i.e., the corona is empty) means that for all $n \in \mathbb{N}$ and $f_1, \ldots, f_n$ satisfying the corona condition, the Bezout equation

$$f_1 g_1 + \cdots + f_n g_n \equiv 1 \tag{2}$$

has a solution $g_1, \ldots, g_n \in H^\infty(X)$. We refer to $\max_{1 \leq j \leq n} ||g_j||_\infty$ as a "bound on the corona solutions". (Here $|| \cdot ||_\infty$ is the norm on $H^\infty(X)$.)

In the present paper we survey some results and methods related to the corona problem for H^∞ on Riemann surfaces.

2 Forelli Projections

In this part we describe one of the methods allowing to prove corona theorems for a wide class of Riemann surfaces.

Let $r : \tilde{X} \to X$ be an unbranched covering of a connected complex manifold X. By $r^*(H^\infty(X)) \subset H^\infty(\tilde{X})$ we denote the pullback by r of the algebra $H^\infty(X)$. A bounded linear projection $P : H^\infty(\tilde{X}) \to r^*(H^\infty(X))$ satisfying

$$P(fg) = P(f)g \quad \text{for any} \quad f \in H^\infty(\tilde{X}) \quad \text{and} \quad g \in r^*(H^\infty(X)) \tag{3}$$

is called a *Forelli projection*.

For X a connected Caratheodory hyperbolic Riemann surface and $r : \mathbb{D} \to X$ the universal covering such a projection P, if exists, solves the corona problem for $H^\infty(X)$. Indeed, if $f_1, \dots, f_n \in H^\infty(X)$ satisfy (1), then their pullbacks $r^* f_1, \dots, r^* f_n \in H^\infty(\mathbb{D})$ satisfy (1) on $\mathbb{D}$. According to the Carleson corona theorem there exist corona solutions $g_1, \dots, g_n \in H^\infty(\mathbb{D})$ of the equation

$$r^* f_1 \cdot g_1 + \cdots + r^* f_n \cdot g_n \equiv 1 \tag{4}$$

with a bound $C(n^{3/2}\delta^{-2} + n^2\delta^{-3})$ for an absolute constant C, see, e.g., [Ga, Ch. VIII]. Applying to (4) the projection P we solve, due to (3), the Bezout equation (2) on X with the bound on the corresponding solutions $C \cdot \|P\| \cdot (n^{3/2}\delta^{-2} + n^2\delta^{-3})$.

If X is an annulus, then the fibre of the covering $r : \mathbb{D} \to X$ is naturally identified with the group of integers $\mathbb{Z}$. Since this group is amenable and abelian, there exists a nonnegative bounded linear functional Λ of norm one on $\ell^\infty(\mathbb{Z})$ invariant under translation by group elements. Then the Forelli projection P is given by the formula

$$P(f)(z) := \Lambda(f_z), \quad z \in \mathbb{D}, \quad f \in H^\infty(\mathbb{D}),$$

where $f_z(g) := f(g \cdot z)$, $g \in \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{D} \to \mathbb{D}$, $g \times z \mapsto g \cdot z$, is the holomorphic action on $\mathbb{D}$ of the deck transformation group $\mathbb{Z}$ of the covering $r : \mathbb{D} \to X$.

By means of this projection the corona theorem for an annulus was established independently by S. Scheinberg [Sc] and E. L. Stout [St2].

If X is a generic connected noncompact Riemann surface, then the fundamental group $\pi_1(X)$ of X is free with the number of generators ≥ 2. In particular, it is not amenable and the previous averaging method is not applicable to the universal covering $r : \mathbb{D} \to X$ in this case. Nevertheless, for certain Riemann surfaces X projections $P : H^\infty(\mathbb{D}) \to r^*(H^\infty(X))$ satisfying (3) still exist.

Forelli [F] was the first to discover that such projections P exist for X being a *finite bordered Riemann surface*. Later, his construction made explicit by Earle and Marden [EM1, EM2]. Subsequently, existence of Forelli projections for certain infinitely connected Riemann surfaces were established by Carleson [C2] for *complements in* $\mathbb{C}$ *of compact homogeneous subsets* E *of* $\mathbb{R}$ (i.e., for which there exist $\varepsilon > 0$ such that $mes\big((x-r, x+r)\cap E\big) \ge \varepsilon r$ for all $r > 0$ and all $x \in E$) and by Jones and Marshall [JM] for *Riemann surfaces* X *such that critical points of Green's functions on* X *or their preimages in* $\mathbb{D}$ *form an interpolating sequence for* $H^\infty(X)$ *or* $H^\infty(\mathbb{D})$*, respectively*. (In particular, the latter class contains surfaces considered previously in [F] and [C2].) In these approaches the projection can be constructed explicitly by means of a function $h \in H^\infty(\mathbb{D})$ with the properties:

(a) $\hat{h}(z) := \sum_{w\in r^{-1}(z)} |h(w)|$, $z \in X$, is a continuous function on X satisfying $\sup_X \hat{h} \le C < \infty$;
(b) $\sum_{w\in r^{-1}(z)} h(w) = 1$ for all $z \in X$.

Having such a function one defines a Forelli projection $P : H^\infty(\mathbb{D}) \to r^*(H^\infty(X))$ of norm $\|P\| \le C$ by the formula

$$P(f)(y) := \sum_{w\in r^{-1}(z)} h(w)f(w), \quad y \in r^{-1}(z), \quad f \in H^\infty(\mathbb{D}). \tag{5}$$

Note that P induces weak* continuous linear functionals on $\ell^\infty(r^{-1}(z))$, $z \in X$, while in the approach with the invariant mean on the deck transformation group such induced functionals are bounded but not weak* continuous.

In [Br4] existence of weak* continuous Forelli projections of the form (5) was established by the author for a wide class of (not necessarily one-dimensional) complex manifolds. The construction is more abstract than the previous ones and uses some techniques of the theory of coherent Banach sheaves over Stein manifolds developed by L. Bungart [B].

Specifically, it was shown in [Br4, Th. 1.1] that such a projection $P : H^\infty(\tilde{X}) \to r^*(H^\infty(X))$ exists for $\tilde{X}$ and X being unbranched coverings of a relatively compact domain N in a connected Stein manifold M such that inclusion $N \hookrightarrow M$ induces an isomorphism of fundamental groups $\pi_1(N) \cong \pi_1(M)$. Moreover, $\|P\|$ is bounded by a constant depending on N only. In particular, this condition is valid for N being a bordered Riemann surface. Thus, due to the covering homotopy theorem, the weak* continuous Forelli projection $P : H^\infty(\mathbb{D}) \to r^*(H^\infty(X))$ exists for X *being a domain in an unbranched covering* R *of a bordered Riemann surface* N *such that inclusion* $X \hookrightarrow R$ *induces a monomorphism of the fundamental groups*, and $\|P\| \le C(N)$, [Br4, Cor. 1.3].

Here is a simple example of such an X:

Example. Consider the standard action of the group $\mathbb{Z} + i\mathbb{Z}$ on $\mathbb{C}$ by translations. The fundamental domain of this action is the square

$$Q := \{z = x + iy \in \mathbb{C} \;:\; \max\{|x|, |y|\} \le 1\}.$$

By Q_t we denote the square homothetic to Q with sidelength t. Let O be the orbit of $0 \in \mathbb{C}$ with respect to the action of $\mathbb{Z} + i\mathbb{Z}$. For any $x \in O$ we will choose some $t(x) \in [\frac{1}{2}, \frac{3}{4}]$ and consider the square $Q(x) := x + Q_{t(x)}$ centered at x. Let $V \subset \mathbb{C}$ be a simply connected domain satisfying the property:

$$\textit{there exists a subset } \{x_i\}_{i\in I} \subset O \textit{ such that } V \cap \left(\bigcup_{x\in O} Q(x)\right) = \bigcup_{i\in I} Q(x_i).$$

We set $X := V \setminus (\cup_{i\in I} Q(x_i))$. Then X satisfies the required conditions. In fact, the quotient space $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ is a torus $\mathbb{CT}$. Let S be the image of $Q_{1/3}$ in $\mathbb{CT}$. Then X is a domain in the regular covering R of $N := \mathbb{CT} \setminus S$ with the deck transformation group $\mathbb{Z}+i\mathbb{Z}$. The condition that the embedding $X \hookrightarrow R$ induces a monomorphism of fundamental groups follows from the construction of X.

Let us recall that a connected non-parabolic Riemann surface X with a Green function G_o is of *Widom type* if

$$\int_0^\infty b(t)\,dt < \infty,$$

where $b(t)$ is the first Betti number of the set $\{x \in X \; : \; G_o(x) > t\}$. This means that the topology of X grows slowly as measured by the Green function. Widom type surfaces are the only infinitely connected ones for which the Hardy theory has been developed to any extent. They have many bounded holomorphic functions. In particular, such functions separate points and directions. We refer to [Ha] for an exposition. It was noted by Jones and Marshall [JM, p. 295] that using results in [W] and [P] it is possible to show that if the Forelli projection $P : H^\infty(\mathbb{D}) \to r^*(H^\infty(X))$ exists for a connected Caratheodory hyperbolic Riemann surface X, then X is of Widom type.

3 Riemann Surfaces with Corona

The first example of a (non-planar) Riemann surface for which the corona theorem fails was found by B. Cole, see, e.g., [G1]. He constructed a sequence of finite bordered Riemann surfaces $R^{(k)}$ and functions $f_1^{(k)}, f_2^{(k)} \in H^\infty(R^{(k)})$ with $\max_j |f_j^{(k)}(z)| \geq \delta > 0$ for all $z \in R^{(k)}$, but where any solution of $f_1^{(k)} g_1^{(k)} + f_2^{(k)} g_2^{(k)} \equiv 1$ must satisfy $\sup_k(\|g_1^{(k)}\|_\infty + \|g_2^{(k)}\|_\infty) = \infty$. (Recently, modifying Cole's example, B. Oh [O1] constructed explicitly Riemann surfaces with large bounds on corona solutions in an elementary way.) Constructing such a sequence for planar domains is equivalent to the failure of the corona theorem for a planar domain, as shown in [G2]. Later, it was shown in [Nak1] that Cole's example can be modified to be of Widom type. Also, it was observed in [JM] that modifying the construction in Gamelin [G2] it is possible to show that if the corona theorem fails for a planar domain, it must fail for a planar domain of Widom type.

Other examples being found later by Barrett and Diller [BD] and Larusson [L]. The construction of the latter paper is based on an interpolation result for holomorphic functions of exponential growth on unbranched coverings of complex projective manifolds which is proved by means of certain vanishing theorems for L_2 cohomology groups on the coverings. In the case of H^∞ functions this result was strengthen in [Br3] to produce the following:

Let Y be an unbranched covering of a complex connected projective manifold $M \subset \mathbb{CP}^N$ of dimension $n \geq 2$. Let C be intersection with M of at most $n-1$ generic hypersurfaces of degree d in $\mathbb{CP}^N$. By the Lefschetz theorem, the preimage X of C in Y is a connected submanifold. Then the restriction $H^\infty(Y) \to H^\infty(X)$ is an isometry for d large enough.

For instance, this can be applied to Y being either an open Euclidean ball or polydisk in $\mathbb{C}^n$, $n \geq 2$, to find a compact Riemann surface S_n and its regular covering $r : \tilde{S}_n \to S_n$ such that $\tilde{S}_n \hookrightarrow Y$ and this embedding induces an isometry of the H^∞ spaces. Thus maximal ideal spaces $\mathscr{M}(H^\infty(\tilde{S}_n))$ and $\mathscr{M}(H^\infty(Y))$ are homeomorphic. In particular, the covering dimension of $\mathscr{M}(H^\infty(\tilde{S}_n))$ is at least n and the corona theorem fails for $\tilde{S}_n$.

A similar L_2 cohomology technique was used in [Br6] by the author to prove the solvability of the corona problem for H^∞ on unbranched coverings of connected Caratheodory hyperbolic Riemann surfaces of finite type.

4 Projective Freeness and Hermiteness for H^∞ on Riemann Surfaces

In this part we describe a matrix version of the corona problem.

A commutative ring with identity R is said to be *projective free* if every finitely generated projective R-module is free. Recall that if M is an R-module, then

1. M is called *free* if $M \cong R^d$ for some integer $d \geq 0$;
2. M is called *projective* if there exists an R-module N and an integer $d \geq 0$ such that $M \oplus N \cong R^d$.

In terms of matrices, the ring R is projective free iff every square idempotent matrix F is conjugate (by an invertible matrix) to a matrix of the form

$$\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix},$$

see [Co1, Prop. 2.6].

From the matricial definition, it follows that any field $\mathbf{k}$ is projective free, since matrices F satisfying $F^2 = F$ are diagonalizable over $\mathbf{k}$. Quillen and Suslin (see [La]) proved, independently, that the polynomial ring over a projective free ring is again projective free. Thus, the polynomial ring $\mathbf{k}[x_1, \dots, x_n]$ is projective free. Also, if R is any projective free ring, then the formal power series ring $R[[x]]$ in a central indeterminate x is again projective free [Co2, Th. 7]. Hence, the ring of formal power series $\mathbf{k}[[x_1, \dots, x_n]]$ is also projective free.

Less is known about projective freeness of topological rings arising in analysis. For instance, from Grauert's theorem [Gra] one obtains that the ring $H(X)$ of holomorphic functions on a connected reduced Stein space X satisfying the property that any complex vector bundle of finite rank on X is topologically trivial, is projective free (cf. the proof of Theorem 1.2 in [BS]). This is the case if, e.g., the space X is contractible or if it is biholomorphic to a connected noncompact (possibly singular) Riemann surface.

For R being a complex commutative unital Banach algebra, certain topological conditions on its maximal ideal space $\mathscr{M}(R)$ (e.g., its contractibility) implying projective freeness of R are presented in [T2] and [BS]. Also, it was established in [BS, Th. 1.5], by means of some results of [Br5], that $H^\infty(X)$ *is projective free for X being a domain in an unbranched covering S of a bordered Riemann surface N such that inclusion $X \hookrightarrow S$ induces a monomorphism of the fundamental groups* (cf. Example in Sect. 2). As a particular case of this result one obtains that $H^\infty(X)$ is projective free for every connected Caratheodory hyperbolic Riemann surface X of finite type. Earlier projective freeness of $H^\infty(\mathbb{D})$ was proved in [Q] by a different method.

The concept of a *Hermite ring* is weaker than that of a projective free ring.

A commutative ring R with identity is said to be *Hermite* if every finitely generated stably free R-module is free.

Recall that a R-module M is called *stably free* if there exist nonnegative integers n, d such that $M \oplus R^n = R^d$.

Clearly every stably free module is projective, and so every projective free ring is Hermite.

A complex commutative unital Banach algebra R is Hermite if and only if for all $k, n \in \mathbb{N}$, $k < n$, each $k \times n$ matrix with entries in R such that all its minors of order k do not belong together to a maximal ideal of R can be completed to an invertible $n \times n$ matrix with entries in R. Moreover, R is Hermite if and only if the algebra $C(\mathscr{M}(R))$ of complex continuous functions on the maximal ideal space of R is Hermite (see, e.g., [Li, Th. 3], [T2, No, Th. 4], [Ta, pp. 179, 196]).

If X is a Riemann surface for which the corona theorem is valid, Hermiteness of $H^\infty(X)$ is equivalent to the following statement (which is stronger than just the solvability of the Bezout equation (2) in the corona theorem for $H^\infty(X)$):

For all $k, n \in \mathbb{N}$, $k < n$, each $k \times n$ matrix $A_{k,n}$ with entries in $H^\infty(X)$ such that all its minors of order k satisfy the corona condition (1) *can be completed to an $n \times n$ matrix $\tilde{A}_{n,n}$ of determinant one with entries in $H^\infty(X)$.*

Hermiteness of $H^\infty(\mathbb{D})$, with estimates of norm $\sup_{z\in\mathbb{D}} \|\tilde{A}_{n,n}(z)\|_{\ell_2^n \to \ell_2^n}$ of the extension $\tilde{A}_{n,n}$ of $A_{k,n}$ depending on δ (in the corona condition for minors of order k of $A_{k,n}$) and k, n only, was first proved by V. Tolokonnikov [T1, Th. 4]. Later, in [T2] he proved Hermiteness of $H^\infty(X)$, where *X is a bordered Riemann surface*, with similar estimates of the norm of the extension $\tilde{A}_{n,n}$ of $A_{k,n}$. The proof is based on the Beurling-Lax-Halmos [Beu,Lax,NF], Forelli [F] and Grauert [Gra] theorems. Recently, in [Br5] Hermiteness of $H^\infty(X)$ was proved by the author for X *being a domain in an unbranched covering S of a bordered Riemann surface N such that inclusion $X \hookrightarrow S$ induces a monomorphism of the fundamental groups*. The proof uses analogs of the Beurling-Lax-Halmos theorem [Br5, Th. 1.7], Grauert theorem [Br5, Th. 1.5] and Forelli theorem [Br4] and allows to estimate the norm of the extension $\tilde{A}_{n,n}$ of $A_{k,n}$ by a constant depending on the corresponding data δ, k, n for $A_{k,n}$ and on N only (in particular, this extends the result of [T2]). Finally, in [Br7] using a topological method, Hermiteness of $H^\infty(X)$ was proved by the author for X *being an unbranched covering of a connected Caratheodory hyperbolic Riemann surface of finite type N* with the norm of the extension $\tilde{A}_{n,n}$ of $A_{k,n}$ bounded by a constant depending on the corresponding data δ, k, n for $A_{k,n}$ and on N only.

5 Remarks about Bounds on the Corona Solutions

All known examples of Caratheodory hyperbolic Riemann surfaces X for which the corona theorem is valid allow bounds on the corona solutions of Bezout equations (2) depending on n and δ (in the corona condition (1) on X), and some inessential parameters only. For such X this, in particular, implies much more stronger corona theorems.

Indeed, let $\mathscr{X} := \{X_s\}_{s\in S}$ be the family of Caratheodory hyperbolic Riemann surfaces. We define $\mathscr{X}_S := \sqcup_{s\in S} X_s$. Then $\mathscr{X}_S$ is a one-dimensional complex manifold (not necessarily second-countable). The Banach algebra $H^\infty(\mathscr{X}_S)$ of bounded holomorphic functions on $\mathscr{X}_S$ is well defined and separates the points of $\mathscr{X}_S$. It is easy to see that

If for each $n \in \mathbb{N}$ and all n-tuples of functions in $H^\infty(X_s)$, $s \in S$, satisfying corona conditions (1)*, Bezout equations* (2) *with these functions admit solutions bounded by a nonnegative function in variables n and δ, then the corona theorem is valid for $\mathscr{X}_S$*, that is, $i(\mathscr{X}_S)$ is dense in the maximal ideal space $\mathscr{M}(H^\infty(\mathscr{X}_S))$ (here $i(\mathscr{X}_S)$ consists of the evaluation homomorphisms at points of $\mathscr{X}_S$).

This is valid, e.g., for a family $\mathscr{X} := \{X_s\}_{s\in S}$, where each X_s is either $\mathbb{D}$ or an open annulus. In turn, if each X_s is a domain in an unbranched covering R of a (fixed) bordered Riemann surface N such that inclusion $X \hookrightarrow R$ induces a monomorphism of the fundamental groups, then $H^\infty(\mathscr{X}_S)$ is Hermite, see estimates in [Br5]. The same is true if each X_s is an unbranched covering of a fixed Caratheodory hyperbolic Riemann surface of finite type, see [Br7]. Further, if all $X_s = \mathbb{D}$ in the definition of $\mathscr{X}$, then estimates obtained in [Tr4] show that $H^\infty(\mathscr{X}_S)$ has stable rank one, etc.

Note that for $\mathscr{X}_S$ being the disjoint union of all bounded finitely connected domains in $\mathbb{C}$, the corona problem is still open and is equivalent to the general corona problem for planar domains, see, e.g., [G2, Han, JM, Nak1, ND] and references therein. In connection with that problem, it might be of interest to study analogs of Gleason parts of points in $\mathscr{M}(H^\infty(\mathscr{X}_S))$ (defined similarly to those in $\mathscr{M}(H^\infty(\mathbb{D}))$, see Sect. 7 below).

6 Operator Corona Problem

Let $H^\infty(L(X,Y))$ be the Banach space of holomorphic functions F on $\mathbb{D}$ with values in the space of bounded linear operators $X \to Y$ between complex Banach spaces X, Y with norm $\|F\| := \sup_{z\in\mathbb{D}} \|F(z)\|_{L(X,Y)}$. As usual, $L(X) := L(X,X)$; by I_X we denote the identity operator $X \to X$.

The operator corona problem is a tentative to generalize the Carleson corona theorem to operator valued functions. It was posed by Sz.-Nagy in 1978 in the following form [Na]:

Suppose that $F \in H^\infty(L(H_1,H_2))$, H_1, H_2 are separable Hilbert spaces, satisfies $\|F(z)x\| \geq \delta\|x\|$ for all $x \in H_1$, $z \in \mathbb{D}$, where $\delta > 0$ is a constant. Does there exist $G \in H^\infty(L(H_2,H_1))$ such that $G(z)F(z) = I_{H_1}$ for all $z \in \mathbb{D}$?

This problem is of importance in operator theory (angles between invariant subspaces, unconditionally convergent spectral decompositions) and in control theory (the stabilization problem). It is also related to the study of submodules of H^∞ and to many other subjects of analysis, see [Ni1,Ni2,Tr2,Tr5,Vi] and references therein.

Obviously, the condition imposed on F is necessary since it implies existence of a uniformly bounded family of left inverses of $F(z)$, $z \in \mathbb{D}$. The question is whether it is sufficient for the existence of a bounded holomorphic left inverse of F.

In general, as it was shown by S. Treil, the answer is negative (see [Tr1, Tr3, Tr6, TW] and references therein). But in some specific cases it is positive. In particular, the Carleson corona theorem [C1] stating that for $\{f_i\}_{i=1}^n \subset H^\infty(\mathbb{D})$ the Bezout equation $\sum_{i=1}^n g_i f_i \equiv 1$ is solvable with $\{g_i\}_{i=1}^n \subset H^\infty$ as soon as $\max_{1\leq i\leq n} |f_i(z)| > \delta > 0$ for every $z \in \mathbb{D}$ means that the answer is positive when $\dim H_1 = 1$, $\dim H_2 = n < \infty$. Later, using some ideas from the T. Wolff's proof of the Carleson corona theorem, M. Rosenblum [R], V. Tolokonnikov [T1] and Uchiyama [U] independently proved that the operator corona problem is solvable if $\dim H_1 = 1$, $\dim H_2 = \infty$. Using a simple linear algebra argument, P. Fuhrmann [Fu] proved that the operator corona problem is solvable if $\dim H_1, \dim H_2 < \infty$, and further V. Vasyunin (see [T1] for the proof) extended this result to the case $\dim H_2 = \infty$ (but still $\dim H_1 < \infty$). Recently, it was established by S. Treil in [Tr7] that the operator corona problem is solvable as soon as F is a "small" perturbation of a left invertible function $F_0 \in H^\infty(L(H_1,H_2))$ (for example, if $F - F_0$ belongs to $H^\infty(L(H_1,H_2))$ with values in the class of Hilbert Schmidt operators).

For a long time there were no positive results in the case $\dim H_1 = \infty$. The first positive results were obtained by P. Vitse in [Vi] where the following more general problem was studied.

Let X_1, X_2 be complex Banach spaces and $F \in H^\infty(L(X_1,X_2))$ be such that for each $z \in \mathbb{D}$ there exists a left inverse G_z of $F(z)$ satisfying $\sup_{z\in\mathbb{D}} \|G_z\| < \infty$. *Does there exist $G \in H^\infty(L(X_2,X_1))$ such that $G(z)F(z) = I_{X_1}$ for all $z \in \mathbb{D}$?*

Since in this general setting the answer is negative, it was suggested in [Vi] to investigate the problem for the case of $F \in H^\infty_{\mathrm{comp}}(L(X_1,X_2))$, the *space of* *holomorphic functions on $\mathbb{D}$ with relatively compact images in* $L(X_1, X_2)$. (The notation $H^\infty_{\rm comp}$ has been introduced in this paper as well.) In particular, it was shown, see [Vi, Th. 2.1], that the answer is positive for F that can be uniformly approximated by finite sums $\sum f_k(z)L_k$, where $f_k \in H^\infty(\mathbb{D})$ and $L_k \in L(X_1, X_2)$. The question of whether each $F \in H^\infty_{\rm comp}(L(X_1, X_2))$ can be obtained in that form is closely related to the still open problem about the Grothendieck approximation property for $H^\infty(\mathbb{D})$. (The strongest result in this direction [BR, Th. 9] states that $H^\infty(\mathbb{D})$ has the approximation property "up to logarithm".)

Another, not involving the approximation property for $H^\infty(\mathbb{D})$, approach was proposed by the author and led to the solution of the above problem for $H^\infty_{\rm comp}$ spaces, i.e., for $F \in H^\infty_{\rm comp}(L(X_1, X_2))$ satisfying the hypothesis of the problem the required left inverse G of F exists and belongs to $H^\infty_{\rm comp}(L(X_2, X_1))$ (see [Br8, Th. 1.5]).

In fact, in [Br8] a particular case of the following *operator completion problem* was investigated (compare with Hermiteness of $H^\infty(\mathbb{D})$):

Let X_1, X_2 be complex Banach spaces and $F \in H^\infty(L(X_1, X_2))$ be such that for each $z \in \mathbb{D}$ there exists a left inverse G_z of $F(z)$ satisfying $\sup_{z\in\mathbb{D}} \|G_z\| < \infty$. *Do there exist functions $H \in H^\infty(L(X_1 \oplus Y, X_2))$ and $G \in H^\infty(L(X_2, X_1 \oplus Y))$, $Y := \operatorname{Ker} G_0$, such that $H(z)G(z) = I_{X_2}$, $G(z)H(z) = I_{X_1\oplus Y}$ and $H(z)|_{X_1} = F(z)$ for all $z \in \mathbb{D}$.*

Seemingly much stronger, this problem is equivalent to the operator corona problem for X_1, X_2 being separable Hilbert cases. This result, known as the Tolokonnikov lemma, is proved in full generality by S. Treil in [Tr7]. (Thus in this case the operator completion problem has a positive solution as soon as the operator corona problem has it.)

In general, the operator completion problem is much more involved. Some sufficient conditions for its solvability for $H^\infty_{\rm comp}$ spaces were given by the author in [Br8, Th. 1.3]. Specifically, it was shown that

If $F \in H^\infty_{\rm comp}(L(X_1, X_2))$ satisfies the conditions of the problem and the group $GL(Y)$ of invertible elements of $L(Y)$ is connected, then the required H and G exist and belong to $H^\infty_{\rm comp}(L(X_1 \oplus Y, X_2))$ and $H^\infty_{\rm comp}(L(X_2, X_1 \oplus Y))$, respectively.

For instance, this is valid in one of the following cases (see [Br10, Mi]):

(a) $\dim_{\mathbb{C}} Y < \infty$; (b) X_2 is isomorphic to a Hilbert space or c_0 or one of the spaces ℓ^p, $1 \le p \le \infty$; (c) X_2 is isomorphic to one of the spaces $L^p[0, 1]$, $1 < p < \infty$, or $C[0, 1]$ and X_1 is not isomorphic to X_2.

Moreover, it is proved by the author in [Br10, Th. 2.5] that if $GL(Y)$ is not connected, then the operator completion problem is not solvable in the class of $H^\infty_{\rm comp}$ spaces.

Proofs of these results are based on an analog of the Cartan-Oka theory for Banach-valued holomorphic functions on the maximal ideal space $\mathscr{M}(H^\infty(\mathbb{D}))$ developed by the author in [Br9]. In this approach one uses some topological properties of this space described in the next section.

7 Topology of the Maximal Ideal Space of H^∞

Recall that the pseudohyperbolic metric on $\mathbb{D}$ is defined by

$$\rho(z,w) := \left|\frac{z-w}{1-\bar{w}z}\right|, \qquad z,w \in \mathbb{D}.$$

For $x, y \in \mathscr{M}(H^\infty(\mathbb{D}))$ the formula

$$\rho(x,y) := \sup\{|\hat{f}(y)|\,;\ f \in H^\infty(\mathbb{D}),\ \hat{f}(x)=0,\ \|f\| \le 1\}$$

gives an extension of ρ to $\mathscr{M}(H^\infty(\mathbb{D}))$ (here $\hat{}: H^\infty(\mathbb{D}) \to C(\mathscr{M}(H^\infty(\mathbb{D})))$ stands for the *Gelfand transform*). The *Gleason part* of $x \in \mathscr{M}(H^\infty(\mathbb{D}))$ is then defined by $\pi(x) := \{y \in \mathscr{M}(H^\infty(\mathbb{D}))\,;\ \rho(x,y) < 1\}$. For $x, y \in \mathscr{M}(H^\infty(\mathbb{D}))$ we have $\pi(x) = \pi(y)$ or $\pi(x) \cap \pi(y) = \emptyset$. Hoffman's classification of Gleason parts [H] shows that there are only two cases: either $\pi(x) = \{x\}$ or $\pi(x)$ is an analytic disk. The latter case means that there is a continuous one-to-one and onto map $L_x : \mathbb{D} \to \pi(x)$ such that $\hat{f} \circ L_x \in H^\infty(\mathbb{D})$ for every $f \in H^\infty(\mathbb{D})$. Moreover, any analytic disk is contained in a Gleason part and any maximal (i.e., not contained in any other) analytic disk is a Gleason part. By M_a and M_s we denote the sets of all non-trivial (analytic disks) and trivial (one-pointed) Gleason parts, respectively. It is known that $M_a \subset \mathscr{M}(H^\infty(\mathbb{D}))$ is open. Hoffman proved that $\pi(x) \subset M_a$ if and only if x belongs to the closure of some interpolating sequence in $\mathbb{D}$.

In [S1] D. Suárez established that the covering dimension of $\mathscr{M}(H^\infty(\mathbb{D}))$ is 2 and the Čech cohomology group $H^2(\mathscr{M}(H^\infty(\mathbb{D}))) = 0$ (the latter also follows directly from projective freeness of $H^\infty(\mathbb{D})$). One of the important ingredients of his proof is a theorem of S. Treil [Tr4] stating that the stable rank of $H^\infty(\mathbb{D})$ is one. Later, in [S2] D. Suárez proved, in addition, that the set M_s of trivial Gleason parts of $\mathscr{M}(H^\infty(\mathbb{D}))$ is totally disconnected (see, e.g., [Nag] for basic topological definitions). The proof uses a modified version of the construction of Garnett and Nicolau [GN] who proved that the interpolating Blaschke products generate $H^\infty(\mathbb{D})$ as a uniform algebra.

Further, in [Br2] the author described the set M_a as a fibre bundle over a fixed compact Riemann surface S of genus ≥ 2 with fibre an open dense subset of the Stone Čech compactification of the fundamental group of S (in particular, this shows that each bounded uniformly continuous with respect to the metric ρ on $\mathbb{D}$ function is extended to a continuous function on M_a). This and the result of [S2] have been used in [Br8, Br9] in the proofs of the basic facts of the theory of Banach-valued holomorphic functions on $\mathscr{M}(H^\infty(\mathbb{D}))$.

It is worth noting that one can use that the covering dimension of $\mathscr{M}(H^\infty(\mathbb{D}))$ is 2 and $H^2(\mathscr{M}(H^\infty(\mathbb{D}))) = 0$ to give another proofs of projective freeness and Hermiteness of $H^\infty(X)$ for X being a Riemann surface of finite type (see, e.g., [Br1, BS, T2]).

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References

[A1] N. L. Alling, A proof of the corona conjecture for finite open Riemann surfaces. Bull. Amer. Math. Soc. **70** (1964), 110–112.

[A2] N. L. Alling, Extensions of meromorphic function rings over non-compact Riemann surfaces, I. Math. Z. **89** (1965), 273–299.

[BD] D. E. Barrett, and J. Diller, A new construction of Riemann surfaces with corona. J. Geom. Anal. **8** (1998), 341–347.

[Be1] M. Behrens, On the corona problem for a class of infinitely connected domains. Bull. Amer. Math. Soc. **76** (1970), 387–391.

[Be2] M. Behrens, The maximal ideal space of algebras of bounded analytic functions on infinitely connected domains. Trans. Amer. Math. Soc. **161** (1971), 358–380.

[Beu] A. Beurling, On two problems concerning linear transformations in Hilbert space. Acta Math. **81** (1948), 17.

[BR] J. Bourgain and O. Reinov, On the approximation properties for the space H^∞. Math. Nachr. **122** (1983) 19–27.

[Br1] A. Brudnyi, Matrix-valued corona theorem for multiply connected domains. Indiana Univ. Math. J. **49** (4) (2000), 1405–1410.

[Br2] A. Brudnyi, Topology of the maximal ideal space of H^∞. J. Funct. Anal. **189** (2002), 21–52.

[Br3] A. Brudnyi, A uniqueness property for H^∞ on coverings of projective manifolds. Michigan Math. J. **51** (2003), no. 3, 503–507.

[Br4] A. Brudnyi, Projections in the space H^∞ and the Corona Theorem for coverings of bordered Riemann surfaces. Ark. Mat. **42** (2004), no. 1, 31–59.

[Br5] A. Brudnyi, Grauert and Lax-Halmos type theorems and extension of matrices with entries in H^∞. J. Funct. Anal. **206** (2004), 87–108.

[Br6] A. Brudnyi, Corona Theorem for H^∞ on Coverings of Riemann Surfaces of Finite Type. Michigan Math. J. **56** (2008), 283–299.

[Br7] A. Brudnyi, Extension of matrices with entries in H^∞ on coverings of Riemann surfaces of finite type. St. Petersburg J. Math. **21** (2009), no. 3, 79–92.

[Br8] A. Brudnyi, Holomorphic Banach vector bundles on the maximal ideal space of H^∞ and the operator corona problem of Sz.-Nagy. Adv. Math. **232** (2013), 121–141.

[Br9] A. Brudnyi, Stein-like theory for Banach-valued holomorphic functions on the maximal ideal space of H^∞. Invent. math. **193** (2013), 187–227.

[Br10] A. Brudnyi, On the completion problem for algebra H^∞, J. Func. Anal. **266** (2014), 4293–4313.

[BS] A. Brudnyi, and A. Sasane, Sufficient conditions for the projective freeness of Banach algebras. J. Funct. Anal. **257** (2009), 4003–4014.

[B] L. Bungart, On analytic fibre bundles I. Holomorphic fibre bundles with infinite dimensional fibres. Topology **7** (1968), 55–68.

[C1] L. Carleson, Interpolation of bounded analytic functions and the corona problem. Ann. of Math. **76** (1962), 547–559.

[C2] L. Carleson, On H^∞ in multiply connected domains. In Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol. II (Beckman, W., Calderón, A. P., Fefferman, R. and Jones. P. W., eds.), pp. 349–372, Wadsworth, Belmont, Calif., 1983.

[Co1] P.M. Cohn, *Free rings and their relations.* Second edition. London Mathematical Society Monographs 19, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1985.

[Co2] P.M. Cohn, Some remarks on projective free rings. Dedicated to the memory of Gian-Carlo Rota. Algebra Universalis, no.2, **49** (2003), 159–164.

[D] W. Deeb, A class of infinitely connected domain and the corona. Trans. Amer. Math. Soc. **31** (1977), 101–106.

[DW] W. Deeb and D.R. Wilken, Δ domains and the corona.Trans. Amer. Math. Soc. **231** (1977), 107–115.

[EM1] C. J. Earle and A. Marden, Projections to automorphic functions. Proc. Amer. Math. Soc. **19** (1968), 274–278.

[EM2] C. J. Earle and A. Marden, On Poincaré series with application to H^p spaces on bordered Riemann surfaces. Illinois J. Math. **13** (1969), 202–219.

[F] F. Forelli, Bounded holomorphic functions and projections. Illinois J. Math. **10** (1966), 367–380.

[Fu] P. A. Fuhrmann, On the corona theorem and its application to spectral problems in Hilbert space. Trans. Amer. Math. Soc. **132** (1968), 55–66.

[G1] T. W. Gamelin, *Uniform algebras and Jensen measures*. London Math. Soc. Lecture Notes Series **32**. Cambridge Univ. Press, Cambridge-New York, 1978.

[G2] T. W. Gamelin, Localization of the corona problem. Pacific J. Math. **34** (1970), 73–81.

[Ga] J. B. Garnett, *Bounded analytic functions*. Academic Press, 1981.

[GJ] J. B. Garnett and P. W. Jones, The corona theorem for Denjoy domains. Acta Math. **155** (1985), 27–40.

[GN] J. B. Garnett and A. Nicolau, Interpolating Blaschke products generate H^∞. Pacific J. Math. **173** no. 2 (1996), 501–510.

[Gra] H. Grauert, Analytische Faserungen über holomorph-vollständigen Räumen. Math. Ann. **135** (1958), 263–273.

[Han] J. Handy, The corona theorem on the complement of certain square Cantor sets. J. Anal. Math. **108** (1) (2009), 1–18.

[HN] M. Hara and M. Nakay, Corona theorem with bounds for finitely sheeted disks. Tohoku Math. J. **37** (1985), 225–240.

[Ha] M. Hasumi, *Hardy classes on infinitely connected Riemann surfaces*. Lecture Notes in Math. **1027**, Springer-Verlag, Berlin Heidelberg, 1983.

[H] K. Hoffman, Bounded analytic functions and Gleason parts. Ann. of Math. **86** (1967), 74–111.

[Ho] L. Hörmander, Generators for some rings of analytic functions. Bull. Amer. Math. Soc. **73** (1967), 943–949.

[JM] P. W. Jones and D. Marshall, Critical points of Green's functions, harmonic measure and the corona theorem. Ark. Mat. **23** (1985), 281–314.

[La] T.Y. Lam, *Serre's conjecture*. Lecture Notes in Mathematics **635**, Springer-Verlag, Berlin-New York, 1978.

[L] F. Lárusson, Holomorphic functions of slow growth on nested covering spaces of compact manifolds. Canad. J. Math. **52** (2000), 982–998.

[Lax] P. D. Lax, Translation invariant spaces. Acta Math. **101** (1959), 163–178.

[Li] V. Lin, Holomorphic fibering and multivalued functions of elements of a Banach algebra. Funct. Anal. and its Appl. English translation, **7** (2) (1973), 122–128.

[Mi] B. S. Mityagin, The homotopy structure of the linear group of a Banach space. Russ. Math. Surv. **25** (5) (1970), 59–103.

[M] C. N. Moore, The corona theorem for domains whose boundary lies in a smooth curve. Proc. Amer. Math. Soc. **100** (1987), no. 2, 266–270.

[Nag] K. Nagami, *Dimension theory*. Academic Press. New York, 1970.

[Na] B. Sz.-Nagy, A problem on operator valued bounded analytic functions. Zap. Nauchn. Sem. LOMI, **81**(1978), 99.

[NF] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*. Translated from the French and revised. North-Holland Publishing Co., Amsterdam, 1970.

[Nak1] M. Nakai, Corona problem for Riemann surfaces of Parreau-Widom type. Pacific J. Math. **103** (1982), 103–109.

[Nak2] M. Nakay, The corona problem on finitely sheeted covering surfaces. Nagoya Math. J. **92** (1983), 163–173.

[Nar] J. Narita, A remark on the corona problem for plane domains. J. Math. Kyoto Univ. **25** (1985), 293–298.

[N] D. J. Newman, Interpolation in $H^\infty(\mathbb{D})$. Trans. Amer. Math. Soc. **92** no. 2 (1959), 502–505.

[ND] B. M. NewDelman, Homogeneous subsets of a Lipschitz graph and the Corona theorem. Publ. Mat. **55** (2011), 93–121.

[Ni1] N. K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 1: Hardy, Hankel, and Toeplitz*. Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002, Translated from the French by Andreas Hartmann.

[Ni2] N. K. Nikolski, *Treatise on the shift operator*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer–Verlag, Berlin, 1986, Spectral function theory, With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller, Translated from the Russian by Jaak Peetre.

[No] M. E. Novodvorskii, On some homotopic invariants of the maximal ideals space. Math. Notes **1** (1967), 487–494. (Russian)

[O1] B. Oh, An explicit example of Riemann surfaces with large bound on corona solutions. Pacific J. Math. **228** (2006), 297–304.

[O2] B. Oh, A short proof of Hara and Nakai's theorem. Proc. Amer. Math. Soc. **136** (2008), 4385–4388.

[P] Ch. Pommerenke, On the Green's function of Fuchsian groups. Ann. Acad. Sci. Fenn. **2** (1976), 409–427.

[Q] A. Quadrat, The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. II. Internal stabilization. SIAM J. Control Optim. **42** no. 1 (2003), 300–320.

[R] M. Rosenblum, A corona theorem for countably many functions. Integral Equations Operator Theory **3** (1980), 125–137.

[Sc] S. Scheinberg, Thesis. Princeton University, 1962.

[St1] E. L. Stout, Two theorems concerning functions holomorphic on multiply connected domains. Bull. Amer. Math. Soc. **69** (1963), 527–530.

[St2] E. L. Stout, Bounded holomorphic functions on finite Riemann surfaces. Trans. Amer. Math. Soc. **120** (1965), 255–285.

[St3] E. L. Stout, On some algebras of analytic function on finite open Riemann surfaces. Math. Z. **92** (1966), 366–379. Corrections to: On some algebras of analytic function on finite open Riemann surfaces. Math. Z. **95** (1967), 403–404.

[S1] D. Suárez, Čech cohomology and covering dimension for the H^∞ maximal ideal space. J. Funct. Anal. **123** (1994), 233–263.

[S2] D. Suárez, Trivial Gleason parts and the topological stable rank of H^∞. Amer. J. Math. **118** (1996), 879–904.

[Ta] J. L. Taylor, Topological invariants of the maximal ideal space of a Banach algebra. Adv. Math. **19** (1976), 149–206.

[T1] V. Tolokonnikov, Estimates for Carleson Corona theorem, ideals of algebra H^∞, Sekefalvi-Nagy problem. Zap. Nauch. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **113** (1981), 178–190. (Russian)

[T2] V. Tolokonnikov, Extension problem to an invertible matrix. Proc. Amer. Math. Soc. **117** (1993), no. 4, 1023–1030.

[Tr1] S. Treil, Angles between co-invariant subspaces, and the operator corona problem. The Szökefalvi-Nagy problem. Dokl. Akad. Nauk SSSR **302** (1988), 1063–1068. (Russian)

[Tr2] S. R. Treil, Geometric methods in spectral theory of vector-valued functions: some recent results, Toeplitz operators and spectral function theory. Oper. Theory Adv. Appl., vol. **42**, Birkhäuser, Basel, 1989, pp. 209–280.

[Tr3] S. Treil, *Geometric methods in spectral theory of vector-valued functions: Some recent results*. In Toeplitz Operators and Spectral Function Theory, Oper. Theory Adv. Appl., vol. 42, Birkhäuser Verlag, Basel 1989, pp. 209–280.
[Tr4] S. Treil, The stable rank of H^∞ equals 1. J. Funct. Anal. **109** (1992), 130–154.
[Tr5] S. R. Treil, Unconditional bases of invariant subspaces of a contraction with finite defects. Indiana Univ. Math. J. **46** (1997), no. 4, 1021–1054.
[Tr6] S. Treil, Lower bounds in the matrix corona theorem and the codimension one conjecture. GAFA **14** 5 (2004), 1118–1133.
[Tr7] S. Treil, An operator Corona theorem. Indiana Univ. Math. J. **53** (2004), no. 6, 1763–1780.
[TW] S. Treil and B. Wick, Analytic projections, Corona Problem and geometry of holomorphic vector bundles. J. Amer. Math. Soc. **22** (2009), no. 1, 55–76.
[U] A. Uchiyama, Corona theorems for countably many functions and estimates for their solutions, (1980) (preprint).
[Vi] M. Vidyasagar, *Control system synthesis: a factorization approach*. MIT Press Series in Signal Processing, Optimization, and Control **7**, MIT Press, Cambridge, MA, 1985.
[W] H. Widom, H^p sections of vector bundles over Riemann surfaces. Ann. of Math. **94** (1971), 304–324.

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