

Remarks on Stability of the Equation of Homomorphism for Square Symmetric Groupoids

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Abstract Let $(G, \star)$ and $(H, \circ)$ be square symmetric groupoids and $S \subset G$ be nonempty. We present some remarks on stability of the following conditional equation of homomorphism

$$f(x \star y) = f(x) \circ f(y) \quad x, y \in S, x \star y \in S,$$

in the class of functions mapping S into H . In particular, we consider the situation where $H = \mathbb{R}$ and

$$-v(x, y) \leq h(x \star y) - h(x) \circ h(y) \leq \mu(x, y) \quad x, y \in S, x \star y \in S,$$

with some functions $\mu, v : S^2 \rightarrow [0, \infty)$.

Keywords Hyers-Ulam stability · Square symmetric groupoids · Homomorphism · Fixed point · Complete metric · Linear equation

1 Introduction

The issue of stability of functional equations has been a very popular subject of investigations for the last nearly 50 years (see, e.g., [6, 11, 23, 25, 31, 32, 35, 36]). The main motivation for it was given by S.M. Ulam (cf. [58]) in 1940 in his talk at the University of Wisconsin, where he presented some unsolved problems and, in particular, the following one.

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Let G_1 be a group and (G_2, d) a metric group. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if $f : G_1 \rightarrow G_2$ satisfies

$$d(f(xy), f(x)f(y)) < \delta \quad x, y \in G_1,$$

then a homomorphism $T : G_1 \rightarrow G_2$ exists with

$$d(f(x), T(x)) < \varepsilon \quad x, y \in G_1?$$

In 1941 D.H. Hyers [21] published a partial answer to it, which can be stated as follows.

Let E and Y be Banach spaces and $\varepsilon > 0$. Then, for every $g : E \rightarrow Y$ with

$$\sup_{x, y \in E} \|g(x + y) - g(x) - g(y)\| \leq \varepsilon,$$

there is a unique solution $f : E \rightarrow Y$ of the Cauchy equation

$$f(x + y) = f(x) + f(y) \tag{1}$$

such that

$$\sup_{x \in E} \|g(x) - f(x)\| \leq \varepsilon.$$

We can describe the result of Hyers by simply saying that *the Cauchy functional Eq. (1) is Hyers-Ulam stable (or has the Hyers-Ulam stability)*. At the moment it is well known (see, e.g., [23]) that in the Hyers result it is enough to assume that $(E, +)$ is an amenable semigroup or a square symmetric groupoid (either of those assumptions is fulfilled when $(E, +)$ is a commutative semigroup).

In the next few years Hyers and Ulam published some further stability results for polynomial functions, isometries, and convex functions in [22, 26–28].

We should mention here that now we are aware of an earlier result concerning stability of functional equations that is due to Gy. Pólya and G. Szegő [40, Teil I, Aufgabe 99] (see also [41, Part I, Chap. 3, Problem 99] and [18, p. 125]) and reads as follows ($\mathbb{N}$ stands for the set of positive integers).

For every real sequence $(a_n)_{n \in \mathbb{N}}$ such that

$$\sup_{n, m \in \mathbb{N}} |a_{n+m} - a_n - a_m| \leq 1,$$

there is a real number ω with $\sup_{n \in \mathbb{N}} |a_n - \omega n| \leq 1$. Moreover,

$$\omega = \lim_{n \rightarrow \infty} \frac{a_n}{n}.$$

A next significant result for the Cauchy equation was obtained by T. Aoki [1], who proved the subsequent (for another extension of the result of Hyers see [55]).

Theorem 1 Assume that E_1 and E_2 are two normed spaces, E_2 is complete, $c \geq 0$ and $0 < p < 1$. Let $f : E_1 \rightarrow E_2$ be a mapping such that

$$\|f(x + y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p) \quad x, y \in E_1. \quad (2)$$

Then there exists a unique solution $T : E_1 \rightarrow E_2$ of (1) with

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{1 - 2^{p-1}} \quad x \in E_1. \quad (3)$$

Unfortunately that result had remained unnoticed by a wider audience for quite a long time. In 1978 it was rediscovered by Th.M. Rassias [47], who came across it while solving a problem communicated to him by Ulam (evidently not familiar with the Aoki outcome, as well); moreover, Th.M. Rassias [47] obtained the linearity of T under the assumption of continuity of f and thus provided an outcome on stability of linear mappings and a method of proving it. In this way and through his numerous further publications (see, e.g., [12, 24, 25, 29, 49–54]) Th.M. Rassias strongly stimulated investigations of that kind of stability (for further information and references see, e.g., [6, 23, 31, 32]). In recognition of this, the results obtained in [47] are quite often referred to as the Hyers-Ulam-Rassias stability of the Cauchy equation (cf., e.g., [16, 31–32]).

In [49] (see also [53, p. 326]) it has been noticed that a result analogous to that of Theorem 1 is also valid for $p < 0$. Next, motivated by the problem raised by Th.M. Rassias (see [48]), Z. Gajda [15] extended the result contained in Theorem 1 for $p > 1$ and gave an example that for $p = 1$ it is not valid (for further such examples see [53]; some complementary result for the case $p = 1$ are given in [17, 56]). Recently, it has been proved in [8] (see also [39]) that, for $p < 0$, each function $f : E_1 \rightarrow E_2$ satisfying (2) (but of course only for $x \neq 0$ and $y \neq 0$) must be additive and the completeness of E_2 is not necessary then.

All those stability results finally yield the following theorem.

Theorem 2 Let E_1 and E_2 be two normed spaces, $c \geq 0$ and $p \neq 1$ be fixed real numbers. Let $f : E_1 \rightarrow E_2$ be a mapping such that

$$\|f(x + y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p) \quad x, y \in E_1 \setminus \{0\}. \quad (4)$$

If $p \geq 0$ and E_2 is complete, then there exists a unique additive function $T : E_1 \rightarrow E_2$ with

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{|2^{p-1} - 1|} \quad x \in E_1 \setminus \{0\}. \quad (5)$$

If $p < 0$, then f is additive.

It has been proved in [7] that estimation (5) is optimum for $p \geq 0$ in the general case. Further extensions and generalizations of the idea of stability, described above, have been proposed in [4, 5, 16, 20, 45, 46] (for more information we refer, e.g., to [6, 11, 13, 23, 31]), where the authors studied mappings f satisfying the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y) \quad (6)$$

for functions φ of some other or more general form than in condition (4). In particular, the function φ of the form

$$\varphi(x, y) \equiv c(\|x\|^p \|y\|^q)$$

has been used in [45, 46], with suitable real c, p, q .

2 An Auxiliary Result

In the sequel we use in the proofs some kind of fixed point approach (for a survey on related results we refer to [10]). The following lemma is very simple, but nevertheless very useful in that ($\mathbb{R}_+$ denotes the set of nonnegative reals).

Lemma 1 *Assume that $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is nondecreasing and*

$$\sum_{n=0}^{\infty} \Gamma^n(t) < \infty \quad t \in \mathbb{R}_+. \quad (7)$$

Then the following three statements are valid.

(a) $\Gamma(0) = 0$.

(b) Γ is continuous at 0 or, for each $t \in \mathbb{R}_+$, there is $m \in \mathbb{N}$ such that

$$\Gamma^n(t) = 0 \quad n > m.$$

(c) If Γ is subadditive, i.e.,

$$\Gamma(t + s) \leq \Gamma(t) + \Gamma(s) \quad t, s \in \mathbb{R}_+,$$

then

$$\Gamma\left(\sum_{n=k}^{\infty} \Gamma^n(t)\right) \leq \sum_{n=k+1}^{\infty} \Gamma^n(t) \quad k \in \mathbb{N}_0, t \in \mathbb{R}_+. \quad (8)$$

Proof The proof seems to be a routine by now, but for the convenience of readers we present it here.

Since Γ is nondecreasing, we have

$$\Gamma(0) \leq \Gamma^n(0) \quad n \in \mathbb{N}.$$

Consequently, it is easily seen that (a) is a consequence of (7).

Next, for the proof of (b), assume that there are $t \in \mathbb{R}_+$ and a sequence $(k_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that

$$\lim_{n \rightarrow \infty} k_n = \infty$$

and

$$\Gamma^{k_n}(t) > 0.$$

Suppose that Γ is not continuous at 0, i.e., according to (a) there is $d > 0$ with

$$\Gamma(c) > d \quad c > 0.$$

Then

$$\Gamma^{k_n+1}(t) = \Gamma(\Gamma^{k_n}(t)) \geq d \quad n \in \mathbb{N},$$

which is a contradiction to (7). Thus we have proved (b).

Finally, we prove (c). So, fix $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$. First, suppose that Γ is not continuous at 0. Then, according to (b), there is $m \in \mathbb{N}$ such that

$$\Gamma^n(t) = 0 \quad n > m.$$

Next, by subadditivity of Γ , we get

$$\begin{aligned} \Gamma\left(\sum_{n=k}^{\infty} \Gamma^n(t)\right) &\leq \sum_{n=k}^m \Gamma^{n+1}(t) + \Gamma\left(\sum_{n=m+1}^{\infty} \Gamma^n(t)\right) \\ &= \sum_{n=k}^m \Gamma^{n+1}(t) + \Gamma(0), \end{aligned}$$

whence, by (a),

$$\Gamma\left(\sum_{n=k}^{\infty} \Gamma^n(t)\right) \leq \sum_{n=k}^m \Gamma^{n+1}(t) = \sum_{n=k+1}^{\infty} \Gamma^n(t).$$

It remains to consider the case when Γ is continuous at 0. Then, analogously as above, the subadditivity of Γ yields

$$\Gamma\left(\sum_{n=k}^{\infty} \Gamma^n(t)\right) \leq \sum_{n=k}^m \Gamma^{n+1}(t) + \Gamma\left(\sum_{n=m+1}^{\infty} \Gamma^n(t)\right) \quad (9)$$

for $m \in \mathbb{N}$, $m > k$. Clearly, (7) and (a) imply that

$$\lim_{m \rightarrow \infty} \Gamma\left(\sum_{n=m+1}^{\infty} \Gamma^n(t)\right) = 0.$$

Hence, letting $m \rightarrow \infty$ in (9), we obtain (8). $\square$

We need yet the following simple observation ($\mathbb{R}$ stands for the set of real numbers).

Proposition 1 *Let S be a nonempty set, $\mathcal{T} : \mathbb{R}^S \rightarrow \mathbb{R}^S$ and $\gamma_n, \delta_n : S \rightarrow \mathbb{R}_+$ for $n \in \mathbb{N}_0$. Assume that*

$$-\delta_{n-1}(t) \leq \mathcal{T}^n(\varphi)(t) - \mathcal{T}^{n-1}(\varphi)(t) \leq \gamma_{n-1}(t) \quad t \in S, n \in \mathbb{N}$$

and

$$\Delta(t) := \sum_{n=0}^{\infty} \delta_n(t) < \infty, \quad \Gamma(t) := \sum_{n=0}^{\infty} \gamma_n(t) < \infty \quad t \in S.$$

Then, for each $t \in S$, the limit

$$\Phi(t) := \lim_{n \rightarrow \infty} \mathcal{T}^n(\varphi)(t)$$

exists and

$$-\Delta(t) \leq \Phi(t) - \varphi(t) \leq \Gamma(t).$$

Proof Since, for every $k, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $k > 0$, $t \in S$, we have

$$\begin{aligned} -\sum_{i=n}^{\infty} \delta_i(t) &\leq \sum_{i=1}^k \mathcal{T}^{n+i}(\varphi)(t) - \mathcal{T}^{n+i-1}(\varphi)(t) \\ &= \mathcal{T}^{n+k}(\varphi)(t) - \mathcal{T}^n(\varphi)(t) \leq \sum_{i=n}^{\infty} \gamma_i(t), \end{aligned}$$

the limit $\Phi(t)$ exists and, with $n = 0$ and $k \rightarrow \infty$, we obtain

$$\Delta(t) \leq \Phi(t) - \varphi(t) \leq \Gamma(t). \quad \square$$

3 Modified Stability on Square Symmetric Groupoids

In the case where f takes values in the set of real numbers $\mathbb{R}$, condition (6) can be written in the form

$$-\varphi(x, y) \leq f(x + y) - f(x) - f(y) \leq \varphi(x, y) \quad (10)$$

and there arises a natural question whether results analogous to those in Theorem 2 can be obtained also for functions $f : E_1 \rightarrow \mathbb{R}$, which satisfy the inequalities

$$-\nu(x, y) \leq f(x + y) - f(x) - f(y) \leq \mu(x, y), \quad (11)$$

with some suitable functions $\mu, \nu : E_1 \times E_1 \rightarrow \mathbb{R}_+$, not necessarily equal. In this section we investigate this issue. Moreover, we do it on a restricted domain and for functions with a domain being a subset of a square symmetric groupoid.

Let us recall that a groupoid $(G, \diamond)$ (i.e., a nonempty set G with a binary operation $\diamond : G \times G \rightarrow G$) is square symmetric provided the operation $\diamond$ is square symmetric, i.e.,

$$(x \diamond x) \diamond (y \diamond y) = (x \diamond y) \diamond (x \diamond y) \quad x, y \in G.$$

Clearly every commutative semigroup is a square symmetric groupoid. Next, let X be a linear space over a field $\mathbb{K}$, $a, b \in \mathbb{K}$, $z \in X$ and define a binary operation $\star : X^2 \rightarrow X$ by:

$$x \star y := ax + by + z \quad x, y \in X.$$

Then it is easy to check that $(X, \star)$ provides a simple example of a square symmetric groupoid. Clearly, $\star$ is commutative only if $a = b$. Moreover, it is easy to check that it is associative only for $a = b = 1$.

If in a nonempty set G we define a binary operation $\star : G^2 \rightarrow G$ by one of the following two conditions:

$$x \star y := x \quad x, y \in G,$$

$$x \star y := y \quad x, y \in G,$$

then it is square symmetric. Also, it is easily seen that $(\mathbb{R}, \circ)$, with

$$x \circ y := A(x - y) \quad x, y \in \mathbb{R},$$

with some fixed mapping $A : \mathbb{R} \rightarrow \mathbb{R}$, is another very simple example of a square symmetric groupoid.

One more example of a square symmetric operations is described in [18, Theorem 12]. Namely, let $(G, \cdot)$ be a groupoid with a right (or left) unit element and $F : G^2 \rightarrow G$ be such that

$$F(z \cdot x, z \cdot y) = z \cdot F(x, y), \quad F(x \cdot z, y \cdot z) = F(x, y) \cdot z \quad x, y, z \in G.$$

Then the operation $\star : G^2 \rightarrow G$, given by: $x \star y := F(x, y)$ for $x, y \in G$, is square symmetric.

Observe yet that, if $(G, \circ)$ is a square symmetric groupoid, H is a nonempty set and $h : H \rightarrow G$ is a bijection, then the operation $\star : H^2 \rightarrow H$, given by:

$$x \star y := h^{-1}(h(x) \circ h(y)) \quad x, y \in H,$$

is also square symmetric.

The square symmetric groupoids have been already considered in several papers investigating the stability of some functional equations (see, e.g., [18, 33, 34, 37, 38, 43, 44, 55, 57]). For a description of square symmetric operations we refer to [14].

Finally, let us mention that $(G, +, d)$ is a complete metric groupoid provided $(G, +)$ is a groupoid, (G, d) is a complete metric space and the operation $+$: $G^2 \rightarrow G$ is continuous, in both variables simultaneously, with respect to the metric d .

In the sequel, given a groupoid $(G, \diamond)$, we define a mapping $\tau_\diamond : G \rightarrow G$ by:

$$\tau_\diamond(x) := x \diamond x.$$

Now we are in a position to prove the next theorem, which generalizes to some extent the main results in [18, 33, 34, 37, 38].

Theorem 3 *Let $(G, \star)$ be a square symmetric groupoid, $\circ : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous square symmetric operation (cf. [14]), $S \subset G$ be nonempty, $\mu, \nu : S^2 \rightarrow \mathbb{R}_+$, $\tau_\star(S) \subset S$, $\Gamma_n : \mathbb{R} \rightarrow \mathbb{R}_+$ be nondecreasing for $n \in \mathbb{N}$, and $h : S \rightarrow \mathbb{R}$ satisfy*

$$-v(x, y) \leq h(x \star y) - h(x) \circ h(y) \leq \mu(x, y) \quad x, y \in S, x \star y \in S. \quad (12)$$

Suppose that one of the following two conditions is valid.

(i) $\tau_\circ$ is bijective and $\sigma := \tau_\circ^{-1}$, $\rho := \tau_\star$ satisfy

$$\sigma^n(x) - \sigma^n(y) \leq \Gamma_n(x - y) \quad x, y \in \mathbb{R}, n \in \mathbb{N}, \quad (13)$$

$$\widehat{\mu}(x) := \sum_{n=0}^{\infty} \Gamma_{n+1}(\mu(\rho^n(x), \rho^n(x))) < \infty \quad x \in S,$$

$$\widehat{\nu}(x) := \sum_{n=0}^{\infty} \Gamma_{n+1}(\nu(\rho^n(x), \rho^n(x))) < \infty \quad x \in S,$$

$$\liminf_{n \rightarrow \infty} \Gamma_n(\mu(\rho^n(x), \rho^n(y))) = 0 \quad x, y \in S, x \star y \in S, \quad (14)$$

$$\liminf_{n \rightarrow \infty} \Gamma_n(\nu(\rho^n(x), \rho^n(y))) = 0 \quad x, y \in S, x \star y \in S. \quad (15)$$

(ii) $\tau_\star|_S$ is a bijection onto S , (13), (14), and (15) hold with $\rho := (\tau_\star|_S)^{-1}$ and $\sigma := \tau_\circ$, and

$$\widehat{\mu}(x) := \nu(\rho(x), \rho(x)) + \sum_{n=1}^{\infty} \Gamma_n(\nu(\rho^{n+1}(x), \rho^{n+1}(x))) < \infty \quad x \in S,$$

$$\widehat{\nu}(x) := \mu(\rho(x), \rho(x)) + \sum_{n=1}^{\infty} \Gamma_n(\mu(\rho^{n+1}(x), \rho^{n+1}(x))) < \infty \quad x \in S.$$

Then the limit

$$F(x) := \lim_{n \rightarrow \infty} \sigma^n(h(\rho^n(x)))$$

exists for each $x \in S$,

$$F(x \star y) = F(x) \circ F(y) \quad x, y \in S, x \star y \in S, \quad (16)$$

and

$$-\widehat{v}(x) \leq F(x) - h(x) \leq \widehat{\mu}(x) \quad x \in S. \quad (17)$$

Moreover, if Γ_1 is subadditive and

$$\liminf_{n \rightarrow \infty} \Gamma_1^n(\widehat{\mu}(\rho^n(x))) = 0 \quad x \in S, \quad (18)$$

$$\liminf_{n \rightarrow \infty} \Gamma_1^n(\widehat{v}(\rho^n(x))) = 0 \quad x \in S, \quad (19)$$

then F is the unique function mapping S into $\mathbb{R}$ such that (16) and (17) are valid.

Proof Write $\Gamma_0(t) = t$, for $t \in \mathbb{R}$ and define $\Delta_n : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Delta_n(x) := -\Gamma_n(-x) \quad x \in \mathbb{R}, n \in \mathbb{N}_0.$$

Then Δ_n is nondecreasing for each $n \in \mathbb{N}_0$ and, by (13),

$$\Delta_n(x - y) \leq \sigma^n(x) - \sigma^n(y) \leq \Gamma_n(x - y) \quad x, y \in \mathbb{R}, n \in \mathbb{N}_0. \quad (20)$$

Let

$$\mathcal{T}(\alpha)(t) := \sigma(\alpha(\rho(t)))$$

for $\alpha \in \mathbb{R}^S$, $t \in S$. Next, (12) with $x = y$ yields

$$-v(x, x) \leq h(\tau_\star(x)) - \tau_\circ(h(x)) \leq \mu(x, x) \quad x \in S. \quad (21)$$

Hence, in the case when (i) holds, by (20), for each $n \in \mathbb{N}_0$ and $z \in S$ we get

$$\begin{aligned} \mathcal{T}^{n+1}(h)(z) - \mathcal{T}^n(h)(z) &= \sigma^{n+1} \circ h \circ \rho^{n+1}(z) - \sigma^n \circ h \circ \rho^n(z) \\ &= \sigma^{n+1} \circ h \circ \rho^{n+1}(z) - \sigma^{n+1} \circ \tau_\circ \circ h \circ \rho^n(z) \\ &\leq \Gamma_{n+1}(h \circ \rho(\rho^n(z)) - \tau_\circ(h(\rho^n(z)))) \\ &\leq \Gamma_{n+1}(\mu(\rho^n(z), \rho^n(z))) \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}^{n+1}(h)(z) - \mathcal{T}^n(h)(z) &= \sigma^{n+1} \circ h \circ \rho^{n+1}(z) - \sigma^n \circ h \circ \rho^n(z) \\ &= \sigma^{n+1} \circ h \circ \rho^{n+1}(z) - \sigma^{n+1} \circ \tau_\circ \circ h \circ \rho^n(z) \\ &\geq \Delta_{n+1}(h \circ \rho(\rho^n(z)) - \tau_\circ(h(\rho^n(z)))) \\ &\geq \Delta_{n+1}(-v(\rho^n(z), \rho^n(z))) \\ &= -\Gamma_{n+1}(v(\rho^n(z), \rho^n(z))). \end{aligned}$$

If (ii) holds, then analogously we obtain

$$\mathcal{T}^{n+1}(h)(z) - \mathcal{T}^n(h)(z) = \sigma^{n+1} \circ h \circ \rho^{n+1}(z) - \sigma^n \circ h \circ \rho^n(z)$$

$$\begin{aligned}
&\leq \Gamma_n(\sigma \circ h(\rho^{n+1}(z)) - h(\rho^n(z))) \\
&= \Gamma_n(\tau_\circ \circ h(\rho^{n+1}(z)) - h \circ \tau_\star(\rho^{n+1}(z))) \\
&\leq \Gamma_n(v(\rho^{n+1}(z), \rho^{n+1}(z))),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}^{n+1}(h)(z) - \mathcal{T}^n(h)(z) &\geq \Delta_n(\sigma \circ h(\rho^{n+1}(z)) - h(\rho^n(z))) \\
&= \Delta_n(\tau_\circ \circ h(\rho^{n+1}(z)) - h \circ \tau_\star(\rho^{n+1}(z))) \\
&\geq \Delta_n(-\mu(\rho^{n+1}(z), \rho^{n+1}(z))) \\
&= -\Gamma_n(\mu(\rho^{n+1}(z), \rho^{n+1}(z)))
\end{aligned}$$

for every $n \in \mathbb{N}_0$ and $z \in S$. So, in view of Proposition 1 with

$$\gamma_n(z) := \begin{cases} \Gamma_{n+1}(\mu(\rho^n(z), \rho^n(z))), & \text{if (i) holds;} \\ \Gamma_n(v(\rho^{n+1}(z), \rho^{n+1}(z))), & \text{if (ii) holds,} \end{cases}$$

and

$$\delta_n(z) := \begin{cases} -\Gamma_{n+1}(v(\rho^n(z), \rho^n(z))), & \text{if (i) holds;} \\ -\Gamma_n(\mu(\rho^{n+1}(z), \rho^{n+1}(z))), & \text{if (ii) holds,} \end{cases}$$

for $z \in S$ and $n \in \mathbb{N}_0$, the limit

$$F(x) := \lim_{n \rightarrow \infty} \mathcal{T}^n(h)(x)$$

exists for every $x \in S$ and (17) holds.

Now we show that F satisfies (16). So take $x, y \in S$ with $x \star y \in S$. Then it is easy to check that, for every $n \in \mathbb{N}$,

$$\rho^n(x), \rho^n(y) \in S,$$

$$\rho^n(x \star y) = \rho^n(x) \star \rho^n(y),$$

$$\sigma^n(h(\rho^n(x))) \circ \sigma^n(h(\rho^n(y))) = \sigma^n(h(\rho^n(x)) \circ h(\rho^n(y))),$$

and consequently

$$\begin{aligned}
&\sigma^n(h(\rho^n(x \star y))) - \sigma^n(h(\rho^n(x))) \circ \sigma^n(h(\rho^n(y))) \\
&= \sigma^n(h(\rho^n(x) \star \rho^n(y))) - \sigma^n(h(\rho^n(x)) \circ h(\rho^n(y))) \\
&\leq \Gamma_n(h(\rho^n(x) \star \rho^n(y)) - h(\rho^n(x)) \circ h(\rho^n(y))) \\
&\leq \Gamma_n(\mu(\rho^n(x), \rho^n(y))),
\end{aligned}$$

$$\sigma^n(h(\rho^n(x \star y))) - \sigma^n(h(\rho^n(x))) \circ \sigma^n(h(\rho^n(y)))$$

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