

Chapter 2

Famous Inequalities

I speak not as desiring more, but rather wishing a more strict restraint.

—Isabella, in *Measure for Measure*, by William Shakespeare

In this chapter we meet three very important inequalities: Bernoulli's Inequality, the Arithmetic Mean–Geometric Mean Inequality, and the Cauchy–Schwarz Inequality. At first we consider only pre-calculus versions of these inequalities, but we shall soon see that a thorough study of inequalities cannot be undertaken without calculus. And really, calculus cannot be thoroughly understood without some knowledge of inequalities. We define Euler's number e by a more systematic method than that of Example 1.32. We'll see that this method engenders many fine extensions.

2.1 Bernoulli's Inequality and Euler's Number e

The following is a very useful little inequality. It is named for the Swiss mathematician Johann Bernoulli (1667–1748).

Lemma 2.1. (Bernoulli's Inequality) *Let $n = 1, 2, 3, \dots$. Then for $x > -1$,*

$$(1 + x)^n \geq 1 + nx.$$

Proof. If $n = 1$, the inequality holds, with equality. For $n = 2$,

$$(1 + x)^2 = 1 + 2x + x^2 \geq 1 + 2x.$$

For $n = 3$, we use the $n = 2$ case:

$$(1 + x)^3 = (1 + x)(1 + x)^2 \geq (1 + x)(1 + 2x) = 1 + 3x + 2x^2 \geq 1 + 3x.$$

For $n = 4$, we use the $n = 3$ case:

$$(1+x)^4 = (1+x)(1+x)^3 \geq (1+x)(1+3x) = 1+4x+4x^2 \geq 1+4x.$$

Etcetera: We could clearly continue this procedure up to any positive integer n , and so our proof is complete. $\square$

Example 2.2. We show that for $|x| < 1$, the sequence $\{x^n\}$ converges to 0. First, for $0 < x < 1$ we can write $x = \frac{1}{1+q}$, for some $q > 0$. Then by Bernoulli's Inequality (Lemma 2.1),

$$(1+q)^n \geq 1+nq,$$

so that

$$0 < x^n = \frac{1}{(1+q)^n} \leq \frac{1}{1+nq}.$$

Letting $n \rightarrow \infty$, the result follows. For $-1 < x < 0$, we simply replace x with $-x$ above. (The case $x = 0$ is trivial.) $\diamond$

Exercise 2.1 contains the fact that for any $x > 0$, $\{x^{1/n}\}$ converges to 1.

Example 2.3. The series

$$1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$

is called a **geometric series**. In it, each term x^k after the first is the *geometric mean* of the term just before it and the term just after it: $x^k = \sqrt{x^{k-1}x^{k+1}}$. Here we find a formula for the sum of a geometric series, when it exists. For $x \neq 1$ the following identity can be found by doing long division on the right-hand side, or simply verified by cross multiplication:

$$\sum_{k=0}^n x^k = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

So if $|x| < 1$ then by Example 2.2, the sequence of *partial sums* $\{S_n\} = \left\{ \sum_{k=0}^n x^k \right\}$ converges to $\frac{1}{1-x}$. Therefore,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for } |x| < 1. \quad \diamond$$

Example 2.4. We write $x = 0.611111 \dots$ as a fraction. Observe that

$$10x = 6 + \sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k = 6 + \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k.$$

The series here is a geometric series, so by Example 2.3,

$$10x = 5 + \frac{1}{1 - 1/10} = \frac{55}{9}.$$

Therefore, $x = 11/18$. $\diamond$

Example 2.5. [2] Here we show that the sequence $\{n^{1/n}\}$ converges to 1. Setting $x = 1/\sqrt{n}$ in Bernoulli's Inequality (Lemma 2.1) we get

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n \geq 1 + \sqrt{n} > \sqrt{n}.$$

Therefore

$$\left(1 + \frac{1}{\sqrt{n}}\right)^2 > (\sqrt{n})^{2/n} = n^{1/n} \geq 1,$$

and the result follows upon letting $n \rightarrow \infty$. $\diamond$

Example 2.6. [12, 16, 55] Using Bernoulli's Inequality (Lemma 2.1) we show again (cf. Example 1.32) that $\{(1 + \frac{1}{n})^n\}$ converges. We have seen that the number to which this sequence converges is Euler's number e. First, observe that

$$\frac{(1 + \frac{1}{n+1})^{n+1}}{(1 + \frac{1}{n})^n} = \left(1 + \frac{1}{n}\right) \left(\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}}\right)^{n+1} = \left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{(n+1)^2}\right)^{n+1}.$$

Then applying Bernoulli's Inequality,

$$\left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \geq \left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{n+1}\right) = 1.$$

Therefore

$$\left(1 + \frac{1}{n+1}\right)^{n+1} \geq \left(1 + \frac{1}{n}\right)^n$$

and so $\{(1 + \frac{1}{n})^n\}$ is increasing. In a very similar way, which we leave for Exercise 2.6 (see also Example 1.43 and Exercise 1.39), one can show that $\{(1 + \frac{1}{n})^{n+1}\}$ is decreasing. Then we have

$$\left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n}\right)^{n+1} \leq \left(1 + \frac{1}{1}\right)^{1+1} = 4,$$

and so $\{(1 + \frac{1}{n})^n\}$ is also bounded above. Therefore, by the Increasing Bounded Sequence Property (Theorem 1.34), this sequence has a limit: Euler's number, e. $\diamond$

Again we emphasize that the basic string of inequalities here, is:

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1} \quad \text{for } n = 1, 2, 3, \dots$$

2.2 The AGM Inequality

We begin with a simple yet incredibly useful fact. It turns out to be a special case of the main result of this section (Theorem 2.10).

Lemma 2.7. *Let a and b be positive real numbers. Then*

$$\sqrt{ab} \leq \frac{a+b}{2}, \quad \text{and equality occurs here} \Leftrightarrow a = b.$$

Proof. It is easily verified that

$$(a+b)^2 - 4ab = (a-b)^2 \geq 0.$$

Therefore,

$$(a+b)^2 \geq 4ab, \quad \text{or} \quad \frac{a+b}{2} \geq \sqrt{ab}.$$

Now if $a = b$, then clearly $\sqrt{ab} = \frac{a+b}{2}$. Conversely, if $\sqrt{ab} = \frac{a+b}{2}$ then in the first line of the proof we must have $(a-b)^2 = 0$, and so $a = b$. $\square$

The average $A = \frac{a+b}{2}$ is known as the *Arithmetic Mean* of a and b . The quantity $G = \sqrt{ab}$ is known as their *Geometric Mean*. A rather satisfying Proof Without Words for Lemma 2.7, which also suggests why $\sqrt{ab}$ is called the Geometric Mean, is shown **Fig. 2.1**. See also Exercise 2.19

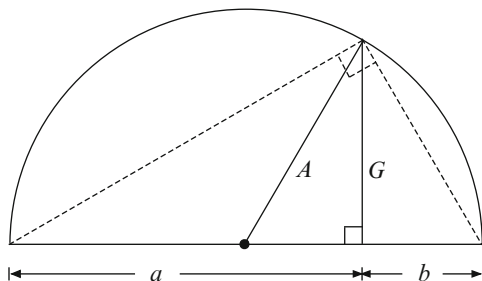


Fig. 2.1 $G = \sqrt{ab} \leq A = \frac{a+b}{2}$

Example 2.8. Suppose we use a balance to determine the mass of an object. We place the object on the left side of the balance and a known mass on the right side, to obtain a measurement a . Then we place the object on the right side of the balance and a known mass on the left side, to obtain a measurement b . By the principle of the lever (or more generally, the principle of moments) the true mass of the object is the Geometric Mean $\sqrt{ab}$. (See also Exercise 2.15.) $\diamond$

Sometimes the most important feature of an inequality is the case in which equality occurs, as the following example illustrates.

Example 2.9. A rectangle with side lengths a and b has perimeter $P = 2a + 2b$ and area $T = ab$. Lemma 2.7 reads $ab \leq (\frac{a+b}{2})^2$, or $T \leq (P/4)^2$. So a rectangle with given perimeter has greatest area when $a = b$, i.e., when the rectangle is a square. Likewise, a rectangle with given area has least perimeter when $a = b$, again when the rectangle is a square. In either case, $T = (P/4)^2$. $\diamond$

The most natural extension of Lemma 2.7 is to allow n positive numbers instead of just two. But we need to know what would be meant by *Arithmetic Mean* and *Geometric Mean* in this case. These turn out to be exactly as one might expect, as follows.

Let $a_1, a_2, \dots, a_n$ be real numbers. Their **Arithmetic Mean** is given by

$$A = \frac{a_1 + a_2 + \dots + a_n}{n} = \frac{1}{n} \sum_{j=1}^n a_j.$$

If these numbers are also nonnegative, then their **Geometric Mean** given by

$$G = ((a_1)(a_2) \cdots (a_n))^{1/n} = \left(\prod_{j=1}^n a_j \right)^{1/n}.$$

A number $M = M(a_1, a_2, \dots, a_n)$ which depends on $a_1, a_2, \dots, a_n$ is called a **mean** simply if it satisfies

$$\min_{1 \leq j \leq n} \{a_j\} \leq M \leq \max_{1 \leq j \leq n} \{a_j\}.$$

However, for practical purposes one often desires other properties, like (i) having $M(a_1, a_2, \dots, a_n)$ independent of the order in which the numbers $a_1, a_2, \dots, a_n$ are arranged, and (ii) having $M(ta_1, ta_2, \dots, ta_n) = tM(a_1, a_2, \dots, a_n)$ for any $t \geq 0$. The reader should agree that A and G each satisfy (i) and (ii).

The Arithmetic Mean–Geometric Mean Inequality below, or what we shall call the **AGM Inequality** for short, extends Lemma 2.7 to n numbers. This inequality is of fundamental importance in mathematical analysis. The great French mathematician Augustin Cauchy (1789–1857) was the first to prove it, in 1821. We provide his proof at the end of this section. (The Scottish mathematician Colin Maclaurin (1698–1746) had an earlier proof, around 1729, which wasn't quite complete.)

The list of mathematicians who have offered proofs of the AGM Inequality over the years is impressive. It includes Liouville, Hurwitz, Steffensen, Bohr, Riesz, Sturm, Rado, Hardy, Littlewood, and Polya. (See, e.g., [5, 9, 18, 32, 49]; the book [9] contains over 75 proofs.) Below we provide the clever 1976 proof given by K.M. Chong [10].

Theorem 2.10. (AGM Inequality) *Let $a_1, a_2, \dots, a_n$ be n positive real numbers, where $n \geq 2$. Then*

$$G \leq A,$$

and equality occurs here $\Leftrightarrow a_1 = a_2 = \dots = a_n$.

Proof. If $n = 2$ then the result is simply Lemma 2.7, so we consider $n \geq 3$. By rearranging the a_j 's if necessary, we may suppose that $a_1 \leq a_2 \leq \dots \leq a_{n-1} \leq a_n$. Then $0 < a_1 \leq A \leq a_n$, and so

$$A(a_1 + a_n - A) - a_1 a_n = (a_1 - A)(A - a_n) \geq 0.$$

That is,

$$a_1 + a_n - A \geq \frac{a_1 a_n}{A}. \quad (2.1)$$

Take $n = 3$ here, and notice that the Arithmetic Mean of the *two* numbers a_2 and $a_1 + a_3 - A$ is A . Now we apply Lemma 2.7 to *these two numbers*, along with (2.1) to get

$$A^2 \geq a_2(a_1 + a_3 - A) \geq a_2 \frac{a_1 a_3}{A}.$$

That is,

$$A^3 \geq a_1 a_2 a_3.$$

Now take $n = 4$. The Arithmetic Mean of the *three* numbers a_2, a_3 and $a_1 + a_4 - A$ is again A , so by what we have just shown applied to *these three numbers*, along with (2.1),

$$A^3 \geq a_2 a_3(a_1 + a_4 - A) \geq a_2 a_3 \frac{a_1 a_4}{A}.$$

That is,

$$A^4 \geq a_1 a_2 a_3 a_4.$$

Clearly we could continue this procedure indefinitely, showing that $A \geq G$ for any positive integer n , and so we have proved the main part of the theorem. Now to address the equality conditions. If $a_1 = a_2 = \dots = a_n$, then it is easily verified that

$G = A$. Conversely, if $A = G$ for some particular n , then in the argument above, $(a_1 - A)(A - a_n) = 0$ so that $a_1 = a_n = A$, and therefore $a_1 = a_2 = \cdots = a_n = A$. $\square$

Example 2.11. Suppose that an investment returns 10 % in the first year, 50 % in the second year, and 30 % in the third year. Using the Geometric Mean

$$[(1.1)(1.5)(1.3)]^{1/3} \cong 1.289,$$

the average rate of return over the 3 years is just under 29 %. The Arithmetic Mean gives an overestimate of the average rate of return, at 30 %. $\diamond$

Example 2.12. We saw in Example 2.9 that a rectangle with given perimeter has greatest area when the rectangle is a square, and that a rectangle with given area has least perimeter when the rectangle is a square. Likewise, using the AGM Inequality (Theorem 2.10), a box (even in n dimensions) with given surface area has greatest volume when the box is a cube, and a box (even in n dimensions) with given volume has least surface area when the box is a cube. $\diamond$

Example 2.13. Named for Heron of Alexandria (c. 10–70 AD), **Heron's formula** gives the area T of a triangle in terms of its three side lengths a, b, c and perimeter P , as follows:

$$16T^2 = P(P - 2a)(P - 2b)(P - 2c).$$

So if we apply the AGM Inequality (Theorem 2.10) to the three numbers $P - 2a$, $P - 2b$ and $P - 2c$, we obtain

$$16T^2 \leq P \left(\frac{(P - 2a) + (P - 2b) + (P - 2c)}{3} \right)^3, \quad \text{or}$$

$$T \leq \frac{P^2}{12\sqrt{3}}.$$

Therefore, for a triangle with fixed perimeter P , its area T is largest possible when $P - 2a = P - 2b = P - 2c$. This is precisely when $a = b = c$, that is, when the triangle is equilateral. Likewise a triangle with fixed area T has least perimeter P when it is an equilateral triangle. In either case, $T = P^2/(12\sqrt{3})$. $\diamond$

Remark 2.14. We saw in Example 2.13 that for a triangle, we have $T \leq P^2/(12\sqrt{3})$. In Example 2.9, we saw that for a rectangle, $T \leq P^2/16$. This latter inequality persists for all quadrilaterals having area T and perimeter P . See Exercise 2.29. These inequalities are called **isoperimetric inequalities**. The isoperimetric inequality for an n -sided polygon is

$$T \leq \frac{P^2}{4n \tan(\pi/n)},$$

and equality holds if and only if the polygon is regular. The isoperimetric inequality for any plane figure with area T and perimeter P is

$$T \leq \frac{P^2}{4\pi}.$$

The famous **isoperimetric problem** was to prove that equality holds here if and only if the plane figure is a circle. The solution of the isoperimetric problem takes up an important and interesting episode in the history of mathematics [37]. For a polished modern solution, see [25]. The reader might find it somewhat comforting that

$$\lim_{n \rightarrow \infty} \left[n \tan \left(\frac{\pi}{n} \right) \right] = \pi.$$

This can be verified quite easily (see Exercise 5.46) using *L'Hospital's Rule*, which we meet in Sect. 5.3. $\circ$

Example 2.15. [26] We show that Bernoulli's Inequality (Lemma 2.1)

$$(1 + x)^n \geq 1 + nx$$

for $x > -1$ and $n = 1, 2, 3, \dots$ follows from the AGM Inequality (Theorem 2.10). First, if $-1 < x \leq -1/n$, then $1 + nx \leq 0 < 1 + x$, and so $1 + nx < (1 + x)^n$. Therefore we assume that $x > -1/n$. We write

$$1 + x = \frac{1 + nx + (n - 1)}{n} = \frac{1 + nx + 1 + 1 + \dots + 1}{n},$$

where there are $n - 1$ 1's to the right of nx in the second numerator. Then applying the AGM Inequality (Theorem 2.10) to the n positive numbers $1 + nx, 1, 1, \dots, 1$, we get

$$(1 + x)^n = \left(\frac{1 + nx + 1 + 1 + \dots + 1}{n} \right)^n \geq (1 + nx)(1)(1) \cdots (1) = 1 + nx,$$

as desired. $\diamond$

Conversely, it happens that the AGM Inequality (Theorem 2.10) follows from Bernoulli's Inequality (Lemma 2.1), and so the two are equivalent. We leave the verification of this for Exercise 2.10.

Example 2.16. For n positive numbers $a_1, a_2, \dots, a_n$, their **Harmonic Mean** is given by

$$H = \left(\frac{1/a_1 + 1/a_2 + \dots + 1/a_n}{n} \right)^{-1}.$$

Replacing a_j with $1/a_j$ in the AGM Inequality (Theorem 2.10) we get

$$H \leq G.$$

Then in $H \leq G \leq A$, the outside inequality can be rewritten rather nicely as

$$\sum_{j=1}^n a_j \sum_{j=1}^n \frac{1}{a_j} \geq n^2. \quad (2.2)$$

Again, equality occurs here if and only if $a_1 = a_2 = \cdots = a_n$. The Harmonic Mean of two numbers $a, b > 0$ is simply

$$H = \frac{2ab}{a+b}.$$

In this case, (2.2) reads

$$(a+b) \left(\frac{1}{a} + \frac{1}{b} \right) \geq 4,$$

and equality occurs here if and only if $a = b$. ◇

To close this section, we supply Cauchy's brilliant 1821 proof of the AGM Inequality (Theorem 2.10) but without addressing the equality conditions—these we leave for Exercise 2.35. The pattern of argument here is powerful and has since been used by mathematicians in many other contexts. (We shall see it applied in one other context in Sect. 8.3.)

Proof. Again, if $n = 2$, this is simply Lemma 2.7. If $n = 4$, we use Lemma 2.7 twice:

$$\begin{aligned} (a_1 \cdot a_2 \cdot a_3 \cdot a_4)^{1/4} &= \left((a_1 \cdot a_2)^{1/2} \right)^{1/2} \left((a_3 \cdot a_4)^{1/2} \right)^{1/2} \\ &\leq \left(\frac{1}{2}(a_1 + a_2) \right)^{1/2} \cdot \left(\frac{1}{2}(a_3 + a_4) \right)^{1/2} \\ &\leq \frac{1}{2} \left(\frac{1}{2}(a_1 + a_2) + \frac{1}{2}(a_3 + a_4) \right) \\ &= \frac{1}{4} (a_1 + a_2 + a_3 + a_4). \end{aligned}$$

If $n = 8$, we use Lemma 2.7 then the $n = 4$ case:

$$\begin{aligned} &(a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdot a_6 \cdot a_7 \cdot a_8)^{1/8} \\ &= \left((a_1 \cdot a_2 \cdot a_3 \cdot a_4)^{1/4} \right)^{1/2} \left((a_5 \cdot a_6 \cdot a_7 \cdot a_8)^{1/4} \right)^{1/2} \\ &\leq \left(\frac{1}{4} (a_1 + a_2 + a_3 + a_4) \right)^{1/2} \cdot \left(\frac{1}{4} (a_5 + a_6 + a_7 + a_8) \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \left(\frac{1}{4} (a_1 + a_2 + a_3 + a_4) + \frac{1}{4} (a_5 + a_6 + a_7 + a_8) \right) \\
&= \frac{1}{8} (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8).
\end{aligned}$$

Clearly we could continue this procedure indefinitely, and so we may assume that we have proved that $G \leq A$ for any n of the form $n = 2^m$. For any (other) n , we choose m so large that $2^m > n$. Now,

$$\frac{a_1 + a_2 + \cdots + a_n + (2^m - n)A}{2^m} = A.$$

The numerator of the left-hand side here has 2^m members in the sum and so we can apply what we have proved so far to see that

$$\left(a_1 \cdot a_2 \cdots a_n \cdot A^{(2^m - n)} \right)^{1/2^m} \leq A.$$

That is,

$$a_1 \cdot a_2 \cdots a_n \cdot A^{2^m - n} \leq A^{2^m} \quad \Rightarrow \quad G^n \leq A^n.$$

□

Remark 2.17. Extending Lemma 2.7 to $n = 4, 8, 16, \dots$ as above is not too hard, just a bit messy. Cauchy's genius lies in being able to extending the result to any n . With this in mind we mention that T. Harriot proved the AGM Inequality (Theorem 2.10) for $n = 3$ around 1,600 [39]. No small feat for the time. ◻

2.3 The Cauchy–Schwarz Inequality

Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers. The Cauchy–Schwarz Inequality provides an upper bound for the sum of products $\sum_{j=1}^n a_j b_j$. The proof we provide below uses Lemma 2.7.

Theorem 2.18. (Cauchy–Schwarz Inequality) *Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers. Then*

$$\left(\sum_{j=1}^n a_j b_j \right)^2 \leq \sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2.$$

Proof. If $\sum_{j=1}^n a_j^2 = 0$ or $\sum_{j=1}^n b_j^2 = 0$ then the inequality holds, with equality.

Otherwise, we set $a_j = \frac{a_j^2}{\sum_{k=1}^n a_k^2}$ and $b_j = \frac{b_j^2}{\sum_{k=1}^n b_k^2}$ in Lemma 2.7 to obtain

$$\frac{a_j}{\sqrt{\sum_{k=1}^n a_k^2}} \frac{b_j}{\sqrt{\sum_{k=1}^n b_k^2}} \leq \frac{1}{2} \left(\frac{a_j^2}{\sum_{k=1}^n a_k^2} + \frac{b_j^2}{\sum_{k=1}^n b_k^2} \right).$$

Then summing from $j = 1$ to n we get

$$\frac{\sum_{j=1}^n a_j b_j}{\sqrt{\sum_{k=1}^n a_k^2} \sqrt{\sum_{k=1}^n b_k^2}} \leq \frac{1}{2} \left(\frac{\sum_{j=1}^n a_j^2}{\sum_{k=1}^n a_k^2} + \frac{\sum_{j=1}^n b_j^2}{\sum_{k=1}^n b_k^2} \right) = \frac{1}{2} (1 + 1) = 1,$$

which is really what we wanted to show. $\square$

Example 2.19. The **Root Mean Square** of the real numbers $a_1, a_2, \dots, a_n$ is:

$$R = \sqrt{\frac{1}{n} \sum_{j=1}^n a_j^2}.$$

For example, suppose that three squares with side lengths a_1, a_2 and a_3 have average area T . Then the single square with area T is the one with side length R . The reader should verify that R is a *mean*. We have seen that $G \leq A$. The Cauchy–Schwarz Inequality (Theorem 2.18) shows that $A \leq R$, on taking $b_1 = b_2 = \dots = b_n = 1/n$. $\diamond$

Remark 2.20. Readers who know some linear algebra might recognize the Cauchy–Schwarz Inequality in the following form. For two vectors $\mathbf{u} = (a_1, a_2, \dots, a_n)$ and $\mathbf{v} = (b_1, b_2, \dots, b_n)$ in $\mathbf{R}^n$, their **dot product** is given by $\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^n a_j b_j$, and the **length** of $\mathbf{u}$ is given by $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$. Then the Cauchy–Schwarz Inequality reads

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

(See [48], for example, for a proof of the Cauchy–Schwarz Inequality in this context.) This says that for non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ we have

$$-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \leq 1,$$

so we may *define* the **angle** θ between such vectors in $\mathbf{R}^n$ as that $\theta \in [0, \pi]$ for which

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

Moreover,

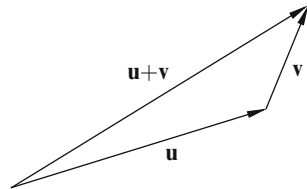
$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| + \|\mathbf{v}\|^2,$$

by the Cauchy–Schwarz Inequality. This last piece equals $(\|\mathbf{u}\| + \|\mathbf{v}\|)^2$, and so we have the **triangle inequality** in $\mathbf{R}^n$:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

So the Cauchy–Schwarz Inequality is fundamental for working in $\mathbf{R}^n$. And, in $\mathbf{R}^n$ it is evident why the triangle inequality is so named—see **Fig. 2.2**. $\circ$

Fig. 2.2 The Triangle Inequality $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$



There are many other proofs of the Cauchy–Schwarz Inequality, a few of which we explore in the exercises. However, we would be remiss if we did not supply what is essentially H. Schwarz’s (1843–1921) own ingenious proof, as follows. (See also Exercises 2.38 and 2.41.) For any real number t ,

$$\sum_{j=1}^n (ta_j + b_j)^2 \geq 0.$$

That is,

$$t^2 \sum_{j=1}^n a_j^2 + 2t \sum_{j=1}^n a_j b_j + \sum_{j=1}^n b_j^2 \geq 0.$$

Now the left-hand side is a quadratic in the variable t and since it is ≥ 0 , it must have either *no* real root or *one* real root. (It cannot have two distinct real roots.) Therefore its discriminant $B^2 - 4AC$ must be ≤ 0 . That is,

$$\left(2 \sum_{j=1}^n a_j b_j\right)^2 - 4 \sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2 \leq 0.$$

Rearranging this inequality yields the desired result. Pretty slick.

Exercises

2.1. Show that for $x > 0$, the sequence $\{x^{1/n}\}$ converges to 1.

Hint: Write $x^{1/n} = \left(1 + n \frac{x-1}{n}\right)^{1/n}$.

2.2. Show that Bernoulli's Inequality (Lemma 2.1), implies Lemma 2.7:

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Hint: Assume that $a \leq A = (a+b)/2$, take $n = 2$, and $x = A/a - 1$.

2.3. [35] Show that Bernoulli's Inequality (Lemma 2.1), holds also for $x \in [-2, -1]$.

2.4. (a) Write $0.\overline{237}$ and $6.45\overline{7132}$ as fractions. (b) Describe how to write any repeating decimal as a fraction.

2.5. [2] Take $x = -1/n^2$ in Bernoulli's Inequality (Lemma 2.1) to show that the sequence $\{(1 + \frac{1}{n})^n\}$ is increasing.

2.6. Show that $\{(1 + \frac{1}{n})^{n+1}\}$ is decreasing, as follows.

(a) Verify that

$$\frac{(1 + \frac{1}{n+1})^{n+2}}{(1 + \frac{1}{n})^{n+1}} = (1 + \frac{1}{n+1}) \left(1 - \frac{1}{(n+1)^2}\right)^{n+1}.$$

(b) Apply Bernoulli's Inequality (Lemma 2.1) to get

$$\left(1 + \frac{1}{n+1}\right) \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \leq \left(1 - \frac{1}{(n+1)^4}\right)^{n+1} < 1.$$

2.7. [47] Find the least positive integer N such that for all $n > N$,

$$\left(\frac{n^{n+1}}{(n+1)^n}\right)^n < n! < \left(\frac{n^{n+1}}{(n+1)^n}\right)^{n+1}.$$

2.8. (e.g., [24, 34, 52, 54]) Show that $x = \left(1 + \frac{1}{u}\right)^u$ and $y = \left(1 + \frac{1}{u}\right)^{u+1}$ (and vice-versa) are solutions to the equation $x^y = y^x$, for $x, y > 0$. (Taking $u = 1, 2, 3, \dots$ in fact yields *all* nontrivial (i.e., $x \neq y$) rational solutions to this equation.)

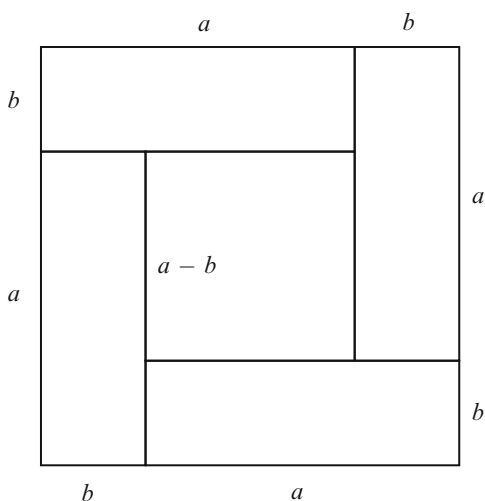
2.9. [28] Denote by $\lfloor x \rfloor$ the greatest integer not exceeding x . This function is often called the *Floor function*. Prove that for $x \geq 1$,

$$\left(1 + \frac{x}{\lfloor x \rfloor}\right)^{\lfloor x \rfloor} \leq 2^x \leq \left(1 + \frac{x}{\lfloor x+1 \rfloor}\right)^{\lfloor x+1 \rfloor}.$$

2.10. [26] In Example 2.15 we showed that the AGM Inequality implies Bernoulli's Inequality. Show that Bernoulli's Inequality implies the AGM Inequality.

2.11. Explain how the inequality $(a+b)^2 - 4ab = (a-b)^2 \geq 0$ in the proof of Lemma 2.7 relates to Fig. 2.3.

Fig. 2.3 For Exercise 2.11



2.12. (a) Fill in the details of another proof of Lemma 2.7, as follows. Let $a, b \geq 0$. Since $(t - \sqrt{a})(t - \sqrt{b})$ has real zeros, conclude that $\sqrt{ab} \leq \frac{a+b}{2}$.

(b) Let $a, c > 0$. Show that if $|b| > a + c$ then $ax^2 + bx + c$ has two (distinct) real roots.

2.13. [19] In Fig. 2.4, ABCD is a trapezoid with AB parallel to DC, and EF is parallel to each of these. Show that m is a weighted average of a and b . That is, $m = \frac{pb+qa}{p+q}$, for some $p, q > 0$.

2.14. (a) Show that for $x > 0$, we have $x + 1/x \geq 2$, with equality if and only if $x = 1$.

(b) Conclude (even though we have not yet officially met the exponential function) that

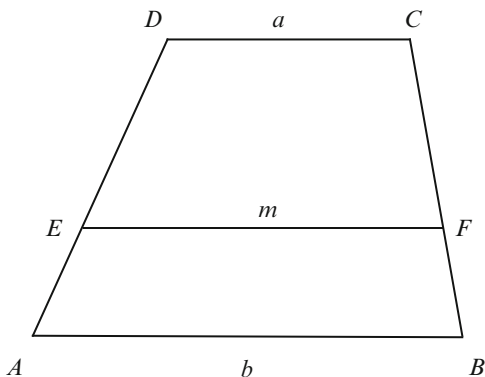
$$\cosh(x) = \frac{e^x + e^{-x}}{2} \geq 1,$$

with equality if and only if $x = 0$.

(c) [31] Show that for $x > 0$,

$$\frac{x^n}{1 + x + x^2 + \cdots + x^{2n}} \leq \frac{1}{2n + 1}.$$

Fig. 2.4 For Exercise 2.13



2.15. [51] The bank in your town sells British pounds at the rate $1\text{£} = \$S$ and buys them at the rate $1\text{£} = \$B$. You and your friend want to exchange dollars and pounds between the two of you, at a rate that is fair to both. Show that the fair exchange rate is $1\text{£} = \sqrt{SB}$, the Geometric Mean of S and B .

2.16. [13] Let $H \leq G \leq A$ denote respectively the Harmonic, Geometric and Arithmetic Means of two positive numbers.

(a) Show that H , G , and A are the side lengths of a triangle if and only if

$$\frac{3 - \sqrt{5}}{2} < \frac{A}{H} < \frac{3 + \sqrt{5}}{2}.$$

(b) Show that H , G , and A are the side lengths of a right triangle if and only if A/H is the *golden mean*:

$$\frac{A}{H} = \frac{1 + \sqrt{5}}{2}.$$

2.17. [56]

(a) Prove that for any natural number n , we have $\sum_{k=1}^n k = \frac{n(n+1)}{2}$.

- (b) For $n = 1$, it is clear that $n! = \left(\frac{n+1}{2}\right)^n$. Use (a) and the AGM Inequality (Theorem 2.10) to prove that for integers $n \geq 2$,

$$n! < \left(\frac{n+1}{2}\right)^n.$$

2.18. [31] Let $a, b > 0$, with $a + b = 1$. Show, as follows, that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

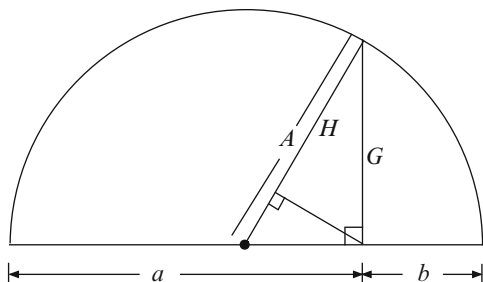
- (a) For such a, b , show that $\frac{1}{ab} \geq 4$.
 (b) Use Lemma 2.7 to show that $(a+1/a)^2 + (b+1/b)^2 \geq 2(a+1/a)(b+1/b)$.
 (c) Combine (a) and (b) to show that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 - 2\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right) \geq (1+4)^2 - \left(a + \frac{1}{a}\right)^2 - \left(b + \frac{1}{b}\right)^2.$$

- (d) Use this to obtain the desired result.

2.19. In Fig. 2.5, which shows a semicircle with diameter $a + b$, we can see that $A > G > H$ as labeled. Use elementary geometry to show that A, G , and H are respectively the Arithmetic, Geometric and Harmonic Means of a and b .

Fig. 2.5 For Exercise 2.19



- 2.20.** (a) Suppose that a car travels at a miles per hour from point A to point B, then returns at b miles per hour. Show that the average speed for the trip is the Harmonic Mean of a and b .
 (b) Show that $G - H < A - G$, where H, G and A the Harmonic, Geometric, and Arithmetic Means of two numbers $a, b > 0$.
 (c) For $a, b > 0$, **Heron's Mean**, named for Heron of Alexandria (c. 10–70 AD), is

$$\hat{H} = \frac{a + \sqrt{ab} + b}{3}.$$

Show that if $a \neq b$, then $G < \hat{H} < A$, where G and A are the Geometric and Arithmetic means of a and b .

2.21. Let $a_1, a_2, \dots, a_n > 0$. We saw in (2.2) that

$$\sum_{j=1}^n a_j \sum_{j=1}^n \frac{1}{a_j} \geq n^2.$$

Apply this to the three numbers $a+b$, $a+c$, and $b+c$ to obtain **Nesbitt's Inequality**:

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

2.22. Denote by H , G and A the Harmonic, Geometric, and Arithmetic Means of two numbers $a, b > 0$. We have seen that $H \leq G \leq A$.

(a) Show that $A - H = (b - a)/2$.

(b) Let $a_1 = H(a, b)$ and $b_1 = A(a, b)$. Then for $n = 1, 2, 3, \dots$, let

$$a_{n+1} = H(a_n, b_n) \quad \text{and} \quad b_{n+1} = A(a_n, b_n).$$

Show that $\{[a_n, b_n]\}$ is a sequence of nested intervals, with $b_n - a_n \rightarrow 0$. Conclude by the Nested Interval Property (Theorem 1.41) that there is c belonging to each of these intervals.

(c) Show that $\sqrt{a_n b_n} = \sqrt{ab} = G$ for all n to conclude that $c = G$.

(For example, if $a = 1$ and $b = 2$, then $\{a_n\}$ is an increasing sequence of rational numbers which converges to $\sqrt{2}$.)

2.23. [43] In Example 2.5 we used Bernoulli's Inequality (Lemma 2.1) to show that $\sqrt[n]{n} \rightarrow 1$ as $n \rightarrow \infty$. Prove this using the AGM Inequality (Theorem 2.10), by setting $a_1 = a_2 = \dots = a_{n-1}$ and $a_n = \sqrt[n]{n}$.

2.24. Show that

$$\lim_{n \rightarrow \infty} \frac{2 + 4 + 6 + \dots + (2n)}{1 + 3 + 5 + \dots + (2n - 1)} = e.$$

2.25. [30]

(a) In Example 2.6 we used Bernoulli's Inequality (Lemma 2.1) to show that $\left\{(1 + \frac{1}{n})^n\right\}$ is an increasing sequence. Show this using the AGM Inequality (Theorem 2.10). Hint: Consider the $n + 1$ numbers $1, \frac{n+1}{n}, \frac{n+1}{n}, \dots, \frac{n+1}{n}$.

(b) Show that $\left\{(1 + \frac{1}{n})^{n+1}\right\}$ is a decreasing sequence using the AGM Inequality (Theorem 2.10). Hint: Consider the $n + 2$ numbers $1, \frac{n}{n+1}, \frac{n}{n+1}, \dots, \frac{n}{n+1}$, apply the AGM Inequality, then take reciprocals.

(c) Use the AGM Inequality (Theorem 2.10) to show that $\left\{(1 - \frac{1}{n})^{-n}\right\}$ is a decreasing sequence (for $n = 2, 3, \dots$).

Hint: Consider the $n + 1$ numbers $1, 1 - \frac{1}{n}, 1 - \frac{1}{n}, \dots, 1 - \frac{1}{n}$.

Note: Many variations of Exercise 2.25 have been discovered and rediscovered over the years (e.g., [15, 22, 27, 29–31, 41, 57]). Other approaches can be found in [3, 17, 42, 44].

2.26. [59] Apply the AGM Inequality (Theorem 2.10) to the $n + k$ numbers

$$\left(1 + \frac{1}{n}\right), \left(1 + \frac{1}{n}\right), \dots, \left(1 + \frac{1}{n}\right), \frac{k-1}{k}, \frac{k-1}{k}, \dots, \frac{k-1}{k}$$

to show that

$$\left(1 + \frac{1}{n}\right)^n < \left(\frac{k}{k-1}\right)^k.$$

So, for example, taking $k = 6$, we may conclude that $e \leq \left(\frac{6}{5}\right)^6 \cong 2.986 < 3$.

2.27. [22] Use $(1 + 1/n)^n < e$ and induction to show that $(n/e)^n < n!$.

2.28. [32, 49] Let

$$p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

be a polynomial with roots $x_1, x_2, \dots, x_n$.

(a) Show that $a_0 = (-1)^n x_1 x_2 \dots x_n$.

(b) Show that $a_{n-1} = -\sum_{j=1}^n x_j$.

(c) Show that $a_1 = (-1)^{n-1} [x_2 x_3 \dots x_n + x_1 x_3 \dots x_n + \dots + x_1 x_2 \dots x_{n-1}]$.

(d) Show that if all of the roots are positive, then $a_1 a_{n-1} / a_0 \geq n^2$.

2.29. Suppose a quadrilateral has side lengths $a, b, c, d > 0$ and denote by s its semi perimeter: $s = (a + b + c + d)/2$. **Bretschneider's formula** says that the area of the quadrilateral is given by

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cos^2(\theta)},$$

where θ is half of the sum of any pair of opposite angles. If the quadrilateral can be inscribed in a circle then elementary geometry shows that $\theta = \pi/2$ and we get **Brahmagupta's formula**

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

(And if $d = 0$ then the quadrilateral is in fact a triangle and we get Heron's formula.) Show that among all quadrilaterals with a given perimeter, the square has the largest area.

2.30. In our proof (that is, K.M. Chong's) of the AGM Inequality (Theorem 2.10) we focused on A and used the inequality (2.1). Fill in the details of the following proof, which focuses instead on G .

(a) Show (assuming again $a_1 \leq a_2 \leq \dots \leq a_n$) that

$$a_1 + a_n - G \geq \frac{a_1 a_n}{G}.$$

(b) Use this to prove the AGM Inequality.

2.31. Fill in the details of H. Dorrie's beautiful 1921 proof of the AGM Inequality (Theorem 2.10), as follows. (This proof was rediscovered by P.P. Korovkin in 1952 [22] and again by G. Ehlers in 1954 [5].) Lemma 2.7 is the case $n = 2$, so we proceed by induction, assuming that the result is true for $n - 1$ numbers. What we want to show is that $\sum_{j=1}^n a_j \geq nG$.

(a) Argue that since $G^n = \prod_{j=1}^n a_j$, at least one a_j must be $\leq G$, and some other a_j must be $\geq G$. So we may assume that $a_1 \leq G$ and $a_2 \geq G$.

(b) Show that $a_1 \leq G$ and $a_2 \geq G$ imply that

$$a_1 + a_2 \geq G + \frac{a_1 a_2}{G}, \quad \text{and so} \quad \sum_{j=1}^n a_j \geq G + \frac{a_1 a_2}{G} + \sum_{j=3}^n a_j.$$

(c) Now apply the assumed result to the $n - 1$ numbers $\frac{a_1 a_2}{G}, a_3, a_4, \dots, a_n$.

2.32. [7, 50] Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers. Show that

$$1 + \prod_{j=1}^n a_j^{1/n} \leq \prod_{j=1}^n (1 + a_j)^{1/n}.$$

Hint: Consider the left-hand side divided by the right-hand side, apply the AGM Inequality (Theorem 2.10), then tidy up.

2.33. [11] Let A, B and C be the interior angles of a triangle. Show that

$$\sin(A) + \sin(B) + \sin(C) \leq \frac{3\sqrt{3}}{2}.$$

Hint:

$$A + B + C = \pi \Rightarrow \sin(A) + \sin(B) + \sin(C) = 4 \cos(A/2) \cos(B/2) \cos(C/2).$$

2.34. [6] Prove that

$$(\sqrt{2} - 1) (\sqrt[3]{6} - \sqrt{2}) \dots \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) < \frac{n!}{(n+1)^n}.$$

2.35. Analyze Cauchy's proof of the AGM Inequality (Theorem 2.10) given at the end of Sect. 2.2 to obtain necessary and sufficient conditions for equality.

2.36. In Cauchy's proof of the AGM Inequality (Theorem 2.10) given at the end of Sect. 2.2, we focused on A and had (for $2^m > n$):

$$A = \frac{a_1 + a_2 + \cdots + a_n + (2^m - n)A}{2^m}.$$

Then we applied the result for the 2^m case. For a proof which focuses instead on G , verify (for $2^m > n$) that

$$G^{2^m} = a_1 \cdot a_2 \cdots a_n \cdot G^{2^m - n},$$

then apply the result for the 2^m case.

2.37. We used Lemma 2.7 to prove the Cauchy–Schwarz Inequality (Theorem 2.18). Then we used the Cauchy–Schwarz Inequality to show that $A \leq R$. Show that $A \leq R$ using Lemma 2.7 directly. When does equality hold?

2.38. Fill in the details of the following proof of the Cauchy–Schwarz Inequality (Theorem 2.18), which is very similar to Schwarz's.

(a) Dispense with the case $a_1 = a_2 = \cdots = a_n = 0$.

(b) Expand the sum in the expression $0 \leq \sum_{j=1}^n (ta_j + b_j)^2$.

(c) Set $t = -\sum_{j=1}^n a_j b_j / \sum_{j=1}^n a_j^2$.

(This is the t at which the quadratic $\sum_{j=1}^n (ta_j + b_j)^2$ attains its minimum.)

2.39. Fill in the details of another proof of the Cauchy–Schwarz Inequality (Theorem 2.18), as follows.

(a) Replace a with a^2 and b with b^2 in Lemma 2.7 to get $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$.

(b) Write $ab = \sqrt{t}a \frac{1}{\sqrt{t}}b$ in (a) to show that for numbers a, b and any $t > 0$,

$$ab \leq \frac{t}{2}a^2 + \frac{1}{2t}b^2.$$

(c) Now write $a_j b_j = \sqrt{t}a_j \frac{1}{\sqrt{t}}b_j$ then sum from $j = 1$ to n to get

$$\sum_{j=1}^n a_j b_j \leq \frac{t}{2} \sum_{j=1}^n a_j^2 + \frac{1}{2t} \sum_{j=1}^n b_j^2.$$

(d) Dispense with the case $a_1 = a_2 = \cdots = a_n = 0$, then set

$$t = \left(\sum_{j=1}^n b_j^2 \right)^{1/2} / \left(\sum_{j=1}^n a_j^2 \right)^{1/2},$$

then simplify. (This is the t at which $\frac{t}{2} \sum_{j=1}^n a_j^2 + \frac{1}{2t} \sum_{j=1}^n b_j^2$ attains its minimum.)

2.40. Apply Schwarz's idea, as in his proof of the Cauchy–Schwarz Inequality (Theorem 2.18), to $\sum_{j=1}^n (a_j + t)^2$. What do you get? Can you prove whatever you got using the Cauchy–Schwarz Inequality?

2.41. [53] Fill in the details of the following proof of the Cauchy–Schwarz Inequality (Theorem 2.18), which is quite possibly just as slick as Schwarz's. Observe that

$$\frac{\sum_{j=1}^n a_j b_j}{\sqrt{\sum_{j=1}^n a_j^2} \sqrt{\sum_{j=1}^n b_j^2}} = 1 - \frac{1}{2} \sum_{j=1}^n \left(\frac{a_j}{\sqrt{\sum_{j=1}^n a_j^2}} - \frac{b_j}{\sqrt{\sum_{j=1}^n b_j^2}} \right)^2.$$

2.42. Fill in the details of the following (ostensibly) different proof of the Cauchy–Schwarz Inequality (Theorem 2.18).

(a) Replace a with a^2 and b with b^2 in Lemma 2.7 to get $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$.

(b) Dispense with the cases $a_1 = a_2 = \cdots = a_n = 0$ or $b_1 = b_2 = \cdots = b_n = 0$.

(c) Set $a = a_j \left(\sum_{j=1}^n a_j^2 \right)^{-1/2}$ and $b = b_j \left(\sum_{j=1}^n b_j^2 \right)^{-1/2}$ in (a), then sum from 1 to n .

2.43. Find necessary and sufficient conditions for equality to hold in the Cauchy–Schwarz Inequality (Theorem 2.18).

2.44. [33] Let $a_1, a_2, \dots, a_n$ be positive real numbers and let $r \leq n$ be a positive integer. Set

$$A_1 = \frac{1}{r} \sum_{j=1}^r a_j, \quad A = \frac{1}{n} \sum_{j=1}^n a_j, \quad \text{and} \quad \sigma^2 = \frac{1}{n} \sum_{j=1}^n (a_j - A)^2.$$

The number σ^2 is called the **variance** of $a_1, a_2, \dots, a_n$. Show that

$$r(A_1 - A)^2 \leq (n - r)\sigma^2.$$

2.45. [58] Let $x_1, x_2, \dots, x_n \in \mathbf{R}$. Show that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \dots + \frac{x_n}{1+x_1^2+x_2^2+\dots+x_n^2} \leq \sqrt{n}.$$

2.46. [14] Show that

$$\left(\sum_{k=1}^n \sqrt{\frac{k - \sqrt{k^2 - 1}}{\sqrt{k(k+1)}}} \right)^2 \leq n \sqrt{\frac{n}{n+1}}.$$

2.47. [23, 38] For n data points $(a_1, b_1), \dots, (a_n, b_n)$,

$$A = \frac{1}{n} \sum_{j=1}^n a_j, \quad \text{and} \quad B = \frac{1}{n} \sum_{j=1}^n b_j,$$

Pearson's coefficient of linear correlation is

$$\rho = \frac{\sum_{j=1}^n (a_j - A)(b_j - B)}{\sqrt{\sum_{j=1}^n (a_j - A)^2} \sqrt{\sum_{j=1}^n (b_j - B)^2}}.$$

Clearly the Cauchy–Schwarz Inequality (Theorem 2.18) implies that $|\rho| \leq 1$. Show that $|\rho| \leq 1$ implies the Cauchy–Schwarz Inequality. (For readers who know a little linear algebra, [21] contains a neat relationship between ρ and something called the *Gram determinant*.)

2.48. [1, 50]

(a) Show that for $x_1, x_2, \dots, x_n \in \mathbf{R}$ and $y_1, y_2, \dots, y_n > 0$,

$$\frac{(x_1 + x_2 + \dots + x_n)^2}{y_1 + y_2 + \dots + y_n} \leq \frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \dots + \frac{x_n^2}{y_n}.$$

(b) Set $x_j = a_j b_j$ and $y_j = b_j^2$ to obtain the Cauchy–Schwarz Inequality (Theorem 2.18).

(c) Let $a, b, c > 0$. Use (a) to obtain **Nesbitt's Inequality**:

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

(d) Use (a) to show that for $a, b > 0$,

$$a^4 + b^4 \geq \frac{1}{8}(a^4 + b^4).$$

2.49. [8, 46] Let $a, b, c > 0$.

(a) Show that if

$$a \cos^2(x) + b \sin^2(x) < c,$$

then

$$\sqrt{a} \cos^2(x) + \sqrt{b} \sin^2(x) < \sqrt{c}.$$

(b) Show that

$$(abc)^{2/3} \leq \frac{ab + ac + bc}{3} \leq \left(\frac{a + b + c}{3} \right)^2.$$

2.50. [20] We saw in Remark 2.14 that the isoperimetric inequality for an n -sided polygon with area T and perimeter P is

$$T \leq \frac{P^2}{4n \tan(\pi/n)}.$$

Show that if $a_1, a_2, \dots, a_n$ are the side lengths of an n -sided polygon, then

$$\sum_{j=1}^n a_j^2 \geq 4T \tan(\pi/n).$$

2.51. Let $a_1, a_2, \dots, a_n$, and $b_1, b_2, \dots, b_n$ be real numbers, with $\sum_{j=1}^n b_j = 0$. Show that

$$\left(\sum_{j=1}^n a_j b_j \right)^2 \leq \left(\sum_{j=1}^n a_j^2 - \left(\sum_{j=1}^n a_j \right)^2 \right) \sum_{j=1}^n b_j^2.$$

2.52. cf. [36] Let $a_1, a_2, \dots, a_n$, and $b_1, b_2, \dots, b_n$ be real numbers with $0 < a \leq a_j \leq A$ and $0 < b \leq b_j \leq B$. Fill in the following details to obtain a **reversed version of the Cauchy-Schwarz Inequality**:

$$\frac{\sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2}{\left(\sum_{j=1}^n a_j b_j \right)^2} \leq \frac{1}{4} \left(\sqrt{\frac{AB}{ab}} + \sqrt{\frac{ab}{AB}} \right).$$

- (a) Verify that $\left(\frac{a_j}{b_j} - \frac{a}{B}\right)\left(\frac{b_j}{a_j} - \frac{A}{b}\right) \leq 0$.
 (b) Use this to verify that

$$a_j^2 + \frac{aA}{bB}b_j^2 \leq \left(\frac{a}{B} + \frac{A}{b}\right)a_jb_j.$$

- (c) Write

$$\left(\sum_{j=1}^n a_j^2\right)^{1/2} \left(\sum_{j=1}^n b_j^2\right)^{1/2} = \sqrt{\frac{Bb}{Aa}} \left(\sum_{j=1}^n a_j^2\right)^{1/2} \left(\sum_{j=1}^n \frac{Aa}{Bb} b_j^2\right)^{1/2},$$

then use Lemma 2.7 and (b) to obtain the desired result.

2.53. Use the Cauchy–Schwarz Inequality (Theorem 2.18) to prove **Minkowski’s Inequality**: Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbf{R}$. Then

$$\left(\sum_{j=1}^n (a_j + b_j)^2\right)^{1/2} \leq \left(\sum_{j=1}^n a_j^2\right)^{1/2} + \left(\sum_{j=1}^n b_j^2\right)^{1/2}.$$

(Notice that if $n = 1$ this is simply the triangle inequality $|a + b| \leq |a| + |b|$.)

Hint: Write $(a_j + b_j)^2 = a_j(a_j + b_j) + b_j(a_j + b_j)$, then sum, then apply the Cauchy–Schwarz Inequality (Theorem 2.18) to each piece.

2.54. [45] The Cauchy–Schwarz Inequality (Theorem 2.18) gives an upper bound for $\sum_{j=1}^n a_j b_j$. Under certain circumstances, a lower bound is given by **Chebyshev’s**

Inequality: Let $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ be sequences of real numbers, with either both increasing or both decreasing. Then

$$\frac{1}{n} \sum_{j=1}^n a_j \cdot \frac{1}{n} \sum_{j=1}^n b_j \leq \frac{1}{n} \sum_{j=1}^n a_j b_j.$$

And the inequality is reversed if the sequences have opposite monotonicity. Fill in the details of the following proof of Chebyshev’s Inequality, for the a'_j s and b'_j s both increasing. (The other case is handled similarly.) First, let $A = \frac{1}{n} \sum_{j=1}^n a_j$.

- (a) Show that there is k between 1 and n such that

$$a_1 \leq a_2 \leq \dots \leq a_k \leq A \leq a_{k+1} \leq \dots \leq a_n.$$

- (b) Conclude that

$$(a_j - A)(b_j - b_k) \geq 0 \text{ for } j = 1, 2, \dots, n,$$

and therefore

$$\frac{1}{n} \sum_{j=1}^n (a_j - A)(b_j - b_k) \geq 0.$$

(c) Expand then simplify the sum on the left hand side in (b).

2.55. [45] (If you did Exercise 2.54.) (a) Use **Chebyshev's Inequality** to prove that for $0 \leq a_1 \leq a_2 \leq \dots \leq a_n$ and $n \geq 1$,

$$\left(\frac{1}{n} \sum_{j=1}^n a_j \right)^n \leq \frac{1}{n} \sum_{j=1}^n a_j^n.$$

(b) Let a, b, c be the side lengths of a triangle with area T and perimeter P . Show that $a^3 + b^3 + c^3 \geq \frac{4\sqrt{3}}{3}PT$ and that $a^4 + b^4 + c^4 \geq 16T^2$.

2.56. [40] (If you did Exercise 2.54.) Use **Chebyshev's Inequality** and the AGM Inequality (Theorem 2.10) to prove that for $0 < a_1 \leq a_2 \leq \dots \leq a_n$,

$$\sum_{j=1}^n a_j^{n+1} \geq a_1 a_2 \dots a_n \sum_{j=1}^n a_j.$$

2.57. For $a_1, a_2, \dots, a_n \in \mathbf{R}$, their **variance** is the number $\sigma^2 = \frac{1}{n} \sum_{j=1}^n (a_j - A)^2$,

where $A = \frac{1}{n} \sum_{j=1}^n a_j$ is their Arithmetic Mean. Suppose that $m \leq a_j \leq M$ for all j .

(a) Verify that

$$\frac{1}{n} \sum_{j=1}^n (a_j - A)^2 = (M - A)(A - m) - \frac{1}{n} \sum_{j=1}^n (M - a_j)(a_j - m),$$

in order to conclude that $\frac{1}{n} \sum_{j=1}^n (a_j - A)^2 \leq (M - A)(A - m)$. (This inequality was obtained differently, and generalized considerably, in [4].)

(b) Show that this inequality is better than, that is, is a *refinement* of **Popoviciu's Inequality**:

$$\frac{1}{n} \sum_{j=1}^n (a_j - A)^2 \leq \frac{1}{4}(M - m)^2.$$

Hint: Show that the quadratic $(Q - x)(x - q)$ is maximized when $x = \frac{1}{2}(Q + q)$.

2.58. [32]

- (a) Extend Exercise 2.57 to prove **Grüss's Inequality**: Let $a_1, a_2, \dots, a_n$, and $b_1, b_2, \dots, b_n$ be real numbers, with $m \leq a_j \leq M$ and $\gamma \leq b_j \leq \Gamma$. Then

$$\left| \frac{1}{n} \sum_{j=1}^n a_j b_j - \frac{1}{n} \sum_{j=1}^n a_j \cdot \frac{1}{n} \sum_{j=1}^n b_j \right| \leq \frac{1}{4} (M - m)(\Gamma - \gamma).$$

Hint: Let $A = \frac{1}{n} \sum_{j=1}^n a_j$, $B = \frac{1}{n} \sum_{j=1}^n b_j$ and begin by applying the Cauchy–Schwarz Inequality (Theorem 2.18) to

$$\left(\frac{1}{n} \sum_{j=1}^n (a_j - A)(b_j - B) \right)^2.$$

- (b) Show by providing an example (take $a_j = b_j$ for simplicity) that the constant $1/4$ in Grüss's Inequality cannot be replaced by any smaller number. That is, the $1/4$ is *sharp*.

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