

Preface

Many college students take a couple of courses in calculus. Afterwards, they either (i) take no more calculus, (ii) take calculus of several variables, or (iii) take real analysis. This book offers another option, not exclusive from options (ii) or (iii).

In the first two college calculus courses, much attention is given (naturally) to preparing students for things to come. But typically there is little time devoted to appreciating the bigger picture or for generally admiring the scenery. Many of the most appealing aspects of the subject are often left for students to pick up on their own. Unfortunately, however, students seldom do so.

I have taught undergraduate real analysis and graduate real analysis for teachers for over 15 years. These courses have evolved in a direction which attempts to address these concerns, and this book is a product of the evolution. The main goal is to see how beautifully things fit together, while admiring the scenery along the way. There are a lot of things in this book that experts in real and classical analysis already know; part of the idea here is that there is no reason why a good calculus student should not know them as well.

The book could be used as a text for a third course in calculus of a single real variable, as a supplementary text for a first course in real analysis, or as a reference for anyone who teaches calculus. The book is almost entirely self-contained, but readers would be best to have already taken the equivalent of two one-semester courses in single variable calculus. Some familiarity with sequences and some experience with proofs would be beneficial, though not entirely necessary.

I have presented ideas in a manner which emphasizes *breadth* as much as *depth*. Throughout the text and in the exercises, alternative approaches to many topics are taken. Such explorations are frequently more meaningful than simply aiming for generalization. Indeed, different arguments offer different insights. But whenever I have diverged from what is customary, I have given the usual treatments their due attention.

Many of the methods, examples, and exercises in the text are adapted from papers in the recent mathematics literature, chiefly: *The American Mathematical Monthly*,

The College Mathematics Journal, *Mathematics Magazine*, *The Mathematical Gazette*, and the *International Journal of Mathematics and Mathematical Sciences in Education and Technology*. I hope that readers will be encouraged to read these and other mathematics journals. At the very least, they will find therein solutions to many of the exercises.

I have also tried to emphasize *pattern*: pattern of development, pattern of proof, pattern of argument, and pattern of generalization. In calculus many threads are related in many ways, but in the end it is a coherent subject. Theorems are often named to emphasize pattern, for example the Cauchy–Schwarz Inequality and the Cauchy–Schwarz Integral Inequality, Jensen’s Inequality and Jensen’s Integral Inequality, the Mean Value Theorem for Sums and the Mean Value Theorem for Integrals, etc.

- The real numbers are introduced carefully, but with an eye on economy. One can be something of an expert in calculus without necessarily knowing all there is to know about the real numbers. As presented here, the “completeness axiom” for the real numbers is the *Increasing Bounded Sequence Property*, rather than the Least Upper Bound Property or Cauchy-completeness, though the latter two notions are explored in some exercises. The Archimedean Property of the real numbers is used freely without explicit mention, but it too is addressed in a few exercises. The word “compact” is never used. The *Nested Interval Property*, a close cousin of the Increasing Bounded Sequence Property, also plays an important part, with bisection algorithms getting their fair attention.
- Important throughout the entire book is the pair of inequalities

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1} \quad \text{for } n = 1, 2, 3, \dots$$

where e is Euler’s number $e \cong 2.71828$. These estimates are frequently revisited, refined, and extended. Inequalities in general play a prominent role as well. The most important of these are Bernoulli’s Inequality, the Arithmetic Mean – Geometric Mean Inequality (the AGM Inequality for short), $1 + x \leq e^x$, and Jensen’s Inequality.

- Considerable emphasis is given to the symbiotic relationship between the exponential function and calculus itself. The former, for example, gives meaning to functions involving real exponents, it enables us to extend Bernoulli’s Inequality and the AGM Inequality, and many of their consequences.
- Considerable attention is devoted to three consequences of the Intermediate Value Theorem which are often overlooked: the Universal Chord Theorem, the Average Value Theorem for Sums and its weighted version, the Mean Value Theorem for Sums. The latter two are so named because of their integral analogues, the Average Value Theorem and the Mean Value Theorem for Integrals. In obtaining these, the Extreme Value Theorem is indispensable. The relationship between sums and integrals is emphasized throughout.

- The definite integral is developed as an extension of the notion of the average value of a continuous function evaluated at N points. Proving that the average value of a continuous function exists is deferred to Appendix A; the proof uses some rather sophisticated ideas which are not used elsewhere. The definite integral's relationship with area is then discussed. Readers who go on to study mathematical analysis will see that the integral as an average is a more enduring theme than the integral as area.
- Chapter 7 (Other Mean Value Theorems) contains some results which have a flavor similar to that of the Mean Value Theorem. Subsequent chapters are independent of this one.
- Chapter 12 (Classic Examples) is also independent of the rest of the book, except that Wallis's product in Sect. 12.1 is used in Chap. 13 to obtain the constant $\sqrt{2\pi}$ which appears in Stirling's formula.
- Some important series are studied, for example, Geometric series, p-series, the Alternating Harmonic series, the Gregory-Leibniz series, and some Taylor series. But series in general are not covered systematically. For example, there is no treatment of power series, tests for convergence, radius of convergence, etc.
- Quadrature rules are studied as means for doing calculus and studying inequalities, rather than being used for conventional numerical methods. Indeed, the quadrature rules are usually applied to a function whose definite integral is known. Particular attention is given to the Trapezoid and Midpoint Rules applied to convex/concave functions.
- Motivated largely by the Mean Value Theorem for the Second Derivative, error terms are studied in Chap. 14. An inequality can often be recast as an equality which contains an error term. Jensen's Inequality, the AGM Inequality, Young's Integral Inequality (among others), and quadrature rules are considered in this way.

Many of the topics in the book, even the better-known ones, have not been collected elsewhere in any single volume. And a good number of these have been published heretofore only in journals. As a result, this book is not in direct competition with any others. Still, there is naturally some overlap between this and other books. Most notably:

A Primer of Real Functions, by R.P. Boas Jr. (Math. Assoc. of America, 1981).

Excursions in Classical Analysis: Pathways to Advanced Problem Solving and Undergraduate Research, by H. Chen (Math. Assoc. of America, 2010).

The Cauchy-Schwarz Master Class, by J.M. Steele (Math. Assoc. of America and Cambridge University Press, 2004).

Inequalities, by G.H. Hardy, J.E. Littlewood & G. Polya (Cambridge Mathematical Library, 2nd edition, 1988).

Analytic Inequalities, by D.S. Mitrinović (Springer-Verlag, New York, 1970).

Mean Value Theorems and Functional Equations, by P.K. Sahoo & T. Riedel (World Scientific, Singapore, 1998).

I have tried to write the sort of book that I would use, that I would like to own as a reference, and that I would fairly recommend to others. To borrow a phrase from G. H. Hardy: I can hardly have failed completely, the subject matter being so attractive that only extravagant incompetence could make it dull.



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