Chapter 2
Classical Banach Spaces and Their Duals

In the next two sections, we will consider the classical sequence and function spaces. The main purpose of these sections is to make the necessary definitions and to identify the dual spaces for these classical spaces. We will therefore take for granted that the various Banach spaces are indeed Banach spaces—putting off until Sect. 2.3 the proofs that they are complete in the given norms.

2.1 Sequence Spaces

In the context of sequence spaces, we denote by e_n the sequence with 1 in the n^{th} coordinate, and 0 elsewhere, so that $e_n = (0, \dots, 0, 1, 0, \dots)$ for all $n \in \mathbb{N}$. Also, we let $e = (1, 1, 1, \dots)$ be the constant sequence with 1 in every coordinate (not to be confused with the base of the natural logarithm $e \approx 2.718$).

Definition 2.1 The set ℓ_p of *p-summable sequences* for $p \in [1, \infty)$ is the collection of sequences

$$\ell_p = \left\{ (\xi_1, \xi_2, \dots, \xi_n, \dots) : \sum_{n=1}^{\infty} |\xi_n|^p < \infty \right\}.$$

Define the *p-norm* on ℓ_p by

$$\|\xi\|_p = \left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{1/p}, \quad \xi = (\xi_n)_{n=1}^{\infty} \in \ell_p.$$

The set ℓ_p is a vector space under component-wise addition and scalar multiplication. (This is a nontrivial fact which we will take as given. [See Theorem A.27.]) Furthermore, ℓ_p is a Banach space when given the p-norm (for $1 \le p < \infty$). We leave the proof of this fact to the exercises. (See Exercise 2.7.)

The next lemma will identify the dual space of ℓ_p for $p \in (1, \infty)$.

Lemma 2.2 *For $p \in (1, \infty)$, the space ℓ_p^* can be identified with ℓ_q, where $\frac{1}{p} + \frac{1}{q} = 1$.*

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Proof For simplicity, we start by assuming the scalars are real. (We will consider the complex case at the end of the proof.)

We wish to identify the dual space of ℓ_p for $p \in (1, \infty)$ as the sequence space ℓ_q, where $\frac{1}{p} + \frac{1}{q} = 1$. First, we will demonstrate how elements in ℓ_q determine linear functionals on ℓ_p. Let $\eta = (\eta_n)_{n=1}^{\infty}$ be an element in ℓ_q and define a scalar-valued function ϕ_η on ℓ_p by

$$\phi_\eta(\xi) = \sum_{n=1}^{\infty} \xi_n \, \eta_n, \tag{2.1}$$

where $\xi = (\xi_n)_{n=1}^{\infty}$ is any sequence in ℓ_p. By Hölder's Inequality (Theorem A.29), this series is absolutely convergent (whence ϕ_η is linear) and

$$|\phi_\eta(\xi)| \le \left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |\eta_n|^q \right)^{1/q} = \|\xi\|_p \|\eta\|_q.$$

It follows that ϕ_η is a bounded linear functional on ℓ_p and $\|\phi_\eta\| \le \|\eta\|_q$.

We claim that $\|\phi_\eta\|$ is in fact equal to $\|\eta\|_q$. In order to show this, it will suffice to find a sequence ξ in ℓ_p such that $\|\xi\|_p = 1$ and $\phi_\eta(\xi) = \|\eta\|_q$. We begin by constructing a sequence $\zeta = (\zeta_n)_{n=1}^{\infty}$ so that $\zeta_n = |\eta_n|^{q-1} \, (\operatorname{sign} \eta_n)$ for each $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} |\zeta_n|^p = \sum_{n=1}^{\infty} |\eta_n|^{(q-1)p} = \sum_{n=1}^{\infty} |\eta_n|^q,$$

where $(q-1)p = q$ follows from the assumption that $\frac{1}{p} + \frac{1}{q} = 1$. Consequently, the sequence ζ is in ℓ_p and $\|\zeta\|_p = \|\eta\|_q^{q/p}$. Observe that

$$\phi_\eta(\zeta) = \sum_{n=1}^{\infty} \zeta_n \cdot \eta_n = \sum_{n=1}^{\infty} |\eta_n|^{q-1} \, (\operatorname{sign} \eta_n) \cdot \eta_n = \sum_{n=1}^{\infty} |\eta_n|^q = \|\eta\|_q^q.$$

Let $\xi = \frac{\zeta}{\|\zeta\|_p}$. Then ξ is a sequence in ℓ_p such that $\|\xi\|_p = 1$ and such that

$$\phi_\eta(\xi) = \frac{\phi_\eta(\zeta)}{\|\zeta\|_p} = \frac{\|\eta\|_q^q}{\|\eta\|_q^{q/p}} = \|\eta\|_q^{q - \frac{q}{p}} = \|\eta\|_q.$$

Therefore, for any $\eta = (\eta_n)_{n=1}^{\infty}$ in ℓ_q, there is a linear functional ϕ_η on ℓ_p such that $\|\eta\|_q = \|\phi_\eta\|$ and such that $\phi_\eta(\xi) = \sum_{n=1}^{\infty} \xi_n \eta_n$ for all $\xi = (\xi_n)_{n=1}^{\infty}$ in ℓ_p.

We have demonstrated that any sequence in ℓ_q determines a bounded linear functional on ℓ_p. Next, we will show that all linear functionals on ℓ_p can be obtained in this way. Let $\psi \in \ell_p^*$ and define a sequence $\eta = (\eta_i)_{i=1}^{\infty}$ by letting $\eta_i = \psi(e_i)$ for each $i \in \mathbb{N}$. We will show that the sequence η is an element of ℓ_q such that $\|\eta\|_q = \|\psi\|$ and such that $\psi(\xi) = \sum_{i=1}^{\infty} \xi_i \eta_i$ for all $\xi = (\xi_i)_{i=1}^{\infty}$ in ℓ_p.

First, we will show that η is in fact an element of ℓ_q. For each $i \in \mathbb{N}$, define $\zeta_i = |\eta_i|^{q-1}\,(\operatorname{sign}\eta_i)$. Then, for any $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} |\zeta_i|^p = \sum_{i=1}^{n} |\eta_i|^{(q-1)p} = \sum_{i=1}^{n} |\eta_i|^q.$$

Computing the ℓ_p-norm of the finite sequence $\sum_{i=1}^{n} \zeta_i\, e_i$, we conclude that

$$\left\| \sum_{i=1}^{n} \zeta_i\, e_i \right\|_p = \left(\sum_{i=1}^{n} |\zeta_i|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} |\eta_i|^q \right)^{\frac{1}{p}}.$$

By assumption, the linear functional ψ is bounded on ℓ_p, and consequently

$$\left| \psi\left(\sum_{i=1}^{n} \zeta_i\, e_i \right) \right| \le \|\psi\| \left\| \sum_{i=1}^{n} \zeta_i\, e_i \right\|_p = \|\psi\| \left(\sum_{i=1}^{n} |\eta_i|^q \right)^{\frac{1}{p}}. \tag{2.2}$$

However, computing directly, we obtain

$$\psi\left(\sum_{i=1}^{n} \zeta_i\, e_i \right) = \sum_{i=1}^{n} \zeta_i\, \psi(e_i) = \sum_{i=1}^{n} \zeta_i\, \eta_i = \sum_{i=1}^{n} |\eta_i|^q. \tag{2.3}$$

From (2.2) and (2.3), it follows that

$$\sum_{i=1}^{n} |\eta_i|^q \le \|\psi\| \left(\sum_{i=1}^{n} |\eta_i|^q \right)^{\frac{1}{p}}.$$

Dividing, we see that

$$\|\psi\| \ge \left(\sum_{i=1}^{n} |\eta_i|^q \right)^{1-\frac{1}{p}} = \left(\sum_{i=1}^{n} |\eta_i|^q \right)^{\frac{1}{q}}.$$

This inequality holds for all $n \in \mathbb{N}$, and so $\eta \in \ell_q$ and $\|\psi\| \ge \|\eta\|_q$.

It remains to show that $\|\psi\| \le \|\eta\|_q$ and that $\psi = \phi_\eta$, where ϕ_η is defined by (2.1). Since we have already demonstrated that $\|\phi_\eta\| \le \|\eta\|_q$, it suffices to show that $\psi = \phi_\eta$.

Suppose $\xi = (\xi_i)_{i=1}^{\infty} \in \ell_p$ and let $\xi^{(n)} = (\xi_1, \ldots, \xi_n, 0, \ldots)$ for each $n \in \mathbb{N}$. We claim that $\xi^{(n)}$ converges to ξ in the norm on ℓ_p. To see this, observe that $\sum_{i=1}^{\infty} |\xi_i|^p < \infty$, by assumption, and consequently

$$\lim_{n\to\infty} \left\| \xi - \xi^{(n)} \right\|_p = \lim_{n\to\infty} \left(\sum_{i=n+1}^{\infty} |\xi_i|^p \right)^{1/p} = 0. \tag{2.4}$$

Now, observe that

$$\psi\left(\xi^{(n)}\right) = \psi\left(\sum_{i=1}^{n} \xi_i\, e_i\right) = \sum_{i=1}^{n} \xi_i\, \psi(e_i) = \sum_{i=1}^{n} \xi_i\, \eta_i.$$

Therefore, by the continuity of ψ,

$$\psi(\xi) = \lim_{n\to\infty} \psi(\xi^{(n)}) = \sum_{i=1}^{\infty} \xi_i\, \eta_i = \phi_\eta(\xi).$$

It follows that $\psi = \phi_\eta$, which is the desired result.

Now assume the scalar field is $\mathbb{C}$. The proof in this case is essentially the same. However, when defining ζ_n, for $n \in \mathbb{N}$, let $\zeta_n = |\eta_n|^{q-1}\rho_n$, where ρ_n is a complex number such that $|\rho_n| = 1$ and $\eta_n\,\rho_n = |\eta_n|$. The argument proceeds as it did in the real case. □

The previous theorem identified the dual space of ℓ_p for $p \in (1, \infty)$ as the space ℓ_q, where $\frac{1}{p} + \frac{1}{q} = 1$, via the *dual space action*

$$\eta(\xi) = \sum_{n=1}^{\infty} \xi_n \eta_n,$$

where $\xi = (\xi_n)_{n=1}^{\infty}$ is in ℓ_p and $\eta = (\eta_n)_{n=1}^{\infty}$ is in ℓ_q. In the above equation, we write $\eta(\xi)$ as shorthand for $\phi_\eta(\xi)$, where ϕ_η is the linear functional corresponding to η that appears in (2.1). The object η is a sequence in the space ℓ_q, but we write $\eta(\xi)$ because we are viewing η as a linear functional in ℓ_p^*.

Next, we wish to identify the dual space of ℓ_1. In order to do this, we must introduce a new space of sequences.

Definition 2.3 The set ℓ_∞ of *bounded sequences* is the collection of sequences

$$\ell_\infty = \left\{(\xi_n)_{n=1}^{\infty} : \sup_{n\in\mathbb{N}} |\xi_n| < \infty\right\}.$$

Define the *supremum norm* on ℓ_∞ by

$$\|\xi\|_\infty = \sup_{n\in\mathbb{N}} |\xi_n|, \quad \xi = (\xi_n)_{n=1}^{\infty} \in \ell_\infty.$$

The set ℓ_∞ is a vector space under component-wise addition and scalar multiplication and is a Banach space when given the supremum norm.

Lemma 2.4 *The space ℓ_1^* can be identified with ℓ_∞.*

Proof The proof is similar to the proof of Lemma 2.2 and is left to the reader. (See Exercise 2.2.) As in Lemma 2.2, the dual space action is

$$\eta(\xi) = \sum_{n=1}^{\infty} \xi_n \eta_n,$$

where now $\xi = (\xi_n)_{n=1}^{\infty}$ is in ℓ_1 and $\eta = (\eta_n)_{n=1}^{\infty}$ is in ℓ_∞. □

Since we identify the dual space of ℓ_p with ℓ_q, it is standard to write $\ell_p^* = \ell_q$. When we write this, however, it is understood that we mean there is a way of identifying the linear functionals in ℓ_p^* with the sequences in ℓ_q and the identification is shown explicitly by the dual space action $\eta(\xi) = \sum_{n=1}^{\infty} \xi_n \eta_n$, where $\xi = (\xi_n)_{n=1}^{\infty}$ is in ℓ_p and $\eta = (\eta_n)_{n=1}^{\infty}$ is in ℓ_q.

We summarize our results in the following theorem.

Theorem 2.5 *Let $p \in [1, \infty)$ and suppose q is such that $\frac{1}{p} + \frac{1}{q} = 1$, with the convention that $q = \infty$ when $p = 1$. Then $\ell_p^* = \ell_q$.*

Proof See Lemmas 2.2 and 2.4.

The relationship between the exponents p and q in Theorem 2.5 motivates the next definition.

Definition 2.6 If $p \in [1, \infty)$ and if q is such that $\frac{1}{p} + \frac{1}{q} = 1$, with the convention that $q = \infty$ when $p = 1$, then p and q are called *conjugate exponents*.

If p and q are conjugate exponents that are both finite, then $\ell_p^* = \ell_q$ and $\ell_q^* = \ell_p$. Since $\ell_1^* = \ell_\infty$, it is natural to ask if ℓ_1 is the dual space of ℓ_∞. While $\ell_1 \subseteq \ell_\infty^*$, the spaces do not coincide. The argument used in the proof of Lemma 2.2 for $p \in (1, \infty)$ fails in the case $p = \infty$ because there exist bounded sequences $(\xi_n)_{n=1}^{\infty}$ that do not satisfy the equation that corresponds to (2.4) for the case $p = \infty$. That is, we can find $\xi \in \ell_\infty$ such that

$$\lim_{n\to\infty} \|\xi - \xi^{(n)}\|_\infty = \lim_{n\to\infty} \left(\sup_{k>n} |\xi_k| \right) \neq 0, \tag{2.5}$$

where $\xi^{(n)} = (\xi_1, \ldots, \xi_n, 0, \ldots)$. As a simple example, let $\xi = e$, the constant sequence having every term equal to 1. We have that $\|e\|_\infty = 1$, and so e is an element of ℓ_∞, but $\|e - e^{(n)}\|_\infty = 1$ for all $n \in \mathbb{N}$.

Let us now consider the space of sequences for which the limit in (2.1.5) is 0.

Definition 2.7 Let c_0 be the space of all sequences converging to 0:

$$c_0 = \left\{ (\xi_n)_{n=1}^{\infty} : \lim_{n\to\infty} \xi_n = 0 \right\}.$$

The space c_0 is a Banach space with the supremum norm

$$\|\xi\|_\infty = \sup_{n\in\mathbb{N}} |\xi_n|, \quad \xi = (\xi_n)_{n=1}^{\infty} \in c_0.$$

Theorem 2.8 $c_0^* = \ell_1$.

The proof is similar to that of Lemma 2.2 and is left as an exercise for the reader. (See Exercise 2.2.) The last sequence space we discuss is the space c.

Definition 2.9 Let c be the space of all convergent sequences:

$$c = \left\{ (\xi_n)_{n=1}^{\infty} : \lim_{n\to\infty} \xi_n \text{ exists} \right\}.$$

The space c is also a Banach space with the supremum norm. Perhaps surprisingly, the dual space of c is also ℓ_1, albeit with a slightly different dual space action. (See

Example 2.22.) It is straightforward to show that c_0 is a closed subspace of c, and in turn c is a closed subspace of ℓ_∞. (See Exercise 2.1.)

2.2 Function Spaces

In this section, let (Ω, Σ, μ) be a measure space, where μ is a positive measure. The theorems in this section are true for positive σ-finite measure spaces, but for simplicity we will assume $\mu(\Omega) < \infty$. As before, $\mathbb{K}$ denotes the underlying scalar field, which is either $\mathbb{R}$ or $\mathbb{C}$. We begin by recalling some definitions from measure theory. (See Appendix A for a more detailed discussion.)

Definition 2.10 If A is a subset of Ω, then the *characteristic function* of A is the function

$$\chi_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

The characteristic function of A is a measurable function if and only if A is a measurable subset of Ω.

Definition 2.11 For $p \in [1, \infty)$, the set of *p-integrable functions* (or *L_p-functions*) on (Ω, Σ, μ) is the collection

$$L_p(\Omega, \Sigma, \mu) = \left\{ f : \Omega \to \mathbb{K} \text{ a measurable function} : \int_\Omega |f|^p \, d\mu < \infty \right\}.$$

We often write this space as $L_p(\Omega, \mu)$ or $L_p(\mu)$ when there is no risk of confusion.

The set $L_p(\mu)$ is actually a collections of *equivalence classes* of measurable functions. Two functions in $L_p(\mu)$ are considered equivalent if they differ only on a set of μ-measure zero. Despite this, we will usually speak of the elements in $L_p(\mu)$ as functions, rather than equivalence classes of functions. We remark that the set $L_p(\mu)$ is a vector space under pointwise addition and scalar multiplication. (As with ℓ_p in the previous section, this is a nontrivial result. [See Theorem A.27.])

Define the *p-norm* on $L_p(\mu)$ by

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}}, \quad f \in L_p(\mu).$$

We will show that $L_p(\mu)$ is a Banach space in the p-norm (for $1 \leq p < \infty$) in the next section. (See Theorem 2.25.)

As in the case of sequence spaces, we must consider the case $p = \infty$ separately.

Definition 2.12 Let (Ω, Σ, μ) be a measure space. The *essential supremum norm* of a measurable function f is defined to be

$$\|f\|_\infty = \inf \{K : \mu(|f| > K) = 0\}.$$

The set of *essentially bounded functions* (or L_∞*-functions*) on (Ω, Σ, μ) is the collection

$$L_\infty(\Omega, \Sigma, \mu) = \{f : \Omega \to \mathbb{K} \text{ a measurable function} : \|f\|_\infty < \infty\}.$$

We often write $L_\infty(\Omega, \mu)$ or $L_\infty(\mu)$ when there is no risk of confusion.

As is the case for $L_p(\mu)$ when p is finite, the set $L_\infty(\mu)$ is a collection of equivalence classes of measurable functions and (as before) we consider two functions to be equal in $L_\infty(\mu)$ if they differ only on a set of μ-measure zero.

The set $L_\infty(\mu)$ is a vector space under pointwise addition and scalar multiplication. The essential supremum defines a norm on $L_\infty(\mu)$ and $L_\infty(\mu)$ is a Banach space when given this norm. (See Theorem 2.26.) We use the terminology "essentially bounded" to describe the functions in $L_\infty(\mu)$ and call $\|f\|_\infty$ the "essential supremum" of $|f|$ in $L_\infty(\mu)$, because the quantity $\|f\|_\infty$ is the smallest number K such that $|f| \le K$ a.e.(μ).

We wish to identify the dual space of $L_p(\mu)$, where $p \in [1, \infty)$. We will see that, analogous to the case of sequence spaces, the dual space of $L_p(\mu)$ is $L_q(\mu)$, where p and q are conjugate exponents. In this case, however, the dual action of $L_q(\mu)$ on $L_p(\mu)$ is given by integration. That is, if $f \in L_p(\mu)$ and $g \in L_q(\mu)$, then the dual action of g on f is given by

$$g(f) = \int_\Omega fg\,d\mu.$$

Notice that g is a function on Ω. Here, however, we write $g(f)$ because we view g as an element of the dual space $L_p(\mu)^*$. As before, we write $L_p(\mu)^* = L_q(\mu)$ to indicate the identification of linear functionals in $L_p(\mu)^*$ with elements of the function space $L_q(\mu)$.

Theorem 2.13 *Let (Ω, Σ, μ) be a positive finite measure space. If p and q are conjugate exponents, where $p \in [1, \infty)$, then $L_p(\mu)^* = L_q(\mu)$.*

Proof Start by assuming the scalars are real. We begin with the case $p \in (1, \infty)$. Let $g \in L_q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, and define a scalar-valued function ϕ_g on $L_p(\mu)$ by

$$\phi_g(f) = \int_\Omega fg\,d\mu, \quad f \in L_p(\mu). \tag{2.6}$$

We will show first that ϕ_g is a linear functional on $L_p(\mu)$ such that $\|\phi_g\| = \|g\|_q$, and then we will show that all linear functionals on $L_p(\mu)$ can be achieved in this way.

We note that ϕ_g is linear (by the linearity of the integral) and $|\phi_g(f)| \le \|f\|_p \|g\|_q$ (by Hölder's Inequality). Thus, ϕ_g is a bounded linear functional and $\|\phi_g\| \le \|g\|_q$. In order to show equality of the norms, it suffices to find a function f in $L_p(\mu)$ such that $\|f\|_p = 1$ and such that $\phi_g(f) = \|g\|_q$. First, define a scalar-valued function h on Ω by letting $h(x) = |g(x)|^{q-1}$ (sign $g(x)$) for each $x \in \Omega$. Then

$$\|h\|_p = \left(\int_\Omega |g|^{(q-1)p}\, d\mu\right)^{\frac{1}{p}} = \left(\int_\Omega |g|^q\, d\mu\right)^{\frac{1}{p}} = \|g\|_q^{q/p},$$

where $(q-1)p = q$ follows from the assumption that $\frac{1}{p} + \frac{1}{q} = 1$. It follows that h is in $L_p(\mu)$. Next, observe that $\phi_g(h) = \|g\|_q^q$. Now, let $f = \frac{h}{\|h\|_p}$. Then $\|f\|_p = 1$ and

$$\phi_g(f) = \frac{\phi_g(h)}{\|h\|_p} = \frac{\|g\|_q^q}{\|g\|_q^{q/p}} = \|g\|_q.$$

Therefore, ϕ_g is a bounded linear functional on $L_p(\mu)$ and $\|\phi_g\| = \|g\|_q$.

We now wish to show that any bounded linear functional in $L_p(\mu)^*$ can be written as in (2.6) for some g in $L_q(\mu)$. To that end, let ψ be a bounded linear functional in the dual space $L_p(\mu)^*$. Define a measure ν on (Ω, Σ) by $\nu(A) = \psi(\chi_A)$ for all $A \in \Sigma$. It is routine to show that ν is finitely additive (by the linearity of ψ), and it is countably additive by the continuity of ψ. We also claim that $\nu \ll \mu$. That is, ν is absolutely continuous with respect to μ. To see this, suppose $A \in \Sigma$ is such that $\mu(A) = 0$. Because ψ is bounded,

$$|\nu(A)| = |\psi(\chi_A)| \leq \|\psi\|\, \|\chi_A\|_{L_p(\mu)} = \|\psi\|\, \mu(A)^{1/p} = 0.$$

By the Radon–Nikodým Theorem (Theorem A.24), there exists a measurable function $g \in L_1(\mu)$ such that $\nu(A) = \int_A g\, d\mu$ for all $A \in \Sigma$. Therefore, for every $A \in \Sigma$,

$$\psi(\chi_A) = \nu(A) = \int_A g d\mu = \int_\Omega \chi_A g d\mu.$$

By linearity, it follows that $\psi(f) = \int_\Omega fg d\mu$ whenever f is a simple measurable function. Let $f \in L_\infty(\mu)$ be a real nonnegative essentially bounded measurable function. Since (Ω, Σ, μ) is a finite measure space, it follows that $f \in L_p(\mu)$. Thus, there exists a sequence of simple measurable functions $(f_n)_{n=1}^\infty$ such that $f_n \geq f_{n-1}$ for all $n \in \mathbb{N}$, and such that $\|f - f_n\|_p \to 0$ as $n \to \infty$. By the continuity of ψ,

$$\psi(f) = \lim_{n\to\infty} \psi(f_n) = \lim_{n\to\infty} \int_\Omega f_n\, g d\mu.$$

Therefore, by Lebesgue's Dominated Convergence Theorem (Theorem A.17),

$$\psi(f) = \int_\Omega fg d\mu, \quad f \in L_\infty(\mu) \cap L_p(\mu),\ f \geq 0.$$

To extend this to an arbitrary real function in $L_\infty(\mu) \cap L_p(\mu)$, let

$$f^+ = f\, \chi_{\{x: f(x) \geq 0\}} \quad \text{and} \quad f^- = -f\, \chi_{\{x: f(x) < 0\}},$$

and observe that $f = f^+ - f^-$.

We claim that $g \in L_q(\mu)$. For each $n \in \mathbb{N}$, define a function h_n on Ω by letting $h_n = \chi_{\{|g| \le n\}} |g|^{q-1} (\operatorname{sign} g)$. Then, for each $n \in \mathbb{N}$, we have $h_n \in L_\infty(\mu) \cap L_p(\mu)$ and

$$\psi(h_n) = \int_{\{|g| \le n\}} |g|^q \, d\mu.$$

By assumption, the linear functional ψ is bounded on $L_p(\mu)$, and so it follows that $|\psi(h_n)| \le \|\phi\| \, \|h_n\|_p$. Computing the L_p-norm of h_n, and once again observing that $(q-1)p = q$, we see that

$$\|h_n\|_p = \left(\int_{\{|g| \le n\}} |g|^{(q-1)p} \, d\mu \right)^{\frac{1}{p}} = \left(\int_{\{|g| \le n\}} |g|^q \, d\mu \right)^{\frac{1}{p}}.$$

Therefore,

$$\int_{\{|g| \le n\}} |g|^q \, d\mu = |\psi(h_n)| \le \|\psi\| \, \|h_n\|_p = \|\psi\| \left(\int_{\{|g| \le n\}} |g|^q \, d\mu \right)^{\frac{1}{p}}.$$

Dividing, we obtain

$$\|\psi\| \ge \left(\int_{\{|g| \le n\}} |g|^q \, d\mu \right)^{1-\frac{1}{p}} = \left(\int_{\{|g| \le n\}} |g|^q \, d\mu \right)^{\frac{1}{q}}.$$

Thus, by Fatou's Lemma (Theorem A.16),

$$\|g\|_q = \left(\int_\Omega \liminf_{n \to \infty} \chi_{\{|g| \le n\}} |g|^q \, d\mu \right)^{1/q} \le \left(\liminf_{n \to \infty} \int_\Omega \chi_{\{|g| \le n\}} |g|^q \, d\mu \right)^{1/q} \le \|\psi\|.$$

Therefore, g is in $L_q(\mu)$ and $\|g\|_q \le \|\psi\|$.

It remains to show that $\psi = \phi_g$. If f is a real nonnegative function in $L_p(\mu)$, then we may choose a sequence $(f_n)_{n=1}^\infty$ of simple measurable functions such that f_n increases to f almost everywhere and such that $f_n \to f$ in the L_p-norm as $n \to \infty$. We have already established that $\psi(f_n) = \int_\Omega f_n \, g \, d\mu$ for all $n \in \mathbb{N}$. We also know that $\psi(f_n) \to \psi(f)$ as $n \to \infty$, because ψ is a continuous linear functional on $L_p(\mu)$. Since $f \in L_p(\mu)$ and $g \in L_q(\mu)$, it follows that $fg \in L_1(\mu)$, by Hölder's Inequality. Thus,

$$\lim_{n \to \infty} \int_\Omega f_n \, g \, d\mu = \int_\Omega fg \, d\mu,$$

by Lebesgue's Dominated Convergence Theorem. Therefore, $\psi(f) = \phi_g(f)$ for all nonnegative functions f in $L_p(\mu)$. As before, we may extend this to all real functions in $L_p(\mu)$ by writing $f = f^+ - f^-$.

We have now proven the theorem for $p \in (1, \infty)$ when the scalar field is $\mathbb{R}$. In order to extend this result to $\mathbb{C}$, we argue as above, but define the function h by the rule $h = |g|^{q-1} \rho$, where $\rho : \Omega \to \mathbb{C}$ is a function such that $|\rho| = 1$ and $g \rho = |g|$. Similarly, we let $h_n = \chi_{\{|g| \le n\}} |g|^{q-1} \rho$ for each $n \in \mathbb{N}$. This argument proves that ϕ_g is a bounded linear functional on $L_p(\mu)$ for all $g \in L_q(\mu)$. It also proves that for any bounded linear functional ψ on $L_p(\mu)$, there exists a function $g \in L_q(\mu)$ such that

$\psi(f) = \phi_g(f)$ for all *real* functions f in $L_p(\mu)$. To extend this result to complex functions f in $L_p(\mu)$, write $f = \Re(f) + i\,\Im(f)$, where $\Re(f)$ and $\Im(f)$ are the real and imaginary parts of f, respectively, and use linearity.

For $p = 1$, the proof is similar to the case when $p \in (1, \infty)$ and is left to the reader. (See Exercise 2.3.) $\square$

As is the case with sequence spaces, the dual of $L_\infty(\mu)$ need not be $L_1(\mu)$.

Theorem 2.13 remains true when μ is a positive σ-finite measure. Such a case can be seen in the following example.

Example 2.14 Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, m)$, where $2^{\mathbb{N}}$ denotes the power set of $\mathbb{N}$ (the collection of all subsets of $\mathbb{N}$), and m is counting measure on $\mathbb{N}$ (i.e., the set function for which $m(A)$ is the cardinality of the set $A \subseteq \mathbb{N}$). Suppose that $f \in L_p(\mathbb{N}, 2^{\mathbb{N}}, m)$, where $p \in [1, \infty)$. Then,

$$\|f\|_p = \left(\int_{\mathbb{N}} |f|^p \, dm \right)^{1/p} = \left(\sum_{n=1}^{\infty} |f(n)|^p \right)^{1/p}.$$

We see that f in $L_p(\mathbb{N}, 2^{\mathbb{N}}, m)$ corresponds to the sequence $(f(n))_{n=1}^{\infty}$ in ℓ_p. The same conclusion holds for $p = \infty$. Therefore,

$$L_p(\mathbb{N}, 2^{\mathbb{N}}, m) = \ell_p, \quad 1 \le p \le \infty.$$

Let us now consider spaces of continuous functions.

Definition 2.15 Let K be a compact metric space. We denote the collection of scalar-valued continuous functions on K by $C(K)$. Define the *supremum norm* on $C(K)$ by

$$\|f\|_\infty = \sup_{t \in K} |f(t)|, \quad f \in C(K).$$

The set $C(K)$ is a vector space under pointwise addition and scalar multiplication and is a Banach space when given the supremum norm. (See Theorem 2.27.) If we wish to emphasize the underlying scalar field, we will write $C_{\mathbb{R}}(K)$ or $C_{\mathbb{C}}(K)$.

Observe that, for $f \in C(K)$, the quantity $\|f\|_\infty$ is actually the maximum of $|f|$, since a continuous function attains its supremum on compact sets.

Remark 2.16 We use the notation $\|\cdot\|_\infty$ to represent both the supremum norm on $C(K)$ (for a compact metric space K) and the essential supremum norm on $L_\infty(\mu)$ (for a measure space $(\Omega, \mathcal{A}, \mu)$). If there is any risk of confusion, we will write $\|\cdot\|_{C(K)}$ and $\|\cdot\|_{L_\infty(\mu)}$ to denote the norm on $C(K)$ and $L_\infty(\mu)$, respectively.

We wish to identify the dual space of $C(K)$. To that end, we consider the following example, where $K = [0, 1]$. In this case, we write $C(K) = C[0, 1]$.

Example 2.17 Consider the following linear functionals on $C[0, 1]$:

(a) *(Integration)* $f \to \displaystyle\int_0^1 f(t)\, dt$.

(b) *(Point evaluation)* $f \to f(s)$ for $s \in K$.

(c) *(Integration against L_1 functions)* $f \to \displaystyle\int_0^1 f(t)\, g(t)\, dt$ for $g \in L_1(0, 1)$.

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