

Non-negative Self-adjoint Extensions in Rigged Hilbert Space

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Abstract. We study non-negative self-adjoint extensions of a non densely defined non-negative symmetric operator \dot{A} with the exit in the rigged Hilbert space constructed by means of the adjoint operator \dot{A}^* (bi-extensions). Criteria of existence and descriptions of such extensions and associated closed forms are obtained. Moreover, we introduce the concept of an extremal non-negative bi-extension and provide its complete description. After that we state and prove the existence and uniqueness results for extremal non-negative bi-extensions in terms of the Kreĭn–von Neumann and Friedrichs extensions of a given non-negative symmetric operator. Further, the connections between positive boundary triplets and non-negative self-adjoint bi-extensions are presented.

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1. Introduction

In order to describe the main ideas and results of the current paper, we first recall the notion of the rigged Hilbert spaces. A triplet $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigged Hilbert space constructed upon a symmetric operator \dot{A} in a Hilbert space \mathcal{H} if $\mathcal{H}_+ = \text{Dom}(\dot{A}^*)$ with an inner product defined by

$$(f, g)_+ = (f, g) + (\dot{A}^* f, \dot{A}^* g), \quad f, g \in \text{Dom}(\dot{A}^*). \quad (1.1)$$

and \mathcal{H}_- is the space of all anti-linear functional on \mathcal{H}_+ that are continuous w.r.t. $\|\cdot\|_+$. An extension theory of symmetric operators in rigged Hilbert spaces was thoroughly covered in [7]. One of the objects of this theory is a self-adjoint bi-extension \mathbb{A} of a symmetric operator \dot{A} whose definition is given below in Preliminaries section. Throughout this entire article, by a non-negative operator in a

rigged Hilbert space we understand an operator \mathbb{T} such that $(\mathbb{T}f, f) \geq 0$ for all $f \in \text{Dom}(\mathbb{T})$. In this paper we put our main focus on non-negative bi-extensions of a non-negative symmetric operator. The theory of extensions of non-negative symmetric operators originates in the works of von Neumann, Friedrichs, and Kreĭn (see survey [12]). That is why most of the main results of the paper are given in terms of the Kreĭn–von Neumann and Friedrichs extensions of a given non-negative symmetric operator that are described in details in Section 3. The existence conditions for non-negative bi-extensions are presented in Section 4 and rely on the concepts of disjointness and transversality of self-adjoint extensions that were introduced in Preliminaries. Here we also give a descriptions of the non-negative self-adjoint bi-extensions and associated closed quadratic forms. Section 5 is solely dedicated to extremal self-adjoint bi-extensions and contains existence and uniqueness results. The connections between non-negative self-adjoint bi-extensions and boundary triplets is established in Section 6.

The results of the current paper complement and enhance the classical results of the theory of extensions of non-negative symmetric operators as well as some new developments of this theory in rigged Hilbert spaces discussed in [7], [8]. Applications of these results may be used in solving realization problems for Stieltjes and inverse Stieltjes functions in infinite-dimensional Hilbert spaces similarly to finite-dimensional cases treated in [13] and [14].

2. Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from \mathcal{H}_1 to \mathcal{H}_2 . Let \dot{A} be a closed, densely defined, symmetric operator in a Hilbert space \mathcal{H} with inner product $(f, g), f, g \in \mathcal{H}$.

Consider the rigged Hilbert space (see [15], [31]) $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, where $\mathcal{H}_+ = \text{Dom}(\dot{A}^*)$ and $(f, g)_+$ is defined by (1.1). Note that by the second representation theorem [20] we have

$$\text{Dom}(I + \dot{A}\dot{A}^*)^{1/2} = \mathcal{H}_+, \quad \text{Ran}(I + \dot{A}\dot{A}^*)^{1/2} = \mathcal{H},$$

and

$$(f, g)_+ = ((I + \dot{A}\dot{A}^*)^{1/2}f, (I + \dot{A}\dot{A}^*)^{1/2}g), \quad f, g \in \mathcal{H}.$$

The Hilbert space \mathcal{H}_+ admits the following (+)-orthogonal decomposition

$$\mathcal{H}_+ = \text{Dom}(\dot{A}) \oplus \mathfrak{N}_{-i} \oplus \mathfrak{N}_i,$$

where $\mathfrak{N}_\lambda := \ker(\dot{A}^* - \lambda I)$, $\text{Im } \lambda \neq 0$ is the defect subspace of \dot{A} . Denote

$$\mathfrak{M} = \mathfrak{N}_{-i} \oplus \mathfrak{N}_i$$

and let

$$P_{\text{Dom}(\dot{A})}^+, P_{\mathfrak{N}_{-i}}^+, P_{\mathfrak{N}_i}^+, P_{\mathfrak{M}}^+$$

be (+)-orthogonal projections in \mathcal{H}_+ onto $\text{Dom}(\dot{A})$, \mathfrak{N}_{-i} , \mathfrak{N}_i , and \mathfrak{M} , respectively.

Recall that \mathcal{H}_- can be identified with the space of all anti-linear functional on \mathcal{H}_+ and continuous w.r.t. $\|\cdot\|_+$. Let \mathcal{R} be the *Riesz–Berezansky operator* (see [7]) which maps \mathcal{H}_- onto \mathcal{H}_+ such that $(f, g) = (f, \mathcal{R}g)_+$ and $\|\mathcal{R}g\|_+ = \|g\|_-$ for all $f \in \mathcal{H}_+$, $g \in \mathcal{H}_-$. Clearly

$$\mathcal{R} \upharpoonright \mathcal{H} = (I + \dot{A}\dot{A}^*)^{-1}.$$

Definition 2.1. Let \mathbb{A} be a linear operator with $\text{Dom}(\mathbb{A})$ dense in \mathcal{H}_+ and with values in \mathcal{H}_- . Then the adjoint operator \mathbb{A}^* is defined as follows:

$$\begin{aligned} \text{Dom}(\mathbb{A}^*) &= \{u \in \mathcal{H}_+ : \exists \psi \in \mathcal{H}_- \mid (u, \mathbb{A}f) = (\psi, f) \text{ for all } f \in \text{Dom}(\mathbb{A})\}, \\ \mathbb{A}^*u &= \psi. \end{aligned}$$

It is easy to see $\mathcal{R}\mathbb{A}^* : \mathcal{H}_+ \supseteq \text{Dom}(\mathbb{A}^*) \rightarrow \mathcal{H}_+$ is the $(+)$ -adjoint operator to $\mathcal{R}\mathbb{A}$ acting in \mathcal{H}_+ .

Definition 2.2. An operator $\mathbb{A} : \mathcal{H}_+ \supset \text{Dom}(\mathbb{A}) \rightarrow \mathcal{H}_-$ is called a *generalized self-adjoint* if $\text{Dom}(\mathbb{A})$ is dense in \mathcal{H}_+ and $\mathbb{A}^* = \mathbb{A}$.

Definition 2.3. A generalized self-adjoint operator $\mathcal{H}_+ \supset \text{Dom}(\mathbb{A}) \rightarrow \mathcal{H}_-$ is called *self-adjoint bi-extension* of a symmetric operator \dot{A} if $\mathbb{A} \supset \dot{A}$.

The formula (see [9], [7])

$$\mathbb{A} = \dot{A}^* + \mathcal{R}^{-1} \left(\mathcal{S} - \frac{i}{2} P_{\mathfrak{N}_i}^+ + \frac{i}{2} P_{\mathfrak{N}_{-i}}^+ \right) P_{\mathfrak{M}}^+ = \dot{A}^* + \mathcal{R}^{-1} \left(\mathcal{S} - \frac{1}{2} \dot{A}^* \right) P_{\mathfrak{M}}^+ \quad (2.1)$$

establishes a one-to-one correspondence between the set of all self-adjoint bi-extensions of \dot{A} and the set of all $(+)$ -self-adjoint operators \mathcal{S} in \mathfrak{M} .

Let \mathbb{A} be a self-adjoint bi-extension of \dot{A} and let the operator \widehat{A} in \mathcal{H} be defined as follows:

$$\text{Dom}(\widehat{A}) = \{f \in \mathcal{H}_+ : \widehat{A}f \in \mathcal{H}\}, \quad \widehat{A} = \mathbb{A} \upharpoonright \text{Dom}(\widehat{A}).$$

The operator \widehat{A} is called a *quasi-kernel* of a self-adjoint bi-extension \mathbb{A} (see [31]). We say that a self-adjoint bi-extension \mathbb{A} of \dot{A} is *twice-self-adjoint* or *t-self-adjoint* (see [7]) if its quasi-kernel \widehat{A} is a self-adjoint operator in \mathcal{H} .

For the existence, description, and analog of von Neumann's formulas for bounded self-adjoint bi-extensions and $(*)$ -extensions see [7] and references therein. In what follows we suppose that \dot{A} has equal deficiency indices. Recall that two self-adjoint extensions A_1 and A_0 of \dot{A} are called *disjoint* if

$$\text{Dom}(A_1) \cap \text{Dom}(A_0) = \text{Dom}(\dot{A}) \quad (2.2)$$

and *transversal* if

$$\text{Dom}(A_1) + \text{Dom}(A_0) = \text{Dom}(\dot{A}^*).$$

Note that it immediately follows from von Neumann formulas that two transversal self-adjoint extensions are automatically disjoint.

The following statements for two self-adjoint extensions A_1 and A_0 of \dot{A} are evident:

$$\begin{aligned} A_1, A_0 \text{ are disjoint} &\iff \overline{\text{Ran}} \left((A_1 - \lambda I)^{-1} - (A_0 - \lambda I)^{-1} \right) = \mathfrak{N}_\lambda, \\ A_1, A_0 \text{ are transversal} &\iff \text{Ran} \left((A_1 - \lambda I)^{-1} - (A_0 - \lambda I)^{-1} \right) = \mathfrak{N}_\lambda \end{aligned}$$

for at least one $\lambda \in \rho(A_1) \cap \rho(A_0)$.

Thus, if the deficiency numbers of \dot{A} are finite (and equal), then two self-adjoint extensions of \dot{A} are transversal if and only they are disjoint.

Let \dot{A} be a closed densely defined symmetric operator and let A_1 be its self-adjoint extension. It has been shown in [2], [9] that any self-adjoint bi-extension \mathbb{A} of \dot{A} such that $\mathbb{A} \supset A_1$ is *generated* by a disjoint to A_1 self-adjoint extension A_0 of \dot{A} via the formulas

$$\begin{aligned} \text{Dom}(\mathbb{A}) &= \text{Dom}(A_1) + \text{Dom}(A_0), \\ \mathbb{A}f &= \dot{A}^*f - \mathcal{R}^{-1}\dot{A}^*\mathcal{P}_Gf, \quad f \in \text{Dom}(\mathbb{A}), \end{aligned}$$

where \mathcal{P}_G is a skew projection operator in $\text{Dom}(\mathbb{A})$ onto G parallel to $\text{Dom}(A_1)$ and G is defined from the (+)-orthogonal decomposition

$$\text{Dom}(A_0) = \text{Dom}(\dot{A}) \oplus G. \quad (2.3)$$

The operator \mathcal{S} corresponding to \mathbb{A} in (2.1) is of the form

$$\begin{aligned} \mathcal{S}f &= \frac{1}{2}\dot{A}^*f, \quad f \in \text{Dom}(A_1) \ominus \text{Dom}(\dot{A}), \\ \mathcal{S}g &= -\frac{1}{2}\dot{A}^*g, \quad g \in \text{Dom}(A_0) \ominus \text{Dom}(\dot{A}). \end{aligned} \quad (2.4)$$

In particular,

$$\mathbb{A}g = (\dot{A}^* - \mathcal{R}^{-1}\dot{A}^*P_{\mathfrak{M}}^+)g, \quad g \in \text{Dom}(A_0).$$

The following formula immediately follows from (2.3)

$$(\mathbb{A}f, f) = (A_1f_1, f_1) + (A_0f_0, f_0) + 2\text{Re}(A_1f_1, f_0), \quad (2.5)$$

where $f = f_1 + f_0$, $f_l \in \text{Dom}(A_l)$, $(l = 0, 1)$.

Let \mathbb{A} be a self-adjoint bi-extension of \dot{A} . We define a *dual* extension \mathbb{A}' on $\text{Dom}(\mathbb{A})$ by the formula

$$(\mathbb{A}'f, g) = (\dot{A}^*f, g) + (f, \dot{A}^*g) - (\mathbb{A}f, g), \quad f, g \in \text{Dom}(\mathbb{A}). \quad (2.6)$$

We note that $\dot{A}^* \in [\mathcal{H}_+, \mathcal{H}] \subset [\mathcal{H}_+, \mathcal{H}_-]$ and the *generalized adjoint* of \dot{A}^* takes the form [7]

$$(\dot{A}^*)^* = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*P_{\mathfrak{M}}^+. \quad (2.7)$$

It follows from (2.1) that if

$$\mathbb{A} = \dot{A}^* + \mathcal{R}^{-1} \left(\mathcal{S} - \frac{i}{2}P_{\mathfrak{N}_i}^+ + \frac{i}{2}P_{\mathfrak{N}_{-i}}^+ \right) P_{\mathfrak{M}}^+$$

is a self-adjoint bi-extension of \dot{A} , then \mathbb{A}' is of the form

$$\mathbb{A}' = \dot{A}^* + \mathcal{R}^{-1} \left(-\mathcal{S} - \frac{i}{2} P_{\mathfrak{N}_i}^+ + \frac{i}{2} P_{\mathfrak{N}_{-i}}^+ \right) P_{\mathfrak{M}}^+.$$

So, if \mathbb{A} is a self-adjoint bi-extension of \dot{A} , then \mathbb{A}' is a self-adjoint bi-extension of \dot{A} as well. It was also shown in [2] that if \mathbb{A} is a t-self-adjoint of \dot{A} , then \mathbb{A}' is also a t-self-adjoint bi-extension of \dot{A} . Moreover, if \hat{A} is a quasi-kernel of \mathbb{A} and \mathbb{A} is generated by a disjoint to \hat{A} self-adjoint extension A , then the quasi-kernel of \mathbb{A}' coincides with A and \mathbb{A}' is generated by \hat{A} . Clearly, $(\mathbb{A}')' = \mathbb{A}$.

Notice that from (2.6) and the inequality

$$2|(\dot{A}^* f, f)| \leq 2\|f\| \|\dot{A}^* f\| \leq \|f\|^2 + \|\dot{A}^* f\|^2 = \|f\|_+^2,$$

we get

$$-\|f\|_+^2 \leq (\mathbb{A}f, f) + (\mathbb{A}'f, f) \leq \|f\|_+^2.$$

3. The Friedrichs and Kreĭn–von Neumann extensions

Let $\tau[\cdot, \cdot]$ be a sesquilinear form in a Hilbert space \mathcal{H} defined on a linear manifold $\text{Dom}(\tau)$. The form τ is called symmetric if $\tau[u, v] = \overline{\tau[v, u]}$ for all $u, v \in \text{Dom}(\tau)$ and non-negative if $\tau[u] := \tau[u, u] \geq 0$ for all $u \in \text{Dom}(\tau)$.

A sequence $\{u_n\}$ is called τ -converging to the vector $u \in \mathcal{H}$ [20] if

$$\lim_{n \rightarrow \infty} u_n = u \quad \text{and} \quad \lim_{n, m \rightarrow \infty} \tau[u_n - u_m] = 0.$$

The form τ is called *closed* if for every sequence $\{u_n\}$ τ -converging to a vector u it follows that $u \in \text{Dom}(\tau)$ and $\lim_{n \rightarrow \infty} \tau[u - u_n] = 0$. The form τ is *closable* [20], i.e., there exists a minimal closed extension (the closure) of τ . We recall that a symmetric operator \dot{B} is called *non-negative* if

$$(\dot{B}f, f) \geq 0, \quad \forall f \in \text{Dom}(\dot{B}).$$

If τ is a closed, densely defined non-negative form, then according to First Representation Theorem [23], [20] there exists a unique self-adjoint non-negative operator T in \mathfrak{H} , associated with τ , i.e.,

$$(Tu, v) = \tau[u, v] \quad \text{for all } u \in \text{Dom}(T) \quad \text{and} \quad \text{for all } v \in \text{Dom}(\tau).$$

According to the Second Representation Theorem [23], [20] the identities hold:

$$\text{Dom}(\tau) = \text{Dom}(T^{1/2}), \quad \tau[u, v] = (T^{1/2}u, T^{1/2}v).$$

Let \dot{B} be a non-negative symmetric operator in a Hilbert space \mathcal{H} . It is known [20] that the non-negative sesquilinear form $\tau_{\dot{B}}[f, g] = (\dot{B}f, g)$, $\text{Dom}(\tau) = \text{Dom}(\dot{B})$, is closable. Following the M. Kreĭn notations we denote by $\dot{B}[\cdot, \cdot]$ the closure of $\tau_{\dot{B}}$ and by $\mathcal{D}[\dot{B}]$ its domain. By definition $\dot{B}[u] = \dot{B}[u, u]$ for all $u \in \mathcal{D}[\dot{B}]$. Because $\dot{B}[u, v]$ is closed, it possesses the property: if

$$\lim_{n \rightarrow \infty} u_n = u \quad \text{and} \quad \lim_{n, m \rightarrow \infty} \dot{B}[u_n - u_m] = 0,$$

then $\lim_{n \rightarrow \infty} \dot{B}[u - u_n] = 0$. For a densely defined \dot{B} , the *Friedrichs extension* B_F of \dot{B} is defined as a non-negative self-adjoint operator associated with the form $\dot{B}[\cdot, \cdot]$ by the First Representation Theorem. If \dot{B} is densely defined then, clearly,

$$\text{Dom}(B_F) = \mathcal{D}[\dot{B}] \cap \text{Dom}(\dot{B}^*), \quad B_F = \dot{B}^* \upharpoonright \text{Dom}(B_F).$$

The Friedrichs extension B_F is a unique non-negative self-adjoint extension having the domain in $\mathcal{D}[\dot{B}]$. Notice that by the Second Representation Theorem [20] one has

$$\mathcal{D}[\dot{B}] = \mathcal{D}[B_F] = \text{Dom}(B_F^{1/2}), \quad \dot{B}[u, v] = (B_F^{1/2}u, B_F^{1/2}v), \quad u, v \in \mathcal{D}[\dot{B}].$$

If \dot{B} is non-densely defined, then its Friedrichs extension B_F is a non-negative linear relation of the form (see [28])

$$B_F = \left\{ \left\langle x, (\dot{B}_0)_F x \right\rangle, x \in \text{Dom}((\dot{B}_0)_F) \right\} \oplus \langle 0, \mathfrak{B} \rangle,$$

where $(\dot{B}_0)_F$ is the Friedrichs extension of the operator $\dot{B}_0 := P_{\overline{\text{Dom}(\dot{B})}} \dot{B}$ in the subspace $\overline{\text{Dom}(\dot{B})}$ and $\mathfrak{B} = \mathcal{H} \ominus \text{Dom}(\dot{B})$.

The Kreĭn–von Neumann extension is defined as follows [1], [16]:

$$\dot{B}_K = ((\dot{B}^{-1})_F)^{-1},$$

where \dot{B}^{-1} is the linear relation inverse to the graph of \dot{B} .

Theorem 3.1 ([1]). *The following relations describing $\mathcal{D}[B_K]$ and $B_K[u]$ hold:*

$$\begin{aligned} \mathcal{D}[B_K] &= \left\{ u \in \mathcal{H} : \sup_{f \in \text{Dom}(\dot{B})} \frac{|\dot{B}f, u|^2}{(\dot{B}f, f)} < \infty \right\}, \\ B_K[u] &= \sup_{f \in \text{Dom}(\dot{B})} \frac{|\dot{B}f, u|^2}{(\dot{B}f, f)}, \quad u \in \mathcal{D}[B_K]. \end{aligned} \tag{3.1}$$

We note the equalities for an arbitrary non-negative self-adjoint operator B in a Hilbert space \mathcal{H} :

$$\begin{aligned} \text{Ran}(B^{1/2}) &= \left\{ g \in \mathcal{H} : \sup_{f \in \text{Dom}(B)} \frac{|(f, g)|^2}{(Bf, f)} < \infty \right\}, \\ \|B^{[-1/2]}g\|^2 &= \sup_{f \in \text{Dom}(B)} \frac{|(f, g)|^2}{(Bf, f)}, \quad g \in \text{Ran}(B^{1/2}), \end{aligned}$$

where $B^{[-1]}$ is the Moore–Penrose inverse. The Kreĭn–von Neumann extension of a non-densely defined non-negative operator \dot{B} is an operator (not just a linear relation) if and only if the domain $\mathcal{D}[B_K]$ is dense in \mathfrak{H} . According to [1] a non-negative operator \dot{B} is called *positively closable* if from $\lim_{n \rightarrow \infty} \dot{B}\varphi_n = g$ and $\lim_{n \rightarrow \infty} (\dot{B}\varphi_n, \varphi_n) = 0$ follows $g = 0$ ($\{\varphi_n\} \subset \text{Dom}(\dot{B})$). Notice that a densely defined \dot{B} is positively closable. A theorem of Ando and Nishio [1] states that \dot{B}

admits non-negative self-adjoint extensions, which are operators, if and only if \dot{B} is positively closable.

A non-negative self-adjoint extension \tilde{B} of \dot{B} is called *extremal* [3], [5], [6] if the relation

$$\inf \left\{ \left(\tilde{B}(u - \varphi), u - \varphi \right) : \varphi \in \text{Dom}(\dot{B}) \right\} = 0$$

holds for every $u \in \text{Dom}(\tilde{B})$. A characterization of the Kreĭn-von Neumann extension B_K is obtained in [5] and [6]: *the Kreĭn-von Neumann extension B_K is the unique extremal non-negative self-adjoint extension of \dot{B} having maximal domain of its closed associated sesquilinear form.*

Theorem 3.2. *Let \tilde{B} be a non-negative self-adjoint extension of \dot{B} . Then*

$$B_K \leq \tilde{B} \leq B_F \quad (3.2)$$

in the sense of quadratic forms. More precisely

$$\begin{aligned} \mathcal{D}[\dot{B}] &\subseteq \mathcal{D}[\tilde{B}] \subseteq \mathcal{D}[B_K], \\ \tilde{B}[u] &\geq B_K[u] \quad \text{for all } u \in \mathcal{D}[\tilde{B}], \\ \tilde{B}[v] &= \dot{B}[v] \quad \text{for all } v \in \mathcal{D}[\dot{B}]. \end{aligned}$$

Besides,

$$\mathcal{D}[\tilde{B}] = \mathcal{D}[\dot{B}] + (\mathcal{D}[\tilde{B}] \cap \mathcal{N}_z), \quad (3.3)$$

where \mathcal{N}_z is the defect subspace of \dot{B} , $z \in \mathbb{C} \setminus [0, +\infty)$.

For a densely defined non-negative \dot{B} inequalities (3.2) in the equivalent form

$$(B_F + I)^{-1} \leq (\tilde{B} + I)^{-1} \leq (B_K + I)^{-1}$$

and equality (3.3) for $z < 0$ were established by M. Kreĭn [23]. For a sectorial operator \dot{B} with vertex at zero and for sectorial linear relations all statements of Theorem 3.2 can be found in [5] and [6].

The next theorem gives a descriptions of all closed forms associated with non-negative self-adjoint extensions of \dot{B} .

Theorem 3.3 ([5]). *If \tilde{B} is a non-negative self-adjoint extension of a non-negative symmetric operator \dot{B} , then the form*

$$(\tilde{B}u, v) - B_K[u, v], \quad u, v \in \text{Dom}(\tilde{B})$$

is non-negative and closable in the Hilbert space $\mathcal{D}[B_K]$. Moreover, the formulas

$$\begin{aligned} \mathcal{D}[\tilde{B}] &= \mathcal{D}[\tau], \\ \tilde{B}[u, v] &= B_K[u, v] + \tau[u, v], \quad u, v \in \mathcal{D}[\tilde{B}] \end{aligned}$$

give a one-to-one correspondence between all closed forms $\tilde{S}[\cdot, \cdot]$ associated with non-negative self-adjoint extensions \tilde{B} of \dot{B} and all non-negative forms $\tau[\cdot, \cdot]$ closed in the Hilbert space $\mathcal{D}[B_K]$ and such that $\tau[\varphi] = 0$ for all $\varphi \in \mathcal{D}[\dot{B}]$.

In addition, the closed forms associated with extremal extensions are closed restrictions of the form $B_K[\cdot, \cdot]$ on the linear manifolds \mathcal{M} such that

$$\mathcal{D}[\dot{B}] \subseteq \mathcal{M} \subseteq \mathcal{D}[B_K].$$

The next theorem can be found in [29], [30], [6], [19].

Theorem 3.4. *Let \dot{B} be a bounded non-densely defined non-negative symmetric operator in a Hilbert space \mathcal{H} , $\text{Dom}(\dot{B}) = \mathcal{H}_0$. Let $\dot{B}^* \in [\mathcal{H}, \mathcal{H}_0]$ be the adjoint of \dot{B} . Put $\dot{B}_0 = P_{\mathcal{H}_0} \dot{B}$, $\mathcal{N} = \mathcal{H} \ominus \mathcal{H}_0$, where $P_{\mathcal{H}_0}$ is an orthogonal projection in \mathcal{H} onto \mathcal{H}_0 . Then the following statements are equivalent*

- (i) \dot{B} admits bounded non-negative self-adjoint extensions in \mathcal{H} ;
- (ii) $\sup_{f \in \mathcal{H}_0} \frac{\|\dot{B}f\|^2}{(\dot{B}f, f)} < \infty$;
- (iii) $\dot{B}^* \mathcal{N} \subseteq \text{Ran}(\dot{B}_0^{1/2})$.

Let \dot{B} be a non-negative closed symmetric operator. Consider the symmetric contractions

$$\dot{S} = (I - \dot{B})(I + \dot{B})^{-1},$$

defined on $\text{Dom}(\dot{S}) = (I + \dot{B})\text{Dom}(\dot{B})$. Notice that the orthogonal complement $\mathfrak{N} = \mathcal{H} \ominus \text{Dom}(\dot{S})$ coincides with the defect subspace \mathfrak{N}_{-1} of the operator \dot{B} . There is a one-to-one correspondence given by the Cayley transform

$$B = (I - S)(I + S)^{-1}, \quad S = (I - B)(I + B)^{-1},$$

between all non-negative self-adjoint extensions B (linear relations in general) of the operator \dot{B} and all self-adjoint contractive (*sc*) extensions S of \dot{S} . As was established by M. Kreĭn in [23], [24] the set of all *sc*-extensions of \dot{A} forms an operator interval $[S_\mu, S_M]$. Following M. Kreĭn's notations we call the endpoints S_μ and S_M by the *rigid* and the *soft* extensions, respectively. They possess the properties

$$\begin{aligned} \inf_{\varphi \in \text{Dom}(\dot{S})} ((I + S_\mu)(f - \varphi), (f - \varphi) = 0, \\ \inf_{\varphi \in \text{Dom}(\dot{S})} ((I - S_M)(f - \varphi), (f - \varphi) = 0, \end{aligned} \quad (3.4)$$

for all $f \in \mathcal{H}$. The operator interval $[S_\mu, S_M]$ can be parameterized as follows

$$S = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2} X (S_M - S_\mu)^{1/2}, \quad (3.5)$$

where X is a self-adjoint contraction in the subspace $\overline{\text{Ran}(S_M - S_\mu)} (\subseteq \mathfrak{N})$.

Notice that for each $S \in [S_\mu, S_M]$ the equalities (3.4) imply

$$\begin{aligned} \inf_{\varphi \in \text{Dom}(\dot{S})} ((I + S)(f - \varphi), (f - \varphi) = ((S - S_\mu)f, f), \\ \inf_{\varphi \in \text{Dom}(\dot{S})} ((I - S)(f - \varphi), (f - \varphi) = ((S_M - S)f, f), \quad f \in \mathcal{H}. \end{aligned} \quad (3.6)$$

Using the relation (see [23])

$$\inf_{\varphi \in \text{Dom}(\dot{S})} ((I + S)(f - \varphi), (f - \varphi)) = \|P_\Omega(I + S)^{1/2}f\|^2,$$

where

$$\Omega = \{g \in \mathcal{H} : (I + S)^{1/2}g \in \mathfrak{N}\},$$

from (3.6) we get the equalities

$$\begin{aligned} (I + S)^{1/2}\Omega &= \text{Ran}((S - S_\mu)^{1/2}), \\ \|(I + S)^{[-1/2]}f\| &= \|(S - S_\mu)^{[-1/2]}f\|^2, \quad f \in \text{Ran}((S - S_\mu)^{1/2}). \end{aligned} \quad (3.7)$$

Let L be a bounded non-negative self-adjoint operator in the Hilbert space \mathcal{H} and let \mathcal{M} be a subspace in \mathcal{H} . The Kreĭn shorted operator $L_{\mathcal{M}}$ [23], [1] is given by the following definition

$$L_{\mathcal{M}} = \max\{X \leq L \mid \text{Ran}(X) \subseteq \mathcal{M}\}.$$

It is shown in [23], that

$$L_{\mathcal{M}} = L^{1/2}QL^{1/2}, \quad (3.8)$$

where Q is an orthoprojection operator onto the subspace $\text{Ran}(Q) = (L^{1/2})^{-1}(\mathcal{M})$. Moreover, [23]

$$(L_{\mathcal{M}}f, f) = \inf_{\varphi \in \mathcal{H} \ominus \mathcal{M}} (L(f - \varphi), f - \varphi), \quad f \in \mathcal{H}. \quad (3.9)$$

Thus, from (3.6) we have

$$(I + S)_{\mathfrak{N}} = S - S_\mu, \quad (I - S)_{\mathfrak{N}} = S_M - S.$$

The next result describes the sesquilinear form $B[u, v]$ by the means of the fractional-linear transformation $S = (I - B)(I + B)^{-1}$. The following proposition can be found in [7].

Proposition 3.5.

- (1) *Let B be a non-negative self-adjoint operator and let $S = (I - B)(I + B)^{-1}$ be its Cayley transform. Then*

$$\mathcal{D}[B] = \text{Ran}((I + S)^{1/2}),$$

$$B[u, v] = -(u, v) + 2 \left((I + S)^{-1/2}u, (I + S)^{-1/2}v \right), \quad u, v \in \mathcal{D}[B].$$

- (2) *Let \dot{B} be a closed densely defined non-negative symmetric operator and let B be its non-negative self-adjoint extension. If $\dot{S} = (I - \dot{B})(I + \dot{B})^{-1}$, $S = (I - B)(I + B)^{-1}$, then*

$$\mathcal{D}[B] = \text{Ran}(I + S_\mu)^{1/2} \dot{+} \text{Ran}(S - S_\mu)^{1/2}. \quad (3.10)$$

We note that $\text{Ran}(B^{1/2}) = \text{Ran}((I - S)^{1/2})$. Now let S_μ and S_M be the rigid and the soft extensions of \dot{S} . Then the Friedrichs and Kreĭn–von Neumann extensions of \dot{B} are given by

$$B_F = (I - S_\mu)(I + S_\mu)^{-1}, \quad B_K = (I - S_M)(I + S_M)^{-1}.$$

4. Non-negative self-adjoint bi-extensions

4.1. Disjointness and transversality of non-negative self-adjoint extensions

Proposition 4.1. *Let \dot{A} be a non-negative closed densely defined operator. Then the following statements hold true for a non-negative self-adjoint extensions A of \dot{A} :*

$$\begin{aligned} A \text{ is disjoint with } A_F &\iff \mathcal{D}[A] \cap \mathcal{H}_+ \text{ is dense in } \mathcal{H}_+, \\ A \text{ is transversal with } A_F &\iff \mathcal{D}[A] \supset \mathcal{H}_+. \end{aligned}$$

Proof. Using equality (3.3) in the form

$$\mathcal{D}[A] = \mathcal{D}[\dot{A}] \dot{+} (\mathfrak{N}_{-1} \cap \mathcal{D}[A])$$

and the relation $\text{Dom}(A_F) = \mathcal{D}[\dot{A}] \cap \text{Dom}(\dot{A}^*)$, we get that

$$\mathcal{D}[A] \cap \mathcal{H}_+ = \text{Dom}(A_F) \dot{+} (\mathfrak{N}_{-1} \cap \mathcal{D}[A]), \quad (4.1)$$

where \mathfrak{N}_λ is the defect subspace of \dot{A} . Taking into account the equality

$$\mathcal{H}_+ = \text{Dom}(A_F) \dot{+} \mathfrak{N}_{-1},$$

we get that $\mathcal{D}[A] \cap \mathcal{H}_+$ is dense in \mathcal{H}_+ if and only if $\mathfrak{N}_{-1} \cap \mathcal{D}[A]$ is dense in \mathfrak{N}_{-1} and

$$\mathcal{D}[A] \cap \mathcal{H}_+ = \mathcal{H}_+ \iff \mathfrak{N}_{-1} \subset \mathcal{D}[A].$$

Put

$$\dot{S} = (I - \dot{A})(I + \dot{A}), \quad S_\mu = (I - A_F)(I + A_F), \quad S = (I - A)(I + A).$$

Then

$$S - S_\mu = (A + I)^{-1} - (A_F + I)^{-1}. \quad (4.2)$$

Now the equality (see (3.10))

$$\mathcal{D}[A] = \mathcal{D}[\dot{A}] \dot{+} \text{Ran}(S - S_\mu)^{1/2} \quad (4.3)$$

implies the validity of the statements in the proposition. \square

From (4.2) and (4.3) we get the following equalities

$$\mathcal{D}[A] \cap \mathcal{H}_+ = \text{Dom}(A_F) \dot{+} \text{Ran}(S - S_\mu)^{1/2} = \text{Dom}(A) \dot{+} \text{Ran}(S - S_\mu)^{1/2}.$$

Notice that the equivalence

$$A_F \text{ and } A_K \text{ are transversal} \iff \text{Dom}(\dot{A}^*) \subseteq \mathcal{D}[A_K]$$

has been shown in [25] (see also [11]). The next statement provides one more criteria for A_F and A_K to be transversal.

Proposition 4.2.

$$A_F \text{ and } A_K \text{ are transversal} \iff \sup_{f \in \text{Dom}(\dot{A})} \frac{\|(I + \dot{A}\dot{A}^*)^{-1/2} \dot{A}f\|^2}{(\dot{A}f, f)} < \infty. \quad (4.4)$$

Proof. Let A be a non-negative self-adjoint extension of \dot{A} . Since $A^{1/2}$ is closed in \mathcal{H} , the closed graph theorem yields that

$$\mathcal{H}_+ \subset \mathcal{D}[A] = \text{Dom}(A^{1/2}) \iff A^{1/2} \upharpoonright \mathcal{H}_+ \in [\mathcal{H}_+, \mathcal{H}],$$

i.e., there exists a number $c > 0$ such that

$$\|A^{1/2}u\|^2 = A[u] \leq c\|u\|_+^2 \quad \text{for all } u \in \mathcal{H}_+.$$

Take $A = A_K$. Then for $u \in \mathcal{D}[A_K] \cap \mathcal{H}_+$

$$\|A_K^{1/2}u\|^2 = \sup_{f \in \text{Dom}(\dot{A})} \frac{|(\dot{A}f, u)|^2}{(\dot{A}f, f)} = \sup_{f \in \text{Dom}(\dot{A})} \frac{|(\mathcal{R}\dot{A}f, u)_+|^2}{(\dot{A}f, f)}.$$

Hence

$$|(\mathcal{R}\dot{A}f, u)_+|^2 \leq \|A_K^{1/2}u\|^2 (\dot{A}f, f).$$

Then

$$\begin{aligned} \mathcal{H}_+ \subset \mathcal{D}[A_K] &\iff |(\mathcal{R}\dot{A}f, u)_+|^2 \leq c\|u\|_+^2 (\dot{A}f, f), \quad \forall u \in \mathcal{H}_+, \quad \forall f \in \text{Dom}(\dot{A}) \\ &\iff \|\mathcal{R}\dot{A}f\|_+^2 = \sup_{u \in \mathcal{H}_+} \frac{|(\mathcal{R}\dot{A}f, u)_+|^2}{\|u\|_+^2} \leq c(\dot{A}f, f), \quad \forall f \in \text{Dom}(\dot{A}) \\ &\iff \sup_{f \in \text{Dom}(\dot{A})} \frac{\|\mathcal{R}\dot{A}f\|_+^2}{(\dot{A}f, f)} < \infty. \end{aligned}$$

Since

$$\|\mathcal{R}g\|_+^2 = \|(I + \dot{A}\dot{A}^*)^{-1/2}g\|^2, \quad g \in \mathcal{H},$$

we arrive at (4.4). □

Notice that due to Theorem 3.4 condition

$$\sup_{f \in \text{Dom}(\dot{A})} \frac{\|(I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}f\|^2}{(\dot{A}f, f)} < \infty$$

means that the operator $\mathcal{R}\dot{A}$ admits (+)-bounded (+)-self-adjoint non-negative extensions. It is not difficult to show that

$$(I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}f = \dot{A}(I + \dot{A}^*\dot{A})^{-1/2}f, \quad f \in \text{Dom}(\dot{A}).$$

This relation implies that if \dot{A} is positively definite, then A_F and A_K are transversal. Indeed,

$$\|(I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}f\|^2 = \|\dot{A}(I + \dot{A}^*\dot{A})^{-1/2}f\|^2 \leq C\|f\|^2 \leq m(\dot{A}f, f), \quad f \in \text{Dom}(\dot{A}).$$

Hence,

$$\sup_{f \in \text{Dom}(\dot{A})} \frac{\|(I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}f\|^2}{(\dot{A}f, f)} < \infty.$$

4.2. Non-negative self-adjoint bi-extensions

Existence. Let $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ be a rigged Hilbert space. If \mathcal{T} is a non-negative, densely defined in \mathfrak{H}_+ and closed sesquilinear form in \mathfrak{H}_+ , then there exists a non-negative generalized self-adjoint operator \mathbb{T} acting from $\text{Dom}(\mathbb{T})$ into \mathcal{H}_- associated with the form \mathcal{T} in the following sense

$$(\mathbb{T}u, v) = \mathcal{T}[u, v] \text{ for all } u \in \text{Dom}(\mathbb{T}) \text{ and all } v \in \text{Dom}(\mathcal{T}). \quad (4.5)$$

Actually, due to the First Representation Theorem, there is a (+)-non-negative self-adjoint operator \mathfrak{T} associated with the form \mathcal{T} in \mathfrak{H}_+ , i.e.,

$$(\mathfrak{T}u, v)_+ = \mathcal{T}[u, v] \text{ for all } u \in \text{Dom}(\mathfrak{T}) \text{ and all } v \in \text{Dom}(\mathcal{T}).$$

If $\mathcal{J} \in [\mathfrak{H}_-, \mathfrak{H}_+]$ is the Riesz–Berezansky operator, then $\mathbb{T} = \mathcal{J}^{-1}\mathfrak{T}$ satisfies (4.5). If a non-negative form is defined on \mathfrak{H}_+ and is bounded in \mathfrak{H}_+ , then, clearly, the associated non-negative self-adjoint operator belongs to $[\mathfrak{H}_+, \mathfrak{H}_-]$.

If $\mathfrak{T} : \mathfrak{H}_+ \supseteq \text{Dom}(\mathfrak{T}) \rightarrow \mathfrak{H}_-$ is a non-negative generalized self-adjoint operator in the rigged Hilbert space $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$, i.e., $(\mathfrak{T}f, f) \geq 0$ for all $f \in \text{Dom}(\mathfrak{T})$ and $\mathfrak{T} = \mathfrak{T}^*$, then the sesquilinear form

$$\mathcal{T}_{\mathfrak{T}}[f, g] = (\mathfrak{T}f, g), \quad \text{Dom}(\mathcal{T}_{\mathfrak{T}}) = \text{Dom}(\mathfrak{T})$$

is closable in \mathfrak{H}_+ . We will denote by $\mathfrak{T}[\cdot, \cdot]$ its closure and by $\mathcal{D}[\mathfrak{T}]$ its domain.

Now we consider a closed non-negative symmetric densely defined operator \dot{A} . Let $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ be the rigged Hilbert space, where $\mathcal{H}_+ = \text{Dom}(\dot{A}^*)$ and (+)-inner product is defined by (1.1). We are going to study non-negative self-adjoint bi-extensions of the operator \dot{A} . Clearly, the operator

$$\dot{B} = \mathcal{R}\dot{A}$$

is non-densely defined in \mathcal{H}_+ , (+)-bounded and (+)-non-negative. Each non-negative (+)-self-adjoint extension B of \dot{B} in \mathcal{H}_+ , which is an operator, determines a non-negative self-adjoint bi-extension of \dot{A} by the formula $\mathbb{A} = \mathcal{R}^{-1}B$. Since

$$\|\dot{B}\varphi\|_+ = \|\mathcal{R}\dot{A}\varphi\|_+ = \|(I + \dot{A}\dot{A}^*)^{-1}\dot{A}\varphi\|_+ = \|(I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}\varphi\|, \quad \varphi \in \dot{A},$$

and $(\dot{B}\varphi, \varphi)_+ = (\dot{A}\varphi, \varphi)$, we can use the Ando and Nishio theorem (see [1]) about positively closable symmetric operator and get the following statement.

Proposition 4.3. *A non-negative densely defined closed symmetric operator \dot{A} admits non-negative self-adjoint bi-extension if and only if from*

$$\lim_{n \rightarrow \infty} (I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}\varphi_n = g \quad \text{and} \quad \lim_{n \rightarrow \infty} (\dot{A}\varphi_n, \varphi_n) = 0$$

follows $g = 0$, where $\{\varphi_n\} \subset \text{Dom}(\dot{A})$.

Theorem 4.4. *Let \dot{A} be a non-negative closed densely defined operator. The following conditions are equivalent:*

- (i) *\dot{A} admits a non-negative self-adjoint bi-extension,*
- (ii) *\dot{A} admits t -self-adjoint bi-extension with quasi-kernel A_K ,*
- (iii) *the Friedrichs and Kreĭn–von Neumann extensions of \dot{A} are disjoint.*

Proof. Clearly (ii) \Rightarrow (i). Let us show that (iii) \Rightarrow (ii). Suppose that the Friedrichs extension A_F and the Kreĭn–von Neumann extension A_K of the operator \dot{A} are disjoint. Then $\text{Dom}(A_F) + \text{Dom}(A_K)$ is $(+)$ -dense in \mathcal{H}_+ or coincides with \mathcal{H}_+ (when A_F and A_K are transversal). Then it follows that $\mathcal{D}[A_K] \cap \mathcal{H}_+$ is $(+)$ -dense in \mathcal{H}_+ or coincides with \mathcal{H}_+ . Clearly, the sesquilinear form

$$A_K[u, v], \quad u, v \in \mathcal{D}[A_K] \cap \mathcal{H}_+,$$

is closed in \mathcal{H}_+ . Because it is at least $(+)$ -densely defined in \mathcal{H}_+ , there is an associated self-adjoint non-negative operator $\mathbb{A}_K: \mathcal{H}_+ \supseteq \text{Dom}(\mathbb{A}_K) \rightarrow \mathcal{H}_-$, i.e.,

$$(\mathbb{A}_K u, v) = A_K[u, v] \text{ for all } u \in \text{Dom}(\mathbb{A}_K) \text{ and for all } v \in \mathcal{D}[A_K] \cap \mathcal{H}_+.$$

Because $(A_K u, v) = A_K[u, v]$ for all $u \in \text{Dom}(A_K)$ and all $v \in \mathcal{D}[A_K]$, we get that $\mathbb{A}_K \supset A_K$, i.e., the quasi-kernel of \mathbb{A}_K is A_K and therefore, A_K is t -self-adjoint bi-extension of \dot{A} .

Let us prove (i) \Rightarrow (iii). Suppose that \dot{A} admits non-negative self-adjoint bi-extensions. Then the Kreĭn–von Neumann extension B_K of the operator $\dot{B} = \mathcal{R}\dot{A}$ in \mathcal{H}_+ is an operator. Due to the formula (3.1) the domain $\mathcal{D}[B_K]$ is at least dense in \mathcal{H}_+ . On the other hand since

$$\frac{|(\dot{B}f, u)_+|^2}{(\dot{B}f, f)_+} = \frac{|(\dot{A}f, u)|^2}{(\dot{A}f, f)},$$

from (3.1) we get

$$\mathcal{D}[B_K] = \mathcal{D}[A_K] \cap \mathcal{H}_+$$

and $B_K[u] = A_K[u]$ for all $u \in \mathcal{D}[A_K] \cap \mathcal{H}_+$. It follows from (4.1) that

$$\mathcal{D}[A_K] \cap \mathcal{H}_+ = \text{Dom}(A_F) \dot{+} (\mathfrak{N}_{-1} \cap \mathcal{D}[A_K]).$$

Therefore, the density of $\mathcal{D}[A_K] \cap \mathcal{H}_+$ implies the density of $\mathfrak{N}_{-1} \cap \mathcal{D}[A_K]$ in \mathfrak{N}_{-1} . Equality (3.10) yields that

$$\overline{\text{Ran}} \left((A_K + I)^{-1} - (A_F + I)^{-1} \right) = \mathfrak{N}_{-1},$$

i.e., A_F and A_K are at least disjoint. \square

Theorem 4.5.

- 1) Let A be a non-negative self-adjoint extension of \dot{A} . Then there exists a t -self-adjoint bi-extension \mathbb{A} of \dot{A} with quasi-kernel A if and only if A is disjoint with A_F .
- 2) If a non-negative self-adjoint extension A of \dot{A} is disjoint with A_F , then t -self-adjoint bi-extension \mathbb{A} with quasi-kernel A and generated by A_F is associated with the sesquilinear form $A[u, v]$, $u, v \in \mathcal{D}[A] \cap \mathcal{H}_+$.

Proof. The form $A[u, v]$ defined on $\mathcal{D}[A] \cap \mathcal{H}_+$ is closed in \mathcal{H}_+ . By Proposition 4.1 A is disjoint with A_F if and only if the linear manifold $\mathcal{D}[A] \cap \mathcal{H}_+$ is dense in \mathcal{H}_+ in which case the non-negative sesquilinear form $A[u, v]$, $u, v \in \mathcal{D}[A] \cap \mathcal{H}_+$ is closed in \mathcal{H}_+ . The latter implies the existence of a non-negative self-adjoint operator $\mathbb{A}: \mathcal{H}_+ \supseteq \text{Dom}(\mathbb{A}) \rightarrow \mathcal{H}_-$ associated with $A[u, v]$, $u, v \in \mathcal{D}[A] \cap \mathcal{H}_+$.

Since $(Au, v) = A[u, v]$ for all $u \in \text{Dom}(A)$ and all $v \in \mathcal{D}[A]$, we get that $\mathbb{A} \supset A$, i.e., the quasi-kernel of \mathbb{A} is A and therefore, A is t-self-adjoint bi-extension of \dot{A} . Further we use the following equality (see [6])

$$A[\varphi, u] = (\varphi, \dot{A}^*u), \quad \varphi \in \mathcal{D}[\dot{A}], \quad u \in \mathcal{D}[A] \cap \mathcal{H}_+.$$

Using (2.7) we get for all $\varphi \in \text{Dom}(A_F)$ and all $u \in \mathcal{D}[A] \cap \mathcal{H}_+$:

$$A[\varphi, u] = (\varphi, \dot{A}^*u) = ((\dot{A}^*)^*\varphi, u) = ((\dot{A}^* - \mathcal{R}^{-1}\dot{A}^*P_{\mathfrak{M}}^+)\varphi, u).$$

Hence, $\text{Dom}(A_F) \subset \text{Dom}(\mathbb{A})$ and

$$\mathbb{A}\varphi = (\dot{A}^* - \mathcal{R}^{-1}\dot{A}^*P_{\mathfrak{M}}^+)\varphi, \quad \varphi \in \text{Dom}(A_F).$$

Since $\text{Dom}(A) \subset \text{Dom}(\mathbb{A})$ and \mathbb{A} is a t-self-adjoint bi-extension of \dot{A} with quasi-kernel A , we get

$$\text{Dom}(\mathbb{A}) = \text{Dom}(A) + \text{Dom}(A_F).$$

Taking into account (2.3), we conclude that \mathbb{A} is generated by A_F . \square

The following statement is an immediate consequence of Theorems 4.4 and 4.5.

Corollary 4.6 ([7]). *The operator \dot{A} admits non-negative self-adjoint bi-extensions in $[\mathcal{H}_+, \mathcal{H}_-]$ if and only if A_K and A_F are transversal.*

It was announced in [26] that the transversality condition in Corollary 4.6 is necessary (and sufficient for the case of finite deficiency indices) for the existence of non-negative self-adjoint bi-extensions in $[\mathcal{H}_+, \mathcal{H}_-]$.

Denote by $\mathcal{P}(\dot{A})$ the set of all non-negative self-adjoint bi-extensions of \dot{A} . As has been proved in Theorem 4.4 the set $\mathcal{P}(\dot{A})$ is nonempty if and only if A_F and A_K are disjoint in which case the set $\mathcal{P}(\dot{A})$ contains the operator \mathbb{A}_K with the following properties:

1. the operator \mathbb{A}_K is associated with the closed form $A_K[u, v]$, $u, v \in \mathcal{D}[A_K] \cap \mathcal{H}_+$, i.e.,

$$\begin{aligned} \mathcal{D}[\mathbb{A}_K] &= \mathcal{D}[A_K] \cap \mathcal{H}_+, \\ \mathbb{A}_K[u, v] &= A_K[u, v], \quad u \in \text{Dom}(\mathbb{A}_K), \quad v \in \mathcal{D}[A_K] \cap \mathcal{H}_+, \end{aligned}$$

2. the operator \mathbb{A}_K is a t-self-adjoint bi-extension of \dot{A} with quasi-kernel A_K and generated by A_F ,
3. $\mathcal{P}(\dot{A}) \ni \mathbb{A} \Rightarrow \mathcal{D}[\mathbb{A}] \subseteq \mathcal{D}[A_K]$, $\mathbb{A}[u] \geq \mathbb{A}_K[u] = A_K[u]$, $u \in \mathcal{D}[\mathbb{A}]$.

Thus, \mathbb{A}_K is the *minimal element* of $\mathcal{P}(\dot{A})$ and is an analog of Kreĭn–von Neumann extension. The minimality property is a consequence of Theorem 3.2. Notice that if A_K and A_F are transversal and the deficiency number of \dot{A} is infinite, then the set $\mathcal{P}(\dot{A})$ contains $+\rightarrow -$ bounded and unbounded operators.

Let A_1 be a non-negative self-adjoint extension of \dot{A} . Let $\mathcal{P}(A_1)$ be the set of all non-negative t-self-adjoint bi-extension of \dot{A} with quasi-kernel A_1 . According

to Theorem 4.5 the set $\mathcal{P}(A_1) \neq \emptyset$ if and only if A_1 is disjoint with A_F . Using Theorem 3.1, and the equalities

$$\mathcal{D}[A_1] = \left\{ f \in \mathcal{H} : \sup_{h \in \text{Dom}(A_1)} \frac{|(A_1 h, f)|^2}{|(A_1 h, h)|} < \infty \right\},$$

$$\frac{|(\mathcal{R}A_1 h, f)_+|^2}{|(\mathcal{R}A_1 h, h)_+|} = \frac{|(A_1 h, f)|^2}{|(A_1 h, h)|}, \quad f \in \mathcal{H}_+,$$

we get: if A_1 and A_F are disjoint, then the operator $\mathbb{A}_{1K} : \mathcal{H}_+ \supseteq \text{Dom}(\mathbb{A}_{1K}) \rightarrow \mathcal{H}_-$, associated with closed in \mathcal{H}_+ non-negative form $A_1[u, v]$, $u, v \in \mathcal{D}[A_1] \cap \mathcal{H}_+$, is the minimal element of the set $\mathcal{P}(A_1)$ in the sense of quadratic forms. According to Theorem 4.5 this operator is generated by A_F . It is an analog of the Kreĭn–von Neumann type extension of A_1 in the rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$. The operator \mathbb{A}_K is the minimal element of the sets $\mathcal{P}(A_K)$ and $\mathcal{P}(\dot{A})$. The next theorem parameterizes the set $\mathcal{P}(A_1)$.

Theorem 4.7. *Let \dot{A} be a non-negative closed symmetric operator with disjoint non-negative self-adjoint extensions A_F and A_K . Suppose \mathbb{A} is a t -self-adjoint bi-extension of \dot{A} with quasi-kernel A_1 and generated by A_0 . Then \mathbb{A} is non-negative if and only if*

$$A_0 \geq A_1 \geq 0.$$

Proof. We will use (2.5)

$$(\mathbb{A}f, f) = (A_1 f_1, f_1) + (A_0 f_0, f_0) + 2\text{Re}(A_1 f_1, f_0),$$

for $f = f_1 + f_0$, $f_1 \in \text{Dom}(A_1)$, $f_0 \in \text{Dom}(A_0)$. It follows that $A_1 \geq 0$ and $A_0 \geq 0$. Replacing f_1 by λf_1 and f_0 by μf_0 we have

$$|\lambda|^2(A_1 f_1, f_1) + |\mu|^2(A_0 f_0, f_0) + \lambda \bar{\mu}(A_1 f_1, f_0) + \mu \bar{\lambda}(f_0, A_1 f_1) \geq 0$$

for all $\lambda, \mu \in \mathbb{C}$. Thus, the 2×2 matrix

$$\begin{pmatrix} (A_1 f_1, f_1) & (A_1 f_1, f_0) \\ (f_0, A_1 f_1) & (A_0 f_0, f_0) \end{pmatrix}$$

is non-negative. Hence

$$|(A_1 f_1, f_0)|^2 \leq (A_1 f_1, f_1)(A_0 f_0, f_0)$$

and

$$\sup_{f_1 \in \text{Dom}(A_1)} \frac{|(A_1 f_1, f_0)|^2}{|(A_1 f_1, f_1)|} \leq (A_0 f_0, f_0). \quad (4.6)$$

This means that

$$f_0 \in \mathcal{D}[A_1] \quad \text{and} \quad A_1[f_1] \leq (A_0 f_0, f_0) \quad (= A_0[f_0]).$$

If $\{f_2^{(n)}\}_{n=1}^\infty \subset \text{Dom}(A_0)$ and A_0 -converges to $\varphi_0 \in \mathcal{D}[A_0]$, then (4.6) yields

$$\sup_{f_1 \in \text{Dom}(A_1)} \frac{|(A_1 f_1, \varphi_0)|^2}{|(A_1 f_1, f_1)|} \leq A_0[\varphi_0].$$

Thus

$$\mathcal{D}[A_0] \subset \mathcal{D}[A_1] \quad \text{and} \quad A_1[\varphi_0] \leq A_0[\varphi_0] \quad \text{for all} \quad \varphi_0 \in \mathcal{D}[A_0],$$

i.e., $A_1 \leq A_0$.

Conversely. Suppose $0 \leq A_1 \leq A_0$. Then for an arbitrary $f_1 \in \text{Dom}(A_1)$, $f_0 \in \text{Dom}(A_0)$ we get

$$\begin{aligned} (\mathbb{A}(f_1 + f_0), f_1 + f_0) &= (A_1 f_1, f_1) + (A_0 f_0, f_0) + 2\text{Re}(A_1 f_1, f_0) \\ &= \|A_1^{1/2} f_1\|^2 + \|A_0^{1/2} f_0\|^2 + 2\text{Re}(A_1^{1/2} f_1, A_1^{1/2} f_0) \\ &= \|A_1^{1/2}(f_1 + f_0)\|^2 + \|A_0^{1/2} f_0\|^2 - \|A_1^{1/2} f_0\|^2 \geq 0. \end{aligned}$$

This proves the theorem. \square

Let A_1 and A_0 be two non-negative self-adjoint extensions of \dot{A} . Consider a form defined on $\text{Dom}(A_1) \times \text{Dom}(A_0)$ as follows

$$\mathcal{B}(f_1, f_0) = (A_1 f_1, f_1) + (A_0 f_0, f_0) + 2\text{Re}(A_1 f_1, f_0), \quad (4.7)$$

where $f_l \in \text{Dom}(A_l)$, ($l = 0, 1$). Let

$$\phi_l = \frac{1}{2}(I + A_l)f_l, \quad S_l \phi_l = \frac{1}{2}(I - A_l)f_l,$$

be the Cayley transform of A_l for $l = 0, 1$. Then

$$f_l = (I + S_l)\phi_l, \quad A_l f_l = (I - S_l)\phi_l, \quad (l = 0, 1). \quad (4.8)$$

Substituting (4.8) into (4.7) we obtain a form defined on $\mathcal{H} \times \mathcal{H}$

$$\tilde{\mathcal{B}}(\phi_1, \phi_0) = \|\phi_1 + \phi_0\|^2 - \|S_1 \phi_1 + S_0 \phi_0\|^2 - 2\text{Re}((S_1 - S_0)\phi_1, \phi_0). \quad (4.9)$$

Let us set

$$F = \frac{1}{2}(S_1 - S_0), \quad G = \frac{1}{2}(S_1 + S_0), \quad u = \frac{1}{2}(\phi_1 + \phi_0), \quad v = \frac{1}{2}(\phi_1 - \phi_0). \quad (4.10)$$

Then

$$\tilde{\mathcal{B}}(\phi_1, \phi_0) = 4H(u, v) := \|u\|^2 + (Fv, v) - (Fu, u) - \|Fv + Gu\|^2. \quad (4.11)$$

Moreover, $F \pm G$ are contractive operators. From the above reasoning we conclude that non-negativity of the form $\mathcal{B}(f_1, f_0)$ on $\text{Dom}(A_1) \times \text{Dom}(A_0)$ is equivalent to non-negativity of the form $H(u, v)$ on $\mathcal{H} \times \mathcal{H}$. The next statement is established in [3], see also [7].

Proposition 4.8. *The form $H(u, v)$ in (4.11) is non-negative for all $u, v \in \mathcal{H}$ if and only if operator F defined in (4.10) is non-negative.*

Proposition 4.8 can be used for another proof of Theorem 4.7 (see [3]).

Let A_1 and A_0 be two disjoint non-negative self-adjoint extensions of \dot{A} . We say that A_1 and A_0 form an *admissible pair* $\langle A_1, A_0 \rangle$ if

$$A_0 \geq A_1 \iff (A_1 + I)^{-1} \geq (A_0 + I)^{-1}.$$

If $S_j = (I - A_j)(I + A_j)^{-1}$, $j = 1, 2$, then the pair $\langle A_1, A_0 \rangle$ is admissible if and only if $\ker(S_1 - S_0) = \text{Dom}(\dot{S})$ and $S_1 \geq S_0$. Let X_j , $j = 0, 1$ be self-adjoint contractions in \mathfrak{N} such that

$$S_j = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2} X_j (S_M - S_\mu)^{1/2}.$$

Then it follows from (3.5) that the pair of non-negative self-adjoint extensions $A_j = (I - S_j)(I + S_j)^{-1}$, $j = 0, 1$ is admissible if and only if

$$\ker(X_1 - X_0) \cap \text{Ran}((S_M - S_\mu)^{1/2}) = \{0\} \quad \text{and} \quad X_1 - X_0 \geq 0.$$

Associated closed forms. The next statement describes $\mathbb{A}[u, v]$ (the closure of the form $(\mathbb{A}f, f)$), where \mathbb{A} is a non-negative t-self-adjoint bi-extension of \dot{A} with quasi-kernel A_1 and generated by A_0 (compare with Theorem 3.3).

Theorem 4.9. *Let $\langle A_1, A_0 \rangle$ be an admissible pair and let \mathbb{A} be a non-negative t-self-adjoint bi-extension of \dot{A} with quasi-kernel A_1 and generated by A_0 . Let $\mathbb{A}[\cdot, \cdot]$ be the closure of the form $(\mathbb{A}f, g)$, $f, g \in \text{Dom}(\dot{A})$. Then*

$$\begin{aligned} \mathcal{D}[\mathbb{A}] &= \text{Dom}(A_1) \dot{+} \text{Ran} \left((S_1 - S_0)^{1/2} \right) = \text{Dom}(A_0) \dot{+} \text{Ran} \left((S_1 - S_0)^{1/2} \right), \\ \mathbb{A}[u] &= \|h\|^2 - \|S_1 h + w\|^2 + 2\text{Re}(h, w) + 2\|(S_1 - S_0)^{-1/2} w\|^2 \\ &= A_1[u] + \|(S_1 - S_0)^{-1/2} w\|^2 - \|(S_1 - S_\mu)^{-1/2} w\|^2, \\ u &= (I + S_1)h + w, \end{aligned} \tag{4.12}$$

where $S_l = (I - A_l)(I + A_l)$, $l = 0, 1$, $h \in \mathcal{H}$, $w \in \text{Ran}((S_1 - S_0)^{1/2})$.

Proof. Let $f = f_1 + f_0$, $f_1 \in \text{Dom}(A_1)$, $f_0 \in \text{Dom}(A_0)$. Then

$$(\mathbb{A}f, f) = (A_1 f_1, f_1) + (A_0 f_0, f_0) + 2\text{Re}(A_1 f_1, f_0).$$

Due to (4.9)

$$(\mathbb{A}f, f) = \tilde{\mathcal{B}}(\phi_1, \phi_0) = \|\phi_1 + \phi_0\|^2 - \|S_1 \phi_1 + S_0 \phi_0\|^2 - 2\text{Re}((S_1 - S_0)\phi_1, \phi_0),$$

where

$$\phi_l = \frac{1}{2}(I + A_l)f_l, \quad S_l \phi_l = \frac{1}{2}(I - A_l)f_l, \quad f_l = (I + S_l)\phi_l, \quad A_l f_l = (I - S_l)\phi_l, \quad l = 0, 1.$$

Represent $f = f_1 + f_0 = (I + S_1)\phi_1 + (I + S_0)\phi_0$ in the form

$$f = (I + S_1)(\phi_1 + \phi_0) - (S_1 - S_0)\phi_0.$$

Then

$$\begin{aligned} (\mathbb{A}f, f) &= \|\phi_1 + \phi_0\|^2 - \|S_1(\phi_1 + \phi_0) - (S_1 - S_0)\phi_0\|^2 \\ &\quad - 2\text{Re}(\phi_1 + \phi_0, (S_1 - S_0)\phi_0) + 2\|(S_1 - S_0)^{1/2}\phi_0\|^2. \end{aligned} \tag{4.13}$$

Suppose that

$$\lim_{n \rightarrow \infty} f^{(n)} = u \text{ in } \mathcal{H}_+, \text{ and } \lim_{n \rightarrow \infty} (\mathbb{A}(f^{(n)} - f^{(m)}), f^{(n)} - f^{(m)}) = 0.$$

We have

$$f^{(n)} = (I + S_1)(\phi_1^{(n)} + \phi_0^{(n)}) - (S_1 - S_0)\phi_0^{(n)}.$$

Due to the direct decomposition

$$\mathcal{H}_+ = \text{Dom}(A_1) \dot{+} \mathfrak{N}_{-1}$$

and inclusions $\{(I + S_1)(\phi_1^{(n)} + \phi_0^{(n)})\} \subset \text{Dom}(A_1)$, $\{(S_1 - S_0)\phi_0^{(n)}\} \subset \mathfrak{N}_{-1}$, we get that the sequences $\{(I + S_1)(\phi_1^{(n)} + \phi_0^{(n)})\}$ and $\{(S_1 - S_0)\phi_0^{(n)}\}$ converge in \mathcal{H}_+ . By definition $\|w\|_+^2 = 2\|w\|^2$, $\forall w \in \mathfrak{N}_{-1}$. Hence $\{(S_1 - S_0)\phi_0^{(n)}\}$ converges in \mathcal{H} . On the other hand convergence of $\{(I + S_1)(\phi_1^{(n)} + \phi_0^{(n)})\}$ in \mathcal{H}_+ yields convergence of $\{\phi_1^{(n)} + \phi_0^{(n)}\}$ in \mathcal{H} . Let

$$h = \lim_{n \rightarrow \infty} (\phi_1^{(n)} + \phi_0^{(n)}) \text{ in } \mathcal{H},$$

$$\text{Dom}(A_1) \ni y = \lim_{n \rightarrow \infty} (I + S_1)(\phi_1^{(n)} + \phi_0^{(n)}) \text{ in } \mathcal{H}_+,$$

$$w' = \lim_{n \rightarrow \infty} (S_1 - S_0)\phi_0^{(n)}.$$

From $\lim_{n \rightarrow \infty} (\mathbb{A}(f^{(n)} - f^{(m)}), f^{(n)} - f^{(m)}) = 0$ and (4.13) we obtain that the sequence $\{(S_1 - S_0)^{1/2}\phi_0^{(n)}\}$ converges in \mathcal{H} . Let

$$g = \lim_{n \rightarrow \infty} (S_1 - S_0)^{1/2}\phi_0^{(n)}.$$

Then $w' = (S_1 - S_0)^{1/2}g$. Set $w = -w'$. Thus

$$u = y + w,$$

where $y = (I + S_1)h \in \text{Dom}(A_1)$, $w \in \text{Ran}((S_1 - S_0)^{1/2})$. We get that

$$\begin{aligned} \lim_{n \rightarrow \infty} (\mathbb{A}f^{(n)}, f^{(n)}) &= \|h\|^2 - \|S_1 h - (S_1 - S_0)^{1/2}g\|^2 \\ &\quad - 2\text{Re}(h, (S_1 - S_0)^{1/2}g) + 2\|g\|^2 \\ &= \|h\|^2 - \|S_1 h + w\|^2 + 2\text{Re}(h, w) + 2\|(S_1 - S_0)^{-1/2}w\|^2. \end{aligned}$$

Now let us prove that the quadratic form

$$\begin{aligned} \eta(u) &= \|h\|^2 - \|S_1 h + w\|^2 + 2\text{Re}(h, w) + 2\|(S_1 - S_0)^{-1/2}w\|^2, \\ u &= (I + S_1)h + w, \quad h \in \mathcal{H}, w \in \text{Ran}(S_1 - S_0)^{1/2} \end{aligned}$$

is non-negative and closed in \mathcal{H}_+ as defined on

$$\text{Dom}(\eta) = \text{Dom}(A_1) \dot{+} \text{Ran}((S_1 - S_0)^{1/2}).$$

Notice that the equality $S_1 - S_0 = 2(A_1 + I)^{-1} - 2(A_0 + I)^{-1}$ yields

$$\text{Dom}(A_1) \dot{+} \text{Ran}((S_1 - S_0)^{1/2}) = \text{Dom}(A_0) \dot{+} \text{Ran}((S_1 - S_0)^{1/2}).$$

First we calculate $A_1[u]$ for $u \in \text{Dom}(A_1) \dot{+} (\mathcal{D}[A_1] \cap \mathfrak{N}_{-1})$. Let us represent u as

$$u = (I + S_1)h + (I + S_1)^{1/2}\omega,$$

where $h \in \mathcal{H}$, $\omega \in \Omega = \{g \in \mathcal{H} : (I + S_1)^{1/2}\omega \in \mathfrak{N}_{-1}\}$. Recall that by (3.7) and (3.10) we have

$$\text{Ran}((S_1 - S_\mu)^{1/2}) = (I + S_1)^{1/2}\Omega = \mathcal{D}[A_1] \cap \mathfrak{N}_{-1}.$$

Using (3.5) we obtain

$$\begin{aligned} A_1[u] &= -\|u\|^2 + 2\|(I + S_1)^{-1/2}u\|^2 \\ &= -\|(I + S_1)h + (I + S_1)^{1/2}\omega\|^2 + 2\|(I + S_1)^{1/2}h + \omega\|^2 \\ &= -\|(I + S_1)h\|^2 - \|(I + S_1)^{1/2}\omega\|^2 - 2\text{Re}((I + S_1)h, (I + S_1)^{1/2}\omega) \\ &\quad + 2\|(I + S_1)^{1/2}h\|^2 + 2\|\omega\|^2 + 4\text{Re}((I + S_1)^{1/2}h, \omega) \\ &= \|h\|^2 - \|S_1 h\|^2 - \|(I + S_1)^{1/2}\omega\|^2 - 2\text{Re}(S_1 h, (I + S_1)^{1/2}\omega) \\ &\quad + 2\text{Re}(h, (I + S_1)^{1/2}\omega) + 2\|\omega\|^2 \\ &= \|h\|^2 - \|S_1 h + (I + S_1)^{1/2}\omega\|^2 + 2\|\omega\|^2 + 2\text{Re}(h, (I + S_1)^{1/2}\omega). \end{aligned}$$

Denoting $w = (I + S_1)^{1/2}\omega$ and using the equality (see (3.7)) $\|(S_1 - S_\mu)^{-1/2}w\| = \|(I + S_1)^{-1/2}w\|^2$, we arrive at the equality

$$A_1[u] = \|h\|^2 - \|S_1 h + w\|^2 + 2\text{Re}(h, w) + 2\|(S_1 - S_\mu)^{-1/2}w\|^2 \geq 0.$$

Furthermore, since $S_1 - S_\mu \geq S_1 - S_0$, we get that

$$\text{Ran}((S_1 - S_\mu)^{1/2}) \supset \text{Ran}((S_1 - S_0)^{1/2})$$

and $\|(S_1 - S_0)^{-1/2}w\|^2 \geq \|(S_1 - S_\mu)^{-1/2}w\|^2$ for all $w \in \text{Ran}((S_1 - S_0)^{1/2})$. So,

$$\begin{aligned} \eta(u) &= A_1[u] + \|(S_1 - S_0)^{-1/2}w\|^2 - \|(S_1 - S_\mu)^{-1/2}w\|^2 \geq 0, \\ u &\in \text{Dom}(A_1) \dot{+} \text{Ran}((S_1 - S_0)^{1/2}) \geq 0. \end{aligned}$$

In addition, one can easily see that the right-hand side of (4.12) is closed on $\text{Dom}(A_1) \dot{+} \text{Ran}((S_1 - S_0)^{1/2})$ in \mathcal{H}_+ . Now we can conclude that (4.12) is valid. \square

Define for $\mathbb{A} \in \mathcal{P}(\dot{A})$ the “dual” quadratic form

$$\mathbb{A}'[u] = 2\text{Re}(\dot{A}^*u, u) - \mathbb{A}[u], \quad u \in \mathcal{D}[\mathbb{A}]$$

and let

$$A'_K[u] = 2\text{Re}(\dot{A}^*u, u) - A_K[u], \quad u \in \mathcal{D}[A_K] \cap \mathcal{H}_+. \quad (4.14)$$

Recall that a linear operator T in a Hilbert space \mathfrak{H} is called *accretive* [20] if $\text{Re}(Tf, f) \geq 0$ for all $f \in \text{Dom}(T)$ and *maximal accretive* (m -accretive) if it is accretive and has no accretive extensions in \mathfrak{H} . The following statements are equivalent [27]:

- (i) the operator T is m -accretive;
- (ii) the operator T is accretive and its resolvent set contains points from the left half-plane;
- (iii) the operators T and T^* are accretive.

Theorem 4.10. *If A_F and A_K are disjoint, then each non-negative self-adjoint bi-extension \mathbb{A} of \dot{A} possess the properties*

$$\mathcal{D}[\mathbb{A}] \subseteq \mathcal{D}[A_K], \quad \mathbb{A}[u] \geq A_K[u], \quad \mathbb{A}'[u] \leq A'_K[u], \quad u \in \mathcal{D}[\mathbb{A}]. \quad (4.15)$$

In addition, if T is quasi-selfadjoint accretive extension of \dot{A} ($\dot{A} \subset T \subset \dot{A}^$), then*

$$A_K[u] \leq \operatorname{Re}(Tu, u) \leq A'_K[u], \quad u \in \operatorname{Dom}(T). \quad (4.16)$$

Proof. As it follows from the proofs of Theorems 4.4 and 4.5 in \mathcal{H}_+ the Kreĭn–von Neumann extension of the operator $\dot{B} = \mathcal{R}\dot{A}$ coincides with the Kreĭn–von Neumann extension of the operator $\dot{B}' = \mathcal{R}A_K$. Therefore, using the minimality of A_K among all non-negative self-adjoint extensions of \dot{A} we arrive at (4.15).

It is established in [4] that for each quasi-self-adjoint accretive extension T of \dot{A} one has

$$\operatorname{Dom}(T) \subset \mathcal{D}[A_K], \quad A_K[u] \leq \operatorname{Re}(Tu, u), \quad u \in \operatorname{Dom}(T).$$

Using the above and (4.14) we get (4.16). \square

Explicit expressions for non-negative t-self-adjoint bi-extensions. Evidently, the linear manifold $\operatorname{Dom}(A_F)$ is a subspace in \mathcal{H}_+ . Let \mathfrak{N}_F be the orthogonal complement to $\operatorname{Dom}(\dot{A})$ in $\operatorname{Dom}(A_F)$ with respect to the inner product $(\cdot, \cdot)_+$ and let $\mathfrak{M}_F = \mathcal{H}_+ \ominus \operatorname{Dom}(A_F)$. Then $\mathfrak{M}_F = A_F \mathfrak{N}_F$. Thus we have the (+)-orthogonal decomposition

$$\mathcal{H}_+ = \operatorname{Dom}(\dot{A}) \oplus \mathfrak{N}_F \oplus \mathfrak{M}_F.$$

Let

$$\mathfrak{N}_0 = \operatorname{Ran}(A_F^{1/2}) \cap \mathfrak{N}_F.$$

Clearly, $A_F^{-\frac{1}{2}}(\mathfrak{N}_0) \subset \operatorname{Dom}(A_F)$. The following equalities take place

$$\dot{A}^* A_F e = -e, \quad e \in \mathfrak{N}_F,$$

$$A_F \dot{A}^* g = -g, \quad g \in \mathfrak{M}_F.$$

Theorem 4.11 ([11]). *The condition $\mathfrak{N}_0 = \{0\}$ is necessary and sufficient for the uniqueness of non-negative self-adjoint extension of \dot{A} . Suppose $\mathfrak{N}_0 \neq \{0\}$. Then the formulas*

$$\operatorname{Dom}(\tilde{A}) = \operatorname{Dom}(\dot{A}) \oplus (I + A_F \tilde{U}) \operatorname{Dom}(\tilde{U}), \quad (4.17)$$

$$\tilde{A}(x + h + A_F \tilde{U} h) = A_F(x + h) - \tilde{U} h, \quad x \in \operatorname{Dom}(\dot{A}), \quad h \in \operatorname{Dom}(\tilde{U})$$

give a one-to-one correspondence between all non-negative self-adjoint extensions \tilde{A} of \dot{A} and all (+)-self-adjoint operators \tilde{U} in \mathfrak{N}_F satisfying the condition

$$0 \leq \tilde{U} \leq W_0^{-1}$$

where W_0^{-1} determines the operator inverse with respect to the (+)-non-negative self-adjoint relation W_0 in \mathfrak{N}_F associated with the (+)-closed in \mathfrak{N}_F non-negative form

$$\omega_0[x, y] = (A_F^{[-1/2]} x, A_F^{[-1/2]} y)_+ = (A_F^{1/2} x, A_F^{1/2} y) + (A_F^{[-1/2]} x, A_F^{[-1/2]} y), \quad x, y \in \mathfrak{N}_0.$$

Here $A_F^{[-1/2]}$ is the Moore–Penrose pseudo-inverse. Operator \tilde{A} coincides with the Kreĭn–von Neumann non-negative self-adjoint extension A_K if and only if $\tilde{U} = W_0^{-1}$.

Moreover,

- the extensions A_F and A_K are disjoint $\iff \mathfrak{N}_0$ is dense in \mathfrak{N}_F ,
- the extensions A_F and A_K are transversal $\iff \mathfrak{N}_0 = \mathfrak{N}_F$.

The associated with \tilde{A} closed form is given by the following equalities:

$$\mathcal{D}[\tilde{A}] = \mathcal{D}[A] + A_F \mathcal{R}(\tilde{U}^{1/2}), \quad (4.18)$$

$$\tilde{A}[\varphi + A_F h] = \|A_F^{1/2} \varphi - A_F^{[-1/2]} h\|^2 + \tilde{U}^{-1}[h] - w_0[h], \quad \varphi \in \mathcal{D}[A], h \in \mathcal{R}(\tilde{U}^{1/2}).$$

Let A_1 and A_0 be two non-negative self-adjoint extensions. From (4.17) and (4.18) it follows that A_1 and A_0 , determined by parameters U_1 and U_0 , respectively, then

- A_1 and A_0 are disjoint if and only if \mathfrak{N}_0 is dense in \mathfrak{N}_F and $\ker(U_1 - U_0) = \{0\}$,
- $A_1 \leq A_0$ if and only if $U_1 \geq U_0$,
- $A_1 \leq A_0$ and A_1 and A_0 are transversal if and only if $\mathfrak{N}_0 = \mathfrak{N}_F$, $\text{Ran}(U_1) = \mathfrak{N}_F$, $U_1 \geq U_0$, and $\text{Ran}(I - U_1^{-1}U_0) = \mathfrak{N}_F$.

Denote by $P_{\mathfrak{N}_F}^+$, $P_{\mathfrak{M}_F}^+$ the orthogonal projection in \mathcal{H}_+ onto \mathfrak{N}_F and $\mathfrak{M}_F = A_F \mathfrak{N}_F$. Notice that

$$\mathfrak{M} = \mathfrak{N}_i \oplus \mathfrak{N}_{-i} = \mathfrak{N}_F \oplus \mathfrak{M}_F.$$

Recall that each self-adjoint bi-extensions of \dot{A} is of the form (2.1), where \mathcal{S} is a (+)-self-adjoint operator in \mathfrak{M} .

Theorem 4.12. *Suppose A_K and A_F are disjoint. Then*

1. *the operator \mathbb{A}_K is of the form*

$$\mathbb{A}_K = \dot{A}^* - \mathcal{R}^{-1} A_F (P_{\mathfrak{N}_F}^+ + W_0 \dot{A}^* P_{\mathfrak{M}_F}^+); \quad (4.19)$$

2. *the operator $\mathbb{A} = \dot{A}^* + \mathcal{R}^{-1}(\mathcal{S} - \dot{A}^*/2)P_{\mathfrak{M}}^+$ belongs to $\mathcal{P}(\dot{A})$ if and only if*

$$\mathcal{S} \geq \mathcal{S}_K = -A_F W_0 \dot{A}^* P_{\mathfrak{M}_F}^+ + \frac{1}{2} \left(\dot{A}^* P_{\mathfrak{M}_F}^+ - A_F P_{\mathfrak{N}_F}^+ \right)$$

in the sense of quadratic forms;

3. *if A_1 is a non-negative self-adjoint extension of \dot{A} disjoint with A_F and if $A_0 \geq A_1$, then the non-negative t -self-adjoint bi-extension of \dot{A} with quasi-kernel A_1 and generated by A_0 is of the form*

$$\mathbb{A} = \dot{A}^* + \mathcal{R}^{-1} \left(\mathcal{S} - \frac{1}{2} \dot{A}^* \right) P_{\mathfrak{M}}^+,$$

where \mathcal{S} is a $(+)$ -self-adjoint operator in \mathfrak{M} given by

$$\begin{cases} \text{Dom}(\mathcal{S}) = (I + A_F U_1) \text{Dom}(U_1) \dot{+} (I + A_F U_0) \text{Dom}(U_0) \\ \mathcal{S}(I + A_F U_1)e = \frac{1}{2}(A_F - U_1)e, \quad e \in \text{Dom}(U_1) \\ \mathcal{S}(I + A_F U_0)g = \frac{1}{2}(-A_F + U_0)g, \quad g \in \text{Dom}(U_0) \end{cases}, \quad (4.20)$$

and U_1, U_0 determine A_1 and A_0 in formulas (4.17). In particular, if $A_0 = A_F$, then

$$\mathcal{S} = -A_F U_1^{-1} \dot{A}^* P_{\mathfrak{M}_F}^+ + \frac{1}{2} \left(\dot{A}^* P_{\mathfrak{M}_F}^+ - A_F P_{\mathfrak{N}_F}^+ \right). \quad (4.21)$$

Proof. From (4.17) we get the equality

$$\text{Dom}(\tilde{A}) \ominus \text{Dom}(\dot{A}) = (I + A_F \tilde{U}) \text{Dom}(\tilde{U})$$

for an arbitrary non-negative self-adjoint extension \tilde{A} of \dot{A} . Then equalities (2.4) yield (4.20). When $A_0 = A_F$, we have $U_0 = 0$. This gives the equality

$$f = (I + A_F U_1)(-U_1^{-1} \dot{A}^*) P_{\mathfrak{M}_F}^+ f + (P_{\mathfrak{N}_F}^+ + U_1^{-1} \dot{A}^* P_{\mathfrak{M}_F}^+) f.$$

Then by virtue of (4.20) we obtain (4.21). The case $A_1 = A_K$ holds true if and only if $U_1 = W_0^{-1}$ and leads to

$$\mathcal{S}_K = -A_F W_0 \dot{A}^* P_{\mathfrak{M}_F}^+ + \frac{1}{2} \left(\dot{A}^* P_{\mathfrak{M}_F}^+ - A_F P_{\mathfrak{N}_F}^+ \right).$$

Then applying (2.1) we get (4.19). Statement (2.) follows from the fact that \mathbb{A}_K is the minimal element of $\mathcal{P}(\dot{A})$. \square

5. Extremal non-negative self-adjoint bi-extensions

Let \dot{S} be a symmetric contraction defined in subspace $\text{Dom}(\dot{S})$. We call a sc-extension S of \dot{S} *extremal* if

$$\inf_{gS \in \text{Dom}(\dot{S})} \|(I - S^2)^{1/2}(g - g_S)\| = 0, \quad \forall g \in \mathcal{H}.$$

We can also offer an equivalent definition of an extremal sc-extension. Let $\mathfrak{N} = \mathcal{H} \ominus \text{Dom}(S)$. We call a sc-extension S of \dot{S} *extremal* if $(I - S^2)_{\mathfrak{N}} = 0$, where $(I - S^2)_{\mathfrak{N}}$ is the Kreĭn shorted operator (see (3.8), (3.9)). The following equality was proved in [10]

$$(I - S^2)_{\mathfrak{N}} = (S_M - S_\mu)^{1/2}(I - X^2)(S_M - S_\mu)^{1/2}, \quad (5.1)$$

where X is corresponding to S (via formula (3.5)) contraction in $\overline{\text{Ran}(S_M - S_\mu)}$. Formula (5.1) implies that S is extremal if and only if X is self-adjoint and unitary, i.e., $X = X^*$ and $X^2 = I$.

Now let \dot{A} be a non-negative closed densely defined symmetric operator. Recall (see Section 3) that a non-negative self-adjoint extension A of \dot{A} is *extremal* [3] if

$$\inf_{\varphi \in \text{Dom}(\dot{A})} (A(h - \varphi), h - \varphi) = 0, \quad \forall h \in \text{Dom}(A).$$

If

$$\dot{S} = (I - \dot{A})(I + \dot{A})^{-1}, \quad S = (I - A)(I + A)^{-1}, \quad (5.2)$$

then $(Ah, h) = ((I - S^2)g, g)$ where $g = (I + S)^{-1}h$. This yields

$$\inf_{\varphi \in \text{Dom}(\dot{A})} (A(h - \varphi), h - \varphi) = \inf_{g_S \in \text{Dom}(\dot{S})} \|(I - S^2)^{1/2}(g - g_S)\|^2,$$

where $\text{Dom}(\dot{S}) = (I + \dot{A})\text{Dom}(\dot{A})$. Therefore, A is extremal non-negative self-adjoint extension of \dot{A} if and only if S is extremal sc-extension of symmetric contraction \dot{S} . The Friedrichs and Kreĭn-von Neumann extensions are extremal.

Let \mathbb{A} be a non-negative self-adjoint bi-extension of the symmetric operator \dot{A} . We call the operator \mathbb{A} an *extremal bi-extension* if

$$\inf_{\varphi \in \text{Dom}(\dot{A})} (\mathbb{A}(f - \varphi), f - \varphi) = 0, \quad \forall f \in \text{Dom}(\mathbb{A}).$$

In what follows we assume that the operators A_F and A_K are disjoint.

Theorem 5.1. *A t -self-adjoint bi-extension \mathbb{A} is extremal if and only if it is generated by an admissible pair $\langle A_1, A_0 \rangle$ of extremal non-negative self-adjoint extensions of \dot{A} .*

Proof. Let A_1 and A_0 be the quasi-kernels of \mathbb{A} and \mathbb{A}' , respectively. Let also \mathbb{A} be an extremal self-adjoint bi-extension. It follows from (2.5) then that

$$(\mathbb{A}f_k, f_k) = (A_k f_k, f_k), \quad \forall f_k \in \text{Dom}(A_k), \quad k = 0, 1.$$

Since \mathbb{A} extends A_1 and is generated by A_0 , it follows from (2.5) that

$$(\mathbb{A}f, f) = (A_1 f_1, f_1) + (A_0 f_0, f_0) + 2\text{Re}(A_1 f_1, f_0) = \mathcal{B}(f_1, f_0),$$

where $f \in \text{Dom}(\mathbb{A})$, $f = f_1 + f_0$, $f_k \in \text{Dom}(A_k)$, $k = 0, 1$. Applying (4.10) and (4.11) we get

$$\begin{aligned} \inf_{\varphi \in \text{Dom}(\dot{A})} (\mathbb{A}(f - \varphi), f - \varphi) &= \inf_{h_S \in \text{Dom}(\dot{S})} H(x - h_S, y) \\ &= \inf_{h_S \in \text{Dom}(\dot{S})} (\|x - h_S\|^2 - (x, Fx) + (y, Fy) - \|Fy + G(x - h_S)\|^2). \end{aligned} \quad (5.3)$$

Since $\inf_{f_A \in \text{Dom}(\dot{A})} (\mathbb{A}(f_k - f_A), f_k - f_A) = 0$ for all $f_k \in \text{Dom}(A_k)$, $k = 0, 1$, the operators A_1 and A_0 are extremal non-negative self-adjoint extensions of \dot{A} .

Hence, the extremality of \mathbb{A} implies that the non-negative self-adjoint extensions A_1 and A_0 are also extremal. Since \mathbb{A} is a non-negative self-adjoint bi-extension, then the pair $\langle A_1, A_0 \rangle$ is an admissible extremal pair.

Conversely, let us assume that $\langle A_1, A_0 \rangle$ is an admissible pair of extremal non-negative self-adjoint extensions of \dot{A} . We are going to prove that the corresponding

non-negative self-adjoint bi-extension \mathbb{A} with quasi-kernel A_1 and generated by A_0 is extremal. The corresponding (via (5.2)) to A_1 and A_0 sc-extensions S_1 and S_0 are extremal. Also, the fact that $\langle A_1, A_0 \rangle$ is an admissible pair, implies that $S_1 - S_0 \geq 0$.

Let

$$S_k = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2} X_k (S_M - S_\mu)^{1/2}, \quad k = 0, 1,$$

where X_k , $k = 0, 1$ are self-adjoint contractions in \mathfrak{N} . Since S_k , $k = 0, 1$ are extremal sc-extensions, then X_k , $k = 0, 1$, are self-adjoint unitary operators and hence $P_k = (I + X_k)/2$, $k = 0, 1$, are orthogonal projections. Also, $X_1 - X_0 \geq 0$ implies that $P_1 - P_0 \geq 0$ and $\text{Ran}(P_1) \supset \text{Ran}(P_0)$. Since $X_k = 2P_k - I$, $k = 0, 1$, then

$$G = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2}(P_1 + P_0 - I)(S_M - S_\mu)^{1/2},$$

and

$$F = (S_M - S_\mu)^{1/2}(P_1 - P_0)(S_M - S_\mu)^{1/2}.$$

Since $I - (P_1 + P_0 - I)^2 = P_1 - P_0$, then (5.1) implies that $(I - G^2) \upharpoonright \mathfrak{N} = F$. Consequently, applying the definition of the operator $(I - G^2) \upharpoonright \mathfrak{N}$ we obtain

$$F = (I - G^2)^{1/2} P_G (I - G^2)^{1/2},$$

where P_G is an orthoprojection onto the subspace

$$\mathcal{H}_G = ((I - G^2)^{1/2})^{-1} \{ \mathfrak{N} \} \cap \overline{\text{Ran}}((I - G^2)^{1/2}).$$

Therefore,

$$\begin{aligned} H(x - h_s, y) &= \|x - h_s\|^2 - (x, Fx) + (y, Fy) - \|Fy + G(x - h_s)\|^2 \\ &= \|(I - G^2)^{1/2}(x - h_s)\|^2 - \|P_G(I - G^2)^{1/2}x\|^2 + \|P_G(I - G^2)^{1/2}y\|^2 \\ &\quad - \|(I - G^2)^{1/2}P_G(I - G^2)^{1/2}y\|^2 - 2\text{Re}((I - G^2)^{1/2}P_G(I - G^2)^{1/2}y, G(x - h_s)) \\ &= \|(I - G^2)^{1/2}(x - h_s)\|^2 - \|P_G(I - G^2)^{1/2}x\|^2 + \|GP_G(I - G^2)^{1/2}y\|^2 \\ &\quad - 2\text{Re}(GP_G(I - G^2)^{1/2}y, (I - G^2)^{1/2}(x - h_s)) \\ &= \|(I - G^2)^{1/2}(x - h_s) - GP_G(I - G^2)^{1/2}y\|^2 - \|P_G(I - G^2)^{1/2}x\|^2. \end{aligned}$$

Thus, since $(I - G^2)^{1/2} \text{Dom}(\dot{S}) \perp \mathcal{H}_G$, then

$$\begin{aligned} \inf_{h_s \in \text{Dom}(\dot{S})} H(x - h_s, y) &= \|P_G(I - G^2)^{1/2}x - P_GGP_G(I - G^2)^{1/2}y\|^2 \\ &\quad - \|P_G(I - G^2)^{1/2}x\|^2, \quad \forall x, y \in \mathcal{H}. \end{aligned} \tag{5.4}$$

Since A_1 and A_0 are extremal non-negative self-adjoint extensions, then the definition of the functional H and (4.11) imply

$$\inf_{h_s \in \text{Dom}(\dot{S})} H(x - h_s, x) = 0, \quad \inf_{h_s \in \text{Dom}(\dot{S})} H(x - h_s, -x) = 0,$$

for all $x \in \mathcal{H}$. Relation (5.4) yields

$$\|P_G(I - G^2)^{1/2}x - P_G G P_G(I - G^2)^{1/2}x\|^2 - \|P_G(I - G^2)^{1/2}x\|^2 = 0,$$

and

$$\|P_G(I - G^2)^{1/2}x + P_G G P_G(I - G^2)^{1/2}x\|^2 - \|P_G(I - G^2)^{1/2}x\|^2 = 0,$$

for all $x \in \mathcal{H}$. Thus, $P_G G P_G(I - G^2)^{1/2}x = 0$ for all $x \in \mathcal{H}$. Applying (5.4) again we get

$$\inf_{h_S \in \text{Dom}(\dot{S})} H(x - h_S, y) = 0, \quad \forall x, y \in \mathcal{H}.$$

Now we can use (5.3) to confirm that

$$\inf_{f_A \in \text{Dom}(\dot{A})} (\mathbb{A}(f - f_A), f - f_A) = 0,$$

which means that \mathbb{A} is an extremal non-negative self-adjoint bi-extension. \square

Recall that the non-negative self-adjoint bi-extension \mathbb{A}_K is associated with the closed in \mathcal{H}_+ form $A_K[u, v]$, $u, v \in \mathcal{D}[A_K] \cap \mathcal{H}_+$. The quasi-kernel of \mathbb{A}_K is the Kreĭn–von Neumann extension A_K and \mathbb{A}_K is generated by A_F . Clearly, \mathbb{A}_K is extremal non-negative self-adjoint bi-extension of \dot{A} .

Theorem 5.2.

- (1) Let A_F and A_K be transversal. Then the operator \mathbb{A}_K is the unique extremal non-negative t -self-adjoint bi-extension.
- (2) Let A_F and A_K be disjoint but not transversal. Then except \mathbb{A}_K there exist infinitely many extremal non-negative t -self-adjoint bi-extensions.

Proof. (1) Suppose that A_F and A_K are transversal. Let also \mathbb{A} be an extremal t -self-adjoint bi-extension with the quasi-kernel A_1 and A_0 be the quasi-kernel of \mathbb{A}' . According to Theorem 5.1 for $S_k = (I - A_k)(I + A_k)^{-1}$, $k = 0, 1$ the following relations hold

$$S_k = S_\mu + (S_M - S_\mu)^{1/2} P_k (S_M - S_\mu)^{1/2}, \quad k = 0, 1, \quad (5.5)$$

where P_k , $k = 0, 1$, are orthoprojections in \mathfrak{N} . Since A_1 and A_0 are disjoint, we have $\ker((S_1 - S_0) \upharpoonright \mathfrak{N}) = \{0\}$. But

$$\ker((S_1 - S_0) \upharpoonright \mathfrak{N}) = \ker((S_M - S_\mu)^{1/2} (P_1 - P_0) (S_M - S_\mu)^{1/2} \upharpoonright \mathfrak{N}).$$

Since $P_1 - P_0 \geq 0$, then $Q = P_1 - P_0$ is an orthoprojection. Also, $\text{Ran}(S_M - S_\mu) = \mathfrak{N}$ implies $\ker(P_1 - P_0) = \{0\}$ or equivalently $P_1 - P_0 = I$. The latter yields $P_1 = I$ and $P_0 = 0$. Consequently, $S_1 = S_M$, $S_0 = S_\mu$ and the quasi-kernels of \mathbb{A} and \mathbb{A}' coincide with A_F and A_K .

(2) Let A_F and A_K be disjoint but not transversal. Then $\text{Ran}((S_M - S_\mu)^{1/2}) \neq \mathfrak{N}$ and $\ker((S_M - S_\mu)^{1/2}) = \{0\}$. We chose a subspace $\mathfrak{L} \subset \mathfrak{N}$ in a way that $\mathfrak{L} \cap \text{Ran}(S_M - S_\mu)^{1/2} = \{0\}$. Let \mathfrak{N}_1 be such that $\{0\} \subseteq \mathfrak{N}_1 \subseteq \mathfrak{L}$. Let also P_1 be an orthogonal projection operator on $\mathfrak{N} \ominus \mathfrak{N}_1$, Q an orthoprojection on $\mathfrak{N} \ominus \mathfrak{L}$, and $P_0 = P_1 - Q$. Then $P_1 - P_0 = Q \geq 0$ and $\ker(P_1) \cap \text{Ran}(S_M - S_\mu)^{1/2} = \{0\}$. Let S_k , $k = 0, 1$, be defined by (5.5). Hence, S_1 and S_0 are extremal sc-extensions and

A_k , $k = 0, 1$ are extremal non-negative self-adjoint extensions of \dot{A} and $\langle A_1, A_0 \rangle$ is an admissible pair. Therefore, according to Theorem 5.1, if $\mathbb{A} \supset A_1$ and \mathbb{A} is generated by A_0 , then \mathbb{A} is extremal t-self-adjoint bi-extension of \dot{A} . It follows from the construction of \mathbb{A} that there is infinite number of these bi-extensions. \square

6. Boundary triplets and self-adjoint bi-extensions

Let \dot{A} be a closed densely defined symmetric operator in \mathcal{H} with equal deficiency numbers.

Definition 6.1 ([21]). The triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ is called a *boundary triplet* for \dot{A}^* if \mathcal{N} is a Hilbert space and Γ_0, Γ_1 are bounded linear operators from the Hilbert space $\mathcal{H}_+ = \text{Dom}(\dot{A}^*)$ (with the inner product (1.1)) into \mathcal{N} such that the mapping

$$\Gamma := \langle \Gamma_0, \Gamma_1 \rangle : \mathcal{H}_+ \rightarrow \mathcal{N} \oplus \mathcal{N},$$

is surjective and the abstract Green identity

$$\left(\dot{A}^* f, g \right) - \left(f, \dot{A}^* g \right) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{N}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{N}},$$

holds for all $f, g \in \mathcal{H}_+$.

It follows from Definition 6.1 (see [17], [18]) that the operators

$$\text{Dom}(A_k) := \ker \Gamma_k, \quad A_k := \dot{A}^* \upharpoonright \text{Dom}(A_k), \quad (k = 0, 1),$$

are self-adjoint extensions of \dot{A} . Moreover, they are transversal, i.e.,

$$\text{Dom}(\dot{A}^*) = \text{Dom}(A_0) + \text{Dom}(A_1).$$

Notice that if $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ is a boundary triplet for \dot{A}^* , then $\Pi' = \{\mathcal{N}, -\Gamma_0, \Gamma_1\}$ is the boundary triplet for \dot{A}^* too.

We are going to provide connections between self-adjoint bi-extensions and boundary triplets [7]. The proposition below follows from Definition 6.1.

Theorem 6.2. *Let \dot{A} be a closed densely defined symmetric operator with equal deficiency indices in the Hilbert space \mathcal{H} . Suppose \mathcal{N} is a Hilbert space, $\Gamma_0, \Gamma_1 \in [\mathcal{H}_+, \mathcal{N}]$, and the operator $\langle \Gamma_0, \Gamma_1 \rangle \in [\mathcal{H}_+, \mathcal{N} \oplus \mathcal{N}]$ is surjective. Then the following statements are equivalent.*

- (i) $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ is the boundary triplet for \dot{A}^* ;
- (ii) the sesquilinear form

$$w(f, g) := (\dot{A}^* f, g) - (\Gamma_1 f, \Gamma_0 g)_{\mathcal{N}}, \quad f, g \in \mathcal{H}_+ = \text{Dom}(\dot{A}^*) \quad (6.1)$$

is Hermitian, i.e., $w(f, g) = \overline{w(g, f)}$;

- (iii) the sesquilinear form

$$w'(f, g) := (\dot{A}^* f, g) + (\Gamma_0 f, \Gamma_1 g)_{\mathcal{N}}, \quad f, g \in \mathcal{H}_+ = \text{Dom}(\dot{A}^*) \quad (6.2)$$

is Hermitian,

If $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigged Hilbert space, \mathcal{N} is a Hilbert space, and $\Gamma \in [\mathcal{H}_+, \mathcal{N}]$, then by Γ^\times we will denote the adjoint operator from $[\mathcal{N}, \mathcal{H}_-]$, i.e., $(\Gamma h, g)_\mathcal{N} = (h, \Gamma^\times g)$ for all $h \in \mathcal{H}_+$ and all $g \in \mathcal{N}$.

The following theorem [7] sets up the connection between boundary triplets and t-self-adjoint bi-extensions.

Theorem 6.3. *Let \dot{A} be a closed densely defined symmetric operator with equal deficiency numbers in the Hilbert space \mathcal{H} . Consider the rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ generated by \dot{A} .*

1. *Let $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ for \dot{A}^* be a boundary triplet for \dot{A}^* . Define operators \mathbb{A} and \mathbb{A}'*

$$\mathbb{A} := \dot{A}^* - \Gamma_0^\times \Gamma_1, \quad \mathbb{A}' := \dot{A}^* + \Gamma_1^\times \Gamma_0,$$

where Γ_0^\times and $\Gamma_1^\times \in [\mathcal{N}, \mathcal{H}_-]$ are the adjoint operators to Γ_0 and Γ_1 , respectively. Then \mathbb{A} and \mathbb{A}' belong to $[\mathcal{H}_+, \mathcal{H}_-]$ and are t-self-adjoint bi-extensions of \dot{A} . Moreover,

$$\mathbb{A} \supset A_1, \quad \mathbb{A}' \supset A_0.$$

2. *If \mathbb{A} is a t-self-adjoint bi-extension of \dot{A} with quasi-kernel A_1 and generated by A_0 , then there exists a boundary triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ for \dot{A}^* such that $\dot{A}^\dagger \upharpoonright \ker \Gamma_1 = A_1$ and $\mathbb{A} = \dot{A}^* - \Gamma_0^\times \Gamma_1$.*

It is shown in the proof of Theorem 6.3 that the form $w(f, g)$ in (6.1) corresponds to \mathbb{A} , the boundary triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$, and $w(f, g) = (\mathbb{A}f, g)$. Similarly, $w'(f, g) = (\mathbb{A}'f, g)$, where $w'(f, g)$ is defined in (6.2), and the boundary triplet is $\Pi' = \{\mathcal{N}, -\Gamma_0, \Gamma_1\}$.

Definition 6.4 ([3]). Suppose that \dot{A} is a non-negative symmetric operator. A boundary triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ is called *non-negative* if

$$w(f, f) = (\dot{A}^*f, f) - (\Gamma_1f, \Gamma_0f)_\mathcal{N} \geq 0 \text{ for all } f \in \mathcal{H}_+.$$

The operator $\mathbb{A} = \dot{A}^* - \Gamma_0^\times \Gamma_1$ corresponding to the boundary triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ is [3] a t-self-adjoint non-negative bi-extension of \dot{A} and belongs to $[\mathcal{H}_+, \mathcal{H}_+]$. If \dot{A} is a positive-definite operator, then for the positive-definite self-adjoint extension A we have $\mathcal{H}_+ = \text{Dom}(\dot{A}^*) = \text{Dom}(A) \dot{+} \ker(\dot{A}^*)$. Consequently, A_F and A_K are transversal. Let P be a projection in \mathcal{H}_+ onto $\text{Dom}(A)$ parallel to $\ker(\dot{A}^*)$, $\Pi = \{\mathcal{N}, \Gamma_K, \Gamma\}$ be a boundary triplet such that $\ker(\Gamma_K) = \text{Dom}(A_K)$. Then

$$(\dot{A}^*f, f) - (\Gamma_Kf, \Gamma f)_\mathcal{N} = (APf, Pf), \quad f \in \mathcal{H}_+,$$

i.e., $\{\mathcal{N}, \Gamma_K, \Gamma\}$ is a positive boundary triplet. The latter equality has been assumed as the definition of a positive boundary triplet (the space of boundary values) in the case of a positive-definite operator \dot{A} in [22].

It was shown in [3] that a positive boundary triplet exists if and only if A_F and A_K are transversal. The following theorem naturally follows from the preceding discussion.

Theorem 6.5. *Let \dot{A} be a closed densely defined non-negative symmetric operator such that A_F and A_K are transversal. Then*

1. *to every non-negative boundary triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ there corresponds a non-negative t -self-adjoint bi-extension $\mathbb{A} = \dot{A}^* - \Gamma_0^\times \Gamma_1$;*
2. *to every non-negative t -self-adjoint bi-extension \mathbb{A} there corresponds (up to equivalence¹) a non-negative boundary triplet.*

Let $\Pi = \{\mathcal{N}, \Gamma_F, \Gamma_K\}$ be a non-negative boundary triplet such that $\text{Dom}(A_K) = \ker \Gamma_K$, and $\text{Dom}(A_F) = \ker \Gamma_F$. In [3] this boundary triplet is called *basic*. It is not hard to see that the corresponding to the basic boundary triplet non-negative t -self-adjoint bi-extension

$$\mathbb{A}_0 = \dot{A}^* - \Gamma_F^\times \Gamma_K \quad (6.3)$$

is such that the quasi-kernel of \mathbb{A}_0 is A_K . At the same time, A_F is the quasi-kernel of the bi-extension $\mathbb{A}'_0 = \dot{A}^* + \Gamma_K^\times \Gamma_F$. It follows that $\mathbb{A}_0 = \mathbb{A}_K$ is the minimal element of $\mathcal{P}(\dot{A})$. The following theorem is established in [3].

Theorem 6.6. *Let $\Pi = \{\mathcal{N}, \Gamma_F, \Gamma_K\}$ be a basic boundary triplet. Then a boundary triplet $\tilde{\Pi} = \{\tilde{\mathcal{N}}, \tilde{\Gamma}_1, \tilde{\Gamma}_0\}$ is non-negative if and only if*

$$\tilde{\Gamma}_1 = X(\Gamma_K - B_1\Gamma_F), \quad \tilde{\Gamma}_0 = X^{*-1}[(I + B_2B_1)\Gamma_F - B_2\Gamma_K],$$

where B_1, B_2 are non-negative bounded operators in \mathcal{H} and X is a linear homeomorphism from \mathcal{H} onto $\tilde{\mathcal{H}}$.

Theorem 6.6 essentially provides us with another way to describe all non-negative t -self-adjoint bi-extensions in $[\mathcal{H}_+, \mathcal{H}_-]$. Namely, if $\Pi = \{\mathcal{N}, \Gamma_F, \Gamma_K\}$ is a basic non-negative boundary triplet, then the formula

$$\mathbb{A} = \dot{A}^* - [\Gamma_F^\times(I + B_1B_2) - \Gamma_K^\times B_2](\Gamma_K - B_1\Gamma_F), \quad (6.4)$$

where B_1, B_2 are non-negative bounded operators in \mathcal{H} , gives that description. Formulas (6.3) and (6.4) yield the following expression for quadratic forms

$$(\mathbb{A}f, f) = (\mathbb{A}_0f, f) + b(f, f), \quad f \in \mathcal{H}_+,$$

where

$$\begin{aligned} b(f, f) &= (B_1\Gamma_F f, \Gamma_F f) + (B_2\Gamma_K f, \Gamma_K f) + (B_1\Gamma_F f, B_2B_1\Gamma_F f) \\ &\quad - 2\text{Re}(B_1\Gamma_K f, B_2\Gamma_K f) \\ &= \|\Gamma_F^{1/2} B_1 f\|_{\mathcal{N}}^2 + \|B_2^{1/2}(B_1\Gamma_F - \Gamma_K)f\|_{\mathcal{N}}^2. \end{aligned}$$

For the corresponding dual self-adjoint bi-extension

$$\mathbb{A}' = \dot{A}^* + (\Gamma_K^\times - \Gamma_F^\times B_1)((I + B_2B_1)\Gamma_F - B_2\Gamma_K),$$

¹Two boundary triplets $\{\mathcal{N}, \Gamma_1, \Gamma_0\}$ and $\{\tilde{\mathcal{N}}, \tilde{\Gamma}_1, \tilde{\Gamma}_0\}$ are called *equivalent* [3] if $\ker \Gamma_k = \ker \tilde{\Gamma}_k$, $k = 0, 1$.

we have

$$(\mathbb{A}'f, f) = (\mathbb{A}'_0f, f) - b(f, f), \quad \forall f \in \mathcal{H}_+.$$

Set

$$\mathcal{N} = \mathfrak{N}_F, \quad \Gamma_0 = -\dot{A}^* P_{\mathfrak{M}_F}^+, \quad \Gamma_1 = P_{\mathfrak{N}_F}^+.$$

One can easily check that $\{\mathcal{N}, \Gamma_1, \Gamma_0\}$ is a boundary triplet for \dot{A}^* . Clearly

$$\ker(\Gamma_0) = \text{Dom}(A_F).$$

Calculating Γ_0^\times and Γ_1^\times one obtains

$$\Gamma_0^\times = \mathcal{R}^{-1} A_F P_{\mathfrak{N}_F}^+, \quad \Gamma_1^\times = \mathcal{R}^{-1} P_{\mathfrak{N}_F}^+.$$

Using Theorem 4.11 we get that the domains of all non-negative self-adjoint extensions \tilde{A} of \dot{A} takes the form

$$\text{Dom}(\tilde{A}) = \{v \in \text{Dom}(\dot{A}^*) : \Gamma_0 v = \tilde{U} \Gamma_1 v\},$$

where \tilde{U} is an arbitrary $(+)$ -self-adjoint and non-negative operator in \mathfrak{N}_F , satisfying $0 \leq \tilde{U} \leq W_0^{-1}$, and

$$\text{Dom}(A_K) = \{v \in \text{Dom}(\dot{A}^*) : \Gamma_0 v = W_0^{-1} \Gamma_1 v\}.$$

Now suppose that A_F and A_K are disjoint (transversal). Then W_0 is a densely defined (everywhere defined) in \mathfrak{N}_F and $(+)$ -self-adjoint and we can rewrite $\text{Dom}(A_K)$ as

$$\text{Dom}(A_K) = \ker(\Gamma_1 - W_0 \Gamma_0).$$

The operator

$$\mathbb{A}_K = \dot{A}^* - \Gamma_0^\times (\Gamma_1 - W_0 \Gamma_0)$$

is t -self-adjoint bi-extension with quasi-kernel A_K and generated by A_F . This is the minimal element of the set $\mathcal{P}(\dot{A})$. Then we get the explicit expressions for \mathbb{A}_K and \mathbb{A}'_K (cf. (4.19)):

$$\begin{aligned} \mathbb{A}_K &= \dot{A}^* - \mathcal{R}^{-1} A_F (P_{\mathfrak{N}_F}^+ + W_0 \dot{A}^* P_{\mathfrak{M}_F}^+), \\ \mathbb{A}'_K &= \dot{A}^* - \mathcal{R}^{-1} (P_{\mathfrak{N}_F}^+ - A_F W_0 P_{\mathfrak{N}_F}^+) \dot{A}^* P_{\mathfrak{M}_F}^+. \end{aligned}$$

If A_F and A_K are transversal, then we set

$$\Gamma_F = \Gamma_0 = -\dot{A}^* P_{\mathfrak{M}_F}^+, \quad \Gamma_K = \Gamma_1 - W_0 \Gamma_0 = P_{\mathfrak{N}_F}^+ + W_0 \dot{A}^* P_{\mathfrak{M}_F}^+.$$

Consequently, we obtain that $\{\mathfrak{N}_F, \Gamma_K, \Gamma_F\}$ is a basic boundary triplet for \dot{A}^* . Applying (6.4) we get a complete description of the set of all t -self-adjoint non-negative bi-extensions of \dot{A} in $[\mathcal{H}_+, \mathcal{H}_-]$ given by the following formula

$$\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1} [A_F (I + (W_0 + B_1) B_2) - B_2] [P_{\mathfrak{N}_F}^+ + (W_0 + B_1) \dot{A}^* P_{\mathfrak{M}_F}^+],$$

where B_1 and B_2 are an arbitrary $(+)$ -bounded and non-negative self-adjoint operators in \mathfrak{N}_F .

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