Chapter 2

Basic elements of the qualitative theory of ordinary differential equations

In this chapter we collect some basic ideas and results from the qualitative theory of ordinary differential equations. We present only the tools needed in our later analysis and the theoretical context where they appear. Most of these results have extensions to more general contexts. To not make our presentation too long we will restrict ourselves to the most relevant facts.

A deeper and more detailed introduction can be found in the following books: A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier [4], [5], M.W. Hirsch and S. Smale [33], V.I. Arnold [7], J. Sotomayor [57], [58], P. Hartman [30], S. Lefschetz [40], L. Perko [53], C. Chicone [14], and recently the book of F. Dumortier, J. Llibre and J.C. Artés [21].

2.1 Differential equations and solutions

2.1.1 Existence and uniqueness of solutions

Let U be a subset of $\mathbb{R}^n$ and W an open subset of U. We say that the function $\mathbf{f} : U \to \mathbb{R}^n$ is *Lipschitz on* W, if there exists a constant $L \in \mathbb{R}$, such that for every $\mathbf{x}$, $\mathbf{y} \in W$

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|| \leq L\,||\mathbf{x} - \mathbf{y}||\,.$$

The constant L is called a *Lipschitz constant for* $\mathbf{f}$ on W. Here and in the sequel $||\cdot||$ denotes the Euclidean norm of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a finite-dimensional vector space, if $\mathbf{f}$ is Lipschitz with respect to a norm of $\mathbb{R}^n$, then $\mathbf{f}$ is Lipschitz with respect to any other norm of $\mathbb{R}^n$. Hence, the definition of Lipschitz functions does not

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depend on the chosen norm. However, this is not true for the Lipschitz constants. For instance, if $\mathbf{f}$ is Lipschitz on W, with Lipschitz constant L with respect to the Euclidean norm of $\mathbb{R}^n$, then $\sqrt{n}L$ is a Lipschitz constant of $\mathbf{f}$ with respect to the *maximum norm* of $\mathbb{R}^n$

$$||\mathbf{x}||_\infty = \max_{1\le k\le n} \{|x_k|\},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, and $(\cdot)^T$ denotes the transposed vector.

In particular when $\mathbf{f}$ is Lipschitz on the whole domain U, we call $\mathbf{f}$ *globally Lipschitz*. On the other hand if for every $\mathbf{x}_0 \in U$ there exists a neighbourhood W of $\mathbf{x}_0$ in U such that $\mathbf{f}$ is Lipschitz on W, then we call $\mathbf{f}$ *locally Lipschitz* on U.

Example 2 (Linear function). Consider the function $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$, where A is a $n\times n$ matrix. Since $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| = \|A\mathbf{x} - A\mathbf{y}\| \le \|A\|\|\mathbf{x} - \mathbf{y}\|$, $\mathbf{f}$ is both locally and globally Lipschitz in $\mathbb{R}^n$, with $L = \|A\|$ as a Lipschitz constant.

Example 3. Consider the quadratic function $f(x) = x^2$. Since

$$|f(x) - f(y)| = |x + y||x - y|, \tag{2.1}$$

for any $x_0 \in \mathbb{R}$ one has $|f(x) - f(y)| < 2(|x_0| + \varepsilon)|x - y|$ in $W = (x_0 - \varepsilon, x_0 + \varepsilon)$. Therefore, f is a locally Lipschitz function in $\mathbb{R}$. However, f is not globally Lipschitz in $\mathbb{R}$. Indeed, assuming that there exists a constant L such that $|f(x) - f(y)| < L|x - y|$ for every $x, y \in \mathbb{R}$, we contradict (2.1).

Example 4 (Piecewise linear function). Consider the piecewise linear function $f(x) = |x|$. From the triangle inequality we have $|f(x)-f(y)| = ||x|-|y|| \le |x-y|$, which implies that f is both locally and globally Lipschitz, with Lipschitz constant equal to 1.

For the purposes of this book it is enough to consider a *differential equation* or a *system of ordinary differential equations* as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{2.2}$$

where $\mathbf{x} = \mathbf{x}(s) \in U$, U is an open subset of $\mathbb{R}^n$ and $\mathbf{f} : U \to \mathbb{R}^n$ is a locally Lipschitz function on U. From now on $\dot{\mathbf{x}}$ denotes the derivative of $\mathbf{x}(s)$ with respect to s. As usual, the domain of $\mathbf{f}$ (the set U) is called the *phase space*, the variable $\mathbf{x}$ is called the *dependent variable*, and s is called the *independent variable* or *time*. We use the variable s instead of the standard variable t because t will denote the trace of some matrices which will appear later on.

In a more general context equation (2.2) is known as an *autonomous ordinary differential equation* (as opposed to *non-autonomous differential equations*), because the function $\mathbf{f}$ does not depend explicitly on the independent variable s.

A smooth function $\phi : I \to U$, where I is an open interval of $\mathbb{R}$, is said to be a *solution* of the differential equation (2.2) if $\dot{\phi}(s) = \mathbf{f}(\phi(s))$ for every $s \in I$.

Geometrically, a differential equation (2.2) assigns to every point $\mathbf{x}$ in the phase space U a vector $\mathbf{f}(\mathbf{x})$ in the tangent space at $\mathbf{x}$. Then a solution of the differential equation is a curve $\phi : I \to U$ whose tangent vector at $\dot{\phi}(s)$ coincides with the vector $\mathbf{f}(\phi(s))$ for any s, see Figure 2.1. From this reason we call the function $\mathbf{f}$ a *vector field*.

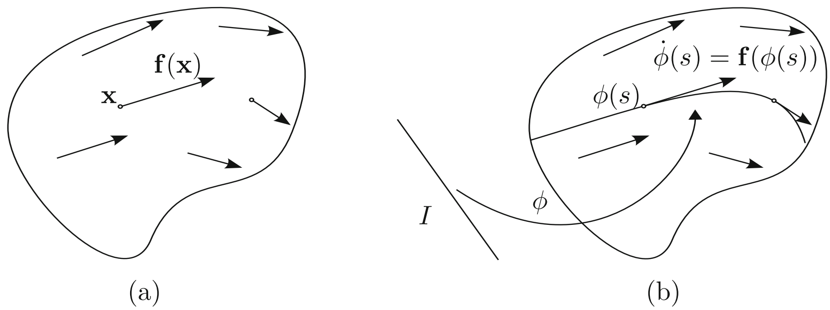


Figure 2.1: (a) Vector field $\mathbf{f}$ defined in the phase space U. (b) A solution $\phi(s)$ of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

The existence of solutions of differential equations (2.2) is not obvious and it depends on some properties of the vector field $\mathbf{f}$. The same is true for the uniqueness of the solution which satisfies the *initial conditions* $(s_0, \mathbf{x}_0)$, i.e. $\phi(s_0) = \mathbf{x}_0$. The following theorem states the basic result in this direction.

Theorem 2.1.1 (Existence and uniqueness)**.** *Let U be an open subset of $\mathbb{R}^n$, $\mathbf{f} : U \to \mathbb{R}^n$ be a locally Lipschitz function on U, $s_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in U$. There exist a constant $c > 0$ and a unique solution $\phi : (s_0 - c, s_0 + c) \to U$ of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ such that $\phi(s_0) = \mathbf{x}_0$.*

For a proof of this theorem we refer the reader to [33].

To emphasize the dependence of the solutions on the *initial conditions* $(s_0, \mathbf{x}_0)$, we denote the solution of the differential equation (2.2) passing through $\mathbf{x}_0$ at time $s = s_0$ by $\phi(s; s_0, \mathbf{x}_0)$.

2.1.2 Prolongability of solutions

From the existence and uniqueness theorem we obtain conditions on the vector field $\mathbf{f}(\mathbf{x})$ so that it has exactly one solution passing through an a-priori fixed point. This solution is defined at least on a sufficiently small open interval. In the next result we find the maximal interval of existence. First, we need to introduce the following definitions.

We say that $\phi : I \to U$, with $\phi = \phi(s; s_0, \mathbf{x}_0)$, is a *maximal solution* of equation (2.2), if for every solution $\psi : J \to U$, with $\psi = \psi(s; s_0, \mathbf{x}_0)$, we have $J \subseteq I$. We call *maximal interval of definition* the interval of definition of the maximal solution $\phi(s; s_0, \mathbf{x}_0)$, and we denote it by $I_{(s_0, \mathbf{x}_0)}$. *From now on we will only consider maximal solutions.* The differential systems (vector fields) such that all their solutions have the maximal interval of definition equal to $\mathbb{R}$ are called *complete*. In the following proposition we present sufficient conditions on the vector field of a differential equation for it to be complete.

Proposition 2.1.2. *Consider the differential equation* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$*, where* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ *is a globally Lipschitz function. Then for every initial conditions* $(s_0, \mathbf{x}_0) \in \mathbb{R} \times \mathbb{R}^n$ *it holds that* $I_{(s_0,\mathbf{x}_0)} = \mathbb{R}$*.*

A proof of Proposition 2.1.2 can be found in [53, Section 3.1, Theorem 3] or in [57, Proposition 4, p. 15]. We emphasize that the differentiability condition imposed on the vector field in the first reference is not essential for the proof and can be removed. Note that the hypothesis in Proposition 2.1.2 is very restrictive. As we will see in Section 3.3, fundamental systems satisfy it.

Example 5. As we saw in Example 2 linear differential systems $\dot{\mathbf{x}} = A\mathbf{x}$ are globally Lipschitz, and hence complete.

2.1.3 Dependence on initial conditions and parameters

Consider the family of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda),$$

where $\mathbf{f} : U \times V \to \mathbb{R}^n$, U is an open subset of $\mathbb{R}^n$, and V is an open subset of $\mathbb{R}^p$. The set V is called the *parameter space* of the differential equation.

Assuming that $\lambda_0 \in V$, $s_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$, there exists exactly one solution of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda_0)$ passing through $\mathbf{x}_0$ at time s_0. We denote this solution by $\phi(s; s_0, \mathbf{x}_0, \lambda_0)$. In the next theorem we summarize the behaviour of the solution $\phi(s; s_0, \mathbf{x}_0, \lambda_0)$ when we vary s_0, $\mathbf{x}_0$ or λ_0. First we introduce some additional definitions.

Let W be an open subset of U. The function $\mathbf{f}(\mathbf{x}, \lambda)$ is said to be *Lipschitz with respect to the first variable in* W, if there exists a positive constant $L \in \mathbb{R}$, such that for every $\mathbf{x}, \mathbf{y} \in W$ and $\lambda \in V$

$$||\mathbf{f}(\mathbf{x}, \lambda) - \mathbf{f}(\mathbf{y}, \lambda)|| \leq L\, ||\mathbf{x} - \mathbf{y}||\,.$$

In particular, if $\mathbf{f}$ is Lipschitz with respect to the first variable in U, then we say that $\mathbf{f}$ is *globally Lipschitz with respect to the first variable*. The function $\mathbf{f}$ is said to be *locally Lipschitz with respect to the first variable* if for every $\mathbf{x}_0 \in U$ there exists a neighbourhood W of $\mathbf{x}_0$ in U such that $\mathbf{f}$ is Lipschitz with respect to the first variable in W. For simplicity we will call $\mathbf{f}$ globally or locally Lipschitz without a reference to the first variable when no confusion can arise.

Theorem 2.1.3 (Dependence on initial conditions and parameters)**.** *Let* U *and* V *be open subsets of* $\mathbb{R}^n$ *and* $\mathbb{R}^p$*, respectively. Let* $\mathbf{f} : U \times V \to \mathbb{R}^n$ *be a locally Lipschitz function with respect to the first variable in* U *and* $\mathbf{f} \in \mathcal{C}^r(U \times V)$ *for some* $r \geq 0$*. Then for every* $(s_0, \mathbf{x}_0, \lambda_0) \in \mathbb{R} \times U \times V$ *the solution* $\phi(s; s_0, \mathbf{x}_0, \lambda_0)$

of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda_0)$ *is* r *times continuously differentiable with respect to* $\mathbf{x}_0$ *and* λ_0 *and* $r+1$ *times continuously differentiable with respect to* s.

A proof of this theorem can be found in Hartman [30, pp. 93–96] or Lefschetz [40, pp. 36–43].

Example 6 (A family of piecewise linear differential equations). Consider the family of differential equations $\dot{x} = |x| + \lambda$, with $\lambda > 0$, which is defined on whole $\mathbb{R}$. With respect to the vector field, the phase space splits into two regions, $\{x < 0\}$ and $\{x > 0\}$, and in both the field is given by a linear function. Moreover, it is continuously differentiable with respect to the parameter λ, but is only globally Lipschitz with respect to the variable x.

Straightforward computations show that the solution $\phi(s; 0, x_0, \lambda)$ of the differential equation passing through $x_0 < 0$ at time $s = 0$ is given by

$$\phi(s; 0, x_0, \lambda) = \begin{cases} \lambda + e^{-s}(x_0 - \lambda), & \text{if } s \leq s^*, \\ \lambda \left(\dfrac{\lambda}{\lambda - x_0} e^s - 1 \right), & \text{if } s > s^* \end{cases}$$

where $s^* = \ln(1 - x_0/\lambda)$ is the time required for the solution to reach the origin, see Figure 2.2. Note that the maximal interval of definition of the solution is $\mathbb{R}$.

Taking the first and the second derivative with respect to s one has that

$$\frac{d^k\phi}{ds^k}(s; 0, x_0, \lambda) = \begin{cases} (-1)^k e^{-s}(x_0 - \lambda), & \text{if } s \leq s^*, \\ \dfrac{\lambda^2}{\lambda - x_0} e^s, & \text{if } s > s^* \end{cases}$$

for $k = 1, 2$. Thus the solution $\phi(s; 0, x_0, \lambda)$ is an analytical function of s in $\mathbb{R} \setminus \{s^*\}$ and once continuously differentiable at $s = s^*$, but it is not twice continuously differentiable at $s = s^*$.

Taking derivatives with respect to λ it is easy to conclude that $\phi(s; 0, x_0, \lambda)$ is once continuously differentiable in $\mathbb{R}$ but is not twice continuously differentiable at $s = s^*$.

This example shows that solutions of piecewise linear differential equations lose regularity at the boundary between the regions where the vector field is linear.

2.1.4 Other properties

We recall now some other properties of the solutions of differential equations. We say that $\phi : \mathbb{R} \to \mathbb{R}^n$ is a *periodic function*, if there exists a positive constant T such that $\phi(s + T) = \phi(s)$ for every $s \in \mathbb{R}$. The smallest value of T satisfying this property is called the *period* of the function ϕ.

Proposition 2.1.4. *Consider the differential equation* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *with* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ *a globally Lipschitz function.*

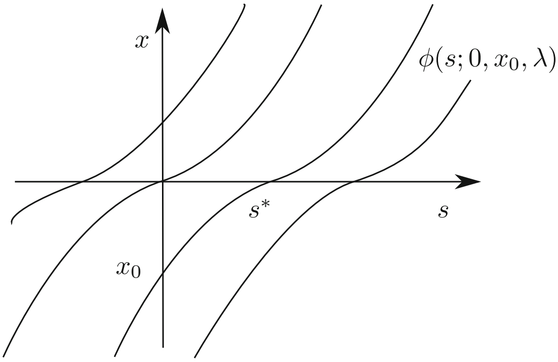


Figure 2.2: Solutions $\phi(s;0,x_0,\lambda)$ of the differential equation $\dot{x} = |x| + \lambda$.

(a) *Let $\phi(s;s_0,\mathbf{x}_0)$ be a solution. Then for every $\tau \in \mathbb{R}$, $\phi(s+\tau,s_0,\mathbf{x}_0)$ is also a solution.*

(b) *Let $\phi(s;s_1,\mathbf{x}_1)$ and $\phi(s;s_2,\mathbf{x}_2)$ be two solutions satisfying $\phi(\tau_1;s_1,\mathbf{x}_1) = \phi(\tau_2;s_2,\mathbf{x}_2)$ for fixed $\tau_1,\tau_2 \in \mathbb{R}$. Then $\phi(s-(\tau_2-\tau_1);s_1,\mathbf{x}_1) = \phi(s;s_2,\mathbf{x}_2)$ for every $s \in \mathbb{R}$.*

(c) *Let $\phi(s;s_0,\mathbf{x}_0)$ be a solution and suppose that there exist $\tau_1,\tau_2 \in \mathbb{R}$, $\tau_1 < \tau_2$, such that $\phi(\tau_1;s_0,\mathbf{x}_0) = \phi(\tau_2;s_0,\mathbf{x}_0)$. Then, $\phi(s;s_0,\mathbf{x}_0)$ is a periodic function whose period is a multiple of $\tau = \tau_2 - \tau_1$.*

For a proof of this result we refer the reader to [60, pp. 8–9]. Note that in this reference the author assumes that the vector field is differentiable, but it is easy to check that this hypothesis can be substituted by requiring the uniqueness of the solutions.

2.2 Orbits

In this section we present some dynamical features of solutions to differential equations. Take $s_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in U$, and let $\phi(s;s_0,\mathbf{x}_0)$ be a maximal solution of the differential equation (2.2). We call the set

$$\gamma(s_0,\mathbf{x}_0) := \left\{\mathbf{x} \in U : \mathbf{x} = \phi(s;s_0,\mathbf{x}_0) \text{ and } s \in I_{(s_0,\mathbf{x}_0)}\right\}$$

the orbit of the solution $\phi(s;s_0,\mathbf{x}_0)$.

When the phase space is the whole $\mathbb{R}^n$ and the vector field $\mathbf{f}$ is globally Lipschitz in $\mathbb{R}^n$, the maximal interval of definition of all solutions is $\mathbb{R}$, see Proposition 2.1.2. Then $\gamma(t_0,\mathbf{x}_0) = \gamma(t_0+\tau,\mathbf{x}_0)$ for every $\tau \in \mathbb{R}$, see Proposition 2.1.4(a).

Hence we will simply use $\gamma(\mathbf{x}_0)$ to denote the orbit through $\mathbf{x}_0$. Moreover, if $\mathbf{x}_1 \in \gamma(\mathbf{x}_0)$, then there exists $s_1 \in \mathbb{R}$ such that $\mathbf{x}_1 = \phi(s_1; s_0, \mathbf{x}_0)$. Applying Proposition 2.1.4(b) to the solutions $\phi(s; s_0, \mathbf{x}_0)$ and $\phi(s; s_1, \mathbf{x}_1)$ one obtains that $\gamma(\mathbf{x}_1) = \gamma(\mathbf{x}_0)$. Therefore orbits are independent on the point of reference, and we can avoid the reference to such point when no confusion can arise.

Suppose that $\mathbf{x}_2 \in \gamma(\mathbf{x}_1) \cap \gamma(\mathbf{x}_0) \neq \emptyset$. Since orbits do not depend on the point of reference, $\gamma(\mathbf{x}_0) = \gamma(\mathbf{x}_1) = \gamma(\mathbf{x}_2)$. Therefore, *if two orbits intersect at a point, then they coincide.*

Example 7. Consider the planar piecewise linear differential system $\dot{x} = x, \dot{y} = |y|$. Since the two variables are decoupled, the corresponding differential equation can be easily solved. Indeed, the solution with initial condition (x_0, y_0) is given by $\phi(s; 0, (x_0, y_0)) = (x(s), y(s))$, where

$$x(s) = e^s x_0, \quad y(s) = \begin{cases} e^s y_0, & \text{if } y_0 \geq 0, \\ e^{-s} y_0, & \text{if } y_0 < 0, \end{cases}$$

see Figure 2.3(a) and (b).

Set $x_0 \in \mathbb{R}$ and $y_0 < 0$. The orbit through the point $\mathbf{p} = (x_0, y_0)$ is defined by $\gamma(\mathbf{p}) = \{(e^s x_0, e^{-s} y_0) : s \in \mathbb{R}\}$, and so $y(s) = x_0 y_0 / x(s)$, which is the branch of an hyperbola passing through $\mathbf{p}$, see Figure 2.3(c).

On the other hand, if $y_0 > 0$, then the orbit through $\mathbf{p}$ is defined by $\gamma(\mathbf{p}) = \{(e^s x_0, e^s y_0) : s \in \mathbb{R}\}$, and so $\gamma(\mathbf{p})$ is a half-line, see Figure 2.3(c).

2.3 The flow of a differential equation

Consider the differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{2.3}$$

where $\mathbf{f} : U \to \mathbb{R}^n$ is locally Lipschitz in an open subset U of $\mathbb{R}^n$. Suppose that for every $\mathbf{x} \in U$, the solution $\phi(s; 0, \mathbf{x})$ is defined on whole $\mathbb{R}$, i.e., $I_{(0,\mathbf{x})} = \mathbb{R}$. The *flow of the differential equation* (2.3) is defined to be the function

$$\Phi : \mathbb{R} \times U \to \mathbb{R}^n$$

given by $\Phi(s, \mathbf{x}) = \phi(s; 0, \mathbf{x})$. The notion of flow introduced here is sometimes referred as *completed flow*. That is because the maximal interval of definition of the solutions is the whole $\mathbb{R}$. Since the differential systems considered in this work are complete, we can use both terms. In particular, if $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz, then the flow of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is complete, see Proposition 2.1.2.

Other authors denote the flow of a differential equation by the pair consisting of the function Φ and the phase space U. It is also usual to denote by $\Phi_s(\mathbf{x})$ the function $\Phi(s, \mathbf{x})$ (see [29] or [53]). Some properties of flows are collected in the following result.

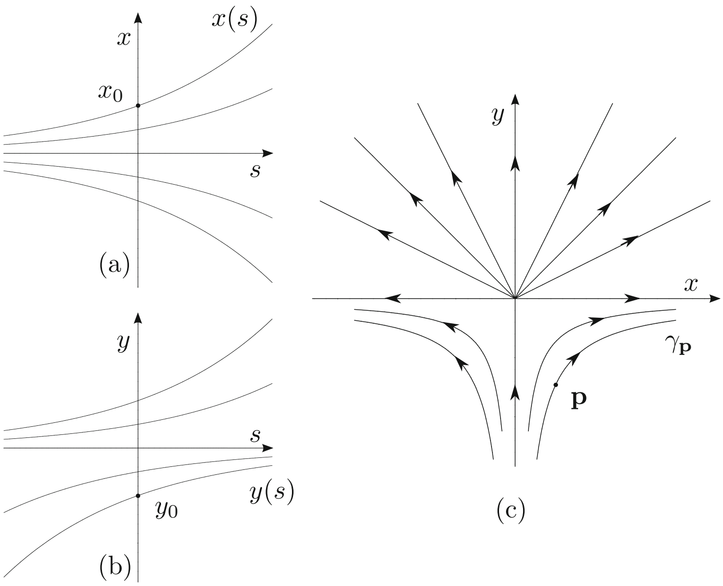


Figure 2.3: Solutions $\phi(s;0,(x_0,y_0)) = (x(s),y(s))$ and orbits of the differential equation $\dot{x} = x$, $\dot{y} = |y|$. (a) Dependence of the first coordinate $x(s)$ of the solution $\phi(s;0,(x_0,y_0))$ on s. (b) Dependence of the second coordinate $y(s)$ of the solution $\phi(s;0,(x_0,y_0))$ on s. (c) Orbit $\gamma_{\mathbf{p}}$ with $\mathbf{p} = (x_0,y_0)$ depicted in the phase space (x,y).

Proposition 2.3.1. *Let $\Phi(s,\mathbf{x})$ be the flow defined by the differential equation* (2.3)*.*

(a) *For every $\mathbf{x} \in U$, $\Phi(0,\mathbf{x}) = \mathbf{x}$.*

(b) *For every $s,t \in \mathbb{R}$ and $\mathbf{x} \in U$, $\Phi(s+t,\mathbf{x}) = \Phi(s,\Phi(t,\mathbf{x}))$.*

(c) *Φ is a continuous function.*

Proof. Statement (a) follows from the definition of Φ. Statement (b) follows by taking $\mathbf{x}_1 = \mathbf{x}$, $\mathbf{x}_2 = \phi(t;0,\mathbf{x})$, $\tau_1 = t$, $\tau_2 = 0$ and $s_1 = s_2 = 0$ and applying Proposition 2.1.4(b). Statement (c) is a consequence of the continuous dependence of the solutions on the initial conditions and parameters, see Theorem 2.1.3. $\square$

In the classical point of view, the objective of the theory of differential equations is to find explicit expressions for the flow $\Phi(s,\mathbf{x})$. In the qualitative theory it is more important to describe the topological properties of the flow and the asymptotic behaviour of its orbits, i.e., the behaviour of the orbits when s tends to $\pm\infty$. The *phase portrait* of a differential equation (2.3) is defined as the union of all the orbits of (2.3).

Let $\Phi(s, \mathbf{x})$ be the flow of the differential equation (2.3) and take $\mathbf{p} \in U$. By the continuous dependence of the solutions on the initial conditions and parameters, the function $\Phi_{\mathbf{p}} : \mathbb{R} \to U$ given by $\Phi_{\mathbf{p}}(s) := \Phi(s, \mathbf{p})$ is continuously differentiable. Furthermore, since $\dot{\Phi}_{\mathbf{p}}(s) = \mathbf{f}(\Phi_{\mathbf{p}}(s))$, if there exists s_0 such that $\dot{\Phi}_{\mathbf{p}}(s_0) = \mathbf{0}$, then (by the uniqueness of the solutions) we have $\Phi_{\mathbf{p}}(s) = \mathbf{p}$ for every $s \in \mathbb{R}$. In this case, the orbit $\gamma(\mathbf{p}) = \{\mathbf{p}\}$ is called a *singular point*. To simplify the notation, if $\gamma(\mathbf{p})$ is a singular point, we denote it by $\mathbf{p}$. Therefore $\mathbb{R}^n \setminus \gamma(\mathbf{p})$, $\mathbb{R}^n \setminus \{\mathbf{p}\}$ and $\mathbb{R}^n \setminus \mathbf{p}$ are identical notations. If $\dot{\Phi}_{\mathbf{p}}(s_0) \neq \mathbf{0}$ for some $s_0 \in \mathbb{R}$, then $\Phi_{\mathbf{p}}(\mathbb{R}) = \gamma(\mathbf{p})$ is a one-dimensional manifold and we call $\mathbf{p}$ a *regular point*. The flow in a sufficiently small neighbourhood of a regular point is said to be *parallel*. For the definition of a parallel flow in a neighbourhood of a singular point see Subsection 2.6.3. By the classification of one-dimensional manifolds (see [38]), $\gamma(\mathbf{p})$ is diffeomorphic either to $\mathbb{R}$, or to $\mathbb{S}^1$. When $\gamma(\mathbf{p})$ is diffeomorphic to $\mathbb{S}^1$ the orbit $\gamma(\mathbf{p})$ is called a *periodic orbit*.

Theorem 2.3.2. *Every orbit of a differential equation* (2.3) *is diffeomorphic either to a point, or to a circle* $\mathbb{S}^1$*, or to a straight line* $\mathbb{R}$.

Example 8. By Example 7, the flow of the piecewise linear differential system $\dot{x} = x$, $\dot{y} = |y|$ is given by $\Phi(s, (x_0, y_0)) = (e^s x_0, e^s y_0)$ when $y_0 \geq 0$ and by $\Phi(s, (x_0, y_0)) = (e^s x_0, e^{-s} y_0)$ when $y_0 < 0$. The corresponding phase portrait is shown in Figure 2.3(c). In this example, each orbit, except the one that passes through the origin, is diffeomorphic to the line $\mathbb{R}$. The orbit through the origin is diffeomorphic to a point. Therefore, it is a singular point.

2.4 Basic ideas in qualitative theory

After analysing the topology of the orbits we present some basic definitions for studying their asymptotic behaviour. Consider the differential equation (2.3) and let E be a subset of U. The set E is said to be *positively invariant (under the flow)* if for every $\mathbf{q} \in E$ we have $\Phi(s, \mathbf{q}) \in E$ for all $s \geq 0$. The set E is said to be *negatively invariant (under the flow)* if for every $\mathbf{q} \in E$ we have $\Phi(s, \mathbf{q}) \in E$ for all $s \leq 0$. A set E is said to be *invariant (under the flow)* when it is both positively and negatively invariant (under the flow).

An invariant set E is *stable*, if for any neighbourhood W of E, there exists a neighbourhood V of E such that, for every $\mathbf{p} \in V$ and $s > 0$ it holds that $\Phi(s, \mathbf{p}) \in W$. An invariant set E is *unstable* when it is not stable.

Given $\mathbf{p}, \mathbf{q} \in U$, the point $\mathbf{q}$ is called an *α-limit point of* $\mathbf{p}$ if there exists a sequence $\{s_n\}_{n=0}^{+\infty}$ satisfying $\lim\limits_{n \nearrow +\infty} s_n = -\infty$ and such that $\lim\limits_{n \nearrow +\infty} \Phi(s_n, \mathbf{p}) = \mathbf{q}$. The point $\mathbf{q}$ is called an *ω-limit point of* $\mathbf{p}$ if there exists a sequence $\{s_n\}_{n=0}^{+\infty}$ satisfying $\lim\limits_{n \nearrow +\infty} s_n = +\infty$ and such that $\lim\limits_{n \nearrow +\infty} \Phi(s_n, \mathbf{p}) = \mathbf{q}$.

The *α-limit set* of a point $\mathbf{p} \in U$, denoted by $\alpha(\mathbf{p})$, is defined as the union of the α-limit points of $\mathbf{p}$. Analogously the *ω-limit set* of a point $\mathbf{p} \in U$, denoted by $\omega(\mathbf{p})$, is defined as the union of the ω-limit points of $\mathbf{p}$.

Let $\gamma(\mathbf{p})$, or simply γ, be the orbit passing through the point $\mathbf{p} \in U$. The *α-limit set of the orbit* γ is the α-limit set of the point $\mathbf{p}$, the *ω-limit set of the orbit* γ is the ω-limit set of $\mathbf{p}$. As it is easy to check, these definitions do not depend on the chosen point $\mathbf{p}$ of the orbit. Therefore, we denote the α- and the ω-limit set of an orbit by $\alpha(\gamma)$ and $\omega(\gamma)$, respectively.

Given an invariant set E, the *stable manifold* of E, denoted by $W^s(E)$, is the set of points in the phase space U whose ω-limit set is contained in E. The *unstable manifold* of E, denoted by $W^u(E)$, is the set of points in U whose α-limit set is contained in E.

A set E is called *asymptotically stable* if its stable manifold $W^s(E)$ is a neighbourhood of E. A set E is called *asymptotically unstable* if its unstable manifold $W^u(E)$ is a neighbourhood of E. In particular, every asymptotically stable (respectively, unstable) set is stable (respectively, unstable).

A *limit cycle* of the differential equation (2.3) is a periodic orbit isolated in the set of all the periodic orbits of (2.3). A limit cycle is called *stable* (respectively, *unstable*) if it is asymptotically stable (respectively, unstable). Another kind of limit cycle, called semistable limit cycle, can be also defined and we will introduce it in Section 2.8.

Example 9. In this example we consider a fundamental system

$$\dot{\mathbf{x}} = \begin{cases} A\mathbf{x} + \mathbf{b}, & \text{if } \mathbf{k}^T\mathbf{x} > 1, \\ B\mathbf{x}, & \text{if } |\mathbf{k}^T\mathbf{x}| \leq 1, \\ A\mathbf{x} - \mathbf{b}, & \text{if } \mathbf{k}^T\mathbf{x} < -1, \end{cases}$$

with parameters $d = \det(A) < 0$, $t = \operatorname{trace}(A) < 0$, $D = \det(B) > 0$ and $T = \operatorname{trace}(B) = 0$. In Section 5.3 we prove that its phase portrait in a neighbourhood of the origin is which is the one shown in Figure 2.4.

Different invariant sets can be easily identified. For instance, invariant sets are present in both the grey and the central white region formed by periodic orbits. This is because every orbit contained in one of these regions does not leave the region, neither in positive time, nor in negative time. Of course, sets formed by singular orbits are also invariant. Hence the singular points $\mathbf{e}_+$, $\mathbf{0}$ and $\mathbf{e}_-$, and the periodic orbit Γ are invariant.

Note that Γ is a stable invariant set. In fact, its stable manifold $W^s(\Gamma)$ is the whole grey region. However, it is not asymptotically stable, because $W^s(\Gamma)$ is not a neighbourhood of Γ. The origin $\mathbf{0}$ is also a stable invariant set which is not asymptotically stable.

On the other hand, the singular point $\mathbf{e}_-$ is the ω-limit set of the orbits γ_1^- and γ_2^-, see Figure 2.4. It is also the α-limit set of the orbits γ_3^- and γ_4^-. The periodic orbit Γ is the ω-limit set of the orbit γ_4^-.

Let γ be an orbit of the flow $\Phi(s, \mathbf{x})$ and $\mathbf{p}$ be a point on γ. We define the positive and negative semiorbit of γ as the sets $\gamma^+(\mathbf{p}) := \{\Phi(s, \mathbf{p}) : s \geq 0\}$ and

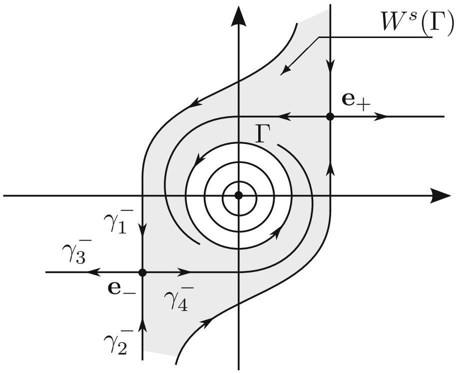


Figure 2.4: Phase portrait of the fundamental system with $D > 0$ and $T = 0$ in a neighbourhood of the origin $\mathbf{0}$. Invariant regions: the singular points $\mathbf{e}_+$, $\mathbf{0}$, $\mathbf{e}_-$; the periodic orbit Γ; and the open region $W^s(\Gamma)$ (in grey) and the open region in the interior of Γ (in white and foliated by periodic orbits).

$\gamma^-(\mathbf{p}) := \{\Phi(s, \mathbf{p}) : s \leq 0\}$, respectively. The orbit γ is called *positively bounded* if there exist a point $\mathbf{p} \in \gamma$ and a compact subset K of U such that $\gamma^+(\mathbf{p}) \subset K$. The orbit γ is called *negatively bounded* if there exist a point $\mathbf{p} \in \gamma$ and a compact subset K of U such that $\gamma^-(\mathbf{p}) \subset K$. Finally, γ is said to be *bounded* if it is positively and negatively bounded.

Proposition 2.4.1. *Let γ be an orbit of the differential system* (2.3). *If γ is positively bounded (respectively, negatively bounded), then $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) is a non-empty set.*

For a proof of this result we refer the reader to [53, p. 191] or [57, p. 245]. Note that in references above, authors require the differentiability of the vector field. It is easy to check that instead of this hypothesis we can require the uniqueness of the solutions and the completeness of the flow.

2.5 Linear systems

Linear systems of differential equations, or briefly, linear systems, are one of the families of differential equations for which there exists a complete theory. We review some of the standard facts on linear systems because, as we will see later, there exists a close relationship between linear and general non-linear differential systems. The nature of this relationship is such that linear systems can be considered as a first natural step in the study of the differential systems.

As usual, $L(\mathbb{R}^n)$ denotes the vector space of the linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $GL(\mathbb{R}^n)$ the group of the invertible linear maps. Consider $T \in L(\mathbb{R}^n)$ and let

A be the matrix representation of T. In the sequel we will identify the linear map T with its matricial representation A, and write $A \in L(\mathbb{R}^n)$. If T is invertible; i.e., $\det(A) \neq 0$, we will write $A \in GL(\mathbb{R}^n)$.

If $A \in L(\mathbb{R}^n)$ we denote by t or trace(A) the trace of A, and by d or $\det(A)$ the determinant of A. This explains our use of the variable s, instead of the more usual one t, to denote the time in the differential equation. Let $A \in L(\mathbb{R}^n)$. Then for every $s \in \mathbb{R}$ we define the *exponential matrix* of the matrix sA as the formal power series

$$e^{sA} := \sum_{k=0}^{\infty} \frac{s^k A^k}{k!},$$

where A^0 denotes the identity matrix Id and $A^k = A^{k-1}A$ for $k \geq 1$. Two matrices $A, B \in L(\mathbb{R}^n)$ are said to be *equivalent* if there exists $P \in GL(\mathbb{R}^n)$ such that $B = PAP^{-1}$. We summarize some properties of the exponential matrix in the following proposition.

Proposition 2.5.1. *Let $A \in L(\mathbb{R}^n)$.*

(a) *For every $s \in \mathbb{R}$, the series*

$$\sum_{k=0}^{\infty} \frac{s^k A^k}{k!}$$

is absolutely convergent. Moreover, if $s_0 > 0$, the series is uniformly convergent in $(-s_0, s_0)$.

(b) *If $A, B \in L(\mathbb{R}^n)$ are equivalent matrices with $B = PAP^{-1}$ for a $P \in \mathrm{GL}(\mathbb{R}^n)$, then $e^{sB} = Pe^{sA}P^{-1}$ for every $s \in \mathbb{R}$.*

(c) *If $B \in L(\mathbb{R}^n)$ is such that $AB = BA$, then $e^{s(A+B)} = e^{sA}e^{sB}$ for every $s \in \mathbb{R}$.*

(d) *For every $s \in \mathbb{R}$, $\left(e^{sA}\right)^{-1} = e^{-sA}$.*

(e) *For every $s \in \mathbb{R}$, $de^{sA}/ds = Ae^{sA}$.*

(f) *Let $\mathbf{v} \in \mathbb{R}^n$ be an eigenvector of A with eigenvalue $\lambda \in \mathbb{R}$. Then $\mathbf{v}$ is an eigenvector of e^{sA} with eigenvalue $e^{s\lambda}$.*

A proof of these results can be found in [7, Chapter 3] or [53, pp. 10–13].

In this section we consider the linear system (more precisely, *the homogeneous linear system*)

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{2.4}$$

where $A \in L(\mathbb{R}^n)$, and denote $d = \det(A)$ and $t = \mathrm{trace}(A)$.

The linear vector field $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ is a globally Lipschitz function with Lipschitz constant $L = ||A||$. From the existence and uniqueness theorem it follows that for every $\mathbf{x}_0 \in \mathbb{R}^n$ there exists a unique solution of system (2.4) passing through $\mathbf{x}_0$ at $s = 0$. Moreover, this solution is defined for all $s \in \mathbb{R}$ (see Proposition 2.1.2). The following result provides an explicit expression for the linear flows.

Theorem 2.5.2 (Linear flow)**.** *The linear differential equation* $\dot{\mathbf{x}} = A\mathbf{x}$*, with* $A \in L(\mathbb{R}^n)$*, defines a flow* $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *given by* $\Phi(s, \mathbf{x}) = e^{sA}\mathbf{x}$.

A proof of this theorem can be obtained as a corollary of Proposition 2.5.1(e).

We denote by $\ker(A)$ the vector subspace formed by the singular points of the linear system (2.4). This subspace is called the *kernel of the linear map* A. Notice that the origin always belong to $\ker(A)$. Moreover, when $A \in GL(\mathbb{R}^n)$, the origin is the unique singular point.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n_s}$ be the generalized eigenvectors corresponding to the eigenvalues of the matrix A with negative real part. The *stable subspace* is the vector subspace generated by the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n_s}$, i.e.,

$$E^s := \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n_s} \rangle .$$

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n_u}$ be the generalized eigenvectors corresponding to the eigenvalues of the matrix A with positive real part. The *unstable subspace* is the vector subspace

$$E^u := \langle \mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_{n_u} \rangle .$$

Let $\mathbf{w}_1, \ldots, \mathbf{w}_{n_c}$ be the generalized eigenvectors corresponding to the eigenvalues of the matrix A with zero real part. The *center subspace* is the vector subspace

$$E^c := \langle \mathbf{w}_1, \mathbf{w}_2 \ldots, \mathbf{w}_{n_c} \rangle .$$

Theorem 2.5.3 (Dynamical behaviour of linear systems)**.** *Consider the linear differential system* $\dot{\mathbf{x}} = A\mathbf{x}$ *with* $A \in GL(\mathbb{R}^n)$*. Then:*

(a) $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$.

(b) $W^s(\mathbf{0}) = E^s$.

(c) $W^u(\mathbf{0}) = E^u$.

For a proof of this result, see [53, Section 1.9].

2.5.1 Non-homogeneous linear systems

Differential systems of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}, \tag{2.5}$$

with $A \in L(\mathbb{R}^n)$ and $\mathbf{b} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ are called *non-homogeneous linear (differential) systems.* By Proposition 2.5.1(e), the flow of systems (2.5) is given by

$$\Phi(s, \mathbf{x}) = e^{sA}\mathbf{x} + \int_0^s e^{(s-r)A}\mathbf{b}\, dr.$$

If the non-homogeneous linear system (2.5) has a singular point $\mathbf{x}^*$, the change of coordinates $\mathbf{z} = \mathbf{x} - \mathbf{x}^*$ transforms it into the homogeneous linear system $\dot{\mathbf{z}} = A\mathbf{z}$. Thus the flow of the non-homogeneous linear system (2.5) is a translation of the flow of a homogeneous linear system, namely $\Phi(s, \mathbf{x}) = e^{sA}(\mathbf{x} - \mathbf{x}^*) + \mathbf{x}^*$. Finally, note that if the non-homogeneous linear system has no singular points, then $\det(A) = 0$.

2.5.2 Planar linear systems

In the following two subsections we restrict our attention to planar linear systems. We begin by showing the following version of the real Jordan normal form theorem [33].

Theorem 2.5.4 (Real Jordan normal form)**.** *Consider a matrix* $A \in L(\mathbb{R}^2)$ *with* $d = \det(A)$ *and* $t = \operatorname{trace}(A)$. *A is equivalent to one of the following matrices J:*

(a) *If* $d = 0$ *and* $t = 0$, *then* $J = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *or* $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(b) *If* $d = 0$ *and* $t \neq 0$, *then* $J = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$.

(c) *If* $d > 0$ *and* $t = 0$, *the eigenvalues of A are complex numbers with zero real part and imaginary part* $\beta > 0$, *and* $J = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}$.

(d) *If* $d > 0$ *and* $t^2 - 4d = 0$, *there exists exactly one real eigenvalue of A with multiplicity two,* λ_1, *and* $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$ *or* $J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$.

(e) *If* $d > 0$ *and* $t^2 - 4d > 0$, *there exist two real eigenvalues of A,* $\lambda_1 > \lambda_2$, *and* $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

(f) *If* $d > 0$, $t \neq 0$ *and* $t^2 - 4d < 0$, *the eigenvalues of A are complex numbers with real part* $\alpha \neq 0$ *and imaginary part* $\beta > 0$, *and* $J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.

(g) *If* $d < 0$, *there exist two real eigenvalues of A,* $\lambda_1 > 0 > \lambda_2$, *and* $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

The matrix J defined in the preceding theorem is called the *real Jordan normal form* of A. Note that, except when $t^2 - 4d = 0$, the real Jordan normal form of A is determined by the parameters t and d. If $t^2 - 4d = 0$, then there exist two possibilities, one diagonal and the other non-diagonal, depending on the coefficients of A.

Consider the linear system

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{2.6}$$

with $A \in L(\mathbb{R}^2)$, and let $P \in GL(\mathbb{R}^2)$ be the matrix which transforms A into its real Jordan normal form J, i.e., $J = PAP^{-1}$. The linear change of coordinates $\mathbf{y} = P\mathbf{x}$ transforms the linear system (2.6) into the system

$$\dot{\mathbf{y}} = J\mathbf{y}. \tag{2.7}$$

To obtain the expression of the linear flow of (2.7) it is enough to consider the following cases:

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \; J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{ and } J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

see Propositon 2.5.1(b) and Theorem 2.5.4

Proposition 2.5.5. *Consider $J \in L(\mathbb{R}^2)$ and $s \in \mathbb{R}$.*

(a) *If* $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, *then* $e^{sJ} = \begin{pmatrix} e^{s\lambda_1} & 0 \\ 0 & e^{s\lambda_2} \end{pmatrix}$.

(b) *If* $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, *then* $e^{sJ} = e^{s\lambda} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

(c) *If* $J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, *then* $e^{sJ} = e^{s\alpha} \begin{pmatrix} \cos(\beta s) & -\sin(\beta s) \\ \sin(\beta s) & \cos(\beta s) \end{pmatrix}$.

For a proof of this proposition see [7], [53], or [57].

Let $\Phi(s, \mathbf{x})$ and $\Psi(s, \mathbf{y})$ be the flows of systems (2.6) and (2.7), respectively. If $\mathbf{x}_0 \in \mathbb{R}^2$, then $\Phi(s, \mathbf{x}_0) = e^{sA}\mathbf{x}_0 = P^{-1}e^{sJ}P\mathbf{x}_0 = P^{-1}\Psi(s, P\mathbf{x}_0)$. Therefore,

$$\Phi(s, \mathbf{x}) = P^{-1}\Psi(s, P\mathbf{x}). \tag{2.8}$$

From this we obtain the expressions of the flow of any planar linear system.

Theorem 2.5.6. *Consider the flow $\Phi(t, \mathbf{x})$ of the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, with $A \in L(\mathbb{R}^2)$, $d = \det(A)$ and $t = \operatorname{trace}(A)$. Let J be the real Jordan normal form of A and P be the matrix such that $J = PAP^{-1}$.*

(a) *If $t^2 - 4d > 0$, then*

$$\Phi(s, \mathbf{x}) = P^{-1} \begin{pmatrix} e^{s\lambda_1} & 0 \\ 0 & e^{s\lambda_2} \end{pmatrix} P\mathbf{x}.$$

(b) *If $t^2 - 4d = 0$, then either $\Phi(s, \mathbf{x}) = e^{s\lambda}\mathbf{x}$ or*

$$\Phi(s, \mathbf{x}) = P^{-1} \begin{pmatrix} e^{s\lambda} & s \\ 0 & e^{s\lambda} \end{pmatrix} P\mathbf{x},$$

depending on whether J is diagonal or not.

(c) *If $t^2 - 4d < 0$, then*

$$\Phi(s, \mathbf{x}) = e^{s\alpha} P^{-1} \begin{pmatrix} \cos(\beta s) & -\sin(\beta s) \\ \sin(\beta s) & \cos(\beta s) \end{pmatrix} P\mathbf{x}.$$

2.5.3 Planar phase portraits

In this subsection we describe the phase portrait of planar linear systems. We also present the notation of singular points of such systems. For a general classification of these singular points, see Subsection 2.7.1.

From relation (2.8) it follows that given $\mathbf{x}_0 \in \mathbb{R}^2$, the orbit of system (2.6) through $\mathbf{x}_0$ and the orbit of system (2.7) through $P\mathbf{x}_0$, $\gamma(\mathbf{x}_0)$ and $\gamma(P\mathbf{x}_0)$, respectively, satisfy $\gamma(\mathbf{x}_0) = P^{-1}\gamma(P\mathbf{x}_0)$. Therefore, the phase portrait of system (2.6) is a linear transformation of the phase portrait of system (2.7). Hence, it is enough to describe the phase portrait of a linear system (2.7), where J is the real Jordan normal form of the matrix A.

Case $d < 0$

If the determinant of the matrix A is strictly negative, then A has two real eigenvalues $\lambda_1 > 0 > \lambda_2$. Hence, the stable and unstable subspaces (E^s and E^u) are each generated by an eigenvector, and the central subspace is the origin, $E^c = \{\mathbf{0}\}$. The real Jordan normal form of A is

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

The phase portrait of the system $\dot{\mathbf{y}} = J\mathbf{y}$ is represented in Figure 2.5, the phase portrait of system $\dot{\mathbf{x}} = A\mathbf{x}$ is a linear transformation of it.

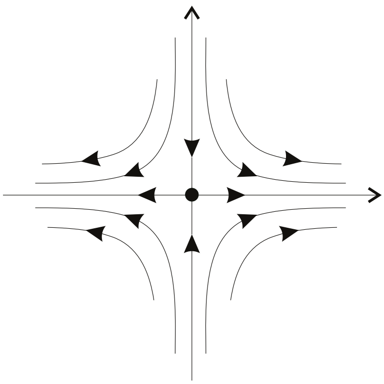
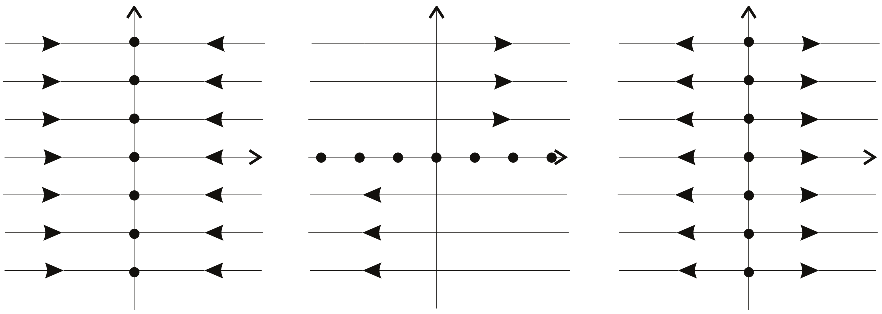


Figure 2.5: A saddle point and its stable and unstable separatrices.

In this case the singular point at the origin is called a *saddle point*. The two orbits in the stable subspace are called the *stable separatrices of the saddle* and the orbits in the unstable subspace are called the *unstable separatrices of the saddle*.

Case $d = 0$

Suppose that A is the zero matrix, i.e., the dimension of $\ker(A)$ is 2. In this case, any point in the phase plane is a singular point, so the case is of no interest. Assume now that $\ker(A)$ has dimension equal to 1, i.e., $\ker(A)$ is a straight line through the origin formed by all the singular points of the system. Hence, the singular points are not isolated. The real Jordan normal form of A changes according to whether $t = \text{trace}(A)$ is equal to zero or not. Thus, when $t = 0$ the matrix J is not diagonal and the straight line $\ker(A)$ is called a *non-isolated nilpotent manifold*, see Figure 2.6(b). When $t < 0$ (respectively $t > 0$) the matrix J is diagonal and the straight line $\ker(A)$ is called a *stable (respectively unstable) normally hyperbolic manifold*, see Figure 2.6(a) (respectively, (c)). The term "normally hyperbolic manifold" is motivated by [34].



(a) (b) (c)

Figure 2.6: Non-isolated singular points: (a) Stable normally hyperbolic manifold for $t < 0$; (b) Non-isolated nilpotent manifold for $t = 0$; and (c) unstable normally hyperbolic manifold for $t > 0$.

Case $d > 0$

We distinguish three cases, depending on the sign of $t^2 - 4d$. When $t^2 - 4d > 0$, the matrix A has two real eigenvalues with the same sign, $\lambda_1 > \lambda_2$. Therefore if $t < 0$, then $E^s = \mathbb{R}^2$ and $E^u = E^c = \{\mathbf{0}\}$; and if $t > 0$, then $E^u = \mathbb{R}^2$ and $E^s = E^c = \{\mathbf{0}\}$. The phase portrait of the system $\dot{\mathbf{y}} = J\mathbf{y}$ is shown in Figure 2.7, depending on t. The corresponding phase portrait of the system $\dot{\mathbf{x}} = A\mathbf{x}$ is obtained by a linear transformation. The origin is called an *asymptotically stable node* if $t < 0$, and an *asymptotically unstable node* if $t > 0$.

When $t^2 - 4d = 0$, there exists a unique eigenvalue λ, which is real, and the real Jordan normal form of A is

$$J = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{or} \quad J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

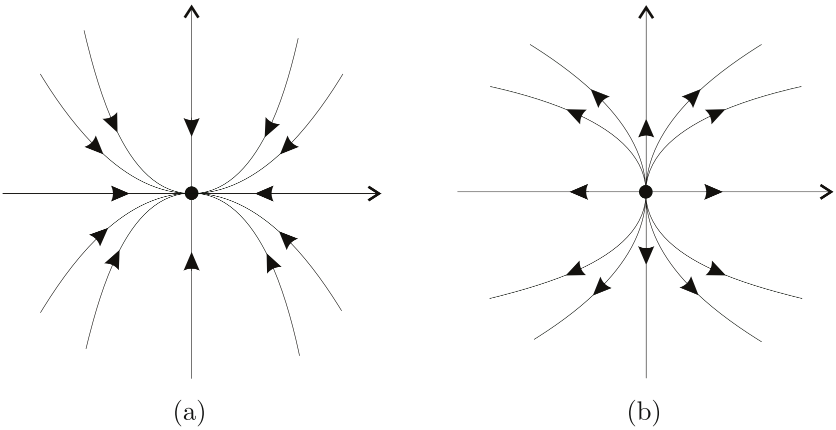


Figure 2.7: (a) Asymptotically stable node. (b) Asymptotically unstable node.

For each of these matrices we have to consider the cases $t < 0$ and $t > 0$. The phase portrait of the system $\dot{\mathbf{y}} = J\mathbf{y}$ is shown in Figure 2.8, depending on t and J. The corresponding phase portrait of the system $\dot{\mathbf{x}} = A\mathbf{x}$ is a linear transformation of it. The origin is called a *degenerated diagonal node* in the first case, and a *degenerated node* in the second one.

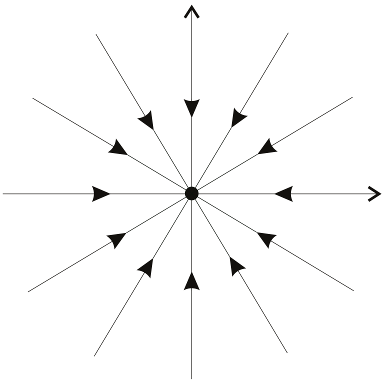
When $t^2 - 4d < 0$, the eigenvalues of A are a pair of conjugate complex numbers and

$$J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

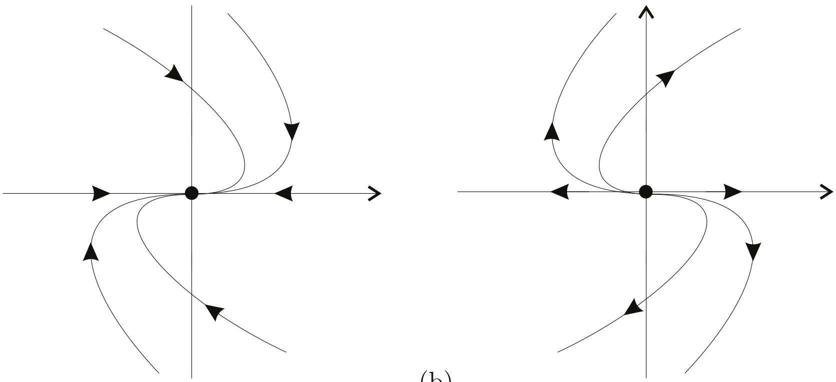
The phase portrait of the system $\dot{\mathbf{y}} = J\mathbf{y}$ is shown in Figure 2.9, depending on the sign of $t = 2\alpha$. The corresponding phase portrait of the system $\dot{\mathbf{x}} = A\mathbf{x}$ is a linear transformation of it. When $t = 0$, the origin is called a *center*. When $t < 0$, the origin is called an *asymptotically stable focus*. When $t > 0$, the origin is called an *asymptotically unstable focus*.

2.6 Classification of flows

Every classification criterion involves appropriate definitions for invariant sets, as specialized to different classes. If the list of the selected invariant sets is large, then the number of elements in each class is small and the classification is not effective. If the list of invariant sets is small, then we can collect systems with different behaviours and assign them to the same class. Thus the first step is to find an optimal classification criterion. In the theory of flows the criterion chosen is the preservation of the "orbit structure", a notion that will be defined in the following subsection.



(a)



(b)

Figure 2.8: (a) Degenerated diagonal nodes. (b) Degenerated nodes.

2.6.1 Classification criteria

We begin by defining equivalence relations for flows, in correspondence to the algebraic, the differentiable and the topological points of view.

Consider the differentiable systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$, with $\mathbf{f} : U \to \mathbb{R}^n$ a locally Lipschitz function defined on $U \subset \mathbb{R}^n$ and $\mathbf{g} : V \to \mathbb{R}^n$ a locally Lipschitz function defined on $V \subset \mathbb{R}^n$. Let $\Phi(s, \mathbf{x})$ and $\Phi^*(s, \mathbf{y})$ be the respective flows. We recall that in this work we consider only complete flows, i.e., the interval of definition of all the solutions is the entire $\mathbb{R}$.

Two flows are said to be *conjugate* if there exists a bijection $\mathbf{h} : U \to V$ (called *conjugacy*), such that $\Phi^*(s, \mathbf{h}(\mathbf{x})) = \mathbf{h}(\Phi(s, \mathbf{x}))$ for every $s \in \mathbb{R}$ and $\mathbf{x} \in U$. The flows are said to be *equivalent* if there exists a bijection $\mathbf{h} : U \to V$ (called *equivalence*), such that γ is an orbit of the first system if and only if $\mathbf{h}(\gamma)$ is an

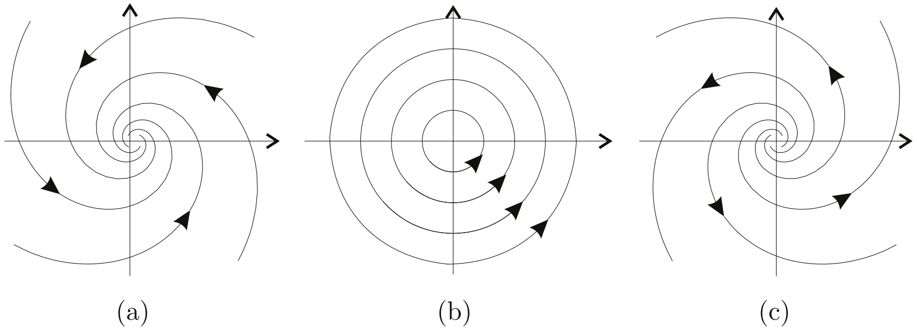


Figure 2.9: (a) Asymptotically stable focus. (b) Center. (c) Asymptotically unstable focus.

orbit of the second one and in addition $\mathbf{h}$ preserves the orientation of the orbit. It is easy to check that if two flows are conjugate, then they are equivalent. The converse is not always true.

An equivalence $\mathbf{h}$ transforms singular points into singular points and periodic orbits into periodic orbits. When $\mathbf{h}$ is a conjugacy, the period of the periodic orbits is also preserved.

Consider two conjugate (respectively equivalent) flows. The flows are said to be *linearly conjugate* (respectively, *linearly equivalent*) if $\mathbf{h}$ is a linear isomorphism. The flows are said to be C^r*-conjugate* (respectively, C^r*-equivalent*), with $r \in \{1, 2, \ldots, \infty, \omega\}$, if $\mathbf{h}$ is a diffeomorphism such that $\mathbf{h}$, $\mathbf{h}^{-1} \in C^r$ (recall here that C^ω denotes the class of analytic functions). The flows are said to be *topologically conjugate* (respectively *topologically equivalent*) if $\mathbf{h}$ is a homeomorphism.

Two differential equations are said to be linearly, C^r, or topologically equivalent (respectively, conjugate) if their flows are linearly, C^r, or topologically equivalent (respectively, conjugate). Further, they are said to present the same *qualitative behaviour* or the same *dynamical behaviour* if they are topologically equivalent.

In the next result we relate the different classification criteria.

Proposition 2.6.1. *Consider two differential equations.*

(a) *If they are linearly conjugate (respectively, equivalent), then they are C^r-conjugate (respectively, C^r-equivalent) for every $r \in \{1, 2, \ldots, \infty, \omega\}$.*

(b) *If they are C^r-conjugate (respectively, C^r-equivalent) with $r \in \{1, \ldots \infty, \omega\}$, then they are topologically conjugate (respectively, equivalent).*

(c) *If they are linearly, C^r, or topological conjugate, then they are linearly, C^r, or topologically equivalent.*

The conjugacy of flows is also a conjugacy of vector fields. In the next lemma we characterize the C^r-conjugacy via the conjugacy of vector fields. As usual, given a diffeomorphism $\mathbf{h} : U \to V$, $D\mathbf{h}(\mathbf{x})$ denotes the *Jacobian matrix* of $\mathbf{h}$ evaluated at the point $\mathbf{x}$.

Lemma 2.6.2. *Consider two differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$, with $\mathbf{f} : U \to \mathbb{R}^n$ and $\mathbf{g} : V \to \mathbb{R}^n$ locally Lipschitz functions on U and V, respectively. Their flows are C^r-conjugate if and only if there exists a diffeomorphism $\mathbf{h} : U \to V$ in C^r such that $D\mathbf{h}(\mathbf{x})\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{h}(\mathbf{x}))$ for every $\mathbf{x} \in U$.*

A proof of this result can be found in [58, p. 19, Lemma 3.4]

2.6.2 Classification of linear flows

Given a linear isomorphism $\mathbf{h} : U \to V$, with U and V open subsets of $\mathbb{R}^n$, there exists a matrix $M \in GL(\mathbb{R}^n)$ such that $\mathbf{h}(\mathbf{x}) = M\mathbf{x}$ for any $\mathbf{x} \in U$.

Lemma 2.6.3. *If the linear map $\mathbf{h}(\mathbf{x}) = M\mathbf{x}$ is constant on an open subset $U \subset \mathbb{R}^n$, then M is the zero matrix.*

Proof. Suppose that M is not the zero matrix. Then there exists a vector $\mathbf{e} \in U$ such that $M\mathbf{e} \neq \mathbf{0}$. Take $\mathbf{x}_0 \in U$. Since U is open, $\mathbf{x}_1 = \mathbf{x}_0 + \delta\mathbf{e} \in U$ for $\delta > 0$ small enough. Therefore, $\delta M\mathbf{e} = M\mathbf{x}_1 - M\mathbf{x}_0 = \mathbf{0}$, a contradiction. $\square$

Proposition 2.6.4 (Linear conjugacy of linear flows)**.** *Consider two linear systems $\dot{\mathbf{x}} = A\mathbf{x}$ and $\dot{\mathbf{y}} = A^*\mathbf{y}$, with $A, A^* \in L(\mathbb{R}^2)$, and denote $d = \det(A)$, $t = \operatorname{trace}(A)$, $d^* = \det(A^*)$ and $t^* = \operatorname{trace}(A^*)$.*

(a) *The systems are linearly conjugate if and only if there exists $M \in GL(\mathbb{R}^2)$ such that $A^* = MAM^{-1}$, i.e., the matrices of the systems are equivalent.*

(b) *If the systems are linearly conjugate, then $d = d^*$ and $t = t^*$.*

(c) *If $d = d^*$, $t = t^*$ and $t^2 - 4d \neq 0$, then the systems are linearly conjugate.*

Proof. (a) Suppose that the given systems are linearly conjugate. By definition there exists a linear map $M \in GL(\mathbb{R}^2)$ such that, for any given solution of the first system, $\mathbf{x}(s) = \phi(s; 0, \mathbf{x}_0)$, the function $\mathbf{y}(s) = M\mathbf{x}(s)$ is a solution of the second one. Moreover, $\dot{\mathbf{y}} = MAM^{-1}\mathbf{y}$. Applying Lemma 2.6.3 to the linear map $\mathbf{h}(\mathbf{y}) = \left(A^* - MAM^{-1}\right)\mathbf{y}$, we conclude that $A^* = MAM^{-1}$.

Conversely, suppose that $A^* = MAM^{-1}$ with $M \in GL(\mathbb{R}^2)$. By Proposition 2.5.1.(b), $e^{sA^*} = Me^{sA}M^{-1}$ for all $s \in \mathbb{R}$. The flows of the linear systems are $\Phi(s, \mathbf{x}) = e^{sA}\mathbf{x}$ and $\Phi^*(s, \mathbf{y}) = e^{sA^*}\mathbf{y}$, respectively, see Theorem 2.5.2. Hence, $\Phi^*(s, M\mathbf{x}) = e^{sA^*}M\mathbf{x} = Me^{sA}\mathbf{x} = M\Phi(s, \mathbf{x})$. Therefore, the systems are linearly conjugate.

Statement (b) follows from statement (a). For a proof of statement (c) see Arnold [7, p. 169]. $\square$

Proposition 2.6.5 (C^r-conjugacy of linear flows)**.** *Two linear flows are C^r-conjugated for $r \in \{1, 2, \ldots, \infty, \omega\}$ if and only if they are linearly conjugate.*

For a proof of the previous proposition see Arnold [7, p. 170].

Corollary 2.6.6. *Consider two linear systems* $\dot{\mathbf{x}} = A\mathbf{x}$ *and* $\dot{\mathbf{y}} = A^*\mathbf{y}$ *and denote* $d = \det(A)$, $t = \operatorname{trace}(A)$, $d^* = \det(A^*)$ *and* $t^* = \operatorname{trace}(A^*)$. *If the flows are* C^r*-conjugate with* $r \in \{1, 2, \ldots, \infty, \omega\}$, *then* $d = d^*$ *and* $t = t^*$.

Proof. The proof follows from Propositions 2.6.5 and 2.6.4 (b). □

In the next result we present a characterization of the topological conjugacy of linear flows.

Proposition 2.6.7 (Topological conjugacy of linear flows)**.** *The flows of two linear systems whose eigenvalues have no zero real part are topologically conjugate if and only if they have the same number of eigenvalues with positive and the same number of eigenvalues with negative real part.*

For a proof of this result see Arnold [7, pp. 172–182].

2.6.3 Topological equivalence of non-linear flows

As we have seen, in the case of linear flows there exists a characterization of the three different classification criteria. To our knowledge a complete characterization of topological equivalence exists only for planar non-linear flows. To introduce it we need some new notations and results analogous to the ones in the previous subsection. Essentially all these definitions and results can be found in [48, pp. 127–148] and [50, pp. 73–81], where they are applied in a more general context. Similar results are due to Peixoto [52].

Consider a differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{f}$ a Lipschitz function defined in $\mathbb{R}^2$, and let $\Phi(s, \mathbf{x})$ be its flow. Following Markus and Neumann, we denote this flow by $(\mathbb{R}^2, \Phi)$. By the continuous dependence of solutions on the initial conditions and parameters, the flow $(\mathbb{R}^2, \Phi)$ is continuous in both variables. The flow $(\mathbb{R}^2, \Phi)$ is said to be *parallel* if it is topologically equivalent to one of the following flows:

(a) The flow defined in $\mathbb{R}^2$ by the differential system $\dot{x} = 1$, $\dot{y} = 0$, called *strip flow*.

(b) The flow defined in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ by the differential system in polar coordinates $\dot{r} = 0$, $\dot{\theta} = 1$, called *annular flow*.

(c) The flow defined in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ by the differential system in polar coordinates $\dot{r} = r$, $\dot{\theta} = 0$, called *spiral* or *radial flow*.

An orbit $\gamma(\mathbf{p})$ of the flow $(\mathbb{R}^2, \Phi)$ is called a *separatrix* if

(a) is a singular point, or

(b) is a limit cycle, or

(c) $\gamma(\mathbf{p})$ is homeomorphic to $\mathbb{R}$ and there is no tubular neighbourhood N of $\gamma(\mathbf{p})$ with the following properties:

 (c.1) Every point $\mathbf{q}$ in N has the same α-limit and ω-limit sets of $\mathbf{p}$, i.e., $\alpha(\mathbf{q}) = \alpha(\mathbf{p})$ and $\omega(\mathbf{q}) = \omega(\mathbf{p})$.

(c.2) The boundary of N, i.e., $\mathrm{Cl}(N) \setminus N$, is formed by $\alpha(\mathbf{p})$, $\omega(\mathbf{p})$ and two orbits $\gamma(\mathbf{q}_1)$ and $\gamma(\mathbf{q}_2)$ such that $\alpha(\mathbf{p}) = \alpha(\mathbf{q}_1) = \alpha(\mathbf{q}_2)$ and $\omega(\mathbf{p}) = \omega(\mathbf{q}_1) = \omega(\mathbf{q}_2)$, see Figure 2.10. As usual $\mathrm{Cl}(N)$ denotes the closure of N, i.e., the smallest closed set containing N.

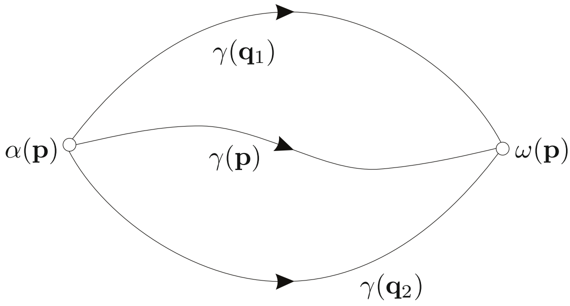


Figure 2.10: The boundary of N.

Let S be the union of the separatrices of the flow $(\mathbb{R}^2, \Phi)$. It is easy to check that S is an invariant closed set. If N is a connected component of $\mathbb{R}^2 \setminus S$, then N is also an invariant set, and the flow $(N, \Phi|_N)$ is called a *canonical region* of the flow $(\mathbb{R}^2, \Phi)$.

Proposition 2.6.8. *Every canonical region of the flow* $(\mathbb{R}^2, \Phi)$ *is parallel.*

For a proof of this proposition see [50].

The *separatrix configuration* of a flow $(\mathbb{R}^2, \Phi)$ is the union of all separatrices of the flow together with an orbit belonging to each canonical region. Given two flows $(\mathbb{R}^2, \Phi)$ and $(\mathbb{R}^2, \Phi^*)$, let S and S^* be the union of their separatrices, respectively. The separatrix configuration C of the flow $(\mathbb{R}^2, \Phi)$ is said to be topologically equivalent to the separatrix configuration C^* of the flow $(\mathbb{R}^2, \Phi^*)$ if there exists an orientation preserving homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$ which transforms orbits of C into orbits of C^*, and orbits of S into orbits of S^*.

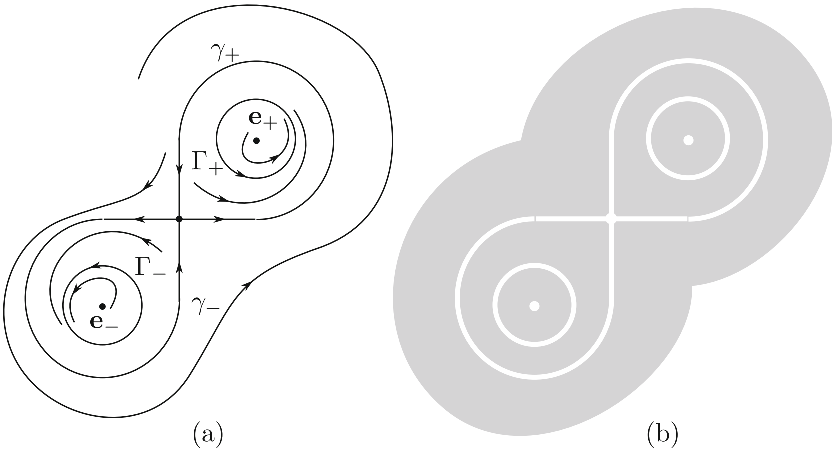
Theorem 2.6.9 (Markus–Neumann–Peixoto)**.** *Let* $(\mathbb{R}^2, \Phi)$ *and* $(\mathbb{R}^2, \Phi^*)$ *be two continuous flows with only isolated singular points. Then they are topologically equivalent if and only if their separatrix configurations are topologically equivalent.*

For a proof of this result we refer the reader to [50].

It follows from the previous theorem that in order to classify the flows of planar differential equations, it is enough to describe their separatrix configuration.

Example 10. Consider the local phase portrait depicted in Figure 2.11(a). The set S of all separatrices is formed by the singular points $\mathbf{e}_+$, $\mathbf{e}_-$ and $\mathbf{0}$, the periodic orbits Γ_+ and Γ_-, and the homoclinic orbits γ_+ and γ_-. Therefore, S is an invariant closed set. In Figure 2.11(b) we represent the set of all canonical regions.

Note that Figure 2.11(a) presents clearly the set of all separatrices together with an orbit for each canonical region which shows the asymptotic behaviour of the orbits contained in its interior. Thus Figure 2.11(a) also represents the separatrix configuration of the phase portrait. From this it is easy to understand that the separatrix configuration is the skeleton of the phase portrait.



(a) (b)

Figure 2.11: (a) Separatrix configuration correspondig to a fundamental system with parameters $D < 0$, $T < 0$ and $t = w_1(d)$, see Section 5.5. (b) Canonical regions associated to the phase portrait.

2.7 Non-linear systems

In this section we return to non-linear flows. Let $U \subseteq \mathbb{R}^n$ be an open subset, $\mathbf{f} : U \to \mathbb{R}^n$ be a locally Lipschitz function in U and $\Phi(s, \mathbf{x})$ be the flow defined by the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Recall that we consider only complete flows, i.e., solutions are defined for every value of time $s \in \mathbb{R}$.

2.7.1 Local phase portraits of singular points

We begin by studying the local behaviour of flows in a neighbourhood of singular points, i.e., points $\mathbf{x} \in U$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.

Theorem 2.7.1 (Lyapunov function)**.** *Consider the differential equation* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, *with* $\mathbf{f} : U \to \mathbb{R}^n$ *a locally Lipschitz function in* U. *Let* $\mathbf{x}_0$ *be a singular point. If there exist a neighbourhood* W *of* $\mathbf{x}_0$ *in* U *and a function* $V : W \to \mathbb{R}$ *satisfying*

(a) $V(\mathbf{x}_0) = 0$ *and* $V(\mathbf{x}) > 0$ *when* $\mathbf{x} \neq \mathbf{x}_0$,

(b) $\frac{dV(\mathbf{x}(s))}{ds} \leq 0$ *in* $W \setminus \{\mathbf{x}_0\}$, *where* $\mathbf{x}(s)$ *is a solution of the differential equation, then* $\mathbf{x}_0$ *is stable. Moreover,*

(c) *if* $\frac{dV(\mathbf{x}(s))}{ds} < 0$ *in* $W \setminus \{\mathbf{x}_0\}$, *then* $\mathbf{x}_0$ *is asymptotically stable.*

The function V figuring in this theorem is called a *Lyapunov function.* For a proof of the Lyapunov function theorem we refer the reader to [33, p. 192].

Now we classify the singular points according to the linear part of the vector field. Let $\mathbf{x}_0$ be a singular point of the differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f}$ is a C^1 function in a neighbourhood of $\mathbf{x}_0$. Let $D\mathbf{f}(\mathbf{x}_0)$ be the Jacobian matrix of $\mathbf{f}$ evaluated at $\mathbf{x}_0$. The point $\mathbf{x}_0$ is said to be a *hyperbolic singular point* if all the eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$ have non-zero real part.

For a planar differential system we say that a singular point $\mathbf{x}_0$ is an *elementary non-degenerate* singular point if the determinant of $D\mathbf{f}(\mathbf{x}_0)$ is not zero. In particular, every hyperbolic singular point is an elementary non-degenerate one. The converse is not true. Since elementary non-degenerate singular points with determinant of $D\mathbf{f}(\mathbf{x}_0)$ less than zero are saddle points, we call *antisaddle* any non-degenerate singular point at which the Jacobian matrix has positive determinant. The singular point $\mathbf{x}_0$ is said to be an *elementary degenerate* singular point if the determinant of $D\mathbf{f}(\mathbf{x}_0)$ is zero and the trace of $D\mathbf{f}(\mathbf{x}_0)$ is non-zero. The singular point $\mathbf{x}_0$ is said to be *nilpotent* if the determinant and the trace of the matrix $D\mathbf{f}(\mathbf{x}_0)$ are both zero and $D\mathbf{f}(\mathbf{x}_0)$ is not the zero matrix.

Since the concept of a flow introduced in our textbook corresponds to the concept of a complete flow used by other authors (see Subsection 2.3), in the following version of the Hartman–Grobman theorem we impose the condition that the maximal interval of definition of all solutions is$\mathbb{R}$.

Theorem 2.7.2 (Hartman–Grobman)**.** *Let* U *be an open subset of* $\mathbb{R}^n$, $\mathbf{f} : U \to \mathbb{R}^n$ *be a* $C^1(U)$ *function,* $\Phi(s, \mathbf{x})$ *be the flow of the differential equation* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, *and* $\mathbf{x}_0$ *be a hyperbolic singular point. Then there exist a neighbourhood* W *of* $\mathbf{x}_0$, *a neighbourhood* V *of the origin, a homeomorphism* $\mathbf{h} : W \to V$ *with* $\mathbf{h}(\mathbf{x}_0) = \mathbf{0}$, *and an interval* $I \subseteq \mathbb{R}$ *containing the origin, such that*

$$\mathbf{h} \circ \Phi(s, \mathbf{x}) = e^{sD\mathbf{f}(\mathbf{x}_0)}\mathbf{h}(\mathbf{x})$$

for every $s \in I$ *and* $\mathbf{x} \in U$.

For a proof of the previous theorem see Section 4.3 in [14] or [51, p. 294].

The Hartman–Grobman theorem asserts that the differential systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{x}} = D\mathbf{f}(\mathbf{x}_0)$ are topologically equivalent in a neighbourhood W of a hyperbolic singular point $\mathbf{x}_0$ and V of the origin. This is why we use the same names for non-linear hyperbolic singular points and for the linear hyperbolic ones. Even for non-hyperbolic singular points, when the system is topologically equivalent to a linear system, we use the same terminology for both singular points. Accordingly, the singular point $\mathbf{x}_0$ of a non-linear differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is said to be a *stable normally hyperbolic singular point* if $\mathbf{f}$ is topologically equivalent to the differential system $\dot{x} = 0$, $\dot{y} = -y$ in a neighbourhood of $\mathbf{x}_0$ and $\mathbf{0}$. The singular point $\mathbf{x}_0$ is said to be an *unstable normally hyperbolic singular point* if $\mathbf{f}$ is topologically equivalent to the differential system $\dot{x} = 0$, $\dot{y} = y$ in a neighbourhood of $\mathbf{x}_0$ and $\mathbf{0}$. The singular point $\mathbf{x}_0$ is said to be a *non-isolated nilpotent singular point* if $\mathbf{f}$ is topologically equivalent to the differential system $\dot{x} = y$, $\dot{y} = 0$.

The standard tool for studying the flow in a neighbourhood of a planar non-hyperbolic singular point is a change of variables called *blow-up*, see [8], [20] and [21] for more details. Here, we summarize a description of this change of variables in the case of planar vector fields $\mathbf{f}(x, y) = (P(x, y), Q(x, y))$, where P and Q are analytic functions. Without loss of generality we can assume that the origin is a singular point of the system (otherwise we can translate the singular point to the origin by a convenient change of variables).

Consider the differentiable function $\mathbf{h}_x : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{h}_x(\bar{x}, \bar{y}) = (\bar{x}, \bar{x}\,\bar{y})$. Using the Jacobian matrix of $\mathbf{h}_x$ and the vector field $\mathbf{f}$ we can define a vector field $\mathbf{f}_x$ on $\mathbb{R}^2$ satisfying the equality

$$D\mathbf{h}_x(\mathbf{f}_x(\bar{x}, \bar{y})) = \mathbf{f}(\mathbf{h}_x(\bar{x}, \bar{y})) = \mathbf{f}(\bar{x}, \bar{x}\,\bar{y}).$$

From here, one obtains the following expression for $\mathbf{f}_x$ when $\bar{x} \neq 0$

$$\mathbf{f}_x(\bar{x}, \bar{y}) = \left(P(\bar{x}, \bar{x}\,\bar{y}), \frac{Q(\bar{x}, \bar{x}\bar{y}) - \bar{y}P(\bar{x}, \bar{x}\bar{y})}{\bar{x}} \right). \tag{2.9}$$

Since the origin is a singular point, i.e., $P(0, 0) = Q(0, 0) = 0$, expression (2.9) can be extended to $\bar{x} = 0$ to yield an analytic vector field on $\mathbb{R}^2$. Such a vector field is called a *blow-up in the x-direction.*

The vector fields $\mathbf{f}$ and $\mathbf{f}_x$ are topologically equivalent in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ and $\mathbb{R}^2 \setminus \{\bar{x} = 0\}$, respectively. Moreover, since $\mathbf{h}_x$ maps the straight line $\bar{x} = 0$ into the origin, the behaviour of the flow of $\mathbf{f}$ in a neighbourhood of the origin can be obtained from the behaviour of the flow of $\mathbf{f}_x$ in a neighbourhood of $\bar{x} = 0$ in the following sense. Let γ be an orbit of the differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ such that the origin is contained in its α- or ω-limit set. If $m = \tan\theta$, with $\theta \in (-\pi/2, \pi/2)$, is the slope of γ at the origin, then the angle θ is called a *characteristic direction* of the origin and the point $(0, m)$ is a singular point of the blow-up system $\dot{\mathbf{u}} = \mathbf{f}_x(\mathbf{u})$. The study of the local phase portrait at the point $(0, m)$ is easier than the one of the origin, because such singular points are less degenerate.

If $m = \pm\infty$, then another change of variables applies. Specifically, consider the function $\mathbf{h}_y : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{h}_y(\bar{x}, \bar{y}) = (\bar{x}\,\bar{y}, \bar{y})$, and the vector field $\mathbf{f}_y$ satisfying $D\mathbf{h}_y(\mathbf{f}_y(\bar{x}, \bar{y})) = \mathbf{f}(\bar{x}\,\bar{y}, \bar{y})$. It follows that

$$\mathbf{f}_y(\bar{x}, \bar{y}) = \left(\frac{P(\bar{x}\,\bar{y}, \bar{y}) - \bar{x}Q(\bar{x}\,\bar{y}, \bar{y})}{\bar{y}}, Q(\bar{x}\,\bar{y}, \bar{y}) \right). \tag{2.10}$$

Thus $(0, 0)$ is a singular point of the blow-up system $\dot{\mathbf{u}} = \mathbf{f}_y(\mathbf{u})$. In general, if $m = \tan\theta$ with $\theta \in (0, \pi)$, then $(1/m, 0)$ is a singular point of the blow-up system

$\dot{\mathbf{u}} = \mathbf{f}_y(\mathbf{u})$. Hence, going back to the original variables a finite number of curves are present, splitting any neighbourhood of the origin into *hyperbolic, elliptic and parabolic sectors*, see Figure 2.12.

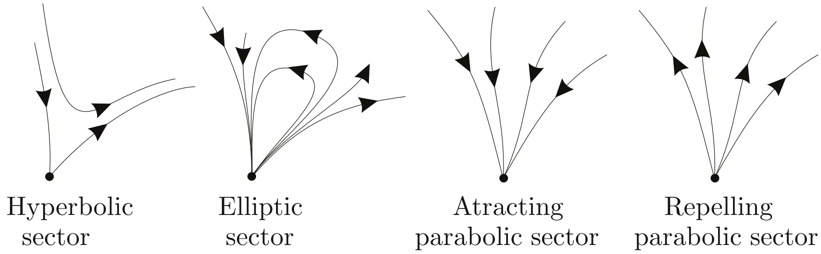


Figure 2.12: Sectors in the neighbourhood of a singular point.

A singular point $\mathbf{x}_0$ is called a *saddle-node* if a neighbourhood of $\mathbf{x}_0$ is the union of a unique parabolic sector and two hyperbolic sectors. Thus a saddle-node has three separatrices: two of them, called the *hyperbolic manifolds* or *separatrices of the saddle-node*, are related to the boundary of the parabolic sector; and the remainder, called the *central manifold* or *separatrice of the saddle-node*, is related to the boundary between the two hyperbolic sectors. Note that this terminology is appropriate only when the singular point is elementary and degenerate. To simplify notation we continue using this terminology not only for nilpotent saddle-nodes, but also for more degenerated saddle points.

Theorem 2.7.3 (Elementary non-degenerate singular points)**.** *Let* $(0,0)$ *be an isolated singular point of the differential system*

$$\dot{x} = X(x,y), \quad \dot{y} = y + Y(x,y),$$

where X *and* Y *are analytic functions in a neighbourhood of the origin and their series expansion involve only terms of second order and higher. Let* $f(x)$ *be a solution of the equation* $y + Y(x,y) = 0$ *in a neighbourhood of the origin and suppose that the function* $g(x) = X(x, f(x))$ *can be written in the form* $g(x) = a_m x^m + O(x^{m+1})$ *where* $O(x^k)$ *stands for an analytic function with terms of order greater or equal than* k *in its series expansion,* $m \geq 2$*, and* $a_m \neq 0$*.*

(a) *If* m *is odd and* $a_m > 0$*, then the origin is topologically equivalent to a stable node.*

(b) *If* m *is odd and* $a_m < 0$*, then the origin is topologically equivalent to a saddle with the stable manifold tangent to the* x*-axis and the unstable manifold tangent to the* y*-axis.*

(c) *If* m *is even, then the origin is a saddle-node. Its hyperbolic manifold is unstable and tangent to the* y*-axis. Its central manifold is tangent to the*

x-axis and when $a_m > 0$ (respectively, $a_m < 0$) it is unstable (respectively, stable) in the 0 direction and stable (respectively, unstable) in the π direction.

For a proof of Theorem 2.7.3 we refer the reader to [4, p. 340] or [21, p. 74].

Theorem 2.7.4 (Nilpotent singular points)**.** *Let $(0,0)$ be an isolated singular point of the system*

$$\dot{x} = y + X(x,y), \quad \dot{y} = Y(x,y),$$

where X and Y are analytic functions in a neighbourhood of the origin and their series expansions involves only terms of second order and higher. Let $y = f(x) = a_2x^2 + a_3x^3 + O(x^4)$ be a solution of the equation $y + X(x,y) = 0$ in a neighbourhood of the origin, and suppose that $F(x) = Y(x, f(x)) = Ax^\alpha(1 + O(x))$ and $\Phi(x) = (\partial X/\partial x + \partial Y/\partial y)(x, f(x)) = Bx^\beta(1 + O(x))$, with $A \neq 0$, $\alpha \geq 2$ and $\beta \geq 1$.

(a) *If α is even, then*

(a.1) *if $\alpha > 2\beta + 1$, the origin is a saddle-node with the three separatrices tangent to the x-axis;*

(a.2) *if $\alpha < 2\beta + 1$ or $\Phi \equiv 0$, then a neighourhood of the origin is the union of two hyperbolic sectors.*

(b) *If α is odd and $A > 0$, then the origin is a saddle whose stable and unstable separatrices are tangent to the x-axis.*

(c) *If α is odd and $A < 0$, then*

(c.1) *if $\alpha > 2\beta+1$ and β even; or $\alpha = 2\beta+1$, β even and $B^2+4A(\beta+1) \geq 0$, then the origin is a node, stable when $B < 0$ and unstable when $B > 0$;*

(c.2) *if $\alpha > 2\beta+1$ and β odd; or $\alpha = 2\beta+1$, β odd and $B^2+4A(\beta+1) \geq 0$, then the origin is the union of a hyperbolic sector and an elliptic sector;*

(c.3) *if $\alpha = 2\beta + 1$ and $B^2 + 4A(\beta + 1) < 0$, then the origin is a focus;*

(c.4) *if $\alpha < 2\beta + 1$; or $\Phi \equiv 0$, then the origin is a center.*

A proof of the previous theorem can be found in [4, pp. 357–362], in [2], or in [21, p. 116].

2.7.2 Periodic orbits: Poincaré map

One of the most important tools in the study of flows in the neighbourhood of periodic orbits is the so called Poincaré map. Consider a locally Lipschitz vector field $\mathbf{f} : U \to \mathbb{R}^n$ and let $\Phi(s, \mathbf{x})$ be the flow defined by the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Let Σ be a hypersurface in $\mathbb{R}^n$ and take a point $\mathbf{p}$ in $\Sigma \cap U$. The flow Φ is said to be *transverse* to Σ at the point $\mathbf{p}$ if $\mathbf{f}(\mathbf{p})$ is not contained in $T_{\mathbf{p}}\Sigma$ (the tangent space to Σ at point $\mathbf{p}$). If $\mathbf{f}(\mathbf{p}) \in T_{\mathbf{p}}\Sigma$, then $\mathbf{p}$ is called a *contact point* of the flow with Σ.

Let V be an open subset of Σ. We say that the flow is *transverse to* Σ at V if the flow is transverse to Σ at every point in V.

Consider now two open hypersurfaces Σ_1, Σ_2 and two points $\mathbf{p}_1 \in \Sigma_1 \cap U$, $\mathbf{p}_2 \in \Sigma_2 \cap U$ such that $\mathbf{p}_2 = \Phi(s_1, \mathbf{p}_1)$. There exist a neighbourhood V_1 of $\mathbf{p}_1$ in $\Sigma_1 \cap U$, a neighbourhood V_2 of $\mathbf{p}_2$ in $\Sigma_2 \cap U$, and a function $\tau : V_1 \to \mathbb{R}$ satisfying $\tau(\mathbf{p}_1) = s_1$ and $\Phi(\tau(\mathbf{q}), \mathbf{q}) \in V_2$ for every $\mathbf{q} \in V_1$. Moreover, if the vector field $\mathbf{f}$ is globally Lipschitz, C^r with $r \geq 1$, or analytic, then the function τ is also continuous, C^r with $r \geq 1$, or analytic, respectively. For more details see [53, pp. 193–194] or [57, pp. 226–227]. In this situation we define the *Poincaré map* as the map $\pi : V_1 \to V_2$ given by

$$\pi(\mathbf{q}) = \Phi(\tau(\mathbf{q}), \mathbf{q}),$$

for every $\mathbf{q} \in V_1$, see Figure 2.13.

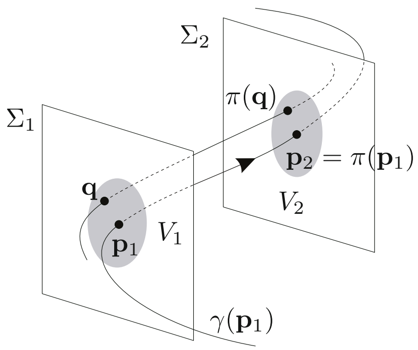


Figure 2.13: Poincaré map π.

When the vector field is globally Lipschitz, C^r with $r \geq 1$, or analytic, the Poincaré map π is also continuous, C^r with $r \geq 1$, or analytic, respectively.

By reversing the sense of the flow it is easy to conclude that the Poincaré map is invertible and the inverse map π^{-1} is continuous, C^r with $r \geq 1$, or analytic, respectively. In the particular case when $\Sigma_1 = \Sigma_2$ the Poincaré map π is called a *return map*.

Consider $\mathbf{p} \in \Sigma_1$ and let $\gamma(\mathbf{p})$ be a periodic orbit. From the continuous dependence of the flow on the initial conditions, it follows that a return map π can be defined in a neighbourhood of $\mathbf{p}$, and $\mathbf{p}$ is a fixed point of π. Conversely, if $\mathbf{p} \in \Sigma_1$ is a fixed point of a return map π, then $\gamma(\mathbf{p})$ is a periodic orbit. Hence, limit cycles are associated to isolated fixed points of return maps. A limit cycle $\gamma(\mathbf{p})$ is called a *hyperbolic limit cycle* if the absolute value of all the eigenvalues of the Jacobian matrix $D\pi(\mathbf{p})$ is different from 1; otherwise $\gamma(\mathbf{p})$ is called a *non-hyperbolic limit cycle*. Note that this definition does not depend on the chosen point $\mathbf{p}$ or on the chosen cross section Σ_1.

Theorem 2.7.5. *Let $\mathbf{f} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function in U, $\gamma(\mathbf{p})$ be a hyperbolic limit cycle of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and π be a return map defined in a neighbourhood of $\gamma(\mathbf{p})$. Suppose that π is differentiable in a neighbourhood of $\mathbf{p}$.*

(a) *If the absolute value of every eigenvalue of $D\pi(\mathbf{p})$ is less than 1, then $\gamma(\mathbf{p})$ is a stable limit cycle.*

(b) *If the absolute value of at least one eigenvalue of $D\pi(\mathbf{p})$ is greater than 1, then $\gamma(\mathbf{p})$ is an unstable limit cycle.*

A proof of this result can be found in [21] or in [57, Chapter IX].

2.8 α- and ω-limit sets in the plane

In this section we deal with the asymptotic behaviour of the remainder orbits. These orbits are diffeomorphic to straight lines, see Theorem 2.3.2. In this section we restrict ourselves to planar flows. In this context the following version of the Jordan curve theorem will be useful later on.

A curve in the plane is said to be a *Jordan curve* if it is homeomorphic to $\mathbb{S}^1$, i.e., if it is a closed curve without autointersections.

Theorem 2.8.1 (Jordan curve)**.** *The complementary set of a Jordan curve γ in the plane is the union of two open, disjoint and connected sets. Furthermore, one of these sets is bounded and its boundary is the curve γ.*

Since orbits of a flow are disjoint, from the Jordan curve theorem it follows that a periodic orbit γ splits the phase plane into two invariant regions, one of which is bounded. This bounded region will be called the *interior* of γ and be denoted by Σ_γ.

Periodic orbits are not the unique Jordan curves formed by solutions. We define a *separatrix cycle* to be a finite union of n singular points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ (some of these points may coincide) and n orbits γ_1, $\gamma_2, \ldots, \gamma_n$, with the property that $\alpha(\gamma_k) = \{\mathbf{p}_k\}$ for $k = 1, 2, \ldots, n$, $\omega(\gamma_k) = \{\mathbf{p}_{k+1}\}$ if $k = 1, 2, \ldots, n-1$, and $\omega(\gamma_n) = \{\mathbf{p}_1\}$, see Figure 2.14. The singular points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ will be called the *vertices of the cycle.*

We define a *homoclinic cycle* to be a separatrix cycle formed by one singular point (*homoclinic point*) and one orbit (*homoclinic orbit*), see Figure 2.14(a). A *double homoclinic cycle* is a separatrix cycle formed by one singular point (in this case $\mathbf{p}_1$ and $\mathbf{p}_2$ are identified) and two orbits, see Figure 2.14(b). Finally, a *heteroclinic cycle* is a separatrix cycle formed by two singular points and two orbits, see Figure 2.14(c).

A periodic orbit γ is said to be *inside asymptotically stable* (respectively, *inside asymptotically unstable*) if there exists a neighbourhood V of γ such that

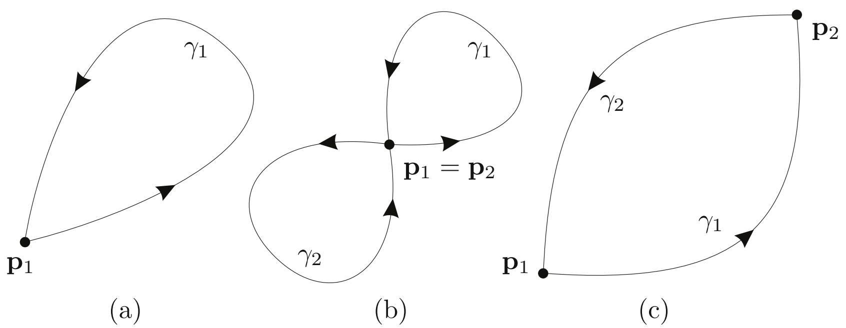


Figure 2.14: Separatrix cycles: (a) homoclinic cycle; (b) double homoclinic cycle; (c) heteroclinic cycle.

$V \cap \Sigma_\gamma \subset W^s(\gamma)$ (respectively, $V \cap \Sigma_\gamma \subset W^u(\gamma)$). A periodic orbit γ is said to be *outside asymptotically stable* (respectively, *outside asymptotically unstable*) if there exists a neighbourhood V of γ such that $V \cap (\mathbb{R}^2 \setminus \mathrm{Cl}(\Sigma_\gamma)) \subset W^s(\gamma)$ (respectively, $V \cap (\mathbb{R}^2 \setminus \mathrm{Cl}(\Sigma_\gamma)) \subset W^u(\gamma)$).

A limit cycle γ is said to be *semistable* if γ is either inside asymptotically stable and outside asymptotically unstable, or inside asymptotically unstable and outside asymptotically stable.

The following result asserts that the α- and ω-limit set of orbits of planar differential systems are simple sets: singular points, periodic orbits, or separatrix cycles.

Theorem 2.8.2 (Poincaré–Bendixson)**.** *Let $\mathbf{f} : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a locally Lipschitz function in the open subset U, and let γ be an orbit of the differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Suppose that γ is positively bounded (respectively, negatively bounded) and the number of singular points in $\omega(\gamma)$ (respectively, in $\alpha(\gamma)$) is finite.*

(a) *If $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) has no singular points, then $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) is a periodic orbit.*

(b) *If $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) has singular points and regular points, then $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) is a separatrix cycle.*

(c) *If $\omega(\gamma)$ (respectively $\alpha(\gamma)$) has no regular points, then $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) is a singular point.*

A proof of this result can be found in the book of Hartman [30, Chapter 7] or in [21]. The following results are corollaries of the Poincaré–Bendixson Theorem, see [21].

Corollary 2.8.3. *Let $\mathbf{f} : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a Lipschitz function in an open set U and let γ be a periodic orbit of the differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. If $\eta, \varrho \subset \Sigma_\gamma$ are orbits* *and* $\omega(\eta) = \gamma$ *(respectively,* $\alpha(\eta) = \gamma$*), then* $\alpha(\varrho) \neq \gamma$ *(respectively,* $\omega(\varrho) \neq \gamma$*).*

Corollary 2.8.4. *Let* $\mathbf{f} : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ *be a Lipschitz function in an open and simply connected set* U *and let* $\gamma \subset U$ *be a periodic orbit of the differential system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. *Then there exists a singular point in* Σ_γ.

2.9 Compactified flows

The aim of this section is to describe the asymptotic behaviour of unbounded orbits, i.e. the behaviour of flows near the infinity.

To do this, we use the so called Poincaré compactification. The French mathematician H. Poincaré was the first to use this technique, in the study of polynomial vector fields. We will only consider some aspects of this technique. More information can be found in [58], [4] and [21].

2.9.1 Poincaré compactification

We define the following sets in $\mathbb{R}^3$

$$\begin{aligned} \mathbb{S}^2 &:= \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\right\}, \\ H_+ &:= \left\{(x,y,z) \in \mathbb{S}^2 : z > 0\right\}, \\ \mathbb{S}^1 &:= \left\{(x,y,z) \in \mathbb{S}^2 : z = 0\right\}, \\ H_- &:= \left\{(x,y,z) \in \mathbb{S}^2 : z < 0\right\}. \end{aligned}$$

$\mathbb{S}^2$ is called the *unit sphere* of $\mathbb{R}^3$, and H_+, $\mathbb{S}^1$ and H_- are called the north hemisphere, the *equator* and the *south hemisphere* of $\mathbb{S}^2$, respectively.

We say that a function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the *Łojasiewicz property at infinity* if there exists a positive integer n such that the function $\mathbf{f}_n$ defined by

$$\mathbf{f}_n(x,y,z) := z^n\, \mathbf{f}\left(\frac{x}{z}, \frac{y}{z}\right) \tag{2.11}$$

can be extended to $z = 0$ and this extension is locally Lipschitz in the whole $\mathbb{S}^2$. Since $\mathbb{S}^2$ is a compact set, if $\mathbf{f}_n$ is locally Lipschitz in $\mathbb{S}^2$, then $\mathbf{f}_n$ is also globally Lipschitz in $\mathbb{S}^2$.

Given a function $\mathbf{f}$, if there exists a non-negative integer n_0 such that the function $\mathbf{f}_{n_0}$ is globally Lipschitz in $\mathbb{S}^2$, then for every $n \geq n_0$ the function $\mathbf{f}_n$ is also globally Lipschitz in $\mathbb{S}^2$. We call the *degree of* $\mathbf{f}$ *at infinity*, and denote it by $n = n(\mathbf{f})$, the least positive integer m such that $\mathbf{f}_m$ is well defined and Lipschitz in $\mathbb{S}^2$.

Lemma 2.9.1. *If the function* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfies the Łojasiewicz property at infinity with degree at infinity equal to* n*, then there exist positive constants* R *and* M*, such that*

$$||\mathbf{f}(\mathbf{x})|| \leq M\, ||\mathbf{x}||^n ,$$

for every $||\mathbf{x}|| > R$.

Proof. Given a point (x, y) in $\mathbb{R}^2$ we consider the point $(\bar{x}, \bar{y}, \bar{z})$ in the north hemisphere H_+, where $\bar{z} = (1 + x^2 + y^2)^{-\frac{1}{2}} > 0$, $\bar{x} = x\bar{z}$, and $\bar{y} = y\bar{z}$. Conversely, for every point $(\bar{x}, \bar{y}, \bar{z}) \in H_+$ the point $(\bar{x}/\bar{z}, \bar{y}/\bar{z})$ belongs to $\mathbb{R}^2$.

By the hypothesis, there exists a positive integer n such that the function $\mathbf{f}_n$ is (globally) Lipschitz in $\mathbb{S}^2$, and consequently $\mathbf{f}_n$ is continuous in $\mathbb{S}^2$. Since the unit sphere is a compact manifold, there exists a positive constant N for which $||\mathbf{f}_n(x, y, z)|| < N$ for every $(x, y, z) \in \mathbb{S}^2$, or, equivalently $||\mathbf{f}_n(x, y, z)|| < N\,||(x, y, z)||^n$. Therefore,

$$|\bar{z}|^n \left|\left|\mathbf{f}\left(\frac{\bar{x}}{\bar{z}}, \frac{\bar{y}}{\bar{z}}\right)\right|\right| < N\ ||(\bar{x}, \bar{y}, \bar{z})||^n .$$

Here $||\cdot||$ denotes the Euclídean norm in $\mathbb{R}^2$ or in $\mathbb{R}^3$, depending on the context.

Dividing by $|\bar{z}|^n$ and returning to the original variables, we obtain $||\mathbf{f}(x, y)|| < N\,||(x, y, 1)||^n$. Taking a positive constant R such that

$$\frac{N}{N+1} < \left(\frac{R}{\sqrt{1+R^2}}\right)^n ,$$

we have $(N + 1)\,||(x, y)||^n > N\,||(x, y, 1)||^n$ for every $||(x, y)|| > R$. The lemma follows by taking $M = N + 1$. □

The inequality in Lemma 2.9.1 justifies the name of the Łojasiewicz property at infinity (see [20] for more information). From this inequality it is also easy to understand the degree of a function at infinity.

The rest of this section is devoted to the compactification of vector fields satisfying the Łojasiewicz property at infinity. We also provide an explicit expression of a flow near infinity and a technique for studying this flow in a neighbourhood of a singular point at infinity.

Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be a local Lipschitz function satisfying the Łojasiewicz property at infinity and let n be the degree of $\mathbf{f}$ at infinity. Consider the diffeomorphisms $\mathbf{h}_+ : \mathbb{R}^2 \to H_+$ and $\mathbf{h}_- : \mathbb{R}^2 \to H_-$ defined by

$$\mathbf{h}_+(x, y) := \frac{1}{\sqrt{1 + x^2 + y^2}}\,(x, y, 1) \quad \text{and} \quad \mathbf{h}_-(x, y) := -\mathbf{h}_+(x, y). \tag{2.12}$$

The functions $\mathbf{h}_+$ and $\mathbf{h}_-$ are the central projections (with center at the origin) of the tangent plane to $\mathbb{S}^2$ at the point $(0, 0, 1)$ onto H_+ and H_-, respectively, see Figure 2.15.

The diffeomorphisms $\mathbf{h}_+$ and $\mathbf{h}_-$ and the vector field $\mathbf{f}$ define two vector fields $\mathbf{f}_+$ and $\mathbf{f}_-$ on the hemispheres H_+ and H_-, respectively, given by

$$\begin{aligned} \mathbf{f}_+(x, y, z) &:= D\mathbf{h}_+\left(\mathbf{h}_+^{-1}(x, y, z)\right)\mathbf{f}\left(\mathbf{h}_+^{-1}(x, y, z)\right), \\ \mathbf{f}_-(x, y, z) &:= D\mathbf{h}_-\left(\mathbf{h}_-^{-1}(x, y, z)\right)\mathbf{f}\left(\mathbf{h}_-^{-1}(x, y, z)\right). \end{aligned} \tag{2.13}$$

Therefore, the rule

$$\widetilde{\mathbf{f}}(x,y,z) := \begin{cases} \mathbf{f}_+(x,y,z), & \text{if } (x,y,z) \in H_+, \\ \mathbf{f}_-(x,y,z), & \text{if } (x,y,z) \in H_-, \end{cases}$$

defines a vector field over $H_+ \cup H_- = \mathbb{S}^2 \setminus \mathbb{S}^1$ which, by (2.12) and (2.13), can be written as

$$\widetilde{\mathbf{f}}(x,y,z) = z \begin{pmatrix} 1-x^2 & -xy \\ -xy & 1-y^2 \\ -xz & -yz \end{pmatrix} \mathbf{f}\left(\frac{x}{z}, \frac{y}{z}\right).$$

In general the vector field $\widetilde{\mathbf{f}}$ cannot be extended to the equator of the sphere. However since $\mathbf{f}$ has degree n at infinity, the vector field $\mathbf{f}_{\mathbb{S}^2}(x,y,z) := z^{n-1}\widetilde{\mathbf{f}}(x,y,z)$ obtained by multiplying by z^{n-1} satisfies

$$\mathbf{f}_{\mathbb{S}^2}(x,y,z) = \begin{pmatrix} 1-x^2 & -xy \\ -xy & 1-y^2 \\ -xz & -yz \end{pmatrix} \mathbf{f}_n(x,y,z). \tag{2.14}$$

Therefore, $\mathbf{f}_{\mathbb{S}^2}$ is defined and Lipschitz on whole $\mathbb{S}^2$. Since $\mathbf{f}_{\mathbb{S}^2}|_{H_+} = z^{n-1}\mathbf{f}_+$ and $\mathbf{f}_{\mathbb{S}^2}|_{H_-} = z^{n-1}\mathbf{f}_-$, the vector field $\mathbf{f}_{\mathbb{S}^2}$ can be understood as an extension to $\mathbb{S}^2$ of the vector field $\widetilde{\mathbf{f}}$ multiplied by the analytic function z^{n-1}. This multiplicative factor is not important in the analysis of the asymptotic behaviour of the flow because it only represents a change in the scale of time. In particular, if we change the variable s to the variable τ by $ds = z^{n-1}d\tau$, the vector field $\mathbf{f}_{\mathbb{S}^2}$ over $\mathbb{S}^2$ can be understood as two copies (each defined on a hemisphere) of the vector field $\mathbf{f}$ defined on $\mathbb{R}^2$. Therefore, the behaviour of $\mathbf{f}$ near infinity follows from the behaviour of $\mathbf{f}_{\mathbb{S}^2}$ in a neighbourhood of the equator. Note that the equator, $z=0$, is invariant under the flow of $\mathbf{f}_{\mathbb{S}^2}$.

For polynomial planar vector fields $\mathbf{f}(x,y) = (P(x,y), Q(x,y))$, with P and Q polynomials, it is easy to prove that $\mathbf{f}$ satisfies Łojasiewicz's property at infinity and $n = \max\{\deg P, \deg Q\}$ is the degree of $\mathbf{f}$ at infinity. Furthermore the vector field $\mathbf{f}_{\mathbb{S}^2}$ is analytic on $\mathbb{S}^2$, see [21] or [58, pp. 57–60] for details.

Consider the *Poincaré disc*, $\mathbb{D} := \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\right\}$, and the so called *gnomonic projection* $\mathbf{p}_+ : H_+ \cup \mathbb{S}^1 \to \mathbb{D}$, given by

$$\mathbf{p}_+(x,y,z) := \frac{1}{1+z}(x,y).$$

The vector field $\mathbf{f}_{\mathbb{S}^2}|_{H_+ \cup \mathbb{S}^1}$ and the diffeomorphism $\mathbf{p}_+$ define a vector field $\mathbf{f}_{\mathbb{D}}$ on $\mathbb{D}$ given by

$$\mathbf{f}_{\mathbb{D}}(x,y) := D\mathbf{p}_+\left(\mathbf{p}_+^{-1}(x,y)\right)\mathbf{f}_{\mathbb{S}^2}\left(\mathbf{p}_+^{-1}(x,y)\right).$$

For a differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f}$ is a locally Lipschitz function in $\mathbb{R}^2$ and satisfies the Łojasiewicz property at infinity with degree n at infinity, we call the differential system $\dot{\mathbf{x}} = \mathbf{f}_{\mathbb{D}}(\mathbf{x})$ *Poincaré's compactification*. The vector fields $\mathbf{f}$ and $\mathbf{f}_{\mathbb{D}}|_{\mathrm{Int}(\mathbb{D})}$ are C^r-equivalent and $\mathbf{h}_{\mathbb{D}} := \mathbf{p}_+ \circ \mathbf{h}_+$ is the equivalence map. Here Int($\mathbb{D}$) denotes the interior of $\mathbb{D}$; that is the biggest open subset contained in $\mathbb{D}$. In this sense we identify the behaviour of $\mathbf{f}_{\mathbb{D}}$ at the boundary $\partial\mathbb{D}$ with the behaviour of $\mathbf{f}$ at infinity.

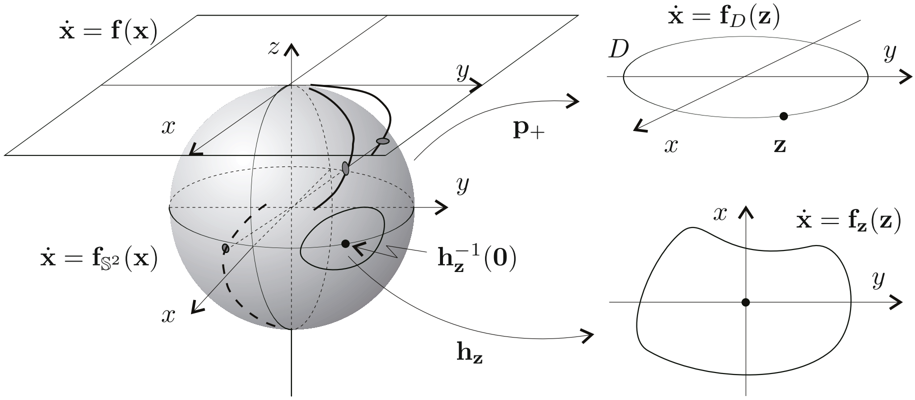


Figure 2.15: Poincaré's compactification.

Finally, for every $\mathbf{x} \in \mathbb{R}^2$ it is easy to prove that

$$\mathbf{h}_{\mathbb{D}}(\mathbf{x}) = \frac{1}{1 + \sqrt{1 + ||\mathbf{x}||^2}}\mathbf{x} \tag{2.15}$$

and

$$\mathbf{f}_{\mathbb{D}}(x, y) = \begin{pmatrix} \frac{1-x^2+y^2}{2} & -xy \\ -xy & \frac{1+x^2-y^2}{2} \end{pmatrix} \mathbf{f}_n \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2} \right). \tag{2.16}$$

2.9.2 The behaviour of a flow at infinity

Since the equator of $\mathbb{S}^2$ is invariant under the flow defined by $\mathbf{f}_{\mathbb{S}^2}$, the boundary of the Poincaré disc $\partial\mathbb{D}$ is invariant under the flow defined by $\mathbf{f}_{\mathbb{D}}$. Then $\partial\mathbb{D}$ is a circle formed by solutions called the *infinity manifold*. A point $\mathbf{p} \in \partial\mathbb{D}$ is said to be a *singular point at infinity* if $\mathbf{f}_{\mathbb{D}}(\mathbf{p}) = \mathbf{0}$. If there are no singular points at infinity, we say that there exists a *periodic orbit at infinity*, or that *the infinity is a periodic orbit*.

Let $\mathbf{p} \in \partial\mathbb{D}$ be a singular point at infinity. As we know, the stable manifold of $\mathbf{p}$, $W^s(\mathbf{p}) \subset \mathbb{D}$, is formed by the orbits γ of the Poincaré compactification satisfying $\mathbf{p} \in \omega(\gamma)$. Consider the subset $\mathbf{h}_{\mathbb{D}}^{-1}(W^s(\mathbf{p}))$ of $\mathbb{R}^2$. For simplicity we call this set the *stable manifold of the singular point at infinity* $\mathbf{p}$ and we also denote it by $W^s(\mathbf{p})$. Note that orbits in $\mathbb{R}^2$ belonging to the stable manifold of a singular point at infinity escape to infinity in forward time. In a similar way we define in $\mathbb{R}^2$ *the unstable manifold of a singular point at infinity* and denote it by $W^u(\mathbf{p})$. Note that orbits in $\mathbb{R}^2$ belonging to the unstable manifold of a singular point at infinity escape to infinity in backward time. In general, we will use the same name for a subset E of $\mathbb{R}^2$ and for the subset $\mathbf{h}_{\mathbb{D}}(E)$ of $\mathrm{Int}(\mathbb{D})$.

When there are no singular points at $\partial\mathbb{D}$, we denote the stable and the unstable manifold of the periodic orbit at infinity by $W^s(\infty)$ and $W^u(\infty)$, respectively.

Let $\mathbf{z} = (x_0, y_0)^T \in \partial\mathbb{D}$ be a singular point at infinity of the differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, that is, a solution of the equation $\mathbf{f}_{\mathbb{D}}|_{\partial\mathbb{D}}(\mathbf{x}) = \mathbf{0}$. To determine the behaviour of the flow in a neighbourhood of $\mathbf{z}$ we use the local chart $(H_{\mathbf{z}}, \mathbf{h}_{\mathbf{z}})$ of $\mathbb{S}^2$, where $H_{\mathbf{z}} = \left\{(x, y, z) \in \mathbb{S}^2 : xx_0 + yy_0 > 0\right\}$ is the hemisphere centered at the point $\mathbf{p}_+^{-1}(\mathbf{z}) = (x_0, y_0, 0)^T$ and

$$\mathbf{h}_{\mathbf{z}}(x, y, z) := \frac{1}{xx_0 + yy_0}(yx_0 - xy_0, z)$$

is the inverse of the central projection (with center at the origin) of the tangent plane to $\mathbb{S}^2$ at the point $(x_0, y_0, 0)^T$. Thus, the vector field $\mathbf{f}_{\mathbb{S}^2}$ and the diffeomorphism $\mathbf{h}_{\mathbf{z}}$ define a locally Lipschitz vector field $\mathbf{f}_{\mathbf{z}} : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\mathbf{f}_{\mathbf{z}}(x, y) := D\mathbf{h}_{\mathbf{z}}\left(\mathbf{h}_{\mathbf{z}}^{-1}(x, y)\right)\mathbf{f}_{\mathbb{S}^2}\left(\mathbf{h}_{\mathbf{z}}^{-1}(x, y)\right).$$

Since $\mathbf{h}_{\mathbf{z}}(x_0, y_0, 0) = \mathbf{0}$, the origin is a singular point of the flow defined by $\mathbf{f}_{\mathbf{z}}$, see Figure 2.15.

The vector fields $\mathbf{f}_{\mathbb{S}^2}$ and $\mathbf{f}_{\mathbf{z}}$ are differentiably conjugate in a neighbourhood of the singular points $\mathbf{p}_+^{-1}(\mathbf{z})$ and $\mathbf{0}$. Therefore, to describe the behaviour of the flow generated by $\mathbf{f}_{\mathbb{D}}$ in a neighbourhood of $\mathbf{z}$ it is sufficient to describe the behaviour of the flow generated by $\mathbf{f}_{\mathbf{z}}$ in a neighbourhood of $\mathbf{0}$ with $y \geq 0$.

We end the section by giving explicit expressions of the vector field $\mathbf{f}_{\mathbf{z}}(x, y)$. From

$$\mathbf{h}_{\mathbf{z}}^{-1}(x, y) = \frac{1}{\sqrt{1 + x^2 + y^2}}(x_0 - xy_0, y_0 + xx_0, y) \tag{2.17}$$

and

$$D\mathbf{h}_{\mathbf{z}}(x, y, z) = \frac{1}{(xx_0 + yy_0)^2}\begin{pmatrix} -y & x & 0 \\ -zx_0 & -zy_0 & xx_0 + yy_0 \end{pmatrix}$$

it follows that

$$\mathbf{f}_{\mathbf{z}}(x, y) = z(x, y)\begin{pmatrix} -y_0 - xx_0 & -xy_0 + x_0 \\ -yx_0 & -yy_0 \end{pmatrix}\mathbf{f}_n\left(\tfrac{x_0 - xy_0}{z(x,y)}, \tfrac{y_0 + xx_0}{z(x,y)}, \tfrac{y}{z(x,y)}\right), \tag{2.18}$$

where $z(x, y) = \sqrt{1 + x^2 + y^2}$.

In particular, for polynomial vector fields $\mathbf{f}(x,y) = (P(x,y), Q(x,y))$, if we take charts centered at the points $\mathbf{z}_x = (1,0)$ and $\mathbf{z}_y = (0,1)$, we obtain

$$\mathbf{f}_{\mathbf{z}_x}(x,y) = y^n m(x,y) \begin{pmatrix} Q\left(\frac{1}{y}, \frac{x}{y}\right) - xP\left(\frac{1}{y}, \frac{x}{y}\right) \\ -yP\left(\frac{1}{y}, \frac{x}{y}\right) \end{pmatrix},$$

$$\mathbf{f}_{\mathbf{z}_y}(x,y) = y^n m(x,y) \begin{pmatrix} -P\left(\frac{-x}{y}, \frac{1}{y}\right) - xQ\left(\frac{-x}{y}, \frac{1}{y}\right) \\ -yQ\left(\frac{-x}{y}, \frac{1}{y}\right) \end{pmatrix},$$

where $m(x,y) = z(x,y)^{1-n}$. These expressions are the usual ones found in the literature, see for instance [58] or [4] or [21]. To obtain the expression of $\mathbf{f}_{\mathbf{z}_y}$ which appears in [58, p. 59] we have to perform the change of variables $(x,y) \to (-x,y)$ which only change the orientation of the base. If we remove $m(x,y)$ from the expressions of $\mathbf{f}_{\mathbf{z}_x}$ and $\mathbf{f}_{\mathbf{z}_y}$ by rescaling the time, these vector fields are polynomial. Note that, in general, $\mathbf{f}_{\mathbb{D}}$ is not C^1.

2.10 Local bifurcations

The qualitative behaviour of a parametric family of differential equations, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda)$, can change by the value of the parameter λ; that is, the qualitative behaviour can change from one topological equivalence class to another. From Theorem 2.6.9, a change of the topological equivalence class implies a change of the separatrix configuration. This change in the separatrix configuration is called a *bifurcation* and the value of the parameter λ in which it takes place is called a *bifurcation value*. In a more general context, the word bifurcation refers not only to other changes in the behaviour of the flow, but also to changes in the topological equivalence class. For details about bifurcation theory see the books of J. Guckenheimer and P. Holmes [29], J. Hale and H. Koçak [31], and S. Chow and J. Hale [17].

In this section we introduce the basic notions of the theory and offer a brief summary of the most usual bifurcations, at least in the context of this book. It is not our purpose to study analytical aspects of bifurcation theory. Here we consider only its geometrical aspects. Some bifurcations described below take place in a neighbourhood of a singular point, hence they are refered as *local bifurcations*.

The set of all bifurcation values in the parameter space is called the *bifurcation set* of the parametric family. When the bifurcation values form a manifold in the parameters space we refer to it as *bifurcation manifold*. The representation in the product space $V \times U$ (where V is the parameter space and U is the phase space) of the invariant sets (singular points, periodic orbits, separatrix cycles, etc. ...) is called *bifurcation diagram*. When the invariant set represented in a bifurcation diagram is a periodic orbit, it is customary to use in the representation the amplitude or the period of the periodic orbit instead of the orbit itself.

2.10.1 Bifurcations from a singular point

Now we describe some of the bifurcations which take place in a neighbourhood of a singular point. We distinguish between uniparametric bifurcations or bifurcations of codimension 1 (saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation and Hopf bifurcation), and biparametric bifurcations or bifurcations of codimension 2 (cusp bifurcation).

For a bifurcation value λ_0 we say that the differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda)$ has a *supercritical saddle-node bifurcation* at the singular point $\mathbf{x}_0$ if

(i) for $\lambda < \lambda_0$, the differential system has no singular points in a neighbourhood U of $\mathbf{x}_0$;

(ii) when $\lambda = \lambda_0$, $\mathbf{x}_0$ is the unique singular point in U and it is a saddle-node;

(iii) for $\lambda > \lambda_0$, the differential system has exactly two singular points at U, one of which is a saddle and the other a node.

In Figure 2.16(a) we represent the bifurcation diagram of the supercritical saddle-node bifurcation. When this bifurcation occurs to the left of the bifurcation value, it is called a *subcritical saddle-node bifurcation*, see Figure 2.16(b).

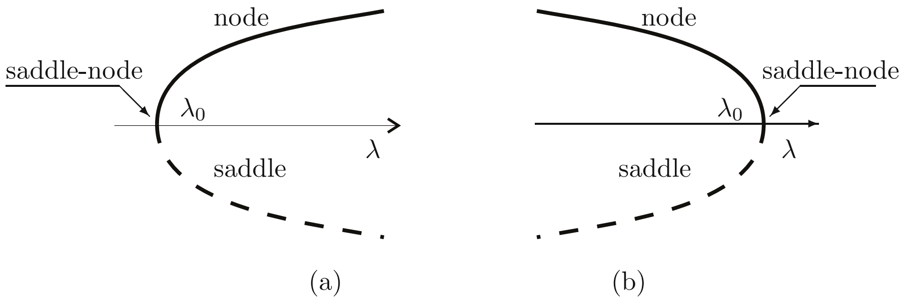


Figure 2.16: Saddle-node bifurcation: (a) supercritical; (b) subcritical.

The differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda)$ is said to have a *transcritical bifurcation* at $\mathbf{x}_0$ for the bifurcation value λ_0 if

(i) for $\lambda < \lambda_0$, there exist exactly two singular points (one stable and one unstable) in a neighbourhood U of $\mathbf{x}_0$;

(ii) for $\lambda = \lambda_0$, the two singular points collapse into one at $\mathbf{x}_0$, which in general is a non-hyperbolic singular point;

(iii) for $\lambda > \lambda_0$, there exist exactly two singular points in U (one stable and one unstable).

In Figure 2.17 we represent the bifurcation diagram of a transcritical bifurcation.

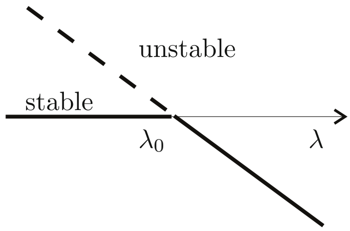


Figure 2.17: Transcritical bifurcation diagram

The differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda)$ is said to have a *supercritical pitchfork bifurcation* at the bifurcation value λ_0 for the singular point $\mathbf{x}_0$ if

(i) for $\lambda < \lambda_0$, there exists exactly one singular point in a neighbourhood U of $\mathbf{x}_0$ and it is a node (respectively, a saddle);

(ii) for $\lambda = \lambda_0$, $\mathbf{x}_0$ is the unique singular point in U;

(iii) for $\lambda > \lambda_0$, there exist exactly three singular points in U. Two of them are nodes (respectively, saddles) and have the same stability as the singular point which exists for $\lambda < \lambda_0$. The other singular point is a saddle (respectively, a node).

When the bifurcation occurs for values of $\lambda < \lambda_0$, it is called a *subcritical pitchfork bifurcation*. In Figure 2.18 we represent the bifurcation diagram of the pitchfork bifurcation. Note that in this bifurcation we can choose different behaviours for the singular points.

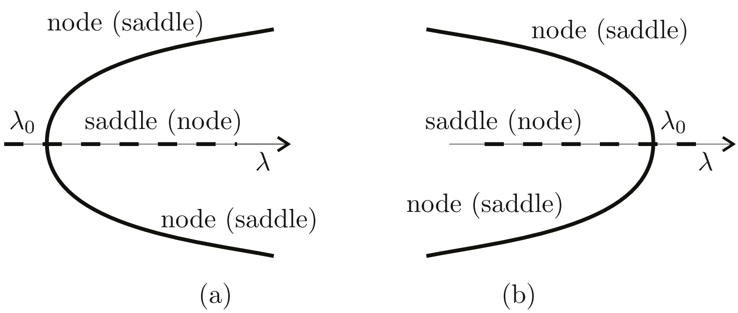


Figure 2.18: Pitchfork bifurcation: (a) supercritical; (b) subcritical. The names in parentheses correspond to the other choice of the singular points.

2.10.2 Bifurcations from orbits

This subsection is devoted to the local bifurcations that involve singular points, periodic orbits and separatrix cycles.

We say that the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\lambda)$ has a *vertical bifurcation* at the singular point $\mathbf{x}_0$ for the bifurcation value λ_0, if

(i) for $\lambda < \lambda_0$, there exists exactly one singular point in a neighbourhood U of $\mathbf{x}_0$;

(ii) for $\lambda = \lambda_0$, $\mathbf{x}_0$ is the unique singular point in U and U is foliated by periodic orbits;

(iii) for $\lambda > \lambda_0$, there exists exactly one singular point in U and it has opposite stability compared with the singular point which appears in (i).

In Figure 2.19(a) we represent the bifurcation diagram of the vertical bifurcation. There the vertical variable corresponds to the amplitude of the periodic orbit.

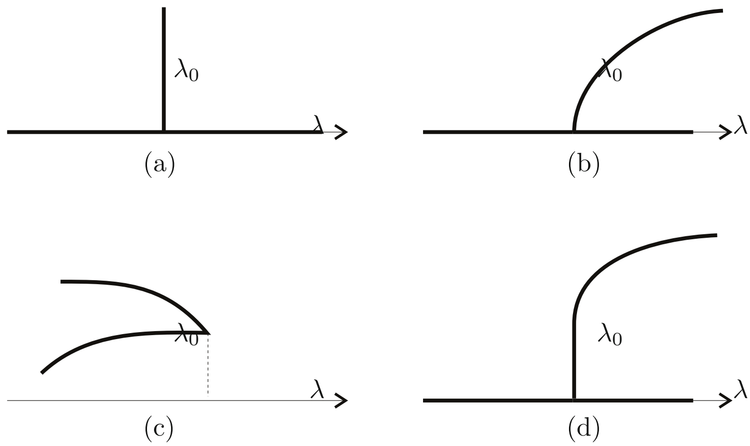


Figure 2.19: Bifurcation diagram involving periodic orbits. The y-axis represent the amplitude of the periodic orbits: (a) vertical bifurcation; (b) Hopf bifurcation; (c) saddle-node bifurcation of limit cycles; and (d) focus-center-limit cycle bifurcation.

The differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda)$ has a *supercritical Hopf bifurcation* at the singular point $\mathbf{x}_0$ for the bifurcation value λ_0, if

(i) for $\lambda < \lambda_0$, there exists exactly one singular point and it is stable (respectively, unstable) in a neighbourhood U of $\mathbf{x}_0$;

(ii) for $\lambda = \lambda_0$, $\mathbf{x}_0$ is the unique singular point in U;

(iii) for $\lambda > \lambda_0$, the system has exactly one singular point $\mathbf{x}_0$ and one limit cycle γ in U. Moreover, the singular point is unstable (respectively stable) the limit cycle is stable (respectively unstable) and the amplitude of γ tends to 0 as λ tends to λ_0.

In Figure 2.19(b) we represent the supercritical Hopf bifurcation diagram. The variable in the vertical axis is the amplitude of the limit cycle γ. When the limit cycle γ appears for $\lambda < \lambda_0$ and disappears for $\lambda > \lambda_0$ we say that it is a *subcritical Hopf bifurcation*.

We say that the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda)$ has a *supercritical saddle-node bifurcation of limit cycles* at λ_0 for the limit cycle γ if

(i) for $\lambda < \lambda_0$, the system has no limit cycles in a neighbourhood U of γ;

(ii) for $\lambda = \lambda_0$, γ is the unique limit cycle in U and it is semistable;

(iii) for $\lambda > \lambda_0$, the system has exactly two limit cycles in U, one stable and the other unstable. Moreover, both limits cycles tend to γ as λ tends to λ_0.

In Figure 2.19(c) we show the supercritical saddle-node bifurcation of limit cycles. When the limit cycles appear for $\lambda < \lambda_0$, we say that a *subcritical saddle-node bifurcation of limit cycles* occurs.

The differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda)$ is said to have a *supercritical focus-center-limit cycle bifurcation* in the periodic orbit γ if

(i) for $\lambda < \lambda_0$, there exists a convex neighbourhood U of γ with exactly one singular point $\mathbf{x}_0$, which is stable (respectively, unstable);

(ii) for $\lambda = \lambda_0$, the singular point $\mathbf{x}_0$ is a local center, with γ in the boundary;

(iii) for $\lambda > \lambda_0$, there exists a unique limit cycle borning at γ and it is stable (respectively, unstable), and there exists exactly one singular point, which is unstable (respectively, stable).

In Figure 2.19(d) we represent the bifurcation diagram of a supercritical Hopf-vertical bifurcation. When the bifurcation takes place for $\lambda < \lambda_0$, it is called *subcritical focus-center-limit cycle bifurcation*.

The differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda)$ is said to have a *homoclinic cycle bifurcation* at point $\mathbf{x}_0$ if

(i) for every $\lambda \neq \lambda_0$, the system has exactly one singular point in a neighbourhood U of $\mathbf{x}_0$ and that point is a saddle;

ii) for $\lambda = \lambda_0$, the system has a saddle point at $\mathbf{x}_0$ and the stable and unstable separatrices of $\mathbf{x}_0$ meet, forming a homoclinic cycle.

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