

## Chapter 2

# Above All with the Greek Alphabet

### 2.1 The Most Ancient of All the Quantitative Physical Laws

I would like to begin with an argument which may be stated most clearly and most forcefully as follows:

Music was one of the primeval mathematical models for natural sciences in the West.

The other model described the movement of the stars in the sky, and a close relationship was postulated between the two: the music of the spheres.

This argument is suggested to us by one of the most ancient events of which trace still remains. It is so ancient that it has become legendary, and has been lost behind the scenes of sands in the desert. A relationship exists between the length of a taut string, which produces sounds when it is plucked and made to vibrate, and the way in which those sounds are perceived by the ear. The relationship was established in a precise mathematical form, that of proportionality, which was destined to dominate the ancient world in general. Given the same tension, thickness and material, the longer the string, the deeper or lower the sound perceived will be; the more it is shortened, the less deep the sound perceived: the length of the string and the depth of the sound are directly proportional. If the former increases, the latter increases as well; if the former decreases, the latter does as well. Or else, the sound could be described as more or less acute, or high. In this case, the length of the string generating it would be described as inversely proportional to the pitch. The shorter the string, the higher the sound produced. None of the special symbols employed in modern manuals were used to express this law, but just common language. If the string is lengthened, the height of the sound is proportionally lowered.

Two thousand years were to pass until the appearance of the formulas to which we are accustomed today. It was only after René Descartes (1596–1650) and subsequently Marin Mersenne (1588–1648), that formulas were composed of the kind

$$v \propto \frac{1}{l}$$

where the height was to be interpreted as the number of vibrations of the string in time, that is to say, the frequency  $\nu$ , and the length was to be measured as  $l$ .

The first volume will accompany us only as far as the threshold of this representation, that is to say, up to the affirmation of a mathematical symbolism increasingly detached from the languages spoken and written by natural philosophers and musicians, and this will be the starting-point for the second volume. Furthermore, it is important to remember that fractions such as  $\frac{1}{7}$  or  $\frac{3}{2}$  were not used in ancient times, but ratios were indicated by means of expressions like '3 to 2', which I will also write as 3:2. The ratio was thus generally fixed by two whole numbers. Whereas a fraction is the number obtained by dividing them, when this is possible.

The same relationship between the length of the string and the height of the sound would appear to have remained stable up to the present day, about 2,500 years later. Is this the only natural mathematical law still considered valid? While others were modified several times with the passing of the years? "... possibly the oldest of all quantitative physical laws", wrote Carl Boyer in his manual on the history of mathematics.<sup>1</sup> That "possibly" can probably be left out.

In Europe, a tradition was created, according to which it was the renowned Pythagoras who was struck by the relationship between the depth of sounds and the dimensions of vibrating bodies, when he went past a smithy where hammers of different sizes were being used. However, the anecdote does not appear to be very reliable, mainly because the above ratio regarded strings.

In any case, the sounds produced by instruments, that is to say, the musical notes perceived by the ear, could now be classified and regulated. How? Strings of varying lengths produced notes of different pitches, with which music could be made. But Pythagoras and his followers sustained that not all notes were appropriate. In order to obtain good music, it was necessary to choose the notes, following a certain criterion. Which criterion? The lengths of the strings must stand in the respective ratios 2:1, 3:2, 4:3. That is to say, a first note was created by a string of a certain length, and then a second note was generated by another string twice as long, thus obtaining a deeper sound of half the height. The two notes gave rise to an interval called *diapason*. Nowadays we would say that if the first note were a *do*, the second one would be another *do*, but deeper, and the interval is called an "octave", and so it is the *do* one octave lower. The same ratio of 2:1 is also valid if we take a string of half the length: a new note twice as high is obtained, that is to say, the *do* one octave higher. But musical notes were to be indicated in this kind of syllabic manner only from Guido D'Arezzo on (early 1000s to about 1050).<sup>2</sup>

The other ratios produced other notes and other intervals. The ratio 3:2 generated the interval of *diapente* (the fifth *do* – *sol*) and 4:3 the *diatessaron* (the fourth *do* – *fa*). Thus the ratios established that what was important for music was not the single isolated sound, but the relationship between the notes. In this way, harmony was born, from the Greek word for 'uniting, connecting, relationship'.

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<sup>1</sup>Boyer 1990, p. 65.

<sup>2</sup>See Sect. 6.2.

At this point, the history became even more interesting, and also relatively well documented, because in the whole of the subsequent evolution of the sciences, controversies were to develop continually regarding two main problems. What notes was the octave to be divided into? Which of the relative intervals were to be considered as consonant, that is to say ‘pleasurable’, and consequently allowed in pieces of music, and which were dissonant? And why? The constant presence of conflicting answers to these questions also allows us to classify sciences immediately against the background of the different cultures: each of them dealt with the problems in its own way, offering different solutions.

Anyway, seeing the surprising success of our original mathematical law model, it was coupled here and there with other regularities that had been identified, and was posited as an explanation for other phenomena. The most famous of these was undoubtedly the movement of the planets and the stars; this gave rise to the so-called music of the heavenly spheres, and connected with this, also the therapeutic use of music in medicine. This original seal, this foundational aporia remained visible for a long time. All, or almost all, of the characters that we are accustomed to considering in the evolution of the mathematical sciences wrote about these problems. Sometimes they made original contributions, other times they repeated, with some personal variations, what they had learnt from tradition. It might be named Pythagorean tradition, so called after the reference to its legendary founder, to whom the original discovery was attributed, or the Platonic or neo-Platonic tradition. This was even to be contrasted with a rival tradition dating back to Aristoxenus. In any case, many scholars felt an obligation to pay homage to tradition in their commentaries, summaries, and sundry quotations, or in their actual theories.

In this second chapter, we shall review the Pythagoreans, and other characters who harked back to their tradition, such as Euclid and Plato, but also significant variations like that of Claudius Ptolemaeus (Ptolemy), or the different conception of Aristoxenus. In Chaps. 6, 8–11, we shall see that the interest in the division of the octave into a certain number of notes, and the interest in explaining consonances passed unscathed, or almost so, through the epochal substitution (revolution?) of the Ptolemaic astronomic system with the Copernican one during the seventeenth century. It might be variously described as musical theory, or acoustics, or as the music of mathematics, or the mathematics of music. All the same, it continued without any interruption in the Europe of Galileo Galilei, Kepler, Descartes, Leibniz, and Newton. It was not completely abandoned, even when, during the eighteenth century, figures like d’Alembert and Euler felt the need to perfect the new symbolic language chosen for the new sciences, and to address them in a general systematic manner.

## 2.2 The Pythagoreans

Pythagoras, . . . constructed his own σοφία [wisdom] πολυμαθία [learning] and κακοτεχνία [art of deception].

Heraclitus.

The mathematical model chosen by the Pythagoreans, with the above-mentioned ratios, selected the notes by means of whole numbers, arranged in a “geometrical” sequence. This means that we pass from one term to the following one (that is to say, from one note to the following one) by *multiplying* by a certain number, which is called the “common ratio” of the sequence. Thus, in the geometrical sequence 1, 2, 4, 8, 16, ... we multiply by the common ratio 2. In “arithmetic” sequences, instead, we proceed by *adding*, as in 1, 2, 3, 4, 5, ... where the common ratio is 1, or in 1, 4, 7, 10, 13, ... where the common ratio is 3. Thus the Pythagoreans had also introduced the “geometrical” or “proportional” mean, with reference to the ratio  $1 : 2 = 2 : 4$ . That is to say, the intermediate term between 1 and 4 in this sequence is obtained by multiplying  $1 \times 4 = 4$  and extracting the square root  $\sqrt{4} = 2$ . The arithmetic mean, on the contrary, is obtained by adding the two numbers and dividing by 2. In other words, in the above arithmetic sequence,  $\frac{1+7}{2} = 4$ .

Lastly, this same kind of music loved by the Pythagoreans also suggested “harmonic” sequences and means. Taking strings whose lengths are arranged in the arithmetic sequence 1, 2, 3, 4, ... notes of a decreasing height are obtained in the harmonic sequence  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ . Consequently, the third mean practised by the Pythagoreans, called the harmonic mean, is obtained by calculating the inverse of the arithmetic mean of the reciprocals.

$$\frac{1}{\frac{1}{2}(2+4)} = \frac{1}{3} \quad \text{or} \quad 2 \frac{\frac{1}{2} \times \frac{1}{4}}{\frac{1}{2} + \frac{1}{4}} = \frac{1}{3}$$

In faraway times, and places steeped in bright Mediterranean sunshine, rather than the pale variety of the Europe of the North Atlantic, the Pythagoreans had thus generally established the arithmetic mean  $a = \frac{b+c}{2}$ , the geometric mean  $a = \sqrt{cb}$  and the harmonic mean  $\frac{1}{a} = \frac{1}{2}(\frac{1}{b} + \frac{1}{c})$ , that is to say,  $a = 2 \frac{b \times c}{b+c}$ . Taking strings whose length is 1, 2, 3 we obtain (if the tension, thickness and material are the same) notes of a decreasing height  $1, \frac{1}{2}, \frac{1}{3}$ , that is to say, the notes that gave unison, the (low) octave, the fifth (which could be transferred to the same octave by dividing the string of length 3 into two parts, thus obtaining  $\frac{2}{3}$ ). The arithmetic sequence (whose common ratio is  $\frac{1}{2}$ )  $1, \frac{3}{2}, 2$  generates the harmonic sequence  $1, \frac{2}{3}, \frac{1}{2}$ . On these bases, the mystic sects that harked back to that character of Magna Graecia (the present-day southern Italy) called Pythagoras, divided the single string of a theoretical musical instrument called the monochord. They believed that the only consonances (symphonies) were unison, the octave, the fifth and the fourth, because they were generated by the ratios 1:1, 2:1, 3:2, 4:3. For them, the fact that music made use of the first four whole numbers, and furthermore that added together, these made  $10 = 1 + 2 + 3 + 4$ , the *tetraktys*, acquired a profound significance. It seemed to be the best proof that everything in the world was regulated by whole numbers and their derivatives.

Games with whole numbers and means were very popular. The preferences for notes became 6, 8, 9, 12. These include the octave 12:6, the fifth 9:6, the fourth 8:6

and the tone 9:8.<sup>3</sup> Furthermore, 9 is the arithmetic mean between 6 and 12, while 8 is the harmonic mean.

$$6 : 9 = 8 : 12$$

In general

$$b : \frac{b+c}{2} = 2 \frac{b \times c}{b+c} : c$$

In other words, the ratio between  $b$ , the arithmetic mean and  $c$  is completed by the harmonic mean.

The points of the *tetraktys* were distributed in a triangle, while 4, 9, and 16 points assumed a square shape. Geometry was invaded by numbers, which were also given symbolic values: odd numbers acquired male values, and even ones female;  $5 = 3 + 2$  represented marriage. And so on.

If it had depended on historical coincidences or on the rules of secrecy practised by initiated members of the Italic sect, then no text written directly by Pythagoras (Samos c. 560–Metaponto c. 480 B.C.) could have been made available to anybody. It is said that only two groups of adepts could gain knowledge of the mysteries: the *akousmatikoi*, who were sworn to silence, and to remembering the words of the master, and the *mathematikoi*, who could ask questions and express their own opinions only after a long period of apprenticeship.

But in time, others (the most famous of whom was Plato) were to leave written traces, on which the narration of our history is based.

Thanks to the ratios chosen for the octave, the fifth and the fourth, the Pythagorean sects rapidly succeeded in calculating the interval of one tone *fa*–*sol*: the difference between the fifth *do*–*sol* and the fourth *do*–*fa*. In the geometric sequence at the basis of the notes, adding two intervals means compounding the relative ratios in the multiplication, whereas subtracting two intervals means compounding the appropriate ratios in the division. Consequently, the Pythagorean ratio for the tone became

$$(3 : 2) : (4 : 3) = 9 : 8.<sup>4</sup>$$

At this point, all the treatises on music dedicated their attention to the question whether it was possible to divide the tone into two equal parts (semitones). The Pythagorean tradition denied it, but the followers of Aristoxenus readily admitted it. Why? Dividing the Pythagorean tone into two equal parts would have meant

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<sup>3</sup>See below.

<sup>4</sup>Even if he is guilty of anachronism, in order to arrive more rapidly at the result, the reader inured by schooling to fractions will easily be able to calculate  $\frac{3}{2} : \frac{4}{3} = \frac{9}{8}$ . However, the use of fractions in music had to await the age of John Wallis (1617–1703), Part II, Sect. 9.2. After all, the Greeks used the letters of their alphabet  $\alpha, \beta, \gamma \dots$  to indicate numbers  $\dots$

admitting the existence of the geometric mean, a ratio between 9 and 8, that is to say,  $9 : \alpha = \alpha : 8$ , where  $9 : \alpha$  and  $\alpha : 8$  are the proportions of the desired semitone. What would the value of  $\alpha$  be, then? Clearly  $\alpha = \sqrt{9 \cdot 8}$ , and therefore  $\alpha = 3.2 \cdot \sqrt{2}$ ! Thus the most celebrated controversy of ancient Greek mathematics, the representation of incommensurable magnitudes by means of numbers, which nowadays are called irrational, acquired a fine musical tone.

The problem is particularly well known, and is discussed in current history books, though it is narrated differently. What is the value of the ratio between the diagonal of a square and its side? In the relative diagram, the diagonal must undoubtedly have a length.

But if we measure it using the side as the natural meter, what do we obtain? In this case, in the end the ratio between the side and the diagonal was called “incommensurable”, for the following reason. If we reproduce the side AB on the diagonal, we obtain the point P, from which a new isosceles triangle PQC is constructed (isosceles because the angle  $\widehat{PCQ}$  has to be equal to  $\widehat{PQC}$ , just as it is equal to  $\widehat{CAB}$ ). By repeating the operation of reproducing QP on the diagonal QC, we determine a new point R, with which the third isosceles right-angled triangle CRS is constructed. And so on, with endless constructions. In other words, this means that it is impossible to establish a part of the side, however small it may be, which can be contained a precise number of times in the diagonal, however large this may be. There is always a little bit left over. The procedure never comes to an end; nowadays we would say that it is infinite.

And yet the problem would appear to be easy to solve, if we use numbers. Because if we assign the conventional length 1 to AB, then by the so-called (in Europe) theorem of Pythagoras (him again!), the diagonal measures  $\sqrt{1 + 1} = \sqrt{2}$ . It would be sufficient, then, to calculate the square root. But, as before, the calculation never comes to an end, producing a series of different figures after the decimal point: 1,414213.... Convinced that they could dominate the world by means of whole numbers, just as they regulated music by means of ratios, the Pythagoreans had hoped to do the same also with the diagonal of the square and  $\sqrt{2}$ . But no whole numbers exist that correspond to the ratio between the diagonal and the side of a square, or which can express  $\sqrt{2}$ , in the same way as we use 10:3. Also the division of 10 by 3 never comes to an end (though it is periodic); however, it can be indicated by two whole numbers, each of which can be measured by 1. Accordingly, the Pythagoreans sustained that  $\sqrt{2}$  was to be set aside, and could not be considered or used like other numbers. Therefore the tone could not be divided into two equal parts. They even produced a logical-arithmetic proof of this diversity.

On the contrary, let us suppose for the sake of argument that  $\sqrt{2}$  can be expressed as a ratio between two whole numbers,  $p$  and  $q$ . Let us start by eliminating, if necessary, the common factors; for example, if they were both even numbers, they could be divided by 2. As

$$\frac{p}{q} = \sqrt{2}, \text{ -- then -- }, p^2 = 2q^2$$

Consequently,  $p^2$  must be an even number, and also  $p$  must be even. It follows that  $q$  must be an odd number, because we have already excluded common factors. But if  $p$  is even, then we can rewrite it as  $p = 2r$ . Introducing this substitution into the hypothetical starting equation, we now obtain  $4r^2 = 2q^2$ , from which  $q^2 = 2r^2$ . In the end, the conclusion that can likewise be derived from the initial hypothesis is that  $q$  should be also even. But how can a number be even and odd at the same time? Is it not true that numbers can be classified in two completely separate classes? It would therefore seem to be inevitable to conclude that the starting hypothesis is not tenable, and that  $\sqrt{2}$  cannot be expressed as a ratio between two whole numbers. Here we come up against the dualism which is a general characteristic, as we shall see, of European sciences.

Maybe it was again due to secrecy, or to the loss of reliable direct sources, but even this question of incommensurability remains shrouded in darkness, as regards its protagonists. Various somewhat inconsistent legends developed, fraught with doubts, and narrated only centuries later, by commentators who were interested either in defending or in denigrating them. Hippasus of Metaponto (who lived on the Ionian coast of Calabria around 450 B.C.) is said to have played a role in identifying the most serious flaw in Pythagoras' construction, and is believed to have been condemned to death for his betrayal, perishing in a shipwreck.<sup>5</sup> A coincidence? The wrath of Poseidon? The revenge of the Pythagorean sect? This was a religious-mathematical murder that deserves to be recorded in the history of sciences, just as Abel is remembered in the *Bible*.

The fundamental property of right-angled triangles, known to everybody and used in the preceding argument, was attributed to the founder of the sect, and from that time on, everywhere, was to be called the theorem of Pythagoras. But this appears to be merely a convention, linked with a tradition whose origins are unknown. The same tradition could sustain, at the same time, that the members of the sect were to follow a vegetarian diet, but also that their master sacrificed a bull to the gods, to celebrate his theorem. And yet he can, at most, have exploited this property of right-angles triangles, like other cultures, e.g. the Mesopotamian one, because he did not leave any proof of it. The earliest proofs are to be found in Euclid.

We are relating the origins of European sciences among the ups and downs and ambiguities of an early conception, sustained by people who lived in the cultural and political context of Magna Graecia. How did they succeed in surviving (apart from Hippasus, the apostate!) and in imposing themselves, and influencing characters who were far better substantiated than them, like Euclid and Plato? Did they do so only on the basis of the strength of their arguments, or did they gain an advantage over their rivals by other means? Because, of course, the Pythagorean theory was not the only one possible, and it had its adversaries.

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<sup>5</sup>*Pitagorici* 1958 and 1962. Boyer 1990, pp. 85–87. Cf. Centrone 1996, p. 84. The Pythagoreans are to be considered as adepts of a religious sect governed by prohibitions and rules, somewhat different from the mathematical community of today, which has other customs.

That a Pythagorean like Archytas lived at Tarentum (fifth century B.C.), becoming tyrant of the city, may perhaps have favoured to some extent the acceptance and the spread of Pythagoreanism? We are inclined to think so. The sect's insistence on numbers, means and music is finally found explicitly in his writings. This Greek offered a first general proof that the tone 9:8 could not be divided into two equal parts, by demonstrating that no geometric mean could exist for the ratio  $n + 1 : n$ . He gave rise to an organisation of culture which was to dominate Europe for the following 2,000 years. The subjects to study were divided into a "*quadrivium*" including arithmetic, geometry, music and astronomy, and a "*trivium*" for grammar, rhetoric and dialectics.

Archytas commanded the army at Tarentum for years, and he is said to have never been defeated. He also designed machines. He is a good example of the contradiction at the basis of European sciences. On the one hand, the harmony of music, and on the other, the art of warfare.<sup>6</sup> How could he expect to sustain them both at the same time, particularly with reference to the education of young people? It is true that in the Greek myths, Harmony is the daughter of Venus and Mars, that is to say, of beauty and war: we shall return to the subject of myths, not to be underestimated, in Plato.

On the other side of the peninsula, on the Tyrrhenian coast, lived Zeno of Elea (Elea 495–430 B.C.): he was not a Pythagorean, but rather drew his inspiration from Parmenides, (Elea c. 520–450 B.C.), the renowned philosopher of a single eternal, unmoved "being". Zeno's paradoxes are famous. How can an arrow reach the target? It must first cover half the distance, then half of the remaining space, and then, again, half of half of half, and so on. The arrow will have to pass through so many points (today we would define them as infinite) that it will never arrive at the target, Zeno concluded. The school of Parmenides taught that movement was an illusion of the senses, and that only thought had any real existence, since it is immune to change. "... the unseeing eye and the echoing hearing and the tongue, but examine and decide the highly debated question only with your thought ...".<sup>7</sup> Zeno's ideal darts were directed not only against the Heraclitus (Ephesus 540–480 B.C.) of "everything passes, everything is in a state of flux", but also against the Pythagoreans, his erstwhile friends, and now the enemies of his master.

Could our world, continually moving and changing, be dominated and regulated by tracing it back to elements which were, on the contrary, stable and sure, because they were believed to be eternal and unchanging? The Pythagoreans were convinced that they could do it by means of numbers; the Eleatics tried to prove by means of paradoxes that this was not possible in the Pythagorean style. Let us translate the paradox of the arrow into the numbers so dearly loved by the Pythagoreans. Let us thus assign the measure of 1 to the space that the arrow must cover. It has covered half,  $\frac{1}{2}$ , then half of half,  $\frac{1}{4}$ , then half of half of half  $\frac{1}{8}$ , and so on,  $\frac{1}{16}$ ,  $\frac{1}{32}$  ...

<sup>6</sup>*Pitagorici* 1958 and 1962. The adjective "harmonic" used for the relative mean, previously called "sub-contrary", is attributed to him.

<sup>7</sup>Thomson 1973, p. 299.



The single terms were acceptable to the Pythagoreans as ratios between whole numbers, but they shied away from giving a meaning to the sum of all those numbers which could not even be written completely; today we would define them as infinite. After all, what other result could have been obtained from a similar operation of adding more and more quantities, if not an increasingly big number? Two thousand years were to pass, with many changes, until a way out of the paradox was found in a style that partly saved, but also partly modified the Pythagorean programme. Today mathematicians say that the sum of infinite terms (a sequence) like  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \dots$  gives as a result (converges to) 1. Thus the arrow moves, and reaches the target, even if we reduce the movement to numbers, but these numbers can no longer be the Pythagoreans' whole numbers; they must include also 'irrationals'.

Anyway, the members of the sect had encountered another serious obstacle to their programme. If whole numbers forced them to imagine an ideal world where space and time were reduced to sequences of numbers or isolated points, then the real world would seem to escape from their hands, because they would not be able to conceive of a procedure to put them together.

There were also some, like Diogenes (of Sinope, the Cynic, 413–327 B.C.), who scoffed at the problem, and proved the existence of movement, simply by walking. Heraclitus started, rather, from the direct observation of a world in continuous transformation; and adopting an opposite approach also to that of the Eleatics, he ignored all the claims of the Pythagoreans, who were often the object of his attacks. "They do not see that [Apollo, the god of the cithara] is in accord with himself even when he is discordant: there is a harmony of contrasting tensions, as in the bow and the lyre." The Pythagoreans combined everything together with their numerical means, whereas among all the things, Heraclitus exalted tension and strife. "Polemos [conflict, warfare] is of all things father and king; it reveals that some are gods, and others men; it makes some slaves, and sets others free." The λόγος logos [discourse, reason] of Heraclitus developed in a completely different way from that of the Pythagoreans. "What can be seen, heard, learnt: that is what I appreciate most."<sup>8</sup>

In the contrasts between the different philosophers, we see the emergence, right from the beginning, of some of the problems for mathematical sciences which are to remain the most important and recurring ones in the course of their evolution. What relationship existed between the everyday world and the creation of numbers with arithmetic, and of points or lines with geometry? By measuring a magnitude in geometry, we always obtain a number? But do numbers represent these magnitudes appropriately?

The whole numbers of the Pythagoreans, or the points of their illustrated models, are represented as separate from each other. We can fit in other intermediate numbers between them,  $\frac{3}{2}$  between 1 and 2, for example, but even if it diminishes, a gap still remains. Thus numerical quantities are said to be "discontinuous" or "discrete". If, on the contrary, we take a line, we can divide it once, twice, thrice, ... as many

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<sup>8</sup>Thomson 1973, pp. 278, 281.

times as you like, obtaining shorter pieces of lines, which, however, can still be further divided. The idea that the operation could be repeated indefinitely was called divisibility beyond every limit. This indicated that the magnitudes of geometry were “continuous”, as opposed to the arithmetic ones, which were “discrete”. And yet there were some who thought that they could find even here something indivisible, that is to say, an atom: the point. Thus in the quadripartite classification of Archytas, music began to take its place alongside arithmetic, seeing that its “discrete” notes appeared to represent its origin and its confirmation in applications. In the meantime, astronomy/astrology displayed its “continuous” movements of the stars by the side of geometry.

So was the everyday world considered to be composed of discrete or continuous elements? Clearly, Zeno’s paradoxes indicated that the supporters of discrete ultimate elements had not found any satisfactory way of reconstructing a continuous movement with them. Could they get away with it simply by accusing those who had not been initiated into their secret activities of allowing their senses to deceive them? Why should numbers, or the only indivisible being, lie at the basis of everything?

Those who, on the contrary, trusted their sight or hearing, and used them for the direct observation of the continuous fabric (the so-called *continuum*) of the world might think that both the Pythagorean numerical models and the paradoxes of the Eleatics were inadequate for this purpose. The process of reasoning needed to be reversed. As the arrow reaches the target, the sum of the innumerable numbers must be equal to 1. But this would have required the construction of a mathematics valued as part of the everyday world, not independent from it. On the contrary, the most representative Greek characters variously inspired by Pythagoreanism generally chose otherwise. Their best model appears to be Plato.

We have already demonstrated above that the discussion about the *continuum*, whether numerical or geometrical, had planted its roots deep down into the field of music, in the division (or otherwise) of the Pythagorean tone into two equal parts. The numerical model of the *continuum* contains a lot of other numbers, besides whole numbers and their (rational) ratios. It does not discriminate those like  $\sqrt{2}$ , which are not taken into consideration by the Pythagoreans, seeing that they do not possess any ratio (between whole numbers), and are thus devoid of *their logos*. Others preferred to seek answers in the practical activity of the everyday world, and thus directly on musical instruments as played by musicians, rather than in the abstract realm of numbers (and soon afterwards, that of Plato’s ideas). They had no doubt that it was possible to put their finger on the string exactly at the point which corresponded to the division into two equal semitones. This string thus became the musical model of the *continuum*. We shall deal below with Aristoxenus, who was their leading exponent.

Here began a history of conflict which was to continue to evolve constantly, without ever arriving at a definite solution. It is also one of the main characteristics of European sciences compared with other cultures, which, as we shall see, represented the question in very different ways.

I have found only one book on the history of mathematics<sup>9</sup> which proposes an exercise of dividing the octave into two equal parts and discussing what the Pythagoreans would have thought of the idea.

## 2.3 Plato

... if poets do not observe them in their invention,  
this must not be allowed.

Plato

The Plato(on) of the firing squad.

Carlo Mazzacurati

Socrates (469–399 B.C.) showed only a marginal interest in the problems of mathematical sciences, with perhaps one interesting exception which we shall see. However, he was not fond of Pythagoreanism. His disciple Plato (Athens 427–Athens 347 B.C.), on the contrary, became its leading exponent. During his travels, the famous philosopher met Archytas, and was deeply influenced by him. Plato was even saved by him when he risked his life at the hands of Dionysius, the tyrant of Siracusa. Thus we again meet up with numbers, means and music in this philosopher, as already presented by the Pythagoreans.

The most reliable text, that believers in the music of the heavenly spheres could quote, now became Plato's *Timaeus*, with the subsequent (much later) commentaries of Proclus (Byzantium 410–Athens 485), Macrobius (North Africa, fifth century) and others. According to the Greek philosopher, when the demiurge arranged the universe in a cosmos, he chose rational thought, rejecting irrational impressions. Consequently, the model was not visible, or tangible; it did not possess a sensible body, but was on the contrary eternal, always identical to itself. Linked together by ratios, the cosmos assumed a spherical shape and circular movements. The heavens thus possessed a visible body and a soul that was “invisible but a participant in reason and harmony”.

Given the dualism between these two terms, the heavens were divided in accordance with the rules of arithmetic ratios, into intervals (like the monochord), bending them into perfect circles. The heavens thus became “a mobile image of eternity . . . , an image that proceeds in accordance with the law of numbers, which we have called time”. “And the harmony which presents movements similar to the orbits of our soul, . . . , is not useful, . . . , for some irrational pleasure, but has been

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<sup>9</sup>Cooke 1997. Although Centrone 1996 is a good essay on the Pythagoreans, he too, unfortunately, underestimates music: he does not make any distinction between their concept of music and that of Aristoxenus. This limitation derives partly from the scanty consideration that he gives to the Aristotelian *continuum* as an essential element, by contrast, to understand the Pythagoreans. Without this, he is left with many doubts, pp. 69, 196 and 115–117. Cf. von Fritz 1940. *Pitagorici* 1958, 1962, and 1964.

given to us by the Muses as our ally, to lead the orbits of our soul, which have become discordant, back to order and harmony with themselves.”

Lastly (on earth) sounds, which could be acute or deep, irregular and without harmony or regular and harmonic, procured “pleasure for fools and serenity for intelligent men, thanks to the reproduction of divine harmony in mortal movements.”<sup>10</sup> Thus for him, the harmony of the cosmos was modelled on the same ratios as musical harmony and the influence of the moving planets on the soul was justified by the similar effects due to sounds.

Together with the ratios for the fifth, 3:2, the fourth, 4:3, and the tone, 9:8, already seen, Plato also indicated that of 256:243 for the “diesis”. This is calculated by subtracting the ditone *do – mi*, 81:64, from the fourth, *do – fa*, that is to say,  $(4:3):(81:64) = 256:243$ . The Pythagorean “sharp” does *not* divide the tone into two equal parts, but it leaves a larger portion, called “apotome”.<sup>11</sup> He even allowed himself a description of the sound. “Let us suppose that the sound spreads like a shock through the ears as far as the soul, thanks to the action of the air, the brain and the blood . . . if the movement is swift, the sound is acute; if it is slower, the sound is deeper . . .”<sup>12</sup>

The classification of the elements according to regular polyhedra is famous in the *Timaeus*. A late commentator like Proclus attributed to the Pythagoreans the ability to construct these five solids, known from then on also as Platonic solids. They are: the tetrahedron made up of four equilateral triangles, the hexahedron, or cube, with six squares, the octahedron with eight equilateral triangles, the dodecahedron with 12 regular pentagons, and the icosahedron with 20 equilateral triangles.

A regular dodecahedron found by archaeologists goes back to the time of the Etruscans, in the first half of the first millennium B.C.<sup>13</sup> In reality, leaving aside the Pythagorean sects and the Platonic schools, which presumed to confine mathematical sciences within their ideal worlds, we find hand-made products, artefacts, monuments, temples, statues, paintings and vases, which undoubtedly testify to far more ancient abilities to construct in the real world what those philosophers then tried to classify and regulate.

On a plane, it is possible to construct regular polygons with any number of sides. But in space, the only regular convex solids with faces of regular polygons are these five. Why? The explanations that have been given are, from this moment on, a part of the history of European sciences. They are an excellent example of how the proofs of mathematical results changed in time and in space, coming to depend on cultural elements like criteria of rigour, importance and pertinence. In other words, with the evolution of history, different answers were given to the questions: when is a proof convincing and when is it rigorous? How important is the theorem? Why does this property provide a fitting answer to the problem?

<sup>10</sup>Plato 1994, pp. 25–27, 31–33, 61, 129–131.

<sup>11</sup>Plato 1994, p. 37.

<sup>12</sup>Plato 1994, p. 103.

<sup>13</sup>Heath 1963, p. 107.

Plato's arguments were based on a breakdown of the figures into triangles and their recombination. He also posited solids which corresponded to the four elements: fire with the tetrahedron, air with the octahedron, earth with the cube, and water with the icosahedron. He justified these combinations by reference to their relative stability: the cube and earth are more stable than the others. The fifth solid, the dodecahedron, represents the whole universe. Over the centuries, Plato's processes of reasoning lost credibility and the mathematical proofs modified their standards of rigour. Analogy became increasingly questionable and weak.

Regular polyhedra were studied by Euclid, Luca Pacioli and Kepler, among others. In one period, these solids were considered important because, with their perfection, they express the harmony of the cosmos. In another, they spoke of a transcendent god who was thought to have created the world, and to have added the signature of his "divine ratio".<sup>14</sup> At the time, this was considered to be necessary for the construction of the pentagon and the dodecahedron: "ineffable", because irrational, and also called "of the mean and the two extremes", or the "golden section".<sup>15</sup> For some, the field of reasoning was to be limited to Euclidean geometry, because the rest would not be germane to the desired solution. Subsequently, however, Euclid's incomplete argument was concluded by the arrival of algebra and group theory. I personally am attached to the relatively simple version offered last century by Hermann Weyl (1885–1955).<sup>16</sup>

In the *Meno*, Plato described Socrates teaching a boy-slave. He led him to recognize, by himself, that twice the area of the square constructed on a given line is obtained by constructing a new square on the diagonal of the first one.

We can interpret the reasoning of Socrates-Plato as an argument equivalent to the theorem of Pythagoras in the case of isosceles right-angled triangles. The first square is made up of two such triangles; the square on the hypotenuse contains four.<sup>17</sup>

The importance of Plato for our history derives from the role that was assigned to mathematical sciences and to music in his philosophy and in Athenian society. He enlarged on what he had learnt from the Pythagorean Archytas, to the point that his voice continues to be heard through the millennia up to today, marking out the evolution of the sciences. The motto, traditionally attributed to him, over the door of his school, the Academy, is famous: let nobody enter who does not know geometry. The fresco by Raffaello Sanzio "Causarum cognitio [knowledge of causes]", in the Vatican in Rome, is also famous; in this painting, together with Plato with his *Timaeus*, indicating the sky, and Aristotle with his *Ethics*, we can find allegories of geometry, astronomy and music.

In his *Politeia [Republic]*, Plato wrote that he wanted to educate the soul with music, just as gymnastics is useful for the body. He was discussing how to prepare the group of people responsible for safeguarding the state by means of warfare,

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<sup>14</sup>Pacioli 1509. See Sect. 6.4.

<sup>15</sup>In the pentagon, the diagonals intersect each other in this ratio.

<sup>16</sup>Weyl 1962.

<sup>17</sup>Heath 1963, p. 178. Fowler 1987, pp. 3–7.

both on the domestic front and abroad. Above all, he criticised poets, who, with their fables about the realm of the dead “do not help future warriors”; the latter risk becoming “emotionally sensitive and feeble”. Laments for the dead are things for “silly women and cowardly men”.<sup>18</sup>

Plato preferred other means to educate soldiers. Music could be useful, provided that languid, limp harmonies like the Lydian mode were eliminated, and the Dorian and Phrygian modes were used, instead. “. . . this will appropriately imitate the words and tones of those who demonstrate courage in war or in any act of violence . . . of those who attend to a pacific, non-violent, but spontaneous action, or intends to persuade or to make a request . . .”. For this reason, the State organisation would not need instruments with several strings, capable of many harmonies [or, even less so, of passing from one to another, that is to say, modulating], and would limit itself to the *lyre*, excluding above all the lascivious breathiness of the *aulos*. Plato made similar comments about the rhythm. “Because the rhythm and the harmony penetrate deeply into the soul, and touch it quite strongly, giving it a harmonious beauty.” Excluding all pleasure and every amorous folly, “the ultimate aim of music is love of beauty”, the philosopher concluded. For the warriors of this state described by Plato, variety in foods for the body was as little recommended as variety in music. “. . . the one who best combines gymnastics and music, and applies them in the most correct measure to the soul, is the most perfect and harmonious musician, much more than the one who tunes strings together.”<sup>19</sup>

In his famous metaphor of the cave, the Greek philosopher explained that with our senses, we can only grasp the shadows of things. We should break the chains, in order to succeed in understanding the true essence and reality, which for him lay in the realm of the ideas. “. . . We must compare the world that can be perceived by sight with the dwelling-place of the prison [the cave where we are imagined to be chained to the wall] . . . the ascent and the contemplation of the world above are equivalent to the elevation of the soul to the intelligible world . . .”.

Thus Plato now presented the discipline that elevated from the “world of generation to the world of being . . .”, and which was suitable to educate young people, who had occupied his attention since the beginning of the book. “Not being useless for soldiers”, then. However, this could not mean gymnastics, which deals “with what is born and dies”, that is to say, the ephemeral body. Nor was it music, which “procured, by means of harmony, a certain harmoniousness, but not science, and with rhythm eurhythm”. It was, instead, the “science of number and of calculation. Is it not true that every art and science must make use of it? . . . And also, maybe, . . ., the art of warfare?” After mocking Homer’s Agamemnon because he did not know how to perform calculations, Socrates-Plato concluded. “And therefore, . . ., should we add to the disciplines that are necessary for a soldier that of being able to calculate and count? Yes, more than anything else, . . ., if he is

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<sup>18</sup>Plato 1999, pp. 117, 119, 125, 145, 149.

<sup>19</sup>Plato 1999, pp. 179, 181, 209, 187, 191, 195, 211.

to understand something about military organizations, or rather, even if he is to be simply a man.”<sup>20</sup>

Calculation and arithmetic are “fit to guide to the truth” because they are capable of stimulating the intellect in cases where it is necessary to discriminate between opposites. According to Plato, here “sensation does not offer valid conclusions”. Thus, “we have distinguished between the intelligible and the visible”. I will return at the end of this chapter to the hallmark of dualism thus impressed by this Greek culture.

“A military man must needs learn them in order to range his troops; and a philosopher because, leaving the world of generation, he must reach the world of being . . .”. Thus he went so far as to impose mathematics by law, in order to be able to “contemplate the nature of numbers”. Not for trading, “but for reasons of war, and to help the soul itself . . . to arrive at the truth of being”, “. . . always rejecting those who reason by presenting it [the soul] with numbers that refer to visible or tangible bodies.” Even if they discussed of visible figures, geometricians would think of the ideal models of which they are copies, “they speak of the square in itself and of the diagonal in itself, but not of the one that they trace . . .”.<sup>21</sup>

Even geometry has an “application in war”. But the philosopher criticised practical geometricians: “They speak of ‘squaring’, of ‘constructing on a given line’ . . .”. Instead, “Geometry is knowledge of what perennially exists.” Even astronomy is presented as useful to generals.<sup>22</sup>

Having rendered homage to the Pythagoreans for uniting astronomy and harmony, Plato criticised those who dealt with music using their ears. “. . . talking about certain acoustic frequencies [vibrations?] and pricking up their ears as if to catch their neighbour’s voice, some claim that they perceive another note in the middle, and define that as the smallest interval that can be used for measuring . . . both the ones and the others give preference to the ears over the mind . . . they maltreat and torture the strings, stretching them over the tuning pegs . . .”.<sup>23</sup>

Still more discourses, that Plato put into the mouth of Socrates, regard subjects that belong to the history of Western sciences. These will be found in numerous books of every kind and of all ages, as sustained by a wide variety of people: philosophers, scientists, educators, historians, professors, professionals and dilettantes. They end up by forming a kind of orthodoxy, which subsequently easily becomes a commonplace, a degraded scientific divulgation, a general mass of nonsense which is particularly suitable to create convenient caricatures, a celebratory advertisement for the disciplines.

Thus we find expressed here the distinction between sciences and opinions, beliefs. “. . . opinion has as its object generation, intellection has being.” Sciences eliminate hypotheses and bring us closer to principles. To understand ideas, these

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<sup>20</sup>Plato 1999, pp. 457, 467, 469, 471.

<sup>21</sup>Plato 1999, p. 447.

<sup>22</sup>Plato 1999, pp. 471, 475, 477, 479, 481, 483.

<sup>23</sup>Plato 1999, pp. 491, 493.

should be isolated from all the rest, and “if by chance he glimpses an image of it, he glimpses with his opinion, not with science . . .”. Young people need to be educated to this, because those responsible for the State cannot be allowed to be extraneous to reason, like irrational lines.”<sup>24</sup>

The discourse undoubtedly has a certain logic, but it is not without clear contradictions. Education, in the State of the warrior-philosophers, would be imposed by law; and yet it was also noted that “no discipline imposed by force can remain lasting in the soul.” [Luckily for us!] Plato often used to repeat when he spoke of young people: “may they be firm in their studies and in war” . . . “. . . assuming the military command and all the public offices . . .” Therefore he was thinking of a State projected for warfare: the defeat suffered by Athens in 404 B.C. in the Peloponnesian War against Sparta weighed like a millstone on the text. It even assumed tones which may, at least for some of us, have hopefully become intolerable: “. . . we said that young children had to be taken to war, as well, on horseback, so that they could observe it, and if there was no danger [how good-hearted of him!], they were to be taken closer, so that they could taste the blood, like little dogs.”<sup>25</sup>

Our none-too-peaceable philosopher seemed to be less worried about armed violence than keeping young people away from pleasure: “habits that produce pleasure, which flatter our soul and attract it to themselves, but which do not persuade people who in all cases are sober”. Young men are to be educated to temperance, and to “remain subject to their rulers, and themselves govern the pleasures of drinking, of eating and of love.”<sup>26</sup> How unsuitable for them, then, Homer became (together with many other poets) who represented Zeus as a victim of amorous passion.

The myth of love, as narrated in the *Symposium* [*The banquet*], appears to be interesting all the same, because it was used to explain medicine, music, astronomy and divination. The first of these was defined as “the science that studies the organism’s amorous movements in its process of filling and emptying”. The good doctor restores reciprocal love when it is no longer present: “. . . creating friendship between elements that are antagonistic in the body and . . . infusing reciprocal love into them . . . a warm coolness, a sweet sourness, a moist dryness . . .” For music, he criticised the Heraclitus quoted above,<sup>27</sup> who would have desired to harmonise what is in itself discordant. “It is not possible for harmony to arise when deep and acute notes are still discordant.” “Music is nothing more and nothing less than a science of love in the guise of harmony and rhythm.” . . . “And such love is the beautiful kind, the heavenly kind; Love coming from the heavenly muse, Urania. There is also the son of Polyhymnia, vulgar love . . .” . . . “men may find a certain pleasure in it, but may it not produce wanton incontinence.” In the seasons, cool heat, and moist

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<sup>24</sup>Plato 1999, pp. 497, 499, 501.

<sup>25</sup>Plato 1999, pp. 505, 507, 513, 506, 507.

<sup>26</sup>Plato 1999, pp. 511, 155.

<sup>27</sup>See above Sect. 2.2.



dryness may find love for each other, and harmony. Otherwise, love combined with violence provokes disorder and damage, like frost, hail and diseases. The science which studies these phenomena “of the movements of stars and of the seasons”, is called astronomy by Plato. Even in the art of prophecy, which concerns relationships between the gods and men, it is love that is dominant: “the task of prophecy is to bear in mind the two types of love”.<sup>28</sup>

Diotima, a woman, then told Socrates how “that powerful demon” called love had originated: first of all, it was one of those demonic beings capable of allowing God to communicate with mortal man. Consequently, thanks to them, the universe became “a complex, connected unit. By means of the agency of these superior beings, all the art that foretells the future takes place . . . the prophetic art in its totality and magic.” . . . “the one who has a sure knowledge of this is a man in contact with higher powers, a demonic figure.” At the party for the birth of Aphrodite, there were also Poros, the son of Metis, and Penia. The latter decided to have a son with Poros, and in this way Love was born. He thus originated from want and his mother, poverty, but he was also generated by the artfulness and the expedients represented by his father. And then he inherited something from his grandmother, Metis, invention, free intuition. In order to reach his aims, in the end, Love must become a sage, a philosopher, an enchanter, a sophist.<sup>29</sup>

With minor modifications to the myth, we can now add that the necessities of life, linked with the capacities of invention, have produced the sciences. However, in the West, and as a result of the interpretation of Plato, these mainly are pushed towards the heavens populated by the ideas of the beautiful, of good and of immortality, causing man to forget that war and death are advancing, on the contrary, on earth.

The extent to which the Pythagorean and Platonic tradition was modified on its passage through the centuries, and was transmitted from generation to generation is narrated in the following history.

## 2.4 Euclid

. . . the theorem of Pythagoras teaches us to discover a *qualitas occulta* of the right-angled triangle; but Euclid’s lame, indeed, insidious proof leaves us without any explanation; and the simple figure [of squares constructed on the sides of an isosceles right-angled triangle] allows us to see it at a single glance much better than his proof does.

Arthur Schopenhauer

A date that cannot be specified more precisely than 300 B.C., and a no-better-defined Alexandria witnessed the emergence of Euclid, one of the most famous mathematicians of all time. We hardly know anything about him, except that he

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<sup>28</sup>Plato 1953, pp. 103–107.

<sup>29</sup>Plato 1953, pp. 128–130 and *passim*.

wrote in Greek, the language of the dominant culture of his period. But what will his mother tongue have been? Maybe some dialect of Egypt?

Euclid's *Elements* were to be handed down from age to age, and translated from one language into another, passing from country to country. For Europe, this was regularly to be the reference text on mathematics in every commentary and every dispute for at least 2,000 years. More or less explicit traces of it are to be found in school books, not only in the West, but all over the world. All books dealing with the history of sciences speak about him. While Plato represents the advertising package for Greek mathematics, Euclid supplies us with the substance. And here we find music again.

This scholar from Alexandria wrote a brief treatise entitled *KATATOMH KANONOS*, traditionally translated into Latin as *Sectio Canonis*, which means *Division of the monochord*. The Pythagorean theory of music is illustrated in an orderly manner: theorem A, theorem B, theorem Γ, ... It was explained in the introduction that sound derives from movement and from strokes. "The more frequent movements produce more acute sounds and the more infrequent ones, deeper sounds ... sounds that are too acute are corrected by reducing the movement, loosening the strings, whereas those that are too deep are corrected by an increase in the movement, tightening the strings. Consequently, sounds may be said to be composed of particles, seeing that they are corrected by addition and subtraction. But all the things that are composed of particles stand reciprocally in a certain numerical ratio, and thus we say that sounds, too, necessarily stand in such reciprocal ratios."<sup>30</sup>

The beginning immediately recalled the Pythagorean ideas of Archytas. The third theorem stated: "In an epimoric interval, there is neither one, nor several proportional means." By epimoric relationship, he meant one in which the first term is expressed as the second term added to a divisor of it. A particular case is  $n + 1 : n$ . From this theorem, after reducing to the form of other theorems the ratios of the Pythagorean tradition translated into segments, Euclid finally derived the 16th theorem which states: "The tone cannot be divided into two equal parts, or into several equal parts."<sup>31</sup> The monochord was divided by Euclid into tones, fourths, fifths and octaves. And, of course, theorem number 14 stated that six tones are greater than the octave, because the ninth theorem had demonstrated that six sesquioctave intervals [9:8] are greater than the double interval [2:1].

Thus Euclid made a decisive contribution, not only to the creation of an orthodoxy for geometry, but also for the theory of music, which was to remain for centuries that of the Pythagoreans. In him, the distinction between consonances and dissonances continued to be justified by ratios between numbers. But here, instead of the *tetractis*, he invoked as a criterion that of the ratios in a multiple, or epimoric form, i.e.  $n : 1$  or else  $n + 1 : n$ , like 2:1; 3:2; 4:3. Such a limp, linear

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<sup>30</sup>We use the 1557 edition of Euclid, with the Greek text and the translation into Latin. An Italian translation is that of Bellissima 2003. Euclid 1557, p. 8 and 14; Bellissima 2003, p. 29. Zanoncelli 1990. Euclid 2007, pp. 677–776, 2360–2379 and 2525–2541.

<sup>31</sup>Euclid 1557, p. 10 and 16; Bellissima 2003, p. 37.

explanation met with a first clear contradiction, which was later to be attacked by Claudius Ptolemaeus. The interval of the octave added to a consonance generates (for the ear) another consonance; thus the octave added to the fourth generates the consonance of the 11th. But its ratio becomes 8:3, which is not among the epimoric forms permitted. The 12th, on the contrary, possesses the ratio 3:1.

Euclid's mathematical model clashed with the reality of music. The theory did not account for all the phenomena that it claimed to explain. Was it an exception? Or was it necessary to substitute the theory? In this way, controversies arose, which were to produce other theories, setting the evolution of science in motion.

Some historians have taken an interest in the pages of Euclid quoted above, partly because they contain an elementary error of logic. One of the first person to realise this was Paul Tannery in 1904. According to Euclid, consonances are determined by those particular kinds of ratios. However in his 11th theorem ("The intervals of the fourth and the fifth are epimoric"), our skilful mathematician wrote that if the double fourth (a seventh) was dissonant, then it must be a non-multiple. As if the inverse implication were true: not consonant implies not multiple and not epimoric. But this is not possible, because it would imply that all epimoric ratios and their ratios are consonant, and so, for example, even the tone 9:8 would become consonant.

For Tannery, this error is sufficient to prove that the treatise on music was not by Euclid. But others are not so drastic; even Euclid may have fallen asleep.<sup>32</sup> After all, errors are commonly found in the work of other famous scientists. Pointing them out and discussing them would appear to be one of the most important tasks of historians.<sup>33</sup> In reality, they are often *lapsus* not noticed during the reasoning, which reveal aspects of their personality that would otherwise remain hidden. They are a precious help to better understand events that are significant for the evolution of the sciences, and not just useless details which become acts of *lèse-majesté* in the pages of hostile historians.

Tannery discovered the error at the beginning of last century, when European mathematical sciences were undergoing a profound transformation. Among other things, modern mathematical logic was developing, and some scholars were even re-considering Euclid in the light of the crisis of the foundations. The most famous of these was David Hilbert (1862–1943), who was polishing him up to make him meet the rigorous standards of the new twentieth-century scientific Europe.<sup>34</sup> The *Elements* were thus interpreted by means of an axiomatic deductive scheme, made up of definitions, postulates and theorems. However, this was an anachronistic reading of the ancient books, amid a dispute about the foundations of mathematics, the *Grundlagenstreit*, which took into consideration other positions, different from the formalistic one of the Hilbertian school of Göttingen.<sup>35</sup>

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<sup>32</sup>Bellissima 2003. Euclid 2007, pp. 691–701.

<sup>33</sup>Tonietti 2000b.

<sup>34</sup>Hilbert 1899.

<sup>35</sup>Tonietti 1982a, 1983a, 1985a, 1990.

For 2,000 years, practically nobody read the works of Euclid from the point of view of a logician. Their importance lay in other fields. However, logic did not enjoy the favour of the Platonic schools, but appeared to be the prerogative of their Aristotelian rivals, with their well-known syllogisms. Now, what the *lapsus*-error betrays is not an apocryphal text falsely attributed to Euclid, but on the contrary, insufficient attention paid by him to the logical structure of the reasoning, and his adhesion (at all costs?) to the Pythagorean-Platonic theories of music.

Naturally, all this can be seen not only in the *Division of the monochord*, but above all in the *Elements*, the books of which (the arithmetic ones) are also used to argue the theorems of music.<sup>36</sup> From the *Elements* we shall extract only a couple of cases, which are most suitable for a comparison between cultures, which is what interests us here.

Euclid was the first to demonstrate here what was subsequently to be regularly called the theorem of Pythagoras, but would be better indicated by his name. In the current editions of the *Elements*, Euclid offered the following proof of the proposition: “In right-angled triangles, the square on the side opposite the right angle is equal to the squares on the sides enclosing the right angle.”

The angle FBC is equal to the angle ABD because they are the sum of equal angles.

The triangle ABD is equal to the triangle FBC because they have equal two sides and the enclosed angle.

The rectangle with the vertices BL is equal to twice the triangle ABD.

The square with the vertices BG is equal to twice the triangle FBC.

Therefore the rectangle and the square are equal, because the two triangles are equal.

The same reasoning may be repeated to demonstrate that the rectangle with the vertices CL is equal to the square with the vertices CH.

As the square with the vertices DC is the sum of the rectangles BL and CL, it is equal to the sum of the squares BG and CH.

Quod erat demonstrandum.<sup>37</sup>

The proof bears the number 47 in the order of the propositions, and is followed by the inverse one (if 47 is true of a triangle, then it must be right-angled), which concludes the first book of the *Elements*. It is obtained by following the chain of propositions, like numbers 4, 35, 37, and 41. The demonstration is based on the other demonstrations; these demonstrations are based, in turn, on the definitions (of angle, triangle, square, . . .), on the postulates (draw a straight line from one point to another, the right angles are all equal to one another, . . .), on common notions (equal angles added to equal angles give equal angles, . . .) and on the possibility of constructing the relative figures. Everything is broken down into shorter arguments,

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<sup>36</sup>Bellissima 2003, p. 31. Euclid 2007.

<sup>37</sup>Euclid 1956, pp. 349–350; Euclid 1970, pp. 146–150. Euclid 2007. Figure on every textbook.

reassembled and well organised in a linear manner; everything seems convincing; everything is well-known to every student.

The earliest commentators, like Proclus, Plutarch or Diogenes Laertius, attributed the theorem to Pythagoras, but none of them are eye-witnesses, indeed, they come many centuries after him, seeing that the period when Pythagoras lived was the fifth century B.C. Pythagoras did not leave anything written, but only a series of disciples and followers. Failing documentary evidence, we can believe or not believe the attribution of the accomplishment to Pythagoras.

Anyway, whoever the author was, Euclid's proof does not appear to be the most direct one, even in the field of the Pythagorean sects.

In the right-angled triangle ABC, by tracing the perpendicular AD to BC, two new triangles, ABD and ADC, are created, which are said to be similar to ABC, because their sides are reduced in the same ratio. In other words, for them,  $BC : AB = AB : BD$ . Consequently, by the rule of proportions,  $BC \times BD = AB \times AB$ . Having demonstrated the equality of square BG and rectangle BL, the argument continues in the same way as Euclid 47.

But Euclid did not follow this route, because he wanted to make his proof independent of the theory of proportions. This appears in the *Elements* only in books five and six. If he had used it (the necessary proposition would have been number eight of book six), he would have broken the linear chain of deductions, forming a circle that he would perhaps have considered vicious. Furthermore, he would have raised the particularly delicate question of incommensurable ratios, necessary to obtain a valid demonstration for every right-angled triangle. Euclid would succeed in avoiding the obstacles, but he would be forced to pay a price: following a route which seems as intelligent as it is artificial.

In his commentary, Thomas Heath, who has left us the current English translation of the *Elements*, considered Euclid's demonstration "... extraordinarily ingenious, ... a veritable *tour de force* which compels admiration, ...".<sup>38</sup> This British scholar compared it with various possibilities proposed in other periods and in other places by other people. At times he erred on the side of anachronism, because he also used algebraic formulas which only came into use in Europe after Descartes; but he seems to be worried above all about preventing some ancient Indian text (coming from the British Empire?) from taking the primacy away from Greece. In the end, he solved the question as follows: "... the old Indian geometry was purely empirical and practical, far removed from abstractions such as irrationals. The Indians had indeed, by attempts in particular cases, persuaded themselves of the truth of the Pythagorean theorem, and had enunciated it in all its generality; but they had not established it by scientific proof."<sup>39</sup>

Thanks to an article by Hieronymus Zeuthen (1839–1920), and to the books of Moritz Cantor (1829–1920) or David E. Smith,<sup>40</sup> Heath had access also to what they

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<sup>38</sup>Euclid 1956, p. 354.

<sup>39</sup>Euclid 1956, p. 364.

<sup>40</sup>Zeuthen 1896; Cantor 1922; Smith 1923.

thought contained an ancient Chinese text: the *Zhoubi* [*The gnomon of the Zhou*]. However, the British historian seems to see, in the Chinese demonstration of the fundamental property of right-angled triangles, only a way to arrive at the discovery of the validity of the theorem in the rational case of a triangle with sides measuring 3, 4, 5. “The procedure would be equally easy for any *rational* right-angled triangle, and would be a natural method of trying to *prove* the property when it had once been *empirically* observed that triangles like 3, 4, 5 did in fact contain a right angle.”<sup>41</sup> Trusting D. E. Smith, he concludes that the Chinese treatises contained “... a statement that the diagonal of the rectangle (3, 4) is 5 and ... a rule for finding the hypotenuse of a ‘right triangle’ from the sides, ...”.<sup>42</sup> But they ignored the proof of that.

It is easier to understand the common defence of Euclid undertaken by Heath, and his underestimation of the Chinese text, even with respect to the Arabs and the Indians. He is less excusable when he writes: “1482. In this year appeared the first printed edition of Euclid, which was also the first printed mathematical book of any importance.”<sup>43</sup> However, Heath was led to his interpretations and judgements by his own Eurocentric prejudices. If we should want to take part in an absurd competition regarding priorities, it would be extremely easy to prove him wrong. We have evidence that *The gnomon of the Zhou* was first printed as long ago as 1084. A 1213 edition of the book is extant today in a library at Shanghai. Heath would only be left with the possibility of sustaining that *The gnomon of the Zhou* is not a book of mathematics, or that it speaks of a mathematics that is not important. Perhaps it is not important for Europe; but what about the world?

In the third chapter of this work, we shall show, on the contrary, that this ancient Chinese book in fact demonstrates the fundamental property of right-angled triangles. In the fourth chapter, we shall discuss other Indian demonstration techniques. It is true, we do not find in the Indian or Chinese texts the theorems that school has accustomed us to, but simply other procedures to convince the reader and help him find the result. Cultures that are different from the Greek one followed different arguments, which, however, are to be considered equally valid.

On the basis of what criterion may we expect to impose a hierarchy of ours from Europe? Unfortunately, we shall see that historical events offer only one. Then it must be the one sustained by Plato: war. But will our moral principles be prepared to accept it?

For more than 2,000 years, in Europe, Euclid will be the model that was generally shared to reason about mathematics. Rivers of ink have been consumed for him. I will not yield here to the temptation of making them more turbid, or better, more limpid, or of deviating them. However, in order to prepare ourselves for the comparison with different models of proof, we need to examine the procedure followed more closely.

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<sup>41</sup>Euclid 1956, pp. 355–356.

<sup>42</sup>Euclid 1956, p. 362.

<sup>43</sup>Euclid 1956, p. 97.

Our famous Hellenistic mathematician wished to convince his readers that: “In right-angled triangles the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle.” To achieve this goal, he had defined the right angle at the beginning of book one. In the list of definitions, the tenth one proclaims: when a straight line conducted to a straight line forms adjacent angles that are equal to each other, each of the equal angles is a right angle, and the straight line conducted on to the other straight line is said to be perpendicular to the one on to which it is conducted. Not satisfied with this, Euclid included among the postulates also the one numbered four: all right angles are equal to one another.<sup>44</sup> He also defined the angle, the triangle and the square. He postulated that a straight line could be drawn from one point to another. Among the common notions, he included the properties of equality. He explained how to construct a square on a given segment of a straight line. All this was either defined, or postulated or demonstrated in the *Elements*. Demonstrating, then, means tracing back to some other property, already defined, or postulated, or demonstrated. And so on.

Euclid scrupulously sought certainty and precision. It seems that he did not want to trust either evidence or his intuition. Who would find it obvious that right angles are equal? Intuition would seem to lead us immediately to see how to draw straight lines, triangles, squares. But what would it be based on? What if it led us to make a mistake? Our Greek mathematician would like to avoid using his eyes, or working with his hands, or believing his ears. For him, the truth of a geometrical proposition should be made independent of the everyday world, practical activities or the senses. The organs of the body would provide us with ephemeral illusions, not properties that are certain and eternal. As in the myth of the cave described by Plato, Euclid would like to detach himself from the distorted shadows of the earth, which are visible on the wall, to arrive at the ideal objects that project them. It was only on these that he based the truths of his geometry, which were thus believed to have descended from the heavens of the eternal ideas. He would like to demonstrate every proposition by describing the procedure to trace it back to them. He would like to, but does he succeed?

Thus Euclid followed this dualism and hierarchy, whereby the earth is subject to the heavens. His model of proof must therefore avoid making reference to material things that are a part of the everyday world, where people use their hands, eyes and ears to live their lives. The truth of a proposition descends from the heavens on high: it is deduced. And yet, luckily for us, even in an abstract scheme like this, limitations filtered through, and echoes could be heard of the ancient origins among men living on the earth.

A line is length without width.<sup>45</sup> Anybody would think of a piece of string which becomes thinner and thinner, or of a stroke made with a pen whose tip gets increasingly thinner. How can we imagine the *Elements* without the numerous

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<sup>44</sup>Euclid 1956, pp. 349, 153–154.

<sup>45</sup>Euclid 1956, p. 153.

figures? But isn't it true that we see the figures? Or only with the eyes of our mind? In any case, it does not seem to have been sufficient for Euclid to think of an object of geometry, or to describe its properties. In order to make it exist in his ideal world as well, it was necessary for him to construct it by means of a given procedure at every step. That of Euclid is an ideal, abstract geometry, but not completely separated from the world. In it, properties are deduced from on high, but they also need to be constructed at certain points. It is made up of immaterial symbols, but they can be represented on the plane that contains them.

Having stated in book seven that a prime number "... is one which is measured by the unit alone",<sup>46</sup> in book nine, Euclid demonstrates proposition 20: "Prime numbers are more than any assigned multitude of prime numbers." Let the prime numbers assigned be represented by the segments A, B, C. Construct a new number measured by A, B, C [the product of the three prime numbers]. Call the corresponding segment obtained DE, which is commensurable with A, B, C. Add to DE the unit DF, obtaining EF. There are two possibilities:

either EF is prime, and then A, B, C, EF are greater than A, B, C,

or EF will be measured by the prime number G. But in this case, it must be different from the prime numbers A, B, C. Otherwise, G would measure both DE and EF, and consequently also their difference. However, the difference is the unit which cannot be measured any further. As this would be "absurd", G must be a new prime number. A, B, C, G form a quantity greater than A, B, C. *Quod erat demonstrandum.*<sup>47</sup>

Note that the numbers are represented by segments, and by ratios between segments. As a result, even this numerical proof is accompanied by a figure.

The previous proof is a procedure that makes it possible to obtain a new prime number. It really constructs the quantity of primes announced in the proposition. Having obtained a new prime number and added it to the previous ones, the procedure can be repeated as many times as is desired. The proof thus constructs, step by step, continually new prime numbers.

Besides the usual anachronistic algebraic translation, Heath concludes in his commentary: "*the number of prime numbers is infinite.*"<sup>48</sup> But in this way, he annuls Euclid's peculiar style, because he transfers the subject among the mathematical controversies of the nineteenth and twentieth centuries. It was not Euclid, but rather these mathematicians who discussed about 'infinite' quantities, which were used in every field of mathematics, above all in analysis.

Only at that time did people like Richard Dedekind (1831–1916), Georg Cantor (1845–1918) and David Hilbert start to use infinity, after defining it formally by means of the characteristics property which cancelled Euclid's fifth common notion: The whole is greater than the part.<sup>49</sup> Up to that moment, it had been considered a paradox that, for example, whole numbers and even numbers could be counted in

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<sup>46</sup>Euclid 1956, II, p. 278.

<sup>47</sup>Euclid 1956, II, p. 412.

<sup>48</sup>Euclid 1956, II, p. 413.

<sup>49</sup>Euclid 1956, I, p. 155.



parallel; because in that way, they would appear to be equally numerous, whereas intuition tells us that even numbers are only a part, there are fewer of them. The paradox became the new definition, which, in the new language that had now entered even primary school textbooks, stated: a set is called infinite when it admits a biunique correspondence with one of its own parts. In other words, when the whole is 'equal' to a part, it is a case of infinity.

Euclid shows what operation to perform in order to construct step by step a increasingly large quantity, but he avoided calling it infinite. Dedekind, Cantor and Hilbert defined as 'infinite' quantities that they could not succeed in constructing. There's a big difference.

Like the Plato painted by Raphael in the Vatican for the Athens school, Euclid pointed his finger upwards; and yet he still maintains some connections with the world, both in pictures and in his constructions. After Hilbert<sup>50</sup> and his undeniable success with the mathematical community of the twentieth century, it became all too common to interpret Euclid axiomatically. And yet we have seen, with a clear example, that this is an anachronistic distortion. Euclid also used other approaches, and not everybody would like to cancel his construction procedures.

Hieronymus Zeuthen saw Euclid better together with the 'problems' *à la* Eudoxus (Cnidos, died c. 355 B.C.)<sup>51</sup> than together with the 'theorems' *à la* Plato. "... Euclid: he is not satisfied with defining equilateral triangles, but before using them, he guarantees their existence by solving the problem of how to construct these triangles: ..."<sup>52</sup> Anyone who believed in the existence of the geometrical object before examining it (in the world of the ideas) would not need to construct it in order to convince himself of its reality (on earth). "But the Greeks used constructions much more widely than we are used to doing, and specifically also in cases where its practical use is wholly illusory. [...] In order to arrive at a certainty on this matter, and at the same time to understand what the theoretical significance of constructions was at that time, they need to be observed from their first appearance in Euclid onwards. The idea will thus be found to be approved that *constructions, with the relative proof of their correctness, served to establish with certainty the existence of what is to be constructed.* Constructions are prepared by Euclid by means of *postulates.*"<sup>53</sup> In ancient geometry, therefore, proofs of existence were supplied by geometrical constructions.

This scholar's interpretations were more or less closely taken up by people like Federico Enriques<sup>54</sup> (1871–1946) and Attilio Frajese.<sup>55</sup> In 1916, also Giovanni Vacca published his translation of Book 1 of the *Elements*, with the parallel Greek text. But the fact that Euclid's proofs were based on constructions was completely

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<sup>50</sup>Hilbert 1899.

<sup>51</sup>Boyer 1990, pp. 78, 99, 105–110.

<sup>52</sup>Zeuthen 1902, pp. 72–73.

<sup>53</sup>Zeuthen 1896, pp. 222–223.

<sup>54</sup>Euclid 1925, pp. 135–141.

<sup>55</sup>Euclid 1970, p. 147.

ignored.<sup>56</sup> Also Alexander Seidenberg stated that Euclid did not practise “the famous axiomatic method”. Even though “he was meticulous in the constructions to abstract from the old ‘peg and cord’ (or ‘straight edge and compass’) constructions.”<sup>57</sup> This scholar dedicated a whole work to rebutting the (anachronistic) idea that Euclid had developed Book 1 of the *Elements* axiomatically.<sup>58</sup> Rather, the ancient Hellenistic geometrician constructed the solution of problems.

It has been confirmed by David Fowler that the historical and real Euclid could not be taken up into the Olympus of orthodox formal axiomatic systematizers without falsifying him: “... their geometry dealt with the features of geometrical thought-experiments, in which figures were drawn and manipulated, ...”.<sup>59</sup> The same line of reasoning is also followed initially by Lucio Russo, who refers directly to Zeuthen. “Mathematicians did not create, ..., new entities by means of pure abstract definitions, but they considered their real geometrical constructibility indispensable, ...”.<sup>60</sup> However, this Italian mathematician, also interested in classical studies, then creates an excessive contrast between the construction procedures and Euclid’s definitions, because he interprets them in a strictly Platonic sense. Thus he makes an effort to show that the latter are not authentic, but added by others. This is possible, considering the long chain of copies and commentaries on the codices that have been handed down to us.<sup>61</sup> But why should the alternative only be between a Platonizing Euclid, for whom the ideas really exist, and one who considers them just conventional names?

Isn’t it true that in the definitions and all the figures, we can already perceive the representation and the inspiration of the everyday world? Russo tries to give the *Elements* a consistency which they do not possess in this sense, in order to assimilate them to his own, modern, post-Hilbertian definition of science, limited to a “rigorously deductive structure.”<sup>62</sup> Luckily for us, the sciences and the arts of demonstration are more varied, as we shall soon see more clearly.

Book 1 of the *Elements* converged towards the proof of the theorem of Pythagoras. We may consider that all 13 books merged together in calculating the angles of regular polyhedra inscribed in a sphere. The 18th proposition of Book 13 reads: “To set out the sides of the five figures and to compare them with one another.” ... “I say next that *no other figure, besides the said five figures, can be constructed which*

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<sup>56</sup>Euclid 1916.

<sup>57</sup>Seidenberg 1960, p. 498.

<sup>58</sup>Seidenberg 1975.

<sup>59</sup>Fowler 1987, p. 21.

<sup>60</sup>Russo 1996, p. 73.

<sup>61</sup>Russo 1996, pp. 235–244.

<sup>62</sup>Russo 1996, p. 32. Mario Vegetti finds that Euclid’s approach is used by Galen and Claudius Ptolemaeus as an axiomatic Platonic model. And yet, even this professor of ancient philosophy, though levelling out the procedure too much, realises that Euclidean rationality has to come to grips with Aristotle: “In the first place, the ontological obligation to consider the forms as transcendent, or at least as external to the empirical, disappears.” Vegetti 1983, p. 155.

*is contained by equilateral and equiangular figures equal to one another.*<sup>63</sup> In this way, the mathematician from Alexandria seemed to have succeeded in making considerable progress in demonstrating what Plato had only outlined with his famous solids. But here, the idea of universal harmony, glimpsed by the philosopher through the dialogues, was now reached by means of a tiring ascent from one theorem to another, from one step to another, less esoteric and more scholastic.

Mathematical sciences were to evolve in Europe with several, sometimes profound, changes. Yet Euclid's *Elements* succeeded in surviving and adapting to the different periods. They represent the backbone of Western history, which the different kinds of sciences inherited from one another. The English logician Auguste de Morgan (1806–1871) could still write in the nineteenth century: “There never has been, and till we see it we never shall believe that there can be, a system of geometry worthy of the name, which has any material departures (we do not speak of *corrections*, or *extensions*, or *developments*) from the plan laid down by Euclid.”<sup>64</sup> But geometry had changed, and was practised with the powerful means of infinitesimal analysis or projection methods. Euclid's geometry was of interest above all as a logical scheme of deductive reasoning, and was to be readjusted, also as such. Only towards the middle of the twentieth century did a group of French mathematicians, united under the pseudonym of Bourbaki, try to substitute the geometrical figures of Euclid's *Elements* with the formal algebraic structures inspired by Hilbert. The new *Eléments de Mathématique*, however, met with far less success than the model whose place they wanted to take. The work remained on the scene for a few decades, and nowadays is found covered with dust mainly on the shelves of Maths Department libraries.<sup>65</sup>

## 2.5 Aristoxenus

Aristoxenus (Tarentum 365/75–Athens? B.C.) is seldom remembered in science history books. When he is mentioned, writers admit that they were forced to include him because in antiquity, the theory of music was a part of the *quadrivium* mentioned above. But it is immediately added that “he turned his back upon the mathematical knowledge of his time, to adopt and propagate a radically ‘unscientific’ approach to the measurement of musical intervals.”<sup>66</sup> This judgement stems from a widespread prejudice. It should be underlined, however, that this Greek from Tarentum left us some important books on harmony and musical rhythm. They continue to be particularly interesting, also for historians of the mathematical

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<sup>63</sup>Euclid 1956, III, pp. 503–509.

<sup>64</sup>Euclid 1956, I, p. v.

<sup>65</sup>Tonietti 1982b, pp. 11–21.

<sup>66</sup>Winnington-Ingram 1970, p. 282.

sciences, precisely because they did not belong to the Pythagorean or Platonic school.

“Tension is the continual movement of the voice from a deeper position to a more acute one, relaxation is the movement from a more acute position to a deeper one. Acuteness is the result of tension, and deepness of relaxation.” Thus Aristoxenus considered four phenomena (tension, acuteness, relaxation and deepness), and not just two, because he distinguished the process from the final result.<sup>67</sup> He criticised “those who reduce sounds to movements and affirm that sound in general is movement”. For Aristoxenus, instead, the voice “moves [when it sings], that is to say, when it forms an interval, but it stops on the note”. Thus Aristoxenus does not appear to be interested in the movement (invisible to the eye) of the string that generates the sound, or to the movement of sound through the air, but only in the movement (perceptible with the ear) with which the passage is made from one note to another.<sup>68</sup>

This last movement has its limits: “The voice cannot clearly convey, nor can the hearing perceive, an interval less than the smallest diesis (*διεσις*, a passage, a quarter of a tone) . . .”<sup>69</sup> After distributing the notes along the steps of the scale, our Greek theoretician listed the “symphonies”, or in other words the consonances, to distinguish them from the “diaphonies”, the dissonances. The former are the intervals the fourth, the fifth, the octave, and their compounds with two or more octaves.<sup>70</sup> “The smallest consonant interval [the fourth] is determined, . . ., by the very nature of the voice.” The largest consonances are not established by theory, but by “our practical usage – by this I mean the use of the human voice and of instruments – . . .”<sup>71</sup>

In his reasonings, Aristoxenus never made any reference to ratios between whole numbers or magnitudes, as the Pythagorean sects, Archytas and Euclid did. He also made a distinction between rational *ρητα* and irrational *αλογα* intervals, but he did not explain the difference in the *Elementa Harmonica* as handed down to us. From his *Ritmica*, it is only possible to infer that by “rational” intervals, he intended those that could be performed in music, assessing their range, whereas the others are “irrational”. Consequently, below a quarter of a tone, the intervals are “irrational”, while all the combinations of quarters of a tone are “rational” for him.<sup>72</sup>

The definition of the tone and its parts now became crucial. “The tone is the difference in magnitude between the first two consonant intervals [between the fifth and the fourth]. It can be divided into three submultiples, one half, one third and one quarter of a tone, because these can be performed musically, whereas it is not

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<sup>67</sup> Aristoxenus 1954, p. 19.

<sup>68</sup> Aristoxenus 1954, pp. 20–21.

<sup>69</sup> Aristoxenus 1954, p. 22.

<sup>70</sup> Aristoxenus 1954, pp. 30–31.

<sup>71</sup> Aristoxenus 1954, p. 31.

<sup>72</sup> Aristoxenus 1954, pp. 24–25.

possible to perform any of the intervals smaller than these.”<sup>73</sup> Euclid, in his 16th theorem, denied the possibility of dividing the tone into equal parts, on the basis of the non-existence of the proportional mean in whole numbers. Here, on the contrary, Aristoxenus calmly performed this division. Was the former “scientific” because he used proportions and ratios in his arguments, and the latter “non-scientific” because, on the contrary, he ignored them following his ear? Certainly not. Rather, these are differences of approach to the problems, which reflect cultural, philosophical and social features, in a word, values, that are very distant from each other.

In Book 2 of the *Elementa Harmonica*, Aristoxenus became explicit. “... the voice follows a natural law in its movement and does not form an interval by chance. And, we shall, unlike our predecessors, try to give proof of this which is in harmony with the phenomena. Because some talk nonsense, disdaining to make reference to sensation, because of its imprecision, and inventing purely abstract causes, they speak of numerical ratios and relative speeds, from which the acute and the deep derive, thus enunciating the most irrelevant theories, totally contrary to the phenomena; others, without any reasoning or proof, passing each of their affirmations off as oracles ...” “Our treatise regards two faculties; the ear and the intellect. By means of the ear, we judge the magnitudes of intervals, by means of the intellect, we realise their value.”<sup>74</sup>

With musical intervals, in his opinion, “it is not possible to use the expressions that are typically used for geometrical figures ... For the geometrician does not use his faculties of sensation, he does not exploit his sight to make a correct, or incorrect evaluation of a straight line, a circle or some other figure, as this is the task of a carpenter, a turner or other craftsmen. For the *μουσικός* [musician], however, the precision of sensible perception is, on the contrary, fundamental, because it is not possible for a person whose sensible perception is defective to give an adequate explanation for phenomena that he has not succeeded in perceiving at all.”<sup>75</sup>

Having chosen the ear as judge, Aristoxenus repeated even more clearly: “as the difference between the fifth and the fourth is one tone, and here it is divided into equal parts, and each of these is a semitone, and is, at the same time, the difference between the fourth and the ditone, it is clear that the fourth is composed of five semitones.”<sup>76</sup>

In the Pythagorean sects, worshippers of whole numbers were trained as adepts; in the Academy, Plato desired to educate the soul of young warriors to eternal being by means of geometry. Now Aristoxenus appealed to musicians, who use their hands and ears to play their instruments. We are faced with a variety of musical scales, modes, melodies, which, however, in practice were difficult to play all on the same instrument, and thus it did not appear possible to pass from one to the other, i.e.

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<sup>73</sup> Aristoxenus 1954, p. 32.

<sup>74</sup> Aristoxenus 1954, p. 47.

<sup>75</sup> Aristoxenus 1954, p. 48.

<sup>76</sup> Aristoxenus 1954, p. 79.

to modulate, either.<sup>77</sup> Plato did not even perceive the problem, because he limited melodies to those (Doric and Phrygian) he considered suitable for the order of his State. He did not tolerate free modulations. Aristoxenus, on the contrary, made them possible with his theory, and facilitated them.

If the fourth were divided into five equal semitones, the octave (the fourth plus the fifth) would be composed, in turn, of 12 equal semitones. On instruments tuned in this way (and not in the Pythagorean manner), semitones, tones, fourths, fifths, and octaves can be freely transposed (transported) along the various steps of the scales, maintaining their value, and thus permitting a full variety of melodies, modes and modulations. It is like what happens today with modern pianos tuned in the equable temperament. But this was to be adopted in Europe only in the eighteenth century, thanks to the efforts of musicians like Johann Sebastian Bach (1685–1750) and Jean Philippe Rameau (1683–1764). Yet even this clear advantage of his theory has recently been denied to Aristoxenus by hostile historians. “. . . although modulation was exploited to some extent by virtuosi of the late fifth century B.C. and after, there is no reason to think that it created a need for a radical reorganization of the system of intervals, or that such could have been imposed upon the lyre players and pipe players of the time.” As he was opposed by the Pythagoreans of his time, our theoretician from Tarentum continues to be judged badly by the Pythagoreans of today.<sup>78</sup>

Some of his other characteristics tend to deteriorate his image in the eyes of certain science historians. Euclid considered sounds as “compounds of particles”.<sup>79</sup> In the *Elementa Harmonica*, on the contrary, sounds appear to form a *continuum*, and accordingly Aristoxenus stated: “. . . we affirm without hesitation that no such thing as a minimum interval exists.”<sup>80</sup> In theory, therefore, the tone could be divided up beyond every limit [*ad infinitum*]. But, guided by his ear, the musician stopped at a quarter of a tone for the requirements of melodies. For him, therefore, music is to be taken out of the group of discontinuous, discrete sciences, and included among the continuous ones, thus disarranging the *quadrivium*. Also in this, the philosophical roots of Aristoxenus are not those of Plato. His whole concept recalls rather the principles of Aristotle (Stagira 384–Calchis 322 B.C.), who was actually mentioned by name at the beginning of Book 2. This offers us a testimony that Aristotle had attended Plato’s lessons, and that Aristoxenus himself had then become a direct pupil of Aristotle: “. . . as Aristotle himself told us, he gave a preliminary account of the contents and method of his topic to his listeners.”<sup>81</sup>

<sup>77</sup> Aristoxenus 1954, pp. 53–55.

<sup>78</sup> Winnington-Ingram 1970, p. 282. This writer shows the origin of her/his prejudices, because she/he immediately adds that “‘temperament’ would distort all the intervals of the scale (except the octave) and, significantly, the fifths and the fourths”. For her/him, the ‘correct’ intervals are, on the contrary, those of Pythagoras. See Part II, Sects. 11.1 and 11.3.

<sup>79</sup> See above Sect. 2.4.

<sup>80</sup> Aristoxenus 1954, p. 67.

<sup>81</sup> Aristoxenus 1954, p. 45.

Thus we have also met the other famous philosopher who, together with Plato, and with alternating fortunes, was to have a significant influence on European culture, profoundly conditioning even its scientific evolution. After representing orthodoxy for centuries in every field of human knowledge, co-opted by Christian and medieval theologians such as Thomas Aquinas (Aquino 1225–Fossanova 1274), with the scientific revolution of the seventeenth century, Aristotle became, at least in the travesty of scholastic philosophy, the idol to be destroyed. Since then, his name has been a synonym in the modern scientific community for error, a process of reasoning based on the authority of books (*ipse dixit*), without any reference to the direct observation of the phenomenon studied, and suffocation of the truth and research by a metaphysics made up of finalistic and linguistic rules, a backward-looking, irrational environment that hinders the progress of knowledge. All these judgements, however, are, on the contrary, ill-founded anachronistic commonplaces. This famous teacher of Alexander the Great displays, together with the usual presumed demerits, also some interesting characteristics for the more attentive historian, though we shall not deal with them in detail. We will recall only his naturalistic writings, which made him worthy of being considered by Charles Darwin (1809–1882) as one of the precursors of evolutionary theory,<sup>82</sup> and his logic based on syllogisms. He would deserve a little attention here, above all because his ideas of the world, of mathematical sciences and of the sciences of life constantly made reference to a continuous substrate: nature does not take jumps, it abhors a void, and so on.

Aristotle criticised indivisibles, sustaining, on the contrary, an infinite divisibility, and tried to confute the paradoxes of Zeno the Eleatic, not just by using common sense. The paradoxes were expressed in the following terms: “Zeno posed four problems about movement, which are difficult to solve. The first concerns the non-existence of movement, because before a body in motion reaches the end of its course, it must reach the half-way point ... The second, called ‘the Achilles’, says that the faster runner will never overtake the slower one, because the one who is behind first has to reach the point from which the one who is ahead had started, and thus the slower runner is always ahead ... The third is ... that the arrow in flight is immobile. This is the result of the hypothesis that time is composed of instants: without this premise, it is impossible to reach this conclusion.” Then Aristotle confuted them. “This is the reason why Zeno’s paradox is incorrect: he supposes that nothing can go beyond infinite things, or touch them one by one on a finite time. Distance and time, and all that is continuous, are called ‘infinite’ in two senses: either as regards division, or as regards [the distance between] the extremes. It is not possible for anything to come into contact in a finite time with objects that are infinite in extension. However, this is possible if they are infinite in subdivision. In this sense, indeed, time itself is infinite.”<sup>83</sup> Aristotle also states that Pythagorean mathematicians [of his time] “do not need infinity, nor do they make use of it.”

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<sup>82</sup>Tonietti 1991.

<sup>83</sup>Sambursky 1959, pp. 182–185.

The Pythagoreans fell into the trap of the paradoxes because they imagined space as made up of points, and time as made up of instants. Aristotle solved the paradoxes with the idea of the continuous which could be divided *ad infinitum*. With their discrete numbers, the Pythagoreans took phenomena to pieces, but then they couldn't put them back together again. Aristotle presents them to us as they appear to our immediate sensibility, maintaining continuity as their essential characteristic.

Nowadays, modern physics deals in its first few chapters with the movement of bodies in an ideal empty space (which becomes the artificial space of laboratories). The physics of Aristotle, on the contrary, dealt with a nature that is in continuous transformation and movement, observed directly and maintained where it is, that is to say, on earth. In the present-day scientific community, only a few heretical members of a minority have dared to sustain positions referable to Aristotle.<sup>84</sup> However, even though for opposite reasons, neither the ancient popularity of Aristotle, nor his current discredit could prevent us from recognizing as valid his contributions to the mathematical sciences: a supporter of continuous models as opposed to the discrete ones of the followers of Democritus and Pythagoras.

Aristotle found contradictions in the Pythagorean reduction of the world to whole numbers: "If everything is to be distributed among numbers, then it must follow that many things correspond to the same number, and that the same number must belong to one thing and to another . . . Therefore, if the same number belonged to certain things, these would be the same as one another, because they would have the same numerical form; for example, the moon and the sun would be the same thing."<sup>85</sup> Here Aristotle manifested the conviction, not only that the essence of things could not be limited to numbers, but also that the world was more numerous than the whole numbers (because it is continuous), thus making it necessary to assign various things to the same number.

As he was connected with Aristotle, and because he did not make any use of numerical ratios, Aristoxenus became the regular target in treatises on music theory. He remained in the history of music, but he was removed from standard books on the history of sciences.<sup>86</sup> As regards these questions, orthodoxy was to be created around the Pythagorean conceptions, and was long maintained. In the next section, we shall see the most famous and lasting variant, so long-lasting that it accompanies us till the nineteenth century.

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<sup>84</sup>Boyer 1990, pp. 116–117. Thom 1980; Thom 2005; Tonietti 2002a.

<sup>85</sup>Aristotle 1982 [Metaphysics] N5, 1093a, 1.

<sup>86</sup>Some followers of Aristoxenus have been listed and studied in Zanoncelli 1990. Aristoxenus remains one of the main sources regarding the Pythagorean sects for many scholars, who, however, curiously seem to avoid accurately the musical writings that are contrary to the Pythagorean scale. von Fritz 1940. *Pitagorici* 1964.



## 2.6 Claudius Ptolemaeus

In looking at Claudius Ptolemaeus (Ptolemy) (Egypt, between the first and second centuries), we shall not start from his best-known book, but from another one, the *APMONIKA*, which would deserve to enjoy the same prestige in the history of science.

Here, from the very start, he opposed “ἀ’κοη’/ Auditus” [hearing] to the “λόγος, Ratio” [reason], criticising the former as only approximate. “... Sensuum proprium est, id quidem invenire posse quod est vero-propinquum; quod autem accuratum est, aliunde accipere: Rationis autem, aliunde accipere quod est vero-propinquum; & quod accuratum est adinvenire. [...] Jure sequitur, Perceptiones sensibiles, a rationalibus, definiendas esse & terminandas: Debere nimirum priores illas (...) istis (...) suppeditare sonituum Differentias; minus quidem accurate sumptas (...) ab istis autem (...) eo perducendas ut accuratae demum evadant & indubitatae.” [“... it is undoubtedly typical of the senses to be able to find what is close to the truth; what is, instead, precise is obtained elsewhere: on the contrary, it is typical of reason to obtain elsewhere what is close to the truth, and to find what is precise. [...] It rightly follows that the perceptions of the senses are established and measured by the rational ones; it is no surprise that the former, rather than the latter, should supply the differences in sounds, but as they are undoubtedly taken less accurately (...), they are to be led back there by these [the rational ones] so that may become sure and undoubted.”]<sup>87</sup>

Ptolemy trusted “Ratio” because it is “... simple ... without any admixture, perfect, well ordered, ... it always remains equal to itself”. Instead, “sensus” depends on “... materia ... mista, & fluxui obnoxia” [“mixed material ... subject to change”], and therefore unstable, which does remain equal, and needs that “Reformatione” [improvement] which is given by reason. Thus the ear, which is imperfect, is not sufficient by itself to judge differences in sounds. Just like the case of dividing a straight line accurately into many parts, a rational criterion is needed for sounds, too. The means used to do this was called the “κανὼν ἁρμονικός, Kanon Harmonicus” [harmonic rule], which was to direct the senses towards the truth. Astrologers were to do the same, maintaining a balance between their more unrefined observations of the stars and reason. “In omnibus enim rebus, contemplantis & scientia utentis munus est, ostendere, Naturae opera secundum Rationem quandam causamque bene ordinatam esse condita, nihilque temere aut fortuito ab ipsa factum esse; & maxime quidem, in apparatus hujusmodi longe pulcherrimis, quales sunt sensuum horum (Rationis maxime participum) Visus atque Auditus.” [“For in all things, it is the duty of the one who contemplates and who

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<sup>87</sup> Ptolemy 1682, pp. 1–3. We follow the edition of John Wallis, extracted from 11 Greek manuscripts compared together, with a parallel Latin translation: *Armoniconum libri tres* [Three books on harmony]. The famous Oxford professor so judged the Venetian edition of 1562 printed by Gogavino: “... versio ... obscura fuerit & perplexa ... a vero saepius aberraverit.” [“... the version is obscure and confused ... it departs from the truth somewhat often.”]

makes use of theory, to present the works of nature as things that have been created by reason, with a certain orderly cause, and nothing is done by nature blindly or by chance; this undoubtedly [happens] above all in the organs that are of a far nobler kind among these senses (which participate in reason at the highest level) sight and hearing.”<sup>88</sup>

But the Pythagoreans speculated above all, and the followers of Aristoxenus were only interested in manual exercises and in following the senses. “... errasse vero utrique.” [“... both the ones and the others [appear] ... to have erred.”] Thus the Pythagoreans adapted “λόγους, Proportiones” [proportions] which often did not correspond to the phenomena. Whereas the Aristoxenians put great insistence on what they perceived with their senses. “... obiter quasi Ratione abusi sunt.” [“... as if they made use of reason [only] on special occasions.”] And for Ptolemy, they succeeded in going both against the nature of reason, and against what was discovered by experience. “... quia Numeros (rationum imagines) non sonituum Differentiis applicant, sed eorum Intervallis. ... quia eos illis adjiciunt Divisionibus quae sensuum testimoniis minime conveniunt.” [“... because they use their numbers (representations of ratios) not for the differences of sounds, but for their intervals. ... because they place them in those divisions which show very little agreement with the testimony of the senses.”]<sup>89</sup>

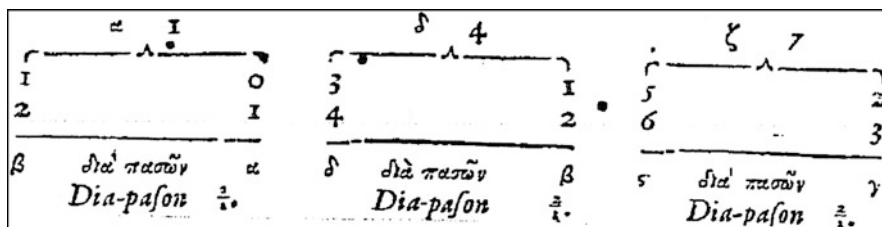
As regards the acuteness and deepness of sounds, Ptolemy described their origin in the quantity of resonant substance. “Adeo ut Sonitus Distantiis (...) contraria ratione respondeant.” [“Such that the sounds correspond to the lengths in an inverse ratio.”] Having made a distinction between continuous and discrete sounds, the former, represented by the lowing of cattle and the howling of wolves, were dismissed as non-harmonic: they would not be liable to being “... nec definitione nec proportionem comprehendere possint: (contra quam scientiarum proprium est.)” [“... to being understood, either by definitions, or by ratios (contrary to what is typical of sciences).”] Among the latter, instead, which he called “Φθόγγοι, Sonos” [tones], it was possible to fix the ratios of the relationships. Then, the combination of these latter ratios gave birth to the “ἁρμονίαι, Concinni” [harmonious] and lastly the “Συμφωνίαι, Consonantias”: Δι' α' τεσσάρων, Diatessaron” [fourth], Δι' α' πεντε, Dia-pente” [fifth] and Δι' α' πασῶν, Dia-pason” [octave]. It was called Dia-pason [through all] and not δι' οκτώ [through eight] because it contained the idea of all the melodies.<sup>90</sup>

The ear perceived as consonances the diatessaron [fourth], the diapente [fifth], the diapason [octave], the diapason united to the diatessaron, the diapason with the diapente and the double diapason. But the “λόγος, ratiocinatio” [reason] of the Pythagoreans excluded the interval of the octave with the fourth from the list of consonances because it did not correspond to the ratios considered as consonant by them: only the ratios termed “ὑπερπαραμέτρων, superparticularium” [superparticular,

<sup>88</sup>Ptolemy 1682, pp. 3–8.

<sup>89</sup>Ptolemy 1682, p. 8.

<sup>90</sup>Ptolemy 1682, pp. 13, 16–18 and 213.



**Fig. 2.1** The numbers used by Ptolemy for the ratios of musical intervals (Picture from Ptolemaei 1682, p. 26)

$n + 1$  to  $n$ ] and “πολλαπλασίῳν, multiplicium” [multiple,  $n$  to 1]. The ratio judged dissonant was represented by the numbers 8 to 4 and 4 to 3, and consequently by the ratio 8 to 3, which is neither multiple nor superparticular. The octave with the fifth, on the other hand, gave 6 to 3 and 3 to 2, and thus 6 to 2, which is equivalent to 3 to 1, a multiple. The double octave was analogously 4 to 1. Therefore, the Pythagoreans’ hypothesis, that adding the octave to the fourth produced a dissonance, became a mistake for Ptolemy, because this was “definitely a clear case of consonance”. Indeed, in general, adding an octave did not change the characteristics of the interval.<sup>91</sup>

“... Prout etiam evidenti experientia compertum est. Non levem autem illis difficultatem creat.” [“... seeing that this is found even by plain experience. This creates a serious difficulty for them [the Pythagoreans] ...”]. Ptolemy found it “absolutely ridiculous” to stop at the first four whole numbers, and ventured to count as far as six, thus arriving at the “senarius” which was to become famous only in the sixteenth century<sup>92</sup> (Fig. 2.1)

Playing with the new numbers, it was not difficult for Ptolemy to recover all the consonances that were pleasurable to his ear. It was thus necessary “... non ipsi [errores] λόγος-Rationis naturae attribueret, sed illis qui eam perperam adhibuerunt.” [“... not to attribute the errors to the nature of reason-ratio-discourse, but to those who erroneously made use of it.”] In the end, therefore, all those consonances were classified as indicated above, without supposing anything “in advance” about multiple or superparticular ratios.<sup>93</sup>

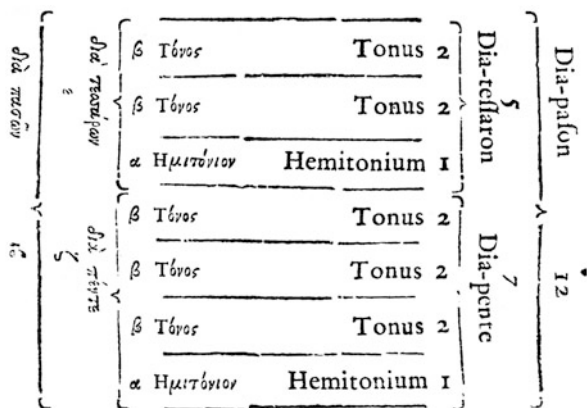
For the λόγος, Ptolemy searched for a “κανόνος, Canonem” [canon, rule], which he found in the “μονοχόρδου, monochordum” [monochord]. The other instruments of sound did not seem to be suitable to avoid the Pythagorean a priori criticisms. He expected “... ad summam accuratorem perduci.” [“... to be conducted to a supreme precision.”] Consequently, he avoided listening to the sounds of the “αὐτὸν, tibia” [flute], or those obtained by attaching weights to strings. “Nam, in tibiis & fistulis, praeterquam quod sit admodum difficile omnem

<sup>91</sup>Ptolemy 1682, pp. 19 and 23–24.

<sup>92</sup>Ptolemy 1682, pp. 25–26. See Sect. 6.6.

<sup>93</sup>Ptolemy 1682, pp. 29–33.





**Fig. 2.3** The division of the octave by Aristoxenus into equal semitones, as related by Ptolemy (Picture from Ptolemaei 1682, p. 41)

Then he went on to criticise the Aristoxenians, much more however than the Pythagoreans. The latter should not have been blamed by the former for studying the ratios of consonances, seeing that these were generally acceptable, but only for their way of reasoning. Instead, the Aristoxenians would not accept them, nor would they invent any better ratios, when they expounded their theory of music. And yet, although these musical impressions touch the hearing, the ratios that express the relationships between sounds should be recognized. However, the Aristoxenians did not explain, or study, how sounds stand in a relationship with one another.

Sed, (...) specierum [εἰδῶν] solummodo Distantias inter se comparant: Ut videantur saltem aliquid numero & proportione facere. Quod tamen plane contrarium est. Nam primo, non definiunt (...) specierum per se quamlibet; qualis sit: (Quomodo nos, interrogantibus, quid est Tonus; dicimus, Differentiam esse duorum Sonorum rationem sesquioctavam continentium). Sed remittunt statim ad aliud quid, quod ad huc indeterminatum est: ut, cum Tonus esse dicunt, Differentiam Dia-tessaron & Dia-pente: (cum tamen Sensus, si velit Tonus aptare, non ante indigeat aut ipso Dia-tessaron, aut alio quovis; sed potius sit, differentiarum istiusmodi quamlibet, per se constituere). [But they compare together only the distances in external aspects, so that they are at least seen to be doing something regarding numbers and ratios. However, this is not really a point in their favour. First of all, they do not define (...) the nature of anything that is, in itself, an external aspect. (As we do when we answer anybody who asks us what a tone is, that it is the difference between two sounds whose ratio is a sesquioctave.) But they invariably make reference to something else which is equally indeterminate for the question: as when they say that the tone is the difference between the diatessaron and the diapente (when, however, the sense that desires to prepare the tone does not need, first of all, the diatessaron, or anything else, but is capable of creating by itself any difference of that kind).]

If they were invited to specify what the above difference is, they would say, if anything, that it is two, and that of the diatessaron is five, and that of the diapason is 12 (Fig. 2.3).

As Aristoxenus had not defined the numerical terms between which the differences were to be calculated, the latter remained uncertain for Ptolemy. The whole procedure to identify the tone by varying the tension of the strings was judged by him as "... inter absurdissima ..." ["... among the most absurd things ..."]. He challenged Aristoxenus' way of measuring the diatessaron as composed of two and a half tones, the diapente of three and a half tones and thus the diapason of six. How? Of course, Ptolemy used the ratios calculated by the numerical procedures of the Pythagoreans, starting from the tone, 9:8. The excess of the diatessaron with respect to the ditone thus became for him the minor semitone. "Quippe cum, in duas aequales rationes (*numeris effabiles*) non dividatur, aut sesquioctava ratio, aut superparticularium quaevis alia: rationes vero duae proxime-aequales, sesquioctavam facientes, sint sesquidecimasexta & sesquidecimasextima: ...". ["As they are not divided into two equal ratios (*that can be expressed with numbers*)<sup>95</sup> or into the sesquioctave ratio [9:8], or any other superparticular ratio; whereas two ratios close to parity which form the sesquioctave are the sesquisixteenth [17:16] and the sesquiseventeenth [18:17]: ..."]<sup>96</sup>

Our renowned Alexandrian mathematician calculated how far the Pythagorean minor semitone, or limma, was lower than a semitone which corresponded to half of a tone. But he did it with whole numbers, without using any roots, probably because he would otherwise have moved music from the discrete side of the *quadrivium* to the continuous side, next to geometry. He obtained such a tiny difference that not even the followers of Aristoxenus, in his opinion, would say that they could hear it with their ears. Therefore, if it could happen that the sense of hearing was likewise mistaken (ignoring the difference), then even greater mistakes would be made in the hotchpotch of many presuppositions to be found in their explanations. The Aristoxenians had demonstrated the tone, 9 to 8, more easily than the ditone, 81 to 64, since the latter was "incompositum, inconcinnum" [without art, not harmonious], while the former was "concinnum" [harmonious]. "Sunt autem sensibus sumptu promptiora quae sunt magis Symmetra." ["After all, those things that are better proportioned can more easily be apprehended by the senses."] <sup>97</sup>

The intentions of the Aristoxenians were made even clearer by the way that they treated the diapason [the octave, considered by them to be exactly six tones], "... (praeterquam ab illa Aurium impotentia)" ["... (as well as by the inability of their ears) ..."]. And Ptolemy demonstrated, on the contrary, with Pythagorean ratios, that the octave contained less than six tones: Aristoxenus had not used numerical ratios to define the diatonic, chromatic and enharmonic genres, but only

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<sup>95</sup>The brackets were added with the italics by Wallis. This enables us to measure the distance between the world of Ptolemy, where it was taken for granted that numbers were only those with a *logos*, rational and expressible, and the sixteenth century, when an equal existence and use would be granted also to non-expressible numbers, the irrationals.

<sup>96</sup>(18:17) combined with (17:16) gives (18:16), which is equivalent to (9:8). Ptolemy 1682, pp. 39–48.

<sup>97</sup>Ptolemy 1682, pp. 49–50.

“διαστημάτων, intervalla” [intervals]. This was the final comment of Ptolemy: “Ipsisque, differentiarum causis, pro non-causis, nihiloque, nudisque extremis, perperam habitis, comparationes suas inanibus & vacuis [intervallis] accomodat. Ob hanc causam, nil pensi habet, ubique fere, bifariam dividere Concinnitates: cum tamen, rationes superparticulares (...) nihil tale patiantur.” [“Having wrongly disposed the causes of the differences in favour of non-causes, and by nothing less arranged simple extremes, he adapts his ratios to empty, baseless [intervals]. For this reason, he does not hesitate to divide the harmonic intervals, in practically all cases, into two parts, when, on the contrary, superparticular ratios do not allow anything of the kind.”]

Instead, the division of the tetrachord [the fourth] of the Pythagorean Archytas of Tarentum was quoted without severe criticism, quite the opposite. Though he, too, deserved to be corrected in certain things, “. . . in plerisque autem, eidem adhaeret, ita tamen ut manifeste recedat ab eis quae sensibus directe sunt comperta . . .”. [“. . . in the majority, on the contrary, he is close to the same [purpose], with the result that he keeps well away from those things that are discovered directly with the senses . . .”].<sup>98</sup>

And yet among all the possible ways of dividing the Greek tetrachord, Ptolemy sought those that were in harmony with the numerical ratios, and with the *φαινόμενον* [apparent, phenomenon]. In short, among the infinite ways of choosing three ratios between whole numbers, which together would give 4 to 3, Ptolemy fixed the superparticular ones to be composed with 5 to 4, 6 to 5, 7 to 6, 8 to 7, 9 to 8. He distributed among these the enharmonic, chromatic and diatonic genres, in turn subdivided into “μαλακός, molle” [soft, effeminate, dissolute] and “συντονός, intensum” [tense], with other intermediate cases. In these markedly Pythagorean games, it remains to be understood what role Ptolemy reserved for hearing, and for the phenomena with which he had stated that he wanted to find an agreement.<sup>99</sup>

Quod autem non modo Rationi congruant praemissae generum divisiones, sed & sensibus sint consentaneae, licebit rursus percipere ex Octachordo canone Diapason continente; sonis, . . ., accurate examinatis, tum respectu aequabilitatis chordarum, tum aequalitatis sonorum. [Furthermore, it will again be understood from the octachord canon containing the diapason, that the above divisions of genres are not only in agreement with reason, but are also compatible with the senses, . . ., after accurately comparing the sounds, with respect both to the uniformity of the strings and to the identity of the sounds.]

He believed that his procedure would stand the test of all the “. . . musices peritissimi . . .” [“. . . most expert musicians . . .”]. “. . . quin potius, in hanc circa aptationem syntaxi [συνταξις] naturam [φύσις] admiremur: Quippe cum, secundum hanc, tum ratio fingat quasi & efformet melodiae conservatrices differentias, tum Auditus quam maxime Rationi obsequatur; Utpote, per ordinem qui inde est, eo adactus; atque agnoscens, . . ., quod sit peculiariter gratum. Quique hujus improbandae partis authores fuerint; neque divisiones secundum rationem aggredi

<sup>98</sup>Ptolemy 1682, pp. 61–62.

<sup>99</sup>Ptolemy 1682, pp. 66–78.

per se potuerint; neque sensu patefactas adinvenire dignati fuerint.” [“... rather, we will admire nature for her availability regarding this adaptation: seeing that in conformity with this, both the ratio practically models and adapts the differences to be maintained to the melody, and, as much as possible, the hearing obeys reason; since it is led to do so by the order that is thus created; and it recognizes, . . ., what is in particular agreeable. And those who would have sustained that this part is to be rejected will neither be able to arrive at the divisions by themselves using reason, nor will they think it worthy to make them known by the senses.”]<sup>100</sup>

Our Ptolemy wrote that he had put the various genres to the test, finding all the diatonic ones suitable for the ears. But in his opinion, they would not be gladdened by the freer modes, such as the soft enharmonic or chromatic ones. “Praeterea, quantum ad totius tetrachordi in duas rationes sectionem, desumitur ea, in hoc genere, ab eis rationibus quae ad aequalitatem proxime accedunt, suntque sibi invicem proximae; nimirum sesquisepta [7 to 6] & sesquiseptima [8 to 7], quae quasi bifariam dividunt totum extremorum excessum. Ipsum igitur, propter ante dicta, tum auditui videtur acceptius, tum & nobis suggerit aliud adhuc genus: Festinantibus utique ab ea concinnitate quae secundum aequalitates jam constituta est, & dispicientibus, siqua haberi poterit, ipsius Dia-tessaron grata compositio, ipsum jam prima vice dividendo in tres rationes prope-aequales, cum aequalibus itidem differentiis.” [“Furthermore, in this [diatonic] genre, as regards the division of the whole tetrachord into two ratios, it is derived from those ratios that are closer to parity, seeing that they are the closest together too. Without any doubt, these are the sesquiseventh [7 to 6] and the sesquiseventh [8 to 7], which divide all the distance between the extremes roughly into two [equal] parts. Thus, on the basis of what has been said above, this genre seems so much the more pleasurable to the hearing, inasmuch as it suggests yet another genre to us: encouraged in particular by that harmoniousness which has already been created on the basis of equalities, and inclined [as we are] to examine what could be considered a pleasurable composition of the diatessaron itself, having already divided it into three almost equal ratios, together with differences that are likewise almost equal.”]

Then Ptolemy reviewed various divisions of the fourth and the fifth into intervals that were constrained to be close to parity in their ratios. He thus came to divide the octave among the numbers 18, 20, 22, 24, 27, 30, 33, 36 (Fig. 2.4).

Sumpta vero aequitonorum, secundum hos numeros sectione, comparebit modus quidem inexpectator & quasi subrusticus, alias autem satis gratus & magis adhuc auribus accommodus, ut haberi despiciatui minime mereatur, tum propter melodiae singulare quid, tum propter bene ordinatam sectionem; tum etiam quia, licet per se canatur, nullam infert sensibus offensionem. [Indeed, having assumed a division of equal tones in accordance with these numbers, a way will appear, which is quite unexpected and somewhat rustic, but otherwise quite pleasant and even more suitable for the ears, such as to deserve not to be at all despised, both because of its particular kind of melody, and its orderly division, and because, even if it is sung, it does not in itself procure any offence to the senses.]

<sup>100</sup>Ptolemy 1682, pp. 78–79.







**Fig. 2.5** Standard monochord of Ptolemy (Picture from Ptolemaei 1682, p. 159)

of producing sounds, but it was even authorised to verify the agreement between the ratios of numbers and the ears. Was this then an experimental apparatus? However, all the ambiguities in him, always solved in favour of rational numerical ratios, and his attitude towards the musical practice of instruments, are made clear in his subject “De incommodo Monochordi Canonis usu” [“On the deleterious use of the monochord canon”]. Here, the previous theoretical monochord became a real instrument in the hands of musicians (Fig. 2.5).

At the time, it was full of defects and inaccuracies, which were covered up or amplified by the event that it was played together with imprecise, and unreliable (for the Pythagorean canon) wind instruments. Experience also allowed our Alexandrian mathematician to criticise Didymus, the musician.<sup>103</sup>

Subsequently, the procedure followed even led him to find pleasure in divisions of the octave, using whole numbers, into tones that were, as far as possible, equal. And yet he lacked that certain something to take a further step along the same road. However, nobody should ever suspect that one of the most influential and famous natural philosophers, and mathematicians, of the ancient world was not able to use square roots for his calculations: the safest and most precise mathematical way, acceptable to the ears, to divide the octave into equal parts. This self-limitation seems to be particularly interesting, because, on the contrary, he calculated the ratios precisely, also by means of geometrical constructions.<sup>104</sup> Geometry was thus to allow him to give, with equal precision, even the proportional mean between 9 and 8: to divide the tone into two exactly equal parts, as the vituperated Aristoxenians claimed to do on their instruments. But, for Ptolemy, harmony was to remain a discrete science, which could use only discrete means, and the thing to avoid was “... sonituum motus continuus (alienissimam ab harmonia speciem continens, ut quae nullum stabilem & terminatum sonum exhibet) ...” [“... the continuous movement of sounds (which contains an aspect that is remote from harmony, like the one that does not manifest any sound that is stable or well specified) ...”].<sup>105</sup> For the equable temperament, Europe and the Western world have to wait until the sixteenth century, but the world is round, and we shall first embark on a voyage to visit other cultures.

<sup>103</sup>Ptolemy 1682, pp. 156–166.

<sup>104</sup>For example, Ptolemy 1682, pp. 97–98ff.

<sup>105</sup>Ptolemy 1682, p. 158.

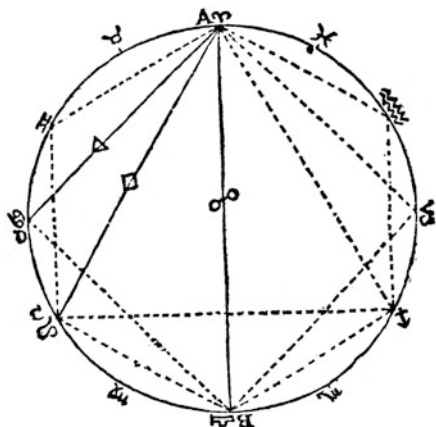
Harmony, in the words of Ptolemy, showed a “*δυνάμις*, facultatem” [power] of its own, and was connected with other things in the world. As sciences above all of ratios, harmony and astronomy were seen as “... cousins generated by the sisters, sight and hearing, and nourished by arithmetic and geometry.” This power was to be typical of movements, especially of those present in the “*οὐρανοῦ*, *divinis corporum coelestium*” [divine elements of heavenly bodies] and in the “*ψυχῆς*, *mortalibus humanarum ... animarum*” [mortal elements of human souls]. But in order to be able to participate in the perfection of mathematical ratios, these new movements should take place in the “*εἶδος*, forma” [ideal form] and not in the “*ὕλη*, materia” [matter], since the power of ratios is not observed “... in eis motibus quibus ipsa materia alteratur, ... cum neque qualitas quae secundum eam sit, neque quantitas, (propter ejus inconstantiam), determinari possit: ...” [“... in those movements by which the matter itself is changed, ... , when neither the quality by virtue of which that happens, nor the quantity can be determined (due to its instability): ...”].<sup>106</sup>

On the basis of these general premises, our renowned ancient astronomer prepared a classification of the effects that harmony should have on souls, and of their relationships with the movements of the heavenly spheres. In his sensitivity to coincidences in numbers, he linked the various faculties of the soul to the different consonances. The diapente [fifth], for example, should correspond to the five senses, the diatessaron [seven notes] to the seven faculties of the intellective soul: “... Imaginationem, ... Mentem, ... Cogitationem, ... Discursum, ... Opinionem, ... Rationem, ... Scientiam ...”. Morals began to come into the question with the diatessaron, which should influence the covetous soul, while the diapente should affect the rational element. Harmonious sounds reveal virtues, non-harmonious ones vices, and so on. “*Animarum Virtus est earum quaedam concinnitas & Vitium inconcinnitas*.” [“The virtue of souls consists of a certain harmoniousness, but vice is found in lack of harmony.”] Consequently, the diatessaron stimulates “... Temperantia, in contemptu voluptatum, Continentia, in sustinendis indigentis, & Verecundia, in vitandis turpibus.” [“temperance, in despising pleasures, continence, in helping the needy, and modesty, in avoiding turpitudes.”] The diapente should regard, on the contrary, “... Mansuetudo, ... Intrepidus animus, ... Fortitudo, ... Tolerantia, ...” [“... meekness, ... bravery of soul, ... fortitude, ... tolerance, ...”], whereas the diatessaron should be linked with a whole series of seven other virtues: “*Acumen, ... Ingenium, ... Perspicacia, ... Judicium, ... Sapientia, ... Prudentia, ... Peritia*” [“shrewdness, ... intelligence, ... perspicacity, ... judgement, ... wisdom, ... prudence, ... competence.”] All this also should make it possible to obtain a good condition of the body.

If the theoretical domain included three parts, the natural, the mathematical and the divine, and the practical realm three more parts, the ethical, the economic and the political, then there must be three harmonic genres, the enharmonic, the chromatic and the diatonic. Ptolemy coupled the enharmonic with nature and ethics,

<sup>106</sup>Ptolemy 1682, pp. 232, 236 and 238.

**Fig. 2.6** Ptolemy's zodiac  
(Picture from Ptolemaei  
1682, p. 254)



the chromatic with mathematics and economy and the diatonic with theology and politics, leaving space, however, for some overlaps. As a consequence of the conditions of life, in their continual alternation between peace and war, or indigence and abundance, our souls are influenced by the different modes, with passages from deep to acute. “Atque hoc ipsum credo Pythagoram considerasse, cum suaserit ut, primo mane exsuscitati, antequam actionem aliquam auspicarentur, musica uterentur & blando cantu.” [“Furthermore I believe that Pythagoras was thinking precisely of this when he gave the advice to make use of music and a sweet song, after waking up early in the morning, before starting any kind of action.”]<sup>107</sup>

Partly for the influence that it had in Europe until the seventeenth century, the closing passage of book 3 of this *APMONIKA* should be remembered. Here the correlations with the “ζωδιῶν κυκλον, Zodiaci circuli” [circle of the zodiac] were based on numbers, and became more precise. “... Coelestium ... corporum hypotheses secundum rationes harmonicas confectas esse.” [“... The principles of the heavenly bodies are composed in accordance with the harmonic ratios.”] The order of sounds and their tension proceed in a straight line, but their power and their constitution are circular. As the revolutions of heavenly bodies are also circular, the ancient astronomer constructed correspondences between the 12 points of the zodiac and the musical notes, dividing the circle according to the proportions established by the musical harmony that had previously been explained<sup>108</sup> (Fig. 2.6).

Deep sounds are compared with the stars in the position where they rise and set, whereas in their highest position at midday, they are closer to acute sounds. And by so doing, Ptolemy distributed musical genres and modes among other astral features, such as the phases of the moon. He divided the circle into 360 parts in order to calculate conjunctions, oppositions and trines in accordance with their relative

<sup>107</sup>Ptolemy 1682, pp. 239–248.

<sup>108</sup>Ptolemy 1682, pp. 249–258. See Part II, Sect. 8.3.

harmonic ratios. He concluded that the sound of Jupiter with those of the Sun and the Moon formed diatessaron and diapason consonances, respectively, while the sound of Venus with the Moon formed one tone. “Evil” planets, like Saturn and Mars, together with those that have “beneficial” effects, like Jupiter and Venus, form a diatessaron consonance. And so on, with various other combinations among planets, consonances and dissonances, variously justified by numbers, by general principles and by dizzy analogies.<sup>109</sup>

In the *Astrological previsions addressed to Sirius*, also known as *Tetrabiblos*, [*The Four Books*], Ptolemy classified the signs of the zodiac not only as male and female, but also based on their reciprocal affinities. And he derived these from their musical ratios, applying to their aspects, (that is to say, to the planets’ angular arrangements with respect to one another), the sesquialtera musical ratio, 3:2, and the sesquithird ratio, 4:3. He obtained that the trine (120°) and the sextile (60°) were then  $\sigma\nu'\mu\phi\omega\nu\iota$  [consonant], whereas the quadratures (90°) and oppositions (180°) were  $\alpha'\sigma\nu'\mu\phi\omega\nu\iota$  [dissonant].<sup>110</sup>

We have dwelt in particular on the *APMONIKA* of our renowned astronomer-astrologer-mathematician from Alexandria, because in general it is wrongly overlooked. On the contrary, other historians have studied, and undoubtedly continue to comment on his most widely used and best known book, the *Syntaxis mathematica* [*Mathematical Order*]. In Europe and the Near East, however, its title was to be completely changed from Greek to Arabic, *Almagest* [*The Greatest*]. The peoples on the southern shores of the Mediterranean were about to break into our history, and were to give this transliterated name to the Greek astronomical collection, *megiste* [greatest].<sup>111</sup> Thanks to its mathematical precision and its accuracy in observing more than one thousand stars, the book was to dominate astronomical and astrological discussion until the seventeenth century. Everybody read it and commented on it, but initially, nobody translated it directly from Greek, but rather from Arabic into Latin.<sup>112</sup>

Ptolemy had to carry out many calculations in order to represent the positions of the stars on the vault of the heavens. He obtained them by means of the chords of the circle, which he measured with great precision, proceeding by half a degree at a time in preliminary tables. However, these are not to be considered truly trigonometric, because *sines* and *cosines* only arrived thanks to the Indians, who made their calculations with semi-chords.<sup>113</sup> Of course, he also needed a good value

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<sup>109</sup>Ptolemy 1682, pp. 260–273. Cf. Barker 2000. He showed that “Ptolemy understood very well what conditions must be met if experimental tests are to be fully rigorous, ...”. However, concerning “... how far the treatise is faithful to the principles it advertises, ...”. There are grounds for some scepticism here, ...”. Therefore, in an independent way, my analysis does not side in Ptolemy’s favour: because, with great probability, the Alexandrian did not test either the attunements of pipes, or Aristoxenus’.

<sup>110</sup>Ptolemy (Tolomeo) 1985, pp. 60–63; translation corrected by me.

<sup>111</sup>See Sect. 5.4.

<sup>112</sup>Boyer 1990, p. 294.

<sup>113</sup>See Sect. 4.3.

for the relationship between the circumference and the diameter ( $\pi$ ), which made an improvement on that of Archimedes, 22 to 7, arriving at 377 to 120, equivalent to 3.1416 in decimal figures, which had not yet been introduced. His generally famous and widely discussed ideas include cycles, epicycles, eccentrics and equants, with which, without foregoing the musical harmony of circular movement, our astronomer-astrologer explained with a good degree of precision the complex movements of heavenly bodies, which were far from uniform and regular.

In his monumental *Geography*, he catalogued thousands and thousands of cities, rivers, and countries, situating them on the surface of the earth with their latitude and longitude. But he underestimated the size of the earth, and consequently overestimated the longitudinal size of his world.<sup>114</sup> No exact system had yet been found to calculate the longitude, as this was to appear only in the eighteenth century. Others after him were likewise to overestimate the size of the Mediterranean, and also the Northern part of the earth with respect to the South, through the projections chosen to represent the terrestrial sphere on the plane of geographic maps.<sup>115</sup> As there is more than one way of projecting a sphere on to a plane, every projection maintains certain characteristics of the figures on the sphere to the detriment of others. Thus the choice becomes subjective, and highlights not so much the geometrical ability as the practical interests and the culture of scholars. In general in that period, they revealed that they considered their own countries as the centre of the world. In the following chapters, we shall expose the limits of a similar Eurocentric vision, not only in geography.

I do not consider it as the goal of historical writing to condense the complexity of historical processes into some kind of digest or synthesis. On the contrary, I see the main purpose of historical studies in the unfolding of the stupendous wealth of phenomena which are connected with any phase of human history and thus to counteract the natural tendency toward oversimplification and philosophical constructions which are the faithful companions of ignorance.

Otto Neugebauer.

## 2.7 Archimedes and a Few Others

So far, we have ignored famous natural philosophers such as Democritus, Eudoxus, Archimedes, Apollonius, Diophantus, Heron, Theon, Hypatia, Pappus and others, because there is no mention of any theory of music in their extant texts which have luckily been handed down to us. Of course, this should not been turned into a value judgement about them, or about anyone else. For them, readers are simply referred to the many other history books that deal with them exhaustively. Let us recall only Democritus of Abdera (460–370 B.C.), who reasoned about fundamental

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<sup>114</sup>Boyer 1990, pp. 193–200.

<sup>115</sup>Peters 1990.

elements, which cannot be broken down any further. These were thought to have formed all the things in the world, moving in a void: the famous atoms. Drawing his inspiration from the numerical atoms of the Pythagoreans, he sustained that even geometrical figures were composed of indivisible fundamental elements. How he would have coped with the *continuum*, incommensurable magnitudes and movement (remembering the paradoxes of the Eleatics), we are unable to say. His writings have been lost, or were treated too negligently by the rival schools of Plato and Aristotle, who did not take enough care to preserve them.<sup>116</sup>

Whether represented or not by means of music, the extent to which the problem of incommensurable ratios was felt to be important in Greek culture would be illustrated in the works written by one of Plato's disciples, Eudoxus of Cnidos (c. 408–c. 355 B.C.), if any were extant. In any case, he was credited with the invention of a method to compare together even ratios of incommensurable magnitudes. Furthermore, he successfully approximated curved figures, such as circles, by means of polygons with a large number of straight sides, obtaining results regarding their ratios of lengths, areas and volumes. In modern times, when the name of the author had been forgotten, as happens all too often, curiously, in the history of sciences, his procedure was to be given a name: Archimedes' exhaustion method. Increasing the number of sides, the polygon comes closer and closer to a circle, until it becomes one with it. To Eudoxus, lastly, we owe a model to represent the movement of stars, made up of concentric spheres in uniform movement, with the Earth immobile at the centre. This cosmology, substantiated by a perfect crystalline matter, was adopted by Aristotle and was to enjoy great success for thousands of years.<sup>117</sup>

Aristarchus of Samos (third century B.C.), on the contrary, said that the Earth was moving and the Sun immobile. But in antiquity, his model did not enjoy the same popularity. The only one who quoted it, for other reasons, was Archimedes, who, however, criticised it for its somewhat imprecise way of dealing with magnitudes.<sup>118</sup> This example will be sufficient to avoid recurring commonplaces about the ancient scientific world, and prepare us rather to understand those selective contexts which made one theory the orthodoxy promoted by the most famous philosophers, while rival theories were heresies worthy only of being forgotten.

As regards the renowned Archimedes of Siracusa (287–212 B.C.), we may recall his experiments on the equilibrium of liquids, his numerous mechanical inventions and his calculating ability, in a style that was not exactly that of Euclid, regarding curvilinear figures and bodies like spheres and cylinders. He was an expert in dealing with levers and balances, and, unlike others who were more theoretical, he was not averse to turning theory into practice. Exploiting his ability in calculations, this natural philosopher invented a procedure to obtain the length of the circumference, knowing the diameter, but the result was only approximate. He inscribed inside the circle a regular hexagon, whose perimeter was easy to calculate, as the figure

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<sup>116</sup>Boyer 1990, pp. 94–96.

<sup>117</sup>Boyer 1990, pp. 105–110.

<sup>118</sup>Archimedes 1974, pp. 447–448.

was made up of six equilateral triangles. He obtained a perimeter whose length was 6, if the diameter of the circle was 2, and a ratio of 6 to 2 ( $\pi = 3$ ), but the circumference, of course, was longer. Then he circumscribed another hexagon around it, but the perimeter was now too long. Then he transformed the hexagon into a regular dodecagon, constructing a triangle in the space that remained between the polygon and the circle.

The perimeter of the new polygon of 12 sides, both inscribed and circumscribed, gave a closer approximation to the circumference. Its side could easily be calculated from the hexagon, using the theorem of Pythagoras. Then the operation could be repeated, obtaining better and better values for the circumference. Archimedes made the calculation automatic, and thus a question of time and patience, using recurring formulas to obtain the new perimeter, by doubling the sides. If  $P_6$  and  $p_6$  indicate the perimeters of the circumscribed and inscribed hexagons, respectively, then  $P_{12}$  and  $p_{12}$  could be obtained by means of the formulas (in post-Cartesian symbolic notation)

$$P_{12} = \frac{2p_6P_6}{(p_6 + P_6)}, \quad p_{12} = \sqrt{p_6P_{12}}.$$

That is to say, he turned them into the famous harmonic and geometric means of the Pythagorean musical tradition and so on. Another ratio made important by the theory of music, 3 to 2, returned in the result to which the mathematician from Siracusa would have liked to consign his remembrance and his fame. He had found that the same ratio held between the volume of a cylinder and that of a sphere inscribed in it, as also between the relative areas. Cicero was to relate that he had seen the figures engraved on his tomb. Three to two was also the ratio between the volume of the paraboloid of revolution and that of a cone with the same base and the same height. He found that 4 to 3 was the ratio between the area of the parabola and that of a triangle with the same base and the same height.<sup>119</sup>

Thus, together with Euclid's style of proof, the Pythagorean tradition continued to make its effects felt on Archimedes. And yet it would not be difficult to find in him also impulses and problems that might have separated him from it. How far would he have to go in multiplying the sides of polygons? When would we reach the final circle with certainty? Wasn't this reminiscent of a certain paradox of Zeno from Elea? Today it would be easy for us to answer: go on to infinity. However, this was the very notion that they tried to avoid in the Greek world of the period. In order to indicate it, they would have made use of the word  $\alpha'\pi\epsilon\iota\lambda'\rho\omega\nu$ , which means "boundless, without limits, unfulfilled, without means", while the verb  $\alpha'\pi\epsilon\iota\lambda'\rho\omega$  means "to give up, to get tired, to succumb, to be forbidden". Our man from Siracusa seems to be reluctant to detach himself from Pythagorean whole numbers or from the geometrical theorems of Euclid. And yet he did so with his procedures to calculate the volume of a sphere, using the system which was later, in

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<sup>119</sup>Napolitani 2001, pp. 21, 32–33, 36–37.



another epoch, defined by others as “exhaustion”:  $\frac{4}{3}\pi r^3$ . But he cannot have been completely sure about it.

Instead of daring to take a bold step off limits into the infinite, he rather used the “*reductio ad absurdum*”. He demonstrated that the magnitude known to him must be greater than a certain value, and at the same time less than the same value, and therefore it must be equal to it. It appeared to be another typical choice between alternatives seen as incompatible, like the choice between even or odd numbers in the Pythagorean argument about the impossibility of measuring  $\sqrt{2}$ . Had he invented, or copied (from Eudoxus) arguments that he chose not to theorise, in order not to come into conflict with the environment, and his points of reference at Alexandria? He had looked beyond whole numbers and Euclid, but it would seem that he preferred not to use his *logos* to talk about it. How would he succeed in knowing in advance the result which he was starting to prove? As he has not left us anything written, historians have advanced various conjectures regarding the ‘mechanical’ heuristics of Archimedes. To these, I now add the theory of music, seeing that those his ratios practically always arrived at 3:2, 4:3, 3:1. Or else, let us consider that as polygons come closer to a parabola, they arrange themselves in a geometrical succession, like the ratios of musical intervals. Lastly, cylinders circumscribed around a paraboloid stand in a relationship to one another like the numbers 1:2:3:4:...

However, he lived in a world that was different from the sectarian mysticism of the Pythagoreans, and from the pure geometrical theories of Euclid. The problems to be solved arrived from a world that was far from being raised to the heavens of Plato. Some of these have remained famous: the hydrostatic force equal to the weight of the liquid moved, levers to launch heavy ships, the equilibrium of paraboloids subject to gravity immersed in liquids as if they could float, devices with the shape of a spiral screw, inclined so as to convey water upwards and distribute it into channels to irrigate fields, estimates of astronomical distances and of differences between metals. And his calculations provided solutions not only for all these various practical problems.

In a text entitled *The Approach*, discovered by chance only at the beginning of the twentieth century, Archimedes recounted that on the contrary, his main mathematical inventions derived from preliminary investigations of a ‘mechanical’ kind. He busied himself with balances, fulcrums and levers, which acted on the paraboloids, triangles, segments, sections of spheres and cones under examination, now treated as composed of heavy matter. He materialised ratios in the law of the lever,  $l_1 : l_2 = p_2 : p_1$ . A small weight  $p_1$  at a great distance  $l_1$  from the fulcrum is in equilibrium with a large weight  $p_2$  at a smaller distance  $l_2$ . Weights and distances are therefore proportional, like the notes and the lengths of strings in music, *mutatis mutandis*. His was thus a terrestrial world subject to gravity, and within certain limits, it was even in movement. He generated the spiral that bears his name today by moving a point along the circumference while at the same time varying its distance from the centre. In this way, he even discovered tangents.

In writing about the law of the lever, his reasoning seems to be less sensible to vicious circles, and consequently less linear than Euclid, but equally indifferent to logic.<sup>120</sup> “Now I am persuaded that this [mechanical] method is no less useful for the proof of propositions; because some of them, which for me were clear from the beginning on the basis of mechanics, were subsequently proved by geometry, since an inquiry conducted with this method excludes a proof.” Mechanics, wrote Archimedes, provided him with the idea of a correct conclusion. “This is why, recognizing by myself that the conclusion is not proved, but with the idea that it is exact, we shall, at the right place, provide a geometrical proof.”

Therefore, in spite of all his inclinations and his faith in the world of mechanics, our renowned man from Siracusa continued to confirm his affirmations in the geometrical language of Euclid. The latter was to maintain his role of matchless warranter of the truth, providing the general scheme of argumentation in orderly lines of propositions, until Galileo Galilei and Isaac Newton. Archimedes lived in a world divided in half between the earth of phenomena and the perfect ideas of geometry, between the necessary approximations of the former and the exactness of the latter. He appears to be uncertain of where to take his stance, because he would like to stand on both sides. His Alexandrian interlocutors, like Eratosthenes (c. 276–c. 194 B.C.), the director of the famous library, had remained in the safe wake of Euclid, and expected from him those theorems that he provided them with. And yet our man from Siracusa also wrote that, thinking of the figures of geometry as part of the world of heavy matter, they would find other, new propositions not yet discovered. “Actually, in favour of this [mechanical] method, once it has been expounded, I am sure that propositions that have not yet appeared to me will be found by others, both those who are now alive, and scholars of the future.”<sup>121</sup>

For various centuries, Archimedes was to remain an unheeded prophet, while everybody else, for one reason or another, (Pythagoreans, followers of Plato or Aristotle, Christians, Neoplatonics or Muslims) continued to study and to comment, with great respect, above all on Euclid’s *Elements*. Then, with the scientific revolution of the seventeenth century, or in some cases even earlier, many scholars started to read the works of Archimedes again, inventing, as he had foretold, procedures, techniques and new mathematical theorems variously connected with astronomy and mechanics. But the text of *The Approach*, containing the prophecy that was being fulfilled, was not extant at that time, and was never to be studied by those who were to draw advantage from it. By a strange quirk of fate, it came to light again only at the beginning of the twentieth century, when the mathematical community had abandoned the heuristic methods of Archimedes, condemned for their lack of rigour, and was constructing a completely different orthodoxy. Going way beyond the Platonising abstraction of Euclid, the formalistic axiomatics of David Hilbert was now following a new criterion of rigour, according to which mathematical arguments were to be expressed by means of pure signs on paper,

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<sup>120</sup>Napolitani 2001, pp. 43–44.

<sup>121</sup>Archimedes 1960, II, pp. 478–479 and 484.

without any meaning posited either by mechanics or by figures.<sup>122</sup> How could certain historians of mathematical sciences not be influenced after a similar change in research and teaching?

Left in his context, therefore, Archimedes could never appear to be an abstract academic, only interested in his theorems, even though a Platonic philosopher like Plutarch (first and second century) tried to depict him as such. According to him, Plato had even rebuked Archytas and Eudoxus because they ruined "... the excellence of geometry, abandoning it with its abstract ideal notions, to pass to sensible objects ... This is how degenerate mechanics was separated from geometry, and, long despised by philosophy, it became one of the military arts."<sup>123</sup> It is only by believing this Greek philosopher that historians of mathematics might succeed in making Archimedes more similar to those mathematicians of our time, inspired by David Hilbert, than he was different from his Alexandrian interlocutors.

On the contrary, he was so present in the reality of his time that he suffered all its tragic consequences. Involved in the Second Punic War, against Rome and on behalf of Carthage, he defended the besieged Siracusa using catapults, powerful winches and his legendary burning-glasses. After the city had fallen into the hand of the enemy, Archimedes is said to have been killed by a Roman soldier. Should the episode be emblematic of the culpable indifference of Roman culture towards the mathematical sciences, to which it never made any significant contribution? It is, on the contrary, a good example not only of the many other faults of war, but also of the declared interest in the sciences, seen as particularly useful in military activities. Marcellus, the victorious general, had taken pains to give orders that the life of the famous natural philosopher should be spared; but in the heat of the looting and the general bloodshed which was the custom of the valiant Roman soldiers, his orders were not obeyed. Subsequently, Cicero ordered his tomb to be traced and repaired with the emblems of the sphere and the cylinder mentioned above. Today, however, undoubtedly as a result of innumerable other similar joyful events, which those in power take pleasure in offering us, it has again been destroyed.<sup>124</sup>

The most curious work by Archimedes would appear to be the *Stomachion* [the word is said to derive from 'stomach', but it is likely to be the name of a puzzle]. In this operation, the renowned mathematician divided a square into 14 pieces, demonstrating that they were commensurable parts, 1:2, 1:4, 1:6, 1:12, 1:24, 1:48. In this way, following the path opened up by Book 10 of Euclid's *Elements*, he was perhaps trying to recover some of the commensurability lost with the relative diagonals.<sup>125</sup>

The pages of Archimedes were treated worse than those of Euclid. Can we not take the extremely limited diffusion of the translations of William of Moerbeke (thirteenth century) or the failure of a printed edition by Johannes Mueller from

<sup>122</sup>Tonietti 1982a, 1988, 1990, 1992b; Napolitani 2001.

<sup>123</sup>Authier 1989, p. 107.

<sup>124</sup>Authier 1989.

<sup>125</sup>Archimedes 1960, pp. 467–473.

Königsberg, nicknamed Regiomontanus (1436–1476), as a judgement also on the little interest shown in the work? In actual fact, Euclid was printed for the first time in 1484, and Archimedes, on the contrary, had to wait until the edition published at Basle in 1544.<sup>126</sup> Of his lost, or missing works, we know a few names, and sometimes the results contained in them. One has come down to us because it was translated into Arabic and then into Latin. The comprehensible part of the *Stomachion* derives from an Arabic manuscript. This may reveal the judgements to which the inventions of Archimedes were subjected. They were moving away from the orthodoxy of the period, if this may be represented by the original ancient *quadrivium*. Why has nothing connected with music remained of such a similar volcanic, polyhedric figure? His indifference towards that part of the *quadrivium* to which the Pythagoreans were most attached is a measurement of his distance from them and from other scholars who in various ways took their inspiration from them. However, it is difficult to exclude that a similar work, if it was written, may have been lost. If a text by Archimedes about music were to re-emerge, like *The Method*, from a palimpsest used for the liturgy of Orthodox Christianity, may we expect significant variants to the division of the diapason?

It is true that chance may guide events to unexpected conclusions. As in the case of the town of Pompeii, which was preserved better than all the others because it was destroyed by Vesuvius, so those who desired to cancel the ancient pagan philosopher, covering his text with edifying prayers, in the end obtained the opposite effect of preserving it. We shall also see later that the spread of Greek scientific culture, not only of Archimedes, in other countries to the east, was the result of the ban to which it was subjected in its original cradle, seeing that the new religion had formed an alliance with imperial power. Chance and heterogenesis of ends, as philosophers rather too obscurely and pompously call the art of achieving results which are totally different from those desired, may sometimes even become the source of happiness and surprising discoveries, not only for historians and not only for the history of sciences.<sup>127</sup>

To Apollonius of Perga (Asia Minor, c. 262–c.190 B.C.), who worked at Alexandria in Egypt for one of the kings named Ptolemy (they were descendants of the first general of Alexander the Great), we owe the terms in current use for conical sections: ellipse [lack], hyperbole [throwing beyond] and parabola [comparing by placing beside]. He drew on a terminology already used also in rhetorical discourse with analogous meanings.

Again, some of his works are extant because although they were lost, they were read with interest by Arabic scholars. Among the many works lost and reconstructed thanks to quotations and subsequent commentaries, there was also *Section of a ratio*, which might have disappointed us if it had dealt exclusively (or mainly?) with ratios between straight lines, ignoring music. But in the second book of the famous *Conics*,

<sup>126</sup>Napolitani 2001, pp. 67–77.

<sup>127</sup>Boyer 1990, pp. 143–165; Napolitani 2001.

we at least find the harmonic division as a ratio between segments distributed along the axis of the ellipse.<sup>128</sup>

Here in the *Conics*, he also left the favourite sentence of many mathematicians, when some profane individual asks them what the use is of all these theorems. “They deserve to be accepted for the sake of the proofs themselves, in the same way as we accept many other things in mathematics for this, and no other reason.” But doesn’t perhaps the answer of this emigrant at Alexandria reveal the problem that is present in a historical context that would have expected much more from its natural philosophers? Did he only dedicate his spare time to his conics? What about the heterogenesis of ends, then? What did he think about Plato’s *Republic*? In this way, would he free himself from all moral responsibility? In any case, supposing that they had not already been stimulated, some of these abstruse properties of conics found a rapid justification in the (military? nautical? commercial? territorial expansion?) art of projecting a sphere on to a plane, for the purpose of making geographical maps.<sup>129</sup>

Everything that other Alexandrian scholars maybe disliked, or tried to hide, Heron of Alexandria (first century), on the contrary, confidently displayed. He dealt with practical problems, giving formulas to solve them, and ignoring theorems to prove them. He constructed machines for warfare, musical instruments such as wind organs, and various devices, and he loved to measure every kind of magnitude, without worrying too much about the theoretical constraints set by his more illustrious colleagues. Being this man far from the usual commonplaces about Greek mathematics, some have even tried to deport him, labelling him as Babylonian or pre-Arabic. Some formulas still bear his name, such as a procedure to extract square roots, already known (of course?) to the ancient Babylonians.<sup>130</sup> We are debtors to him for the following definition of mathematics. “Mathematics is a theoretical science of things understood by the mind and by the senses, which fall into its traps. Someone has said shrewdly and rightly of mathematics what Homer says of Eris, the goddess of strife. . . . Thus mathematics starts from a point and a line, but then its action extends to the heavens, to earth and to all the beings of the universe.”<sup>131</sup>

Another umpteenth inhabitant of the same city was Diophantus of Alexandria (maybe third century). Projecting, as usual, their own idea of the mathematical sciences on to the ancient character, or, even worse, in order to belittle the Arabs, some would already consider him to be an “algebraist”. Like others, only half of his works have come down to us, but unlike the majority, he dedicated himself to the theory of numbers using non-geometrical procedures: original results are to be found in his *Arithmetic*. While everybody else discussed the subjects under examination with discourses and words taken from everyday Greek, even if loaded

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<sup>128</sup>Cf. Fano & Terracini 1957, pp. 356–360.

<sup>129</sup>Boyer 1990, pp. 166–184.

<sup>130</sup>Boyer 1990, pp. 201–204.

<sup>131</sup>Heron, Heiberg edition, IV, 162.

with particular technical meanings, Diophantus, on the contrary, used “syncopated” words (from the Greek for “to break, to shorten”) to indicate the powers of numbers and the number to be sought (the unknown). We may see in these the beginning of a special symbology for calculations in mathematics, separating it from the common expressions of daily life. In this way, he wrote sequences of terms and numbers that were almost the equivalent of modern polynomials. However, his *Arithmetic* appears to be a list of numerical problems for which he was trying to find complete, or rational, solutions. Seeing the rigorously numerical spirit that animated him, this other Alexandrian might seem to be a genuine Pythagorean who was totally unaware of problems with incommensurable magnitudes, and consequently did not need to have recourse to geometry in order deal with them. For this reason, we might also wonder if there may have been, among his lost works, even a numerical theory of musical intervals.<sup>132</sup>

As regards another mathematician and philosopher who used the Greek language, Nicomachus of Gerasa (first century), we may more confidently say that he was an orthodox Pythagorean. In his *Introduction to arithmetic*, we find the complete tradition of this sect: from division into even and odd numbers to ratios between whole numbers used for music. The following generations of Pythagorean musical theoreticians took their inspiration from him.<sup>133</sup>

Something musical re-emerged in the last great Alexandrian mathematician, Pappus (fourth century). He commented on Book 10 of Euclid’s *Elements* in a work which would have been lost, as usual, if it had not been of interest for the Arabs, who translated it and preserved it. Here we find the problem of incommensurable ratios, though it is discussed with the idea that magnitudes are rational, or otherwise not rational, only by convention, and not as a result of their intrinsic nature. Euclid had chosen a segment with respect to which he measured the rationality of other segments. And he had also deliberately broadened the notion of “rationality” to that of “potential rationality”, when the squares of segments proved to be rational. In this way, the side and the diagonal of the square became “potentially rational”, in the ratio 1:2, taking the side as the measurement. Then he had classified the other irrational segments in various categories, which he then treated with additions and subtractions. He gave the name “apotome” to the difference between two magnitudes which were only potentially commensurable. Commenting on this, Pappus associated the apotome with harmony, whereas the other irrational segments were correlated with arithmetic and geometry. Thus he harked back to the *quadrivium* of the Pythagoreans, and for the rest, references were not lacking to Plato. In this way of dealing with irrational magnitudes only by means of geometry, Pappus remained in the wake of Euclid, and to leave this course, it will be necessary to wait for some time, until the arrival of subsequent contributions made by Arabic scholars.<sup>134</sup>

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<sup>132</sup>Boyer 1990, pp. 211–215.

<sup>133</sup>Boyer 1990, pp. 210–211.

<sup>134</sup>Ben Miled 2002, pp. 351–352. See Chap. 5.

The *Collection* of this other natural philosopher from Alexandria is also rich in precious historical details about a world that was fading away, together with interesting new theorems; even though it remains a work of classical geometrical accuracy. In book 3, our musical means were represented in an original manner on the same semicircle. DO is the banal arithmetic mean between AB and BC, and DB a well-known geometrical mean, whereas the representation of the harmonic in DF appears to be an original idea of Pappus.

The geometer generalised the theorem of Pythagoras to include also all kinds of non-right-angled triangles, on sides where he constructed all kinds of parallelograms. He attributed a certain mathematical intuition to bees, seeing that they were capable of literally constructing hexagonal prisms, with which they realised an economy of material: given the same perimeter, the hexagon includes a larger area than polygons with a smaller number of sides. The largest area would be that of the circle. He studied curves created in relationship to distances from a growing number of sides. Taking his cue from this problem, Descartes will arrive at his *Geometry* in the seventeenth century. With Pappus, a “back to front” method of proof called “analysis” became explicit. In this method, we start from the property that is sought, and we derive other consequences from it. If these include the starting premise, then the property is considered as proved, but if properties considered impossible are obtained, then also the property sought is considered impossible.

In the cultural context of Alexandria, many other singular figures were born. Among them, we may mention Theon (fourth century), who wrote commentaries on some of the above-mentioned books, including Euclid, and we owe to him and to this activity of his the existence of the most ancient editions of the *Elements*. His daughter Hypatia (fourth and fifth century) continued her father’s work, but in 415 she was lynched by a crowd of Christians, who did not tolerate that she had maintained such a great admiration for those aspects of classical Greek culture which they hated so much. Furthermore, it must be significant that she was one of the very few members of the female sex in our history.<sup>135</sup> This tragic episode brought to light the contrasts between ancient tradition and the new form of Christian religion, which was changing the historical context. Episodes of intolerance and censure towards disapproved cultural aspects were to assume a formal character in the edict of the Christian emperor Justinian, who officially closed the pagan schools of Athens in 529. Also Proclus (410–485), a scholar who studied Plato, has left us a commentary on Euclid, together with historical details about ancient mathematicians, which the new context was cancelling.<sup>136</sup>

In our history, Harmony is not only the daughter of Venus, but also of a father like Mars. War, soldiers or political powers that were born from wars have already appeared several times, and cannot be omitted without compromising an understanding of events.

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<sup>135</sup>Cf. Boyer 1990, though here at p. 209 the Italian translator turned her into a man.

<sup>136</sup>Boyer 1990, pp. 215–225.

The commander and tyrant, Architas, gave the Pythagorean mark which was to continue to the end, passing through Plato's *Republic*, as an essential element to educate young soldiers. After the death of Alexander the Great in 323 B.C., his general Ptolemy seized the kingdom of Egypt and transformed Alexandria into the centre of the cultural and scientific world, making it particularly powerful in many other ways. Also Aristotle died in 322 B.C.

All the greatest natural philosophers that we have discussed were there or thereabouts they would have passed along. Archimedes and Heron arrived at the explicit design of war machines. Apollonius worked directly for Ptolemy Philadelphus as his Treasurer General. Yet, based on the little that we know, not all of them were born there, quite the opposite. But at Alexandria they reached their maturity and worked, becoming captivated by the place. How can we define a capacity like this, which attracted famous figures from the four corners of the Mediterranean? If the term 'scientific policy' seems too anachronistic, what should we think of the resources placed at the disposal of scholars here, the meetings that they expected to benefit from, the circulation of writings contained in the famous library? As king of Egypt, Ptolemy set up for this purpose the *Museum* [Casket of the Muses] and collected hundreds of thousands of papyri. Directing the great library was a prestigious task that was carried out by famous scholars.<sup>137</sup>

These scholars, though not always closely linked with Pythagorean ideas, were at least under the influence of Platonic philosophies, and underlined the ideal qualities of their research. Then, as today, scholars claimed their independence, guided only by a love for the truth. This, of course, freed them from many other concerns, including, not to be overlooked, the assumption of responsibility for what they were doing, like all other common mortals. They often did not let their values become evident, and ignored, above all, moral values. We, on the contrary, shall follow the priceless advice of Albert Einstein (1879–1955), and contemplate not only what natural philosophers wrote on the subject, but above all the way that they acted and how they behaved during their lives.

## 2.8 The Latin Lucretius

Titus Lucretius Carus (c. 98–54 B.C.) is, like Aristoxenus, another figure famous for his absence from current histories of the sciences. His *De rerum natura* [*On the nature of things*] is generally excluded from them, with the exception that we shall see, because it does not correspond to the recurring models found in other writings on sciences. Lucretius composed verses in Latin instead of listing propositions in Greek. He described natural phenomena visible to everybody instead of proving geometrical theorems that could only be imagined. He did not refer back to the Pythagoreans, or to Plato, or to Euclid, but to Epicurus (fourth century B.C.).

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<sup>137</sup>Napolitani 2001, p. 9.



In his poem, we do not find any figures, or numbers, or ratios, but “primordia” [primordials, fundamentals] and “inane” [void].

Corpora sunt porro partim primordia rerum  
partim concilio quare constant principiorum.  
[Bodies are indeed partly primordia of things  
partly they are unions composed of fundamentals.]

These “primordia” are often translated by the “atoms” of Democritus and Epicurus.<sup>138</sup> “... nequeunt oculis rerum primordia cerni.” [“... the primordia of things cannot be seen with the eyes.”]<sup>139</sup>

This natural philosopher and Latin poet allowed himself to be guided by his common sense and above all by his senses.

Corpus enim per se communis dedicat esse  
sensus; cui nisi prima fides fundata valebit,  
haud erit occultis de rebus quo referentes  
confirmare animi quicquam ratione queamus.  
[For the event that the material body exists by itself is shown  
by common sense; a basic trust in this will act as a foundation, otherwise there will be no  
way to speak about hidden things  
in order to confirm something reasonable to the mind.]<sup>140</sup>

... Quid nobis certius ipsis  
sensibus esse potest, qui vera ac falsa notemus?  
[... What can there be more sure for us  
than our very senses by which we distinguish true and false things?.]<sup>141</sup>

These Latin verses reveal extraordinary intuitions, which only entered into the thinking of modern physics centuries later.

Tempus item per se non est, sed rebus ab ipsis  
consequitur sensus, transactum quid sit in aevo,  
tum quae res instet, quid porro deinde sequatur.  
Nec per se quemquam tempus sentire fatendumst  
semotum ab rerum motu placidaque quiete.  
[Time in itself does not exist, but from things themselves  
derives its meaning, what has been accomplished in time,  
what thing still persists, and what will follow after.  
It must be admitted that nobody feels time by itself,  
separated from the movement of things and from peaceful repose.]<sup>142</sup>

What could seem to return to an Aristotelian time, as a measurement of movement that is, was to come back again in the idea of Albert Einstein: that time depends on the matter distributed in the universe. And isn’t his attempt to prove the existence of atoms (not just a mathematical make-believe ad hoc) with the Brownian

<sup>138</sup>Lucretius I, 483–484; 1969, p. 32. The translations are mine, and Ron Packham’s.

<sup>139</sup>Lucretius I, 268; 1969, p. 48.

<sup>140</sup>Lucretius I, 422–425; 1969, p. 28.

<sup>141</sup>Lucretius I, 699–700; 1969, p. 44.

<sup>142</sup>Lucretius I, 459–463; 1969, p. 30.

movement of pollen reminiscent of these verses of the Latin natural philosopher? Clearly, “primordia” cannot be seen so clearly. And yet, we see

... corpora quae in solis radiis turbare videntur,  
quod tales turbae motus quoque materiai  
significant clandestinos caecosque subesse.  
Multa videbis enim plagis ibi percita caecis  
commutare viam retroque repulsa reverti  
[...]  
scilicet hic a principiis est omnibus error.  
[... specks that can be seen in the sunrays, moving confusedly,  
where this confusion indicates that there are  
also hidden, invisible movements of matter behind it.  
Here you will see many things, driven by invisible collisions,  
change their direction and turn back, repelled.  
[...]  
in other words, this movement derives from all the fundamentals. [the primordia]<sup>143</sup>

Reading the following verses, what else could come to our mind, other than Galileo Galilei and falling bodies?

... omnia quapropter debent per inane quietum  
aeque ponderibus non aequis concita ferri.  
[... all things, therefore, albeit unequal in weight, must be  
borne through the still void at equal speed.]<sup>144</sup>

Lucretius spoke enthusiastically of a rich variety of phenomena displayed on the stage of an infinite world.

Tantum elementa queunt permutato ordine solo.  
At rerum quae sunt primordia, plura adhibere  
possunt unde queant variae res quaeque creari.  
[This is what elements [common letters in words or verses] can do simply by changing their  
order.  
But those that are the primordia of things can unite many things  
so that all the other various things can be created.]<sup>145</sup>  
... usque adeo, quem quisque locum possedit, in omnis  
tantundem partis infinitum omne relinquit.  
[... to the point that, whatever place anyone occupies,  
he still leaves all the infinite, equally large in every direction.]<sup>146</sup>

The infinite void space, where the poet made his “primordia” move, was boundless.

... omne quidem vero nil est quod finiat extra.  
[... in truth, indeed, nothing exists that limits everything from the outside.]<sup>147</sup>

<sup>143</sup>Lucretius II, 126–132; 1969, p. 78.

<sup>144</sup>Lucretius II, 238–239; 1969, p. 84.

<sup>145</sup>Lucretius I, 827–829; 1969, p. 52.

<sup>146</sup>Lucretius I, 966–967; 1969, pp. 60–62.

<sup>147</sup>Lucretius I, 987; 1969, p. 62.

Consequently,

... nam medium nil esse potest, quando omnia constant  
 infinita. ...  
 [... nothing can stand at the centre when everything  
 is infinite. ...].<sup>148</sup>

Forcing things just a little, isn't this a description of a spherical surface without any border and without any centre?

With material primordia in the infinite timeless void, Lucretius formed the world of phenomena without creation.

... de nilo quoniam fieri nil posse videmus.  
 [... for we see that nothing can be born from nothing.]<sup>149</sup>

Our Latin natural philosopher shows too much trust in the senses, and too great an admiration for

Aeneadum genetrix, hominum divumque voluptas,  
 alma Venus, ...  
 [Mother of the Romans, delight of men and gods,  
 life-giving Venus, ...].<sup>150</sup>

... tactus enim, tactus, pro divum numina sancta,  
 corporis est sensus, vel cum res externa sese  
 insinuat, vel cum laedit quae in corpore natast  
 aut iuvat egrediens genitalis per Veneris res,  
 [... for touch, indeed, touch, by the sacred gods,  
 is the sense of the body, both when something external  
 penetrates, and when that which is born in the body wounds  
 or delights, passing by the route of procreating Venus,].<sup>151</sup>

Nec tamen hic oculos falli concedimus hilum.  
 [...]  
 hoc animi demum ratio discernere debet,  
 nec possunt oculi naturam noscere rerum.  
 Proinde animi vitium hoc oculis adfingere noli.  
 [However we do not agree that the eyes be at all deceived.  
 [...]  
 after all, it is the reasoning of the soul that must discern,  
 and eyes cannot know the nature of reality.  
 Therefore, do not attribute to the eyes this fault of the mind.]<sup>152</sup>

Among the examples given of illusions due to the mind, Lucretius included ships, suns, moons, stars, horses, columns, clouds that are sometimes still, and sometimes

<sup>148</sup>Lucretius I, 1070–1071; 1969, p. 68.

<sup>149</sup>Lucretius II, 287; 1969, p. 88.

<sup>150</sup>Lucretius I, 1–2; 1969, p. 3.

<sup>151</sup>Lucretius II, 434–437; 1969, p. 96.

<sup>152</sup>Lucretius IV, 379–386; 1969, p. 232.

in movement, which today might seem to be considerations about the principle of relativity and perspective.

Nam nil aegrius est quam res secernere apertas  
ab dubiis, animus quas ab se protinus addit.

[...]

... , cum in rebus veri nil viderit ante,

[...]

Invenies primis ab sensibus esse creatam  
notitiam veri neque sensus posse refelli.

[For there is nothing more onerous than distinguishing clear things  
from doubts, those things that the mind always adds by itself.

[...]

... when nothing true has previously been seen in things,

[...]

You will find that it is from the senses that  
the knowledge of truth is first created, and the senses cannot be disproved.]<sup>153</sup>

Besides taste, smell and sight, the poet first spoke of sounds and hearing.

Asperitas autem vocis fit ab asperitate  
principiorum et item levior levore creatur.  
Nec simili penetrant auris primordia forma,

[Furthermore, the harshness of sound derives from the roughness  
of the primordia, and likewise soft sounds are created from smoothness;  
nor do the primordia enter into the ear with the same form.]<sup>154</sup>

Praeterea partis in cunctas dividitur vox,  
ex aliis aliae quoniam gignuntur, ubi una  
dissiluit semel in multas exorta, ...

[...]

At simulacra viis directis omnia tendunt  
ut sunt missa semel ...

[Furthermore, sound is shared out everywhere,  
because other sounds are generated from one another, when  
a voice, once emitted, is divided into many, ...

[...]

Images, on the contrary, all proceed in straight lines  
once they have been projected.]<sup>155</sup>

Thus Lucretius could not dwell in the Pythagorean-Platonic tradition. However, music supplied him with ideas to narrate the variety of the world.

... ne tu forte putes serrae stridentis acerbum  
horrorem constare elementis levibus aequae  
ac musaeae mele, per chordas organici quae  
mobilibus digitis expergefata figurant;

[... lest you may believe that the rough vibration of the rasping saw  
is composed of smooth elements in the same way as

<sup>153</sup>Lucretius IV, 467–479; 1969, p. 236.

<sup>154</sup>Lucretius IV, 542–544; 1969, p. 240.

<sup>155</sup>Lucretius IV, 604–605 and 609; 1969, p. 244.

the musical melodies, which musicians create on their strings  
modulating them with their agile fingers.]<sup>156</sup>

However, the different forms of *primordia* could not be infinite for Lucretius. Otherwise his world would become too unstable.

... cycnea mele Phoebeaque daedala chordis  
carmina consimili ratione oppressa silerent.  
namque aliis aliud praestantius exoreretur.  
[... the melodies of swans and the artistic songs of Apollo on the  
strings would become silent, suffocated by perfectly similar rules.  
For another song, more excellent than the others, would be created.]<sup>157</sup>

Was our poet afraid that a variety without limits in music might lead to an excessive, paralysing uncertainty in the choice of melodies?

The celebrations for Mother-Earth were accompanied by the sound of drums, cymbals, horns,

... et Phrygio stimulat numero cava tibia mentis, ...  
[... and the hollow flute excites their minds with the Phrygian rhythm, ...].<sup>158</sup>

We do not find any tendency in the book to reduce sounds to numbers by means of *primordia*. On the contrary, it was excluded that they could possess sensible properties, like smell or taste; they were also "... sonitu sterila ..." ["... devoid of sound ..."].<sup>159</sup> And [Pythagorean] "harmony" was rejected as an influence on the soul, necessary for "feeling", because for Lucretius, the spirit, mind and soul formed "*unam naturam*" ["a single nature"] with the parts of the body.<sup>160</sup> Thus, for him, music was not born from strings, but from flutes and shepherd's pipes.

Et zephyri, cava per calamorum, sibila primum  
agrestis docuere cavas inflare cicutas.  
[And the whistling of the wind through the empty reeds first  
taught peasants to blow into hollow hemlock reed-pipes.]<sup>161</sup>

The result was melodies to excite bodies in Bacchic dances. These Muses did not come down from Apollo's Helicon, but lived in the countryside; they did not bring the music of the spheres, but cultivated that of Mother Earth.

Tum caput atque umeros plexis redimire coronis  
floribus et foliis lascivia laeta monebat,  
atque extra numerum procedere membra moventis  
duriter et duro terram pede pelleret matrem;  
unde oriebantur risus dulcesque cachinni,  
omnia quod nova tum magis haec et mira vigeabant.

<sup>156</sup>Lucretius II, 410–413; 1969, p. 94.

<sup>157</sup>Lucretius II, 505–507; 1969, p. 100.

<sup>158</sup>Lucretius II, 618–620; 1969, p. 106.

<sup>159</sup>Lucretius II, 845; 1969, p. 120.

<sup>160</sup>Lucretius III, 117–160; 1969, pp. 150–152.

<sup>161</sup>Lucretius V, 1382–1383; 1969, p. 366.

Et vigilantibus hinc aderant solacia somno,  
 ducere multimodis voces et flectere cantus  
 et supera calamos unco percurrere labro;  
 unde etiam vigiles nunc haec accepta tuentur  
 et numerum servare genus didicere, neque hilo  
 maiorem interea capiunt dulcedini' fructum  
 quam silvestre genus capiebat terrigenarum.  
 [Then joyful lasciviousness prompted them to adorn their heads  
 and shoulders with crowns of intertwined flowers and leaves,  
 and to advance shaking their members out of time  
 clumsily, and to stamp on mother earth with vigorous feet;  
 this gave rise to laughter and sweet peals of mirth,  
 these were all things that were new then and surprising.  
 And the sleepless found comfort for their rest  
 producing sounds in various ways and modulating tunes  
 and running their puckered lips over the fifes.  
 Thus also in our times watchmen stand guard over these traditions  
 and have learnt to observe the genre of melodies,  
 nor they take a fruit a whit sweeter  
 than that the country race of earth-dwellers used to pick up.]<sup>162</sup>

The music that our philosopher-cum-poet enjoyed was clearly the opposite of the kind that Plato considered suitable for young soldiers.

Lucretius presented us with a world that was continually changing, and described it through the transformations that he observed in the rain-soaked earth, inhabited by herbs and plants, where animals, herds and human beings roamed. Here, life became food, and food, life.

... praeterea cunctas itidem res vertere sese.  
 [... in the same way, then, all things are transformed one into another.]<sup>163</sup>

Iamne vides igitur magni primordia rerum  
 referre in quali sint ordine quaeque locata  
 et commixta quibus dent motus accipiantque?  
 [Can you not see, therefore, it is of great importance in what order  
 the primordia of things stand and how they are distributed  
 and mixed together, in order to produce and undergo changes?]<sup>164</sup>

In this world, made up of material primordia continually jumbled up together in the void,

... scire licet gigni posse ex non sensibu' sensus.  
 [... it is possible to understand how senses can be born from non-senses.]<sup>165</sup>

Our Latin natural philosopher avoided creation and a creator. His divinities appear to be poetical metaphors for sensible phenomena. Religious beliefs were presented by him as sources of suffering and unhappiness: the sacrifice of Iphigenia

<sup>162</sup>Lucretius V, 1399–1411; 1969, p. 368.

<sup>163</sup>Lucretius II, 874; 1969, p. 120.

<sup>164</sup>Lucretius II, 883–885; 1969, p. 122.

<sup>165</sup>Lucretius II, 930; 1969, p. 124.

for her father. A man could as well speak of Neptune, Ceres, Bacchus and the other gods,

... , dum vera re tamen ipse  
religione animum turpi contingere parcat.  
[... , provided that in reality, he himself, on the contrary,  
beware of contaminating his mind with foul religion.]<sup>166</sup>

He terminated his celebration of his master, Epicurus, with these words:

Quare religio pedibus subiecta vicissim  
obteritur, nos exaequat victoria coelo.  
[Consequently, religion was trampled down underfoot,  
while the victory raises us to the level of the heavens.]<sup>167</sup>

“... desiperest ...” [“... it is folly ...”] to believe in the gods.<sup>168</sup> Those who appealed to them did not understand “... coeli rationes ordine certo ...” [“... the reasons in the fixed order of the heavens ...”]. Then ensued a mankind that was unhappy because: “Nec pietas ...” [“It was not devotion ...”] to shed the blood of animals on altars, “... sed mage pacata posse omnia mente tueri.” [“... but there would be more piety, if everything could be considered with a serene mind.”]<sup>169</sup>

Cum praesertim hic sit natura factus, ut ipsa  
sponte sua forte offensando semina rerum  
multimodis temere incassum frustra coacta  
tandem coluerunt ea quae coniuncta repente  
magnarum rerum fierent exordia semper,  
terrai maris et caeli generisque animantum.  
[Especially using that [serene mind] with nature, just as the seeds of  
things themselves bumping into one another spontaneously by chance,  
blindly driven in various ways fruitlessly and in vain,  
in the end grew those things which, thrown suddenly together,  
always became the beginnings of great things,  
the earth, the sea, the sky, and living creatures.]<sup>170</sup>

Nunc et seminibus si tanta est copia quantam  
enumerare aetas animantum non queat omnis, ...  
[And now there is such a great abundance in the seeds that  
a whole life of living creatures would not suffice to count them, ...].<sup>171</sup>

The same force would then have produced in other parts of the terrestrial globe various other generations of living creatures, plants, animals and different kinds of men.

<sup>166</sup>Lucretius II, 659–660; 1969, p. 108.

<sup>167</sup>Lucretius I, 78–79; 1969, p. 6.

<sup>168</sup>Lucretius V, 146–165; 1969, pp. 292–294.

<sup>169</sup>Lucretius V, 1183–1203; 1969, pp. 354–356; cf. Lucretius VI, 54; 1969, p. 376.

<sup>170</sup>Lucretius II, 1058–1063; 1969, p. 132.

<sup>171</sup>Lucretius II, 1070–1071; 1969, p. 134.

As regards the soul and the spirit, our Latin poet does not appear to have suffered from the dualisms typical of Platonic philosophies, which were in the following centuries to become the orthodoxies of the Jewish and Christian religions in Europe.

Haec eadem ratio naturam animi atque animai  
corpoream docet esse. . . .  
[This same reason teaches us that the nature  
of the mind and of the soul is corporeal. . . .].<sup>172</sup>

For him, those vital and spiritual elements are kinds of fluids contained in the body, as in a vase.

Haec igitur natura tenetur corpore ab omni  
ipsaque corporis est custos et causa salutis;  
[. . .]  
discidium [ut] nequeat fieri sine peste maloque;  
ut videas, quoniam coniunctast causa salutis,  
coniunctam quoque naturam consistere eorum.  
[“This nature [the spirit] is thus contained in every body  
and is the guard of the body and the cause of its good health;  
[. . .]  
as no separation can take place without illness or ruin;  
you see, as the cause of its well-being is united,  
so also their nature remains linked.”]<sup>173</sup>

Quippe etenim corpus, quod vas quasi constitit eius,  
cum cohibere nequit conquassatum ex aliqua re.  
[For the body, clearly, which almost is like the vase [of the mind],  
cannot detain it when it is damaged by something].<sup>174</sup>

Thus for Lucretius, as the soul, mind and spirit are born, so they die together with the body.

Trusting his senses, our Latin natural philosopher observed the phenomena of the atmosphere and the earth, trying serenely to find the reasons. He liked the clouds, and described their formation in the water cycle from the sea to rain. Having freed himself from Jupiter the Rain-bringer, and the Tyrrhenian [Etruscan] haruspices, he made thunder and lightning spring from friction between the clouds. “. . . ut omnia motu percalefacta vides ardescere, . . .” [“. . . as you see that all things, heated up by movement, catch fire, . . .”]. He wrote that in this way even lead bullets could be liquefied, if they flew for a long distance.<sup>175</sup>

The colours of the rainbow were produced by the sunrays filtered through the vapours of the clouds.<sup>176</sup> He evoked in detail and in poetic tones the cyclones that formed over the sea, due to the winds that created a whirlwind, calling them by their

<sup>172</sup>Lucretius III, 163; 1969, p. 152.

<sup>173</sup>Lucretius III, 323–324 and 347–349; 1969, p. 162.

<sup>174</sup>Lucretius III, 440–441; 1969, p. 168. Cf. III, 554–557; 1969, p. 174 and III, 579; 1969, p. 176.

<sup>175</sup>Lucretius VI, 177–179; 1969, p. 382.

<sup>176</sup>Lucretius VI, 524–526; 1969, p. 404.



Greek name of “prester”.<sup>177</sup> He tried to explain how the magnet sticks to iron by emitting tiny invisible particles, which are, however, capable of shifting the air and creating voids, subsequently filled by the body attracted. For him, even earthquakes were produced by the swirling of air in the caves of the earth.<sup>178</sup>

The natural world of Lucretius was dominated by the phenomena that move in eddies, and those for which friction is important. He extended his models to the movements of heavenly bodies, without following the greatest Greek philosophers in their classic separation between terrestrial and astral phenomena.

... quanto quaeque magis sint terram sidera propter,  
tanto posse minus cum caeli turbine ferri.  
[... the closer each star is to the earth  
the less it will be attracted by the whirling of the sky.]<sup>179</sup>

The senses faithfully transmitted a complex variety of that world which our Latin poet was concerned to preserve for us.

Nam veluti tota natura dissimiles sunt  
inter se genitae res quaeque, ita quamque necessest  
dissimili constare figura principiorum;  
[For, just as in nature all things generated are  
different from one another, so it is necessary that the fundamentals  
[primordia] be different in shape;]<sup>180</sup>

Variously moved in the void by their own weight, colliding, mingling, separating and recombining in countless ways, particles of matter gave shape to a world in continuous change.

Scilicet haec ideo terris ex omnia surgunt,  
multa modis multis multarum semina rerum  
quod permixta gerit tellus discretaque tradit.  
[That is to say, all these things thus arise from the earth,  
many seeds of many things in many ways  
the earth bears in itself, mixed together, and gives forth separate.]<sup>181</sup>

The repeated *m*'s of the attractive alliteration recall to our mind the *Alma mater Venus* as a general model of this universal generation without creation.

This natural philosopher *sui generis* indicated the particles of matter sometimes as “praecordia” [viscera], sometimes as “semina” [seeds]; here they became “materia” or “materies” [matter], there “principia” [primordia]: he never used the more contemporary term, current for us, of “atomos”, as others did, for example Cicero. He observed particles of water in the clothes left on the sea shore, particles of a

<sup>177</sup>Lucretius VI, 423–450; 1969, p. 398.

<sup>178</sup>Lucretius 1969, pp. 404–408 and 426–434.

<sup>179</sup>Lucretius V, 623–624; 1969, p. 320.

<sup>180</sup>Lucretius II, 720–722; 1969, p. 112.

<sup>181</sup>Lucretius VI, 788–790; 1969, p. 420.

plague killed the inhabitants of Athens, particles of matter carried the smell of things to the nostrils.

Among these, the various ways of distributing the empty spaces further increased the variety.

Multa foramina cum variis sint reddita rebus,  
dissimili inter se natura praedita debent  
esse et habere suam naturam quaeque viasque.  
[As the many spaces are assigned to various things,  
they must possess a nature that is different one from the other,  
and have each one its own nature and its own paths.]<sup>182</sup>

This variety was regenerated by a continual changing, which does not encounter any reduction to a limited number of ultimate elements in the book.

... omniparens eadem rerum commune sepulcrum,  
ergo terra tibi libatur et aucta recrescit.  
... assidue quoniam fluere omnia constat.  
[... the earth, mother of all things, and also common grave of things,  
thus loses something, and grows again, richer for you.]  
[... because it is recognized that all things are continually in a state of flux.]<sup>183</sup>

Usque adeo omnibus ab rebus res quaeque fluenter  
fertur et in cunctas dimittitur undique partis  
nec mora nec requies interdatur ulla fluendi,  
perpetuo quoniam sentimus, et omnia semper  
cernere odorari licet et sentire sonare.  
[To such an extent is everything carried forward by everything else  
in a continual flow, and is dispatched everywhere in every direction  
that there is no respite or rest in the flow,  
because we perceive them incessantly, and we can always  
see, smell and hear the sounds of everything.]<sup>184</sup>

The substances that flow in *De rerum natura* are material; even if they were invisible, Lucretius offered indirect evidence that can be perceived by means of the senses.

... , ut aestus  
pervolet intactus, nequeunt impellere usquam;  
[... , when the exhalation  
flies past without contact, they cannot push in any direction;]<sup>185</sup>

Thus the magnet does not attract either gold or wood, because the force that it emanates passes through their interstices without touching them.

<sup>182</sup>Lucretius VI, 981–983; 1969, p. 430.

<sup>183</sup>Lucretius V, 259–260 and 280; 1969, pp. 298 and 300.

<sup>184</sup>Lucretius VI, 931–935; 1969, p. 428.

<sup>185</sup>Lucretius VI, 1059–1060; 1969, p. 434.

To convince his readers about the nature of things, Lucretius expounded his reasons in a captivating poetic guise: "... musaeo dulci contingere melle, ..." ["... to spread over them the sweet honey of the Muses ..."].<sup>186</sup> He recounted

... quibus ille modis congressus materiai  
fundarit terram caelum mare sidera solem  
lunaique globum; ...  
[... in what ways that meeting of matter  
founded the earth, the sky, the sea, the stars, the sun  
and the globe of the moon; ...].<sup>187</sup>

The thunder that shakes the sky and earth together was presented as an argument for the unity of the world.

... quod facere haud ulla posset ratione, nisi esset  
partibus aeriis mundi caeloque revincta.  
Nam communibus inter se radicibus haerent  
ex ineunte aevo coniuncta atque uniter apta.  
[... in no case could this happen for any reason if [the earth] were  
not connected with the airy regions of the world and the sky.  
For they have been attached together with common roots  
ever since the beginning of the centuries, joined and linked in unity.]<sup>188</sup>

Lucretius saw this same unity between the soul and the body; and at times he did not fail to elaborate analogies between certain phenomena and the human body. The water cycle is similar to the circulation of fluids in the body, the earthquake is like trembling caused by the cold.<sup>189</sup>

The things of the world follow an order, and are repeated, like the seasons, or the movement of the sun and the moon.<sup>190</sup> Our poet-cum-natural philosopher sought the "ratio" [reason] for this, which he sometimes called the "causa". He reserved the term "lex" [law] for the social rules needed to maintain a life in common among people. As regards the magnet, he wondered "... quo foedere fiat naturae ..." ["... by means of what pact of nature it happens ..."].<sup>191</sup> And, with poetic sensitivity, he admitted his doubt about the possibility of always finding the reasons for a phenomenon, because there might be many of them.

Sunt aliquot quoque res quarum unam dicere causam  
non satis est, verum pluris, unde una tamen sit;  
[There are also various things for which it is not sufficient to indicate  
only one cause, but several, of which one, however, is the real one;]

Here he was referring to the floods caused by the Nile.<sup>192</sup>

<sup>186</sup>Lucretius I, 147; 1969, p. 60.

<sup>187</sup>Lucretius V, 67–69; 1969, p. 288.

<sup>188</sup>Lucretius V, 552–555; 1969, p. 316.

<sup>189</sup>Lucretius VI, 498–503; 1969, p. 402. Lucretius VI, 591–595; 1969, p. 408.

<sup>190</sup>Lucretius 1969, pp. 324, 352, 370.

<sup>191</sup>Lucretius V, 906–907; 1969, p. 426.

<sup>192</sup>Lucretius VI, 703–704; 1969, p. 414.

However, for Lucretius, the particles of matter did not follow a deterministic order fixed by absolute laws.

Nam certe neque consilio primordia rerum  
ordine se suo quaeque sagaci mente locarunt  
nec quos quaeque darent motus pepigere profecto  
sed quia multa modis multis primordia rerum  
ex infinito iam tempore percita plagis  
ponderibusque suis consuerunt concita ferri  
omnimodisque coire atque omnia pertemptare  
quaecumque inter se possent congressa creare,  
propterea fit uti magnum vulgata per aevum  
omne genus coetus et motus experiundo  
tandem conveniant ea quae convecta repente  
magnarum rerum fiunt exordia saepe,  
terrai maris et caeli generisque animantum.  
[For undoubtedly the primordia of things did not arrange themselves  
in order, each on the basis of its own decision, with shrewd judgement,  
nor did they negotiate, undoubtedly, what movements to cause,  
but as several primordia of things, in many ways  
set in motion already from time immemorial by collisions  
and by their own weight, have been used to be transported grouped  
together, and to join up in every way and to try all possibilities,  
whatever they could create by uniting together;  
thus it comes to pass that when they are diffused for a long time,  
by trying every kind of union and movement  
finally they merge, forming things suddenly brought together,  
which often become the beginnings of great things,  
the earth, the sea, the sky, animals and the human race.]<sup>193</sup>

Even though, as he often repeated, everything was just a mixture of matter, which would inevitably fall through the void, sooner or later, as a result of the collisions that continue for a long period of time, every thing observed would find its occasion to be born. Lucretius conceived of the world as unstable, and therefore free: free both from a divine destiny, and from any absolute law which might determine movement once and for all.

... corpora cum deorsum rectum per inane feruntur  
ponderibus propriis, incerto tempore ferme  
incertisque locis spatio depellere paulum,  
tantum quod momen mutatum dicere possis.  
[... when the bodies are dragged down in a straight line through  
the void, by their own weight, at some unspecified moment,  
and in places not established in space, they deviate a little,  
enough for you to call it a change in movement.]<sup>194</sup>

Otherwise collisions could not take place, and matter would not have the chance to generate and regenerate continually. Therefore,

<sup>193</sup>Lucretius V, 419–431; 1969, pp. 308–310.

<sup>194</sup>Lucretius II, 217–220; 1969, p. 84.

... paulum inclinare necessest  
 corpora; nec plus quam minimum, ne fingere motus  
 obliquos videamur et id res vera refutet.  
 [... it is necessary for bodies to incline a little;  
 no more than the minimum that is sufficient, so that we will not seem  
 to invent oblique movements which are refuted by reality.]<sup>195</sup>

Furthermore, for our Latin philosopher, this would give rise to

... exiguum clinamen principiorum  
 nec regione loci certa nec tempore certo.  
 [... a minimal inclination of the primordia  
 neither in a sure place nor at a sure time.]<sup>196</sup>

Denique si semper motus conectitur omnis  
 et vetere exoritur (semper) novus ordine certo  
 nec declinando faciunt primordia motus  
 principium quoddam quod fati foedera rumpat,  
 ex infinito ne causam causa sequatur,  
 libera per terras unde haec animantibus exstat,  
 unde est haec, inquam, fatis avulsa voluntas  
 per quam progredimur quo ducit quemque voluptas,  
 declinamus item motus nec tempore certo  
 nec regione loci certa, sed ubi ipsa tulit mens?  
 [Lastly, if every movement is always connected  
 and the new (always) arises with certainty from the old order,  
 and the primordia do not, in their deviations, by movements make  
 some kind of principle that breaks the bonds of destiny,  
 so that cause does not follow cause everlastingly,  
 where does this free will come from for living creatures in the world?  
 And from where, I repeat, comes this will separated from destiny  
 by which we go wherever our desire leads each of us,  
 and also we modify our movements, not at certain moments,  
 nor in certain places, but where our mind itself has brought us?]<sup>197</sup>

Lucretius has restored to us the freedom of *voluptas* [pleasure] and with this, man returned to the best guide for his life.

... ipsaque deducit dux vitae dia voluptas  
 et res per Veneris blanditur saecula propagent,  
 ne genus occidat humanum.  
 [... the very guide of life, divine pleasure, has led  
 and attracts by the ways of Venus, and generations are perpetuated  
 so that the human race does not die out.]<sup>198</sup>

By making them incapable of uniting *per Veneris res*, incapable of feeding, we were released from monsters that any strange hotchpotch perhaps might

<sup>195</sup>Lucretius II, 243–245; 1969, p. 84.

<sup>196</sup>Lucretius II, 292–293; 1969, p. 88.

<sup>197</sup>Lucretius II, 251–260; 1969, p. 86.

<sup>198</sup>Lucretius II, 172–174; 1969, p. 80.

have been produced.<sup>199</sup> Whereas, instead, "... Venus in silvis iungebat corpora amantum;" ["... In the woods, Venus united the bodies of lovers;"].<sup>200</sup> In this way, a varied, multiform life had been perpetuated on earth, starting from the senseless collisions between primordia. The world had then been populated with every kind of phenomenon and every living creature.

An, credo, in tenebris vita ac maerore iacebat,  
donec diluxit rerum genitalis origo?  
Natus enim debet quicumque est velle manere  
in vita, donec retinebit blanda voluptas.  
[And, I believe, did life not pine in darkness and sorrow  
until the sexual origin of things shone forth?  
For once born, everyone must desire to remain  
alive, as long as a pleasant desire attracts him.]<sup>201</sup>

Linquitur ut merito maternum nomen adepta  
terra sit, e terra quoniam sunt cuncta creata.  
[It remains that the name of 'maternal' be assigned deservedly  
to the earth, because all things were born from the earth.]<sup>202</sup>

And yet, for Lucretius, this earth that was so happy and joyful already seemed to be starting to decline.

Iamque adeo fracta est aetas effetaque tellus . . .  
[Indeed, the age is already broken and the earth worn out . . .].<sup>203</sup>

. . . hic natura suis refrenat viribus auctum.  
[... here nature curbs the growth with its own forces.]<sup>204</sup>

Sed quia finem aliquam pariendi debet habere,  
destitit, ut mulier spatio defessa vetusto.  
Mutat enim mundi naturam totius aetas  
ex alioque alius status excipere omnia debet,  
nec manet ulla sui similis res: omnia migrant,  
omnia commutat natura et vertere cogit.  
[...]  
Sic igitur mundi naturam totius aetas  
mutat et ex alio terram status excipit alter,  
quod, tulit ut nequeat, possit quod non tulit ante  
[But, as there must be some end to generating,  
[the earth] desisted, like a woman tired by old age.  
For age changes the nature of the whole world,  
another state must receive everything from yet another,  
and nothing remains similar to itself: everything changes,  
nature transforms all things, and forces them to vary.

<sup>199</sup>Lucretius 1969, p. 334.

<sup>200</sup>Lucretius V, 962; 1969, p. 340.

<sup>201</sup>Lucretius V, 175–178; 1969, p. 294.

<sup>202</sup>Lucretius V, 795–796; 1969, p. 330.

<sup>203</sup>Lucretius II, 1150; 1969, p. 138.

<sup>204</sup>Lucretius II, 1121; 1969, p. 136.

[...]

Thus, therefore, age changes the nature of the whole world,  
and from one condition another one rules the earth, so that  
what it bore should be negated, and it can bear what it had not before.]<sup>205</sup>

Yet in the incessant dance of the primordia, in the succession of changing generations, our Latin natural philosopher introduced another dramatic protagonist, for which he also expressed his own moral judgement.

Denique tantopere inter se cum maxima mundi  
pugnent membra, pio nequaquam concita bello,  
nonne vides aliquam longi certaminis ollis  
posse dari finem?

[In the end, when to such a labour the most mighty members of the  
world fight among themselves, engaged in a thoroughly unjust war,  
do you not see that some close may be put to their long struggle?]<sup>206</sup>

Unfortunately, for him and for us, mankind was to enter into another epoch, after an initial period of peace.

At non multa virum sub signis milia ducta  
una dies dabat exitio nec turbida ponti  
aequora libebant [?] navis ad saxa virosque.

[But [in those times] many thousands of men led under the banners  
were not slaughtered in one single day, nor did the surging  
waters of the sea sacrifice men and ships on the rocks.]<sup>207</sup>

Then the Iron Age arrived, when men laboured increasingly to invent new arms.

Sic alid ex alio peperit discordia tristis,  
horribile humanis quod gentibus esset in armis,  
inque dies belli terroribus addidit augmen.

[Then deadly discord generated one thing from another  
which was to be terrifying for the nations of men in arms  
and daily added an increase to the terrors of war.]<sup>208</sup>

Tunc igitur pelles, nunc aurum et purpura curis  
exercent hominum vitam belloque fatigant;

[...]

Ergo hominum genus incassum frustraue laborat  
semper et (in) curis consumit inanibus aevum,  
nimirum quia non cognovit quae sit habendi  
finis et omnino quoad crescat vera voluptas.

Idque minutatim vitam provexit in altum  
et belli magnos commovit funditus aestus.

[So then it was skins, now it is gold and purple that  
tire the life of men with cares and torment them with war.

[...]

<sup>205</sup>Lucretius V, 826–836; 1969, p. 332.

<sup>206</sup>Lucretius V, 380–383; 1969, p. 306.

<sup>207</sup>Lucretius V, 999–1001; 1969, p. 342.

<sup>208</sup>Lucretius V, 1305–1307; 1969, p. 362.

Thus mankind labours uselessly and in vain,  
 and spends his age always worrying about nothing,  
 because, clearly, he has not learnt to recognize the purpose  
 of possessing, and above all how far true pleasure may grow.  
 And this has gradually dragged his life down to the depths  
 and has aroused great outbursts of war down inside him.]<sup>209</sup>

This Latin poet, who was a witness of several wars, condemned the age of mankind, both his and ours.

... nequaquam nobis divinitus esse paratam  
 naturam rerum: tanta stat praedita culpa.  
 [... in no way has the nature of things been divinely  
 prepared for us: so great is the blame that it stands accused of.]<sup>210</sup>

Then Lucretius imagined the end of the world. Was he perhaps somewhat relieved?

... una dies dabit exitio, multosque per annos  
 sustentata ruet moles et machina mundi.  
 Nec me animi fallit quam res nova miraque menti  
 accadat exitium caeli terraeque futurum,  
 [...]  
 succidere horrissona posse omnia victa fragore.  
 [... a single day will give over to perdition, and the mass  
 with the machinery of the world, sustained for many years, will fall.  
 Or it does not escape me how new and surprising for the mind  
 the future ruin of the sky and the earth will be,  
 [...]  
 all things can be overcome and destroyed with a terrible-sounding din.]<sup>211</sup>

The only scholar who has recently taken Lucretius into consideration for the evolution of sciences was Michel Serres. This Frenchman underlined the model based on the flow of water and liquids, with the consequent whirlpools. He made of it "... in contrast with the enterprises of Mars ... a science of Venus, without violence or guilt, in which the thunderbolt is no longer the wrath of Zeus, ...".<sup>212</sup> He reinterpreted Lucretius, considering in particular the problems of stability brought to the attention of scholars by René Thom (1923–2002) during the '1970s of last century. But then he contaminated everything with an excessive dose of anachronism, setting it in a one-dimension history of sciences without any internal conflicts. Also for him, the *primordia* became the usual atoms; the deviations of the *clinamen* [inclination] were assimilated to Newton's fluxions and to the infinitesimals of Leibniz. In his opinion, Lucretius anticipated the combination of letters, numbers and notes, typical of this German philosopher. Our French scholar followed the ceaseless rhythm of the text of Lucretius. But unfortunately, he ignored

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<sup>209</sup>Lucretius V, 1423–1435; 1969, pp. 368–370.

<sup>210</sup>Lucretius V, 198–199; 1969, p. 296.

<sup>211</sup>Lucretius V, 95–109; 1969, p. 290.

<sup>212</sup>Serres 1980, p. 127.



its complex variety, confusing music with the arithmetic of the Pythagoreans. Thus he made it reversible, as if were subject to a fixed pre-existent Newtonian concept of time.<sup>213</sup> On the contrary, it is the music of things and sounds that creates its own various rhythms and times.

Having connected Lucretius somehow with Archimedes, Serres relegated him to a precursor who invented modern physics. To me, on the contrary, Lucretius seems rather to be a poet and natural philosopher, distant, in his age, from the traditions of Pythagoras and Euclid, but similar in some ways to some non-orthodox figures of today.<sup>214</sup>

Among his arguments open to criticism, however, Serres scattered a few gems, which are worth the whole of the rest of the book: “Scientists foresee the exact time of an eclipse, but they cannot foresee whether it will be visible to them. Meteorology is a repressed part of history. [. . .] This interests only those people that scholars are not interested in: farmers and seamen. [. . .] For it is the weather of clouds where people should not have their heads, and which should not exist in their heads.”<sup>215</sup> Even today, organisers can plan races between sailing-boats out on the sea, spending millions of euros or dollars, without succeeding in disputing them, due to lack of wind.

As our Frenchman underlined, the world of Lucretius was one which was continually renewed, out in the open, come rain, come shine. On the contrary, modern orthodox mathematical sciences have been separated from the world, closed inside laboratories, and fixed in the rigid formulas of laws.

In the end, according to the interpretation of Serres, the wholly justified pessimism of the Latin poet took this form: “Culture is the continuation of barbarism, using other instruments.”<sup>216</sup> In our great epoch of wars and violence, this sentence should undergo just a tiny *clinamen* to be appropriate: orthodox sciences are the continuation of barbarism, using arms that are more powerful and more destructive.

And there were Phobos [Fear], Deimos [Terror] and with them the restless Eris [Strife], the sister and companion of murderous Mars, who, though small at first, raises her crest high, and then points her head towards the heavens, while her feet are still on the earth.

Homer, *Iliad* IV, 442–443.

Nec me animi fallit Graiorum obscura reperta  
difficile inlustrare latinis versibus esse  
multa novis verbis praesertim cum sit agendum  
propter egestatem linguae et rerum novitatem  
sed tua me virtus tamen et sperata voluptas  
suavis amicitiae quemvis efferre laborem  
suadet et inducit noctes vigilare serenas.

<sup>213</sup>Serres 1980, pp. 159–163.

<sup>214</sup>Tonietti 2002a; Tonietti 1983b, pp. 279–280.

<sup>215</sup>Serres 1980, pp. 75–76.

<sup>216</sup>Serres 1980, p. 200.

[Nor am I unaware that it is difficult to illustrate the obscure discoveries of the Greeks in Latin verse, especially since we must use unfamiliar words, due to the poverty of our language and the unfamiliarity of the theme; however, your skill and the expected pleasure of your sweet friendship persuade me to bear any toil and induce me to spend serene nights awake.]

Titus Lucretius Carus, *De rerum natura*, I, 136–142.

## 2.9 Texts and Contexts

We would prefer to keep as far away as possible from well-known philosophies of history, whichever side they come from. But this is no reason to avoid asking ourselves general questions connected with these environments shared by many of our protagonists. After all, even the texts of mathematicians, in spite of the considerable efforts made by some of them, acquire sense only if they are set in their context, without which it would be impossible to understand them.

Unfortunately, while it is customary and required for every other cultural product to reconstruct its context, for the sciences, on the contrary, this appears to be curiously uncommon, and even discouraged, if not openly opposed, by the guardians of the disciplines. Why else do those mythical divinities become characters in the history once again, and, worse still, responsible for what they do?

A famous physicist like Erwin Schroedinger (1887–1961) asked the question “Do the sciences depend on the environment?” and presented arguments in favour of this thesis.<sup>217</sup> Paul Forman reconstructed the hostile environment that surrounded the German scientific community after the defeat in 1918, drawing from it some surprising effects for the invention of quantum mechanics.<sup>218</sup> In another period, from 1979 to 1983, a journal like *Testi & contesti* [*Texts & Contexts*] succeeded in coming to light and growing around similar ideas, until a hostile environment suppressed it. This does not appear to be only a necessity of mine, then. Recently, Nathan Sivin suggested “doing away with the border between foreground and context, and studying scientific change as an integral whole, what I call a cultural manifold.”<sup>219</sup>

The historical context allows us to consider possible features that are shared by the characters we are studying, without masking their uniqueness and without reducing them to some arbitrary disembodied philosophical concept. The context also reveals the heretical minority positions, and if we can succeed in preserving

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<sup>217</sup>Schroedinger 1963.

<sup>218</sup>Forman 2002.

<sup>219</sup>*Testi & contesti* 1979–1983. Sivin 2005, p. 58. The context was described as “essential for understanding” also by Vegetti 1983, pp. 11–12ff.

their detail, it would explain the reasons for their lower esteem compared with the formation of orthodoxy.

Let us then see what we should do, together with other histories of the Greek and Latin sciences, in order to achieve this purpose, bearing in mind, however, that if we had ignored music, we would have preserved a context that would be distorted in various ways. Furthermore, we should be careful not to confuse features that are attributed to the origins of European mathematical sciences with presumed universal characteristics, which rather serve to exclude other cultures that do not possess them.

Almost all our scholars, regardless of their various mother tongues, ended up by writing in Greek, because this was the dominant language accepted for culture, even in the Roman Empire. We may ask ourselves, therefore, if and how this language opened up spaces for other common characteristics of that cultural context, which ended up by being concentrated in Alexandria: with Euclid, Ptolemy and many others, medical doctors included. Athens remained the seat of the great schools of philosophy, for as long as they lasted: Plato's Academy, Aristotle's Lyceum and the Garden of Epicurus.

In the *ΣΤΟΙΧΕΙΑ* [Letters, elements, principles, shadows of the gnomon], Euclid predicated innumerable properties for one figure or another. In this way, he expressed his thought succinctly, listing one geometrical characteristic after another. "... τετραγωνον ἴσον ἔστι ..." ["... the square is equal ..."]. The series of theorems is built up, sustained by a continuous interplay of copulas ἔστι [is], εἶσιν [are]. It is difficult to imagine the text without the possibility of conjugating the verb 'to be': "Let the right-angled triangle be ... the square is ... is straight ... is equal ... is twice ... are on the same parallels ..." <sup>220</sup>

In the *ΚΑΤΑΤΟΜΗ ΚΑΝΟΝΟΣ* [*Division of the monochord*], whereas, Euclid predicated regarding musical intervals and the relative ratios ἔστω ... [let ... be ...], ἔστι [is]. And he continued to range his theorems always with an underlying structure of copulas: "... διαπασῶν ... ἔστι ..." ["... the diapason (the octave) ... is ..."]. <sup>221</sup>

Also Latin expresses properties easily, using the verb 'to be', conjugated in the forms *est*, *sunt*, *esse*. We, who are their heirs in Europe, have grown so used to this event that we forget it, and ignore it. But this *is* (as I was saying), on the contrary, one of the characteristics of our culture, and our geographical area: it *is not* (again) universal. In the following chapter, we shall see another area where it *is* missing.

The term for the verb 'to be' also indicated the 'being' of existence. It was opposed to the γίγνομαι of becoming, being born, being generated or created. Thus ὁὢν [the one who is, the being] effectively represented the uncreated eternal being. This was predicated of a God outside the world of mortal creatures, and thus a transcendent God. Parmenides developed the affirmation "he is" to an even more stable "he is and he cannot not be." "The being is"; "the god is" and "he is one". Plato later wrote: "that discourse is true which says things that are". Aristotle put his

<sup>220</sup>Euclide 1916, pp. 110–114.

<sup>221</sup>Euclid 1557.

seal on it: “saying that being is not, or that non-being is, is false; saying that being is, and that non-being is not, is true: thus whoever says ‘it is’ or ‘it is not’ either tells the truth or speaks falsely.” “. . . being is common to all things”. Vitruvius will translate this idea into Latin and into practice in the motto: “fabros esse” [be originators].<sup>222</sup> Another aspect of the Greek language which is important here can be found in the letter  $\alpha$  [a]; when it is placed in front of certain words, it indicates their absence: ‘ $\alpha'$ πειρον [without end], ‘ $\alpha'$ λογος [without words, inexpressible, without reason, without a ratio].

I still continue to reject all forms of determinism that are traced back to language, together with every other kind of determinism or reductionism that simplifies the complex events of history. However, how can we not suspect that those continual discussions, of which we read in every field of that context, achieved a particularly convincing flow and sound in Greek? In their political assemblies, in their tribunals, in the schools of philosophy, among scholars of all subjects, there was a continual  $\delta\alpha\lambda\omicron\gamma\eta'$  [dialogue, evaluating, arguing], a debate between opposing positions.<sup>223</sup> This is generally the form of the books written by Plato.

Also the Greek scientific environment was animated by countless disputes, in which everything was divided in half, confronted and in the end discriminated. The Graeco-Roman culture appears to us to be largely dualistic in its prevailing forms. Exceptions were rare and to find examples, there are a plethora to choose from.

Typical dualisms of this culture were those between the sky and the earth, and between rational and irrational. There was a desire to distinguish “. . . friend from enemy, to know the one, and not to know the other.” For the alternative between good and evil, Plato, as usual, criticised Homer, who on the contrary, made “. . . a hotchpotch of them”. The true was to be separated from the false, the eternal from the ephemeral and contingent. The former were the attributes of divinity. Parmenides contrasted the Way of truth to the Way of opinion, seen as misleading.<sup>224</sup> Euclid’s theorems enjoyed the only alternative between true and false. The famous saying *tertium non datur* [there’s no third way] became a part of logic: a theorem is either true or false. This was employed to derive an extremely useful final inversion in proofs by *reductio ad absurdum*. There is no need to insist on the dualism between soul and body, seeing that it has entered into the everyday language of Western culture, revived by various people in different epochs. Aristotle classified animals by dividing them in a dualistic manner. He distinguished the *logos* between meaning and truth; for the latter, only *apophantic*, or affirmative, discourses are valid, as distinct from the *epos* [poetic word]. Where was primacy to be assigned in the body? To the heart or to the brain? In either case, the soul would command the body. Greek society was reflected in a dualistic anthropology made up of the couples: “reason/desire; soul/body; male/female; master/slave; man/animal; . . . Greek/barbarian.” They also included the opposition sky/earth,

<sup>222</sup>Lloyd 1978, p. 38. Vegetti 1979, pp. 61–69, 76, 93, 102.

<sup>223</sup>Lloyd 1978. Lloyd & Vallance 2001.

<sup>224</sup>Lloyd 1978, p. 38.

parallel to male/female and upper/lower. Exceptions to this dualism, such as the monkey, created embarrassment.<sup>225</sup>

Among the Pythagorean sects, numbers could be only even or odd. Hence, as in the argument against  $\sqrt{2}$ , derived the alternative between *logos* and *alogos*. Empedocles (500–430 B.C.) wrote of the couple love/hate, attraction/repulsion, which generated the one or the multiple. Tangible qualities were distributed by Aristotle into couples of opposites such as hard or soft, rough or smooth, bitter or sweet. This gave rise to a medicine dominated by hot or cold, and wet or dry humours. For the Stoics, the basic opposition was between active and passive. The *pneuma* [spirit of life] was divided between *psyche* [soul anima] and *nous* [reason]. Human activities, above all the cognitive ones, were differentiated between theory and practice. We have already encountered discussions between reality and appearance, that is to say, between essence (being) and the *φαίνόμενον* [what is visible, what appears]. For the Pythagoreans, mathematics discriminated between the possible and the impossible. The opposition between continuous and discrete is not to be underestimated, as it divided the schools of Aristotle and the Stoics from those of Pythagoras and Democritus. Anaximander and Anaxagoras made the world begin with a separation between hot and cold, light and dark, dry and wet.<sup>226</sup>

Here we shall follow above all the dualism between truth and error. We have seen above that this derived from the discussion on 'being'. How could the *ᾠληθεία* [truth, reality] be attained, then? Everybody had something to say about this. Curiously, for this term two different etymologies are proposed: not latent, from *λανθάνω* [to remain hidden] or not forgotten, from *ληθῆ* [oblivion]? A Pythagorean wrote: "No lies penetrate into numbers; for lies are adversaries and enemies of nature, just as the truth is innately typical of the species of numbers." Democritus sentenced: "We know nothing according to truth; because truth is in the depths." Talking about him, Galen (second century) quoted: "Opinion is the colour, opinion the sweet, opinion the bitter, truth the atoms and the void." Everybody, wrote Aristotle, "... has posited contraries as principles, as if they were constrained by truth itself." He described Empedocles in these terms: "... guided by the truth itself, he is forced to admit that natural realities are only the essence." "Rather than as a historian, Aristotle behaves like an anatomist. The systems of thought undergo a double treatment of dissection." The very principles for making distinctions are in turn classified. "The principle may be (a) one or (b) multiple; if it is one, it may be (a') immobile or (a'') mobile. If there are many, they may be (b') finite in number or (b'') infinite in number; in the second case, they may be (b''') equal in kind or (b''') different in kind; and so on." From doctors, Galen expected "... a loving folly for truth."<sup>227</sup>

<sup>225</sup> Plato 1999, pp. 123, 133, 143, 471, 733, 781–782 ... Vegetti 1979, pp. 20–22, 61, 101, 113, 121, 125, 132. Vegetti 1983, pp. 53, 59ff., 85, 94, 122.

<sup>226</sup> Sambursky 1959, pp. 13, 23–24, 37, 113, 124ff., 163ff., 228, 235. Lloyd 1978, pp. 105, 171, 173, 239–240, 264, 307.

<sup>227</sup> Sambursky 1959, pp. 39, 160–162. Vegetti 1979, pp. 59–62, 85–86. Vegetti 1983, p. 118.

The weight of truth in Greek culture is to found also among their poets, such as Sophocles. His Tiresias is said to be “the only man in whom truth is innate”. Jocasta, on the contrary, a woman, saw the world as dominated rather by  $\tau\upsilon'\chi\eta$  [chance, fate, destiny].<sup>228</sup> In the myth, Tiresias received the ability to foresee the future as a series of violations and condemnations. For striking and disturbing two snakes tightly entwined in their love, he was transformed into a woman. Having thus learnt also the nature of female sexuality, he then became the arbiter of the dispute between Jupiter and Juno about which of the two experienced more delight in their married union. “Maior vestra profecto est quam quae contingit maribus . . . voluptas”. [“Your delight [of females] is undoubtedly greater than that of males”].<sup>229</sup> For lifting the veil from the most intimate mystery in history, Tiresias was punished by the angry goddess, who struck him blind. But the king of the gods recompensed him with the gift of prophecy. The myth is a good representation of the *aporia* on which the problem of knowledge was being founded in the West. Are things understood by separating them or by uniting them? And what other secrets would we like to uncover? Tiresias separated, and the Greeks ended up by choosing in most cases the former of the two routes. Those who confuse them will be punished with ignorance, those who make distinctions will know their future, even if they may not be able to bear it. Pleasure was removed far from knowledge, as if it were an obstacle.

The alternatives needed to be separated. The presumed truth should stand only on one side. The only person who admitted a kind of pluralism of explanations seems to have been Epicurus. “. . . for these [the heavenly phenomena], many kinds of origins are proposed, and in accordance with the witness of the senses, different explanations can be given of their mode of existence. [. . .] unless, for the sake of the method of a single explanation, all the others are senselessly disregarded, without understanding what it is, or is not, possible for man to know, yearning thus to glimpse what can not be seen.” On the contrary, Plato has been interpreted as follows: “Reason is an Apollinean impulse which introduces order, making distinctions and dividing things.” The philosopher, whether it was Pythagoras or Empedocles, Parmenides or Plato, is “. . . a man who is at the same time able to wield a dissecting knife like a butcher, the *mageiros* – both the profane one used at the market, and the sacred one of the sacrifice and the hieroscopy.”<sup>230</sup>

For Greek culture in general, knowing meant solving controversies by using the  $\delta\iota\chi\alpha'\iota\omicron\nu$  [that which settles]. For this reason, it is necessary that ‘ $\text{I } \delta\iota\chi\alpha'\zeta\omega$ ’ [I divide into two, I separate]. In Latin, the word used was *discriminare* [which has

<sup>228</sup>Vegetti 1983, pp. 30–31.

<sup>229</sup>Ovidio 1988, pp. 164–167.

<sup>230</sup>Sambursky 1959, pp. 205–208. The faith in progress, towards our physical sciences of the twentieth century, continually led Sambursky to make anachronistic comparisons between the ancient Greeks and us, taken as touchstones. Here, as regards poor Epicurus, who is presented in a contradictory manner, as if he had been afraid of religion, his judgement was: “. . . he abolishes any possibility of arriving at a comprehensive scientific conclusion.” and “. . . ‘scientific failure’ . . .”. Lloyd 1978, p. 169; Vegetti 1979, pp. 92, 94.

passed unchanged into English, to discriminate]. Even if equally cruel, the knife was sometimes symbolic, like the *logos* that separated the Greeks from the barbarians, like a transcendent religious philosophy that separated the soul from the body, like a male-chauvinistic society afflicted by sex phobia, that separated the male from the female. Plato confessed in what kind of historical context he had elaborated his condemnation of mingling, and his cult of an Apollinean purity. “If the decision of the Athenians and the Spartans had not rejected the impending slavery, now almost all the Greek races would be mixed together, and the barbarians would be mixed with the Greeks and vice versa, just like the nations nowadays under the domination of the Persians, who are dispersed and mingled, and confusedly scattered.”<sup>231</sup>

In a discipline unfortunately ignored by us like medicine, divisions were actually made literally, with the bloody dissection of animals, and sometimes of living creatures, not only of human corpses, but also of living people. In this way, classifications were made even of dead animals. Only when he saw himself as a fisherman or a hunter did Aristotle consider them as alive. But otherwise his rationality killed them. Alexandrian doctors like Herophilus or Erasistratus (third century B.C.) and Galen (second century) investigated live patients as if they were dead.<sup>232</sup>

An equally cutting instrument, if not even more so, because the blood, hypocritically, was not visible, was the law. The Graeco-Latin culture was the culture of the law. The *κόσμος* [order, decoration, cosmos] was separated from the *χάος* [yawning abyss, chaos] because it was ordered and guided by rules: *νόμος* [tradition, law, rule, norm].<sup>233</sup> It is possible that ‘*ἰ λέγω*’ [I pronounce, declare, prescribe them]: *νόμος λέγει* [the law prescribes]. This was the derivation of the word *lex* in Latin, and of the word ‘legge’ in Italian.

Ever since the time of Thales (sixth century B.C.) and Solon, the laws of the cosmos were developed consistently with those of the city-state, like *The constitution of Athens*, which was edited by Aristotle.<sup>234</sup> When studies were carried out on the movements of the stars, with them both regularities and irregularities were discovered. But these, in turn, had to be explained by means of new regular movements, among which the spherical ones along circumferences enjoyed most prestige and success. In the *Laws*, Plato wrote that the movement of the sphere around the centre and the “... circular movement of intelligence ...” were similar “... in accordance with the same principle and the same order. [...] Every course and movement of the sky and of all the bodies in the sky is of a similar nature to the revolution and the calculations of intelligence.” “Time itself seems to be a kind of circle”, wrote Aristotle.<sup>235</sup> In the end, the famous epicycles and deferents

<sup>231</sup>Vegetti 1979, p. 133.

<sup>232</sup>Lloyd 1978, pp. 223–225; Lloyd & Vallance 2001, p. 552. Vegetti 1979, pp. 111, 113, 125. Vegetti 1979, pp. 14, 23, 27, 33, 37–40. Vegetti 1983, pp. 116ff.

<sup>233</sup>Lloyd 1978, pp. 20, 120.

<sup>234</sup>Lloyd 1978, pp. 16, 119–120.

<sup>235</sup>Sambursky 1959, pp. 85 and 296.

arrived. Through music (which is thus not to be ignored), reduced to norm in turn by the theories of Pythagoras, Euclid and Ptolemy, the law for the stars could be extended to the souls of human beings, and their relative behaviour, which could thus be ordered as well.

However, all this love of laws often had the consequence of imposing a certain disregard for discrepancies between the theories and the phenomena. The latter could not always be “saved” as such. Archimedes ignored friction in his machines.<sup>236</sup> But it could also be sustained that the imperfections derived from the corrupted earth, and that in the pure sky of the Platonic ideas, everything was perfectly regular. Ptolemy wrote: “... every study that deals with the quality of matter is hypothetical.”<sup>237</sup> In this way, the laws could succeed in expressing truths that were eternal and universal, that is to say, that transcended contingent historical circumstances, which depended rather on living people.

Parmenides accompanied his truth with necessity, persuasion and  $\delta\iota\kappa\eta$  [justice]. One of Plato’s followers explained the perfection of Greek astronomy compared with that of the barbarians, “because the Greeks possess the prescriptions thanks to the oracle of Delphi, and all the complex of divine worship set up by the laws.” For Aristotle, the relationship was so close that it could be inverted. Then the law became “... reason free from desire” based on a divine order. The *logos* expressed the truth, the order and the law of the world. Plato presumed to control the inevitable carnal impulses for food and sex by means of “... fear, the law and true discourse”. For him, children were bad because they followed their instincts and their nature. “The guardian should keep watch carefully, and pay particular attention to the education of the little ones, correcting their nature and always guiding it towards the good, in accordance with the laws.” The judicial conception of science became explicit with Ptolemy. “Therefore, continuing the comparison [...] of the criterion with the tribunal, the sensible realities can be likened to those who are on trial; the contingent aspects of these realities are like the actions of the defendants; the sensor, like the trial documents; sensation, like the lawyers; [...] the intellect, like the judges; [...] reason, like the law [...]. Opinion can be compared to a sentence which is in a certain sense uncertain and dubious, against which it is possible to lodge an appeal; science, on the contrary, can be compared to a sentence that is absolutely certain and unanimous. And above all the purpose of truth is similar to that of society.”<sup>238</sup>

In the project of controlling seeming phenomena by means of laws, therefore, the mathematical sciences played the leading role. But they did not succeed yet in achieving a general mastery over everything. They seemed to work best above all in the field of music and in certain areas of astronomy and astrology. Laws that were firmly anchored to eternal, universal truths could be shown above all in the

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<sup>236</sup>Lloyd 1978, pp. 219–220, 269, 278, 327.

<sup>237</sup>Lloyd 1978, p. 282.

<sup>238</sup>Sambursky 1959, pp. 197ff. Vegetti 1979, pp. 71–73, 90, 104–107, 110. Vegetti 1983, pp. 71ff., 170.



mathematical sciences. Orthodoxy grew up around these, though sceptical heretical characters remained, such as Xenophanes of Colophon (sixth century B.C.), with his “nobody knows or will ever know the truth about the gods, or about all the things of which I speak.”<sup>239</sup>

Our Greek and Latin scholars arrived at the truth that they sought by means of those forms of reasoning called *θεορρημα* [vision, theory, theorem, demonstration]. But what kinds of demonstrations? The ones that followed the schemes of reasoning according to Aristotle, by means of logical syllogisms? Or those invented by Archimedes who took his inspiration from his machines and balances? No. It was above all Euclid’s proofs that enjoyed success; the famous theorems of his *Elements* were to represent, in the following centuries (and even in a different culture like that of the Arabs<sup>240</sup>) the law for every process of reasoning that claimed, with the due authority, to arrive at the certainty of eternal, universal truths. The pathway followed in order to arrive at them was considered subjective and insignificant. In general, it appears to be absent from Greek texts. The essential thing was to expound the final result in the form of a theorem that could be deduced from other truths. The physician-cum-anatomist Galen prescribed: “. . . demonstration is to be learnt from Euclid and then, after learning that, come back to me; I will show you these two straight lines on the animal”; “. . . lastly, we will try to prove the theorems, not assuming anything other than what was established at the beginning.” Euclid’s way of defining and distinguishing with his propositions was taken as a general criterion by Galen, who opened up bodies (not always dead ones) with his sharpened knives. “Where will the proof come from, then? From no other source than dissection?” In their different fields, the stylus and the knife were the instruments used to make distinctions and to arrive at the truth.<sup>241</sup> For this reason, it was necessary to transcend the uncertain instabilities of life on this earth. Did they take their inspiration, then, to a certain extent from those divinities that were venerated by some religion?

At Delphi, on the pediment of the temple dedicated to Apollo, the *E* of *Eĩ* [“You are”] was visible. Parmenides thus identified ‘being’ with a god. In the tradition of the Hebrew *Bible*, this was “Ego sum qui sum” [“I am who I am”].<sup>242</sup> Not only myth, therefore, to subject the people to the laws, in Aristotle this god became “the prime mover”. Philosophers often presented themselves as priests. Aristotle called metaphysics “the science of divine things”. For him, in rising up towards Heaven, man is “. . . among the animals known to us, either the only one that participates in the divine, or the one that participates to the greatest extent.” In the end, certain philosophers began to believe themselves divine: because the activity of thought was dedicated to the divine, and because thought itself has a divine nature.<sup>243</sup>

<sup>239</sup>Lloyd 1978, pp. 264–265, 308 and 323.

<sup>240</sup>See Chap. 5.

<sup>241</sup>Lloyd 1978, pp. 190–191. Vegetti 1983, pp. 113ff., 151ff., 162, 167–168.

<sup>242</sup>*Bible*, “Exodus” III, 14.

<sup>243</sup>Vegetti 1979, pp. 66, 73, 91, 94, 95, 116, 138, 142–143.

Mathematics and religion appear to be closely linked by the followers of the Pythagorean sects. Also a famous physician like Galen (129–c. 199) presented himself as capable of revealing mysteries written in sacred books. His purpose was to build up a “rigorous theology”, and to lift up his “hymn to the gods”. Ptolemy justified his own astronomy: “... it can especially open the way to the theological field, seeing that it alone can correctly come close to a motionless, separate activity.” A disregard for the body and for the material world, together with a belief in immortality, were aspects to be found not only in Greek philosophy, but also in the relative religion. Plato wrote: “Every soul is immortal. For everything that is always in motion is immortal”. For him, incontrovertible proof was offered by the movement of the stars, and the relative music of the spheres. Some of these ideas were later to be found even among Christian writers. Origen (third century) expressed the wish: “I hope that you will learn from Greek philosophy things that will be of use for your general or preparatory studies for Christianity, and from geometry or astronomy things that may be of use for the interpretation of the holy scriptures.” Even Augustine of Hippo (354–430) respected the Platonism of the period, while condemning a Christian’s search for causes as vain and useless, since for him it was sufficient to have faith in the Creator.<sup>244</sup>

Doubtless, there were infinite discussions and diatribes about the truth of this or the other position. With equal certainty, scholars were divided about who, or what, should guarantee this truth. But the event that it seemed to descend from heaven could convince many, even if not all, of them. Why all this anxiety to attribute to others what they had invented? Why not enjoy all the merits themselves? Was it that they were afraid that otherwise they would have to assume also the defects, and thus be fully and solely responsible?

Eratosthenes (third century B.C.) worked for the Ptolemy family at Alexandria, taught their children and directed their library; he once wrote to a customer of his: “My invention [a machine for doubling the volume of a solid] may be useful also for those who desire to increase the size of catapults and martinets, because everything has to be increased proportionately, if we want the shot to be proportionately longer. This cannot be achieved without calculating the means.” However, it was his more famous correspondent, Archimedes, who invented the most renowned war machines, capable of keeping at bay the might of the Romans during the siege of Siracusa.<sup>245</sup> Even the leader of the attacking forces, Marcellus, had his own devices. These included one enormous machine called the “sambuca”.<sup>246</sup>

In any case, this represents one of the clearest episodes in which we can see that a context of war was capable of polarising everything, including the interests of people devoted to the sciences. Among the titles that we know of the books written by Democritus, there was also one on the technique of warfare. We have already dwelt on the way Plato supported to educate his young men through the mathematical

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<sup>244</sup>Lloyd 1978, pp. 134, 303, 319. Vegetti 1983, p. 174. Samburky 1959, pp. 67ff.

<sup>245</sup>See above Sect. 2.7.

<sup>246</sup>Authier 1989, pp. 108, 116 and 123. Lloyd 1978, p. 194.

sciences in order to prepare them better for war.<sup>247</sup> We must note first of all that the Hellenistic age of the Ptolemies, when such important results were achieved, was not at all pacific. The first of the Ptolemies was a general of Alexander the Great.

In his *On the construction of the artillery*, Philo of Byzantium (third and second century B.C.) described the relative problems, contrasting the mistakes of the early archaic attempts with the successes achieved by the engineers of Alexandria. For their war machines, the latter calculated the proportions of the various parts, and verified the results experimentally. They "... received considerable help from sovereigns who were in search of glory, and amenable to the arts and crafts." In his *De Architectura*, Marcus Vitruvius Pollio (first century B.C.) projected war machines that he used in the imperial army of Octavian Augustus. The Heron of Alexandria (first century) that we have already encountered sponsored *On the construction of the artillery*.

Pappus of Alexandria (fourth century) was later to write in his *Mathematical Collection*: "The most necessary of the mechanical arts, from the point of view of everyday requirements, are as follows: (1) The art of pulley makers .... (2) The art of makers of war machines, who are also called mechanics. Missiles of stone, iron, or similar materials are projected for great distances by the catapults that they construct. (3) The art of makers of machines ...". In defining mechanics as "... the study of material objects that can be perceived by the senses ...", even Proclus (fifth century) included, under the first point, the construction of devices that were useful in war. "The priority assigned to the projecting and construction of war machines" may surprise only those ingenuous ones who continue to believe, by faith, in the sublime purity of disinterested scientific research performed by natural philosophers, motivated only by a love for truth. In his arguments, Aristotle would indulge in military comparisons. The *ταξις* [array, battle formation, order] of the world had to be guaranteed, like that of an army. "An army is in good conditions when it is in order, and when it has a general, and in particular when it has a general." Pliny the Elder (24–79) represented animals as a war or post-war spectacle, which was put on show during triumphs and in circuses. He besides wrote books on the military art, and on the wars in Germany.<sup>248</sup>

Plato described the *σωμα* [body] as the site of battles between humours. These give rise to illnesses, including those of the soul. Among these, we find *ἡφροδισία* [sexual pleasure]. The same image was used by Hippocratic medicine.<sup>249</sup> Even the famous physician Galen, who cured gladiators, and followed the Roman soldiers in their campaigns against the Germans, declared: "What is more useful for a doctor in curing a war wound, extracting missiles, amputating bones ... than a detailed knowledge of all the parts of the arms and legs ...?". For him, scorpions, tarantulas and vipers were to be suppressed, because they were "... evil by nature, and not of their own free will. Logically, therefore, we hate evil men ... and we kill those

<sup>247</sup> See above Sect. 2.3.

<sup>248</sup> Lloyd 1978, pp. 47, 144–147, 240–248. Vegetti 1979, pp. 87, 95. Vegetti 1983, pp. 97ff.

<sup>249</sup> Plato 1994, pp. 142–143. Vegetti 1983, pp. 47–51.

who are irremediably evil for three good reasons: so that they will not commit evil, remaining alive; so that they will arouse the fear in their fellow-men that they will be punished for the evils that they commit, and thirdly, it is better for them to die, as they are so corrupt in their souls that they cannot be educated by the Muses, or improved by Socrates or by Pythagoras.” In a treatise of Hippocratic medicine, *Airs, waters, places*, the writer intended to explain the weakness of Asian peoples. As they are “. . . subject to despots, they do not think of how to train themselves for warfare, but rather how to seem unsuitable for fighting. The dangers are clearly not the same. It is natural that in their case, they are forced to fight, suffer and die on behalf of their masters, . . .”. In order to curb the “wild beast” that urges man towards food and sex, Plato placed some “sentinels” in the heart, just as his Republic needed soldiers. Aristotle subjected all plants and animals to man. “. . . also the art of war will by nature be, in a certain sense, a technique of acquisition (and the art of hunting is a part of this), and it must be practised against those beasts and men that refuse to allow anyone to command them, even if they were born for this: because by nature such a war is right.” “. . . dominating the barbarians is befitting for the Greeks.”<sup>250</sup>

In general, science historians modestly avoid recalling the links between mathematical sciences and military arts, perhaps so that they will not have to admit the influence of war contexts on their evolution. And yet Giovanni Vacca did acknowledge it in the introduction to his edition and translation of “Book 1” of Euclid’s *Elements*. He noted the “progress of mechanics” due to the military arts, and quoted Plato’s *Republic* for “. . . the manifest usefulness of geometry in the art of war . . .”. He even identified in this the origin of the speculations dedicated by Tartaglia and Galileo Galilei to movement. This exception can easily be explained by the date of the edition: 1916. In that period, a part of the Italian population was labouring under the illusion that by entering into the world war and achieving an easy, rapid victory, the Italian Risorgimento would be rhetorically completed. In those years, therefore, this mathematician and historian, above all of Chinese matters,<sup>251</sup> considered war as a factor of patriotic, civil and social progress.<sup>252</sup> But as this did not happen then, leading to fascism and resuming in full force worse than before in 1939, neither will it happen today, now that the century of warfare and violence is continuing into the new millennium.

Far be it from us to fall into a totally pessimistic or consolatory philosophy of history, because we continue not to want to exclude from our history the heretics, chance (like Jocasta) and the heterogenesis of ends. However, senators, kings and emperors, for one reason or another, amplified the probability of obtaining the results that they desired by favouring these researches, compared with other forms of culture. We have been able to tell the story of the former together with the relative

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<sup>250</sup>Lloyd 1978, pp. 287 and 302. Plato 1994, pp. 108–111. Vegetti 1979, pp. 105, 112, 120, 134.

<sup>251</sup>See Chap. 3.

<sup>252</sup>Euclid 1916, pp. xiii–xiv. Sambursky 1959, pp. 283 and 292, on the contrary, complained that all those war machines had not produced a “serious, multifarious technological development”.

circumstances. Who knows what happened to the latter, if they ever existed? The traces that have remained are undoubtedly scarce, and more difficult to find.

The name *sambuca* attributed to the war machine of the Roman general Marcellus is interesting, because in Greek, *σαμβύχη* had not only that meaning, but it also indicated a musical instrument with four strings, a kind of triangular harp. As in the myth of the birth from Venus and Mars, or in Plato, harmony and war were presented as variously linked.<sup>253</sup> Arms and harmony have a common origin in the Greek language. *ᾠροδίζω* meant 'to join, to regulate, to govern, to be in agreement'. *ᾠροδνία* derived from this, with the meaning of 'connection, agreement, concord, musical harmony'. The same verb, with the same meaning, also gave rise to *ᾠροα*, which was not so much the amorous union of Aphrodite as a war chariot, and also to *ᾠρομενον*, an instrument, and (in the plural) the equipment of a ship, its rig. For sailing-boats, still today, the Italian *armamento* [armament] means the way in which shrouds, halyards and sheets are connected to the mast and to the sails.

It is equally rare, however, for historians of the sciences to give due importance to musical harmony. We have shown, on the contrary, how much importance Greek scholars of the mathematical disciplines dedicated to it. But this is not just a pedantic, bureaucratic question of completeness. Only the music (of the spheres) explains the insistence on considering our souls as part of the heavens, making human beings similar to stars in the astrological and astronomic picture, and trying to represent the strokes of the pulse by numbers. How could rhythm be measured in that period apart from by numbers? Galen wrote: "... as musicians establish their rhythms in accordance with certain precise combinations of periods of time, contrasting the *ᾠροσις* [lifting, raising, arsis] to the *θεσις* [downstroke, beat, thesis], so Herophilus [third century B.C.] supposed that the dilation of the artery corresponded to the arsis, and its contraction to the thesis."<sup>254</sup> Even in the seventeenth century, we shall find one of the main protagonists of the modern scientific revolution, who turns to music in order to measure the time of a physical phenomenon. Socrates was to compare Plato to a swan that sings and then flies away.<sup>255</sup>

By means of music, it is easier to understand how many, and what kinds of obstacles the Greek and Roman natural philosophers had created between

<sup>253</sup> Authier 1989, p. 116.

<sup>254</sup> Lloyd 1978, pp. 216, 219 and 228. Sambursky 1959, pp. 45–46ff., wrote that musical harmony was "... the first example of the application of mathematics to a basic physical phenomenon". Unfortunately, however, he added that the Pythagoreans had carried out "... authentic quantitative measurements, using wind instruments and instruments with strings of different lengths ..." This does not transpire from the completely different tradition that built up around them. Furthermore, if they had really done so, they would not have been able to maintain the ratios that were so dear to them; because reed-pipes and strings are tuned in accordance with different numbers, as will be seen in Sects. 3.2 and 6.7 below. It is clear that Sambursky does not seem to have had any direct experience with his ears, either.

<sup>255</sup> See Part II, Sect. 8.2. Vegetti 1979, p. 73.

mathematical sciences and the world of the senses. Free access to this world was forbidden by the orthodoxy that grew up around the Pythagorean-Platonic-Euclidean-Ptolemaic axis. They judged the harmony of Aristoxenus to be heretical, with its divisions of musical intervals into equal parts, which were attuned to the ears of the musicians. The prohibition of lascivious, effeminate music, which distracted young men from the virile military arts, was extended to the kind of theory of music that permitted it, thus offering a better justification in practice for micro-intervals. Consequently, the famous question of denying any numerical representation for incommensurable ratios, which was almost equivalent to the division of the tone into equal parts, also assumed the nature of a prohibition, and not just that of a distinction between ratios. Nowadays we would say that diversity was transformed into discrimination and inferiority. And it would be sufficient, then, to read Plato's *Republic* to discover the historical context responsible for discriminating between the two positions: the defeat of Athens (404 B.C.) in the Peloponnesian wars. Music is thus able to offer us new material, in order to re-discuss the *vexata questio* about the invention, or otherwise, of so-called experimental methods, and their relationship, or otherwise, with mathematics.

Aristoxenus was after to be taken into consideration again in Europe only by musicians centuries later, and before the division of the octave into equal parts was given its due importance by scholars of exact sciences. We can maintain for Greek culture the important place it deserves in the evolution of the sciences. We can likewise recognize that it took advantage of its characteristic inclination for discussions, facilitated by its language. But we must also add that we have identified in it powers and instruments of discrimination.

Not everything was left free to develop. In the Museum and the Library at Alexandria, the Ptolemies organised their explicit 'scientific policy', which did not range in all possible directions. Plato made Socrates say that not all sciences were equal, and that a hierarchy existed among them. Those at the basis of the art of building and trading were different from geometry and pure calculations. "Among them, the arts and sciences practised in the search for knowledge by true philosophers are far superior, in precision and truth in measurements and numbers." On that point, Eratosthenes, Galen, and above all Claudius Ptolemy referred to Plato. Ptolemy wrote that he had dedicated himself "... to mathematical theory as a whole, but with a particular preference for that part of it which deals with divine and heavenly things, because this branch alone investigates the things which always exist without changing." We are thus forced to conclude: how? "Well, in reality, science acts as a powerful device for the censure and exclusion of possibilities of discourse, and therefore the control of imaginable universes and images of the world: so much the more powerful, because it does not speak in the name of this or the other option of values, but in the name of the truth itself, ...". Some of these censures are well-known, like the movement of the Earth around the Sun for Aristarchus, or the infinite nature of the universe, and the intelligence of animals. Here I have added to these the

division of the tone and the octave into equal parts, following the ear, as suggested by Aristoxenus.<sup>256</sup>

“... law also means obeying the will of one alone.” Even in the *polis*, Aristotle distinguished men like kings, so perfect in virtues and capable in politics as to be like gods. “For them, given their nature, there is no law: they are the law, and it would be ridiculous to try to draw up a set of laws for them.” There was a hierarchy between those who commanded and those who owed obedience, and had to submit. “Commanding and being commanded are not only necessary, but also beneficial; [...] this happens among living creatures in all nature; and there is a principle of command also in things that do not participate in life, like musical harmony.” Thus, for Aristotle: “As the race of the Greeks occupies geographically a central position, so it participates in the character of both, because it has courage and intelligence, and thus it continually lives in freedom; it has the best political institutions and the possibility of dominating all others, provided that it maintains constitutional unity.”<sup>257</sup>

Diogenes the Cynic preferred to eat like dogs, and laughed at Plato’s definitions, “man is an unfledged biped”, showing a cock that had been plucked.<sup>258</sup> As memories of him, we have, above all, anecdotes and caricatures.

Partly to understand better to what extent that historical context acted as a filter, we shall study in the following chapters how other great written cultures behaved in this regard. Let us start from the one that is most different and most distant: China.

He turns the bow round and round in his hands!  
[...]  
Like a skilful singer who, having fastened  
The twisted catgut of his new lyre  
At both ends, without any difficulty  
Stretches the string by turning the peg;  
So he effortlessly strung the great bow.  
Then he decided to test the string: he opened  
His hand, and the string sang an acute note,  
Like a chirruping swallow’s song.

Homer, *Odyssey*, XXI, 480–493.

<sup>256</sup>Sambursky 1959, pp. 55–56. Vegetti 1983, pp. 151ff., 156, 169ff., 175ff. Paul Tannery (1843–1904) did not contrast Aristoxenus sufficiently with the Pythagoreans and Platonics, putting them all together. But to the Frenchman should be recognized his great merit in attributing the correct role to music in the development of Greek mathematics. He went so far as to write: “... l’origine de la conception grecque de la mesure du rapport est essentiellement musicale, ...” [... the origin of the Greek idea of measuring the ratio is essentially musical, ...]; Tannery 1915 (1902), p. 73. Cf. Mathiesen 2004 who did not attribute *Sectio Canonis* to Euclid. Cf. Barker 2007 who believes that *Sectio canonis* is Euclid’s.

<sup>257</sup>Vegetti 1979, pp. 108, 141, 119–121, 134.

<sup>258</sup>Vegetti 1979, pp. 43, 51. Vegetti 1983, p. 86.

And Yet It Is Heard

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