Chapter 2
How Long and How Much Endowing One Commodity

Real economic problems are complex and involve too many variables and constraints to be both computable and reasonably relevant. Mathematics cannot be "applied" to "economics," at best, economics can motivate mathematics, in other ways than physical sciences did, by offering at least "qualitative" mathematical metaphors of economic evolutions, not quantitative ones[4–11]. But the access time to these statements is forbidding. Consequently, we suggest to begin our study by "illustrating" the general concepts in the context of the simplest example of the evolution of a single commodity. These illustrations are quite instructive, provide another way than general mathematical theorems to grasp what the concepts attempt to mean and to assess their economic relevance. However, in any case these simple systems are far too simplistic for their conclusions to be taken seriously. But if a given property is not economically validated for such an example, it is "falsified" in the Popperian sense in the general case.

Since we deal with evolutionary economic systems, we need to start with some considerations on time and evolution, on prediction and anticipation, by restricting the polysemy of these concepts as much as we can.

2.1 Provider Financing One Commodity

We study here the evolution of (only) one commodity $t \mapsto x(t) \in \mathbb{R}$ (for instance, the number of units of a numéraire).

Assume that $\mathbf{k} \in \mathbb{R}$ is a *minimal threshold* (or, in this economic setting, a *viability threshold*.[1]) above which the commodity must remain.

[1]In which case $\mathbf{k} \geq 0$.

J.-P. Aubin, *Time and Money*, Lecture Notes in Economics and Mathematical Systems 670, DOI 10.1007/978-3-319-00005-3_2,

The simplest example of viability condition requires that there exists a propitious temporal window $[T - \Omega, T]$ of *aperture* Ω such that the *viability property*

$$\forall\, t \in [T - \Omega, T], \quad x(t) \ \geq\ \mathbf{k} \tag{2.1}$$

holds true. This provides in particular the *initial value* $x(T - \Omega)$ at the beginning of the temporal window whenever any terminal condition $x(T) = x$ is required at its end.

In economics and finance, the velocity[2] $x'(t) = \lim_{h \mapsto 0+} \dfrac{x(t) - x(t-h)}{h}$ is interpreted as a *transaction*.[3] In this chapter, we require that the transaction is equal to the *endowment* needed to finance it. The coevolution of the commodity and of the endowment is described by the system of two differential equations[4]

$$\begin{cases} (i)\ x'(t) \ =\ y(t) \\ (ii)\ y'(t) \ =\ \varrho(y(t)) \end{cases} \tag{2.2}$$

satisfying the terminal condition

$$x(T) \ =\ x \tag{2.3}$$

where $x \geq \mathbf{k}$ is a parameter denoting the objective to be reached at time T.

In this illustration of otherwise general concepts avoiding unnecessary cumbersome analytical details, we assume that $\varrho = 0$, so that the variables y denote *constant endowments*, which vary as physical parameters do in deterministic dynamical systems: they remain constant on the temporal widow $[T - \Omega, T]$ and wait for the evolution to end at the temporal window before being changed.[5]

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[2]According to a suggestion of *Efim Galperin*, derivatives from the right $x'(t) = \lim_{h \mapsto 0+} \dfrac{x(t+h) - x(t)}{h}$ are "physically non-existent" since time $t + h$ is not yet known, following the Czech Count Jiri Buquoy, who in 1812, formulated the equation of motion of a body with variable mass, which retained only the attention of *Poisson* before being forgotten.

[3]If we regard the number $x(t)$ of units of numéraire as a *value*, then $x'(t)$ is regarded as an *impetus*, which is the derivative of a value, in the same way than in mechanics, the force is the derivative of a potential, the analog of value.

[4]Stating that the acceleration $x''(t) = \rho(x'(t))$.

[5]This issue, at the origin of the gap between specialists of dynamical systems and those on control theory (where the parameters evolve), is rarely explicitly dealt with. This is the reason for mentioning the following quotation of the book [127, *Introduction to Applied Nonlinear Systems and Chaos*] by *Stephen Wiggins*: ***On the Interpretation and Application of Bifurcation Diagrams: A Word of Caution***: *"[...] In all of our analysis thus far the parameters have been constant. The point is that we cannot think of the parameter as varying in time, even though this is what happens in practice. Dynamical systems having parameters that change in time (no matter how slowly!) and that pass through bifurcation values often exhibit behavior that is very different from the analogous situation where the parameters are constant."*

Recall that a *set-valued map* $F : X \rightsquigarrow Y$ associates with any $x \in X$ a subset $F(x) \subset Y$. This subset $F(x)$ can be empty, reduced to a singleton $F(x) := \{y\}$ or equal to a subset containing more than one element. It is said to be *single-valued* if for any $x \in X$, $F(x) := \{f(x)\}$ contains only one element $f(x)$.

Definition 2.1. [**Regulation Maps**] A *regulation map* $R : (t, x) \rightsquigarrow R(t, x)$ (regarded as a *transaction map* in economics and a *management rule* in finance) associated with the system of differential equations (2.2), p. 32, is a map such that the evolution governed by the *regulation law*

$$\forall\, t \in [T - \Omega, T], \quad x'(t) \;=\; R(t, x(t)) \tag{2.4}$$

is also governed by the system (2.2), p. 32, but satisfies additional requirements (such as viability and other ones).

Happily, it may happen that either $R(t, x)$ is single-valued or that $R(t, x) = R(x)$ does not depend on time, or both: in this case, the regulation map is called a *feedback* (in automatics).

1. In the framework of a *direct approach*[10–21], the regulation map R is given as a part of the *assumptions* of the "model" (and most often, assumed to be single-valued).

   For instance, we could just assume that, as *Thomas Malthus* did in his celebrated *An essay on the principle of population* (1798) in the framework of population dynamics,[6] that $x'(t) = -rx(t)$, providing exponential $x(t) = e^{-r(T-t)}x$ which violate the viability constraint when $\mathbf{k} > 0$: it is viable on the temporal window $\left[T - \frac{\log\left(\frac{x}{\mathbf{k}}\right)}{r}, T\right]$.

   Or, to overcome this sad property, to follow *Pierre-François Verhulst* who proposed in 1838 the *"logistic"* equation $x'(t) = -rx(t)(x(t) - \mathbf{k})$. It provides the *logistic* (or "sigmoid") function $x(t) = \frac{\mathbf{k}x}{x + (\mathbf{k} - x)e^{-r(T-t)}}$, arriving at x at time T, remaining above $\mathbf{k}$ on the temporal window $]-\infty, T]$, decreasing and converging to it asymptotically.

   Or the thousands macro-economic models a priori cleverly designed, but quite often not viable.

[6]The sign of the growth rate is negative instead of positive, since we want to bound below the number of commodity units instead of bounding above the members of the population: "*Population, when unchecked, increases in a geometrical ratio"* and his solution: *"By moral restraint, I mean a restraint from marriage [. . .]"*. In economics, this would be *a restraint from consumption*. See Chaps. 6 and 7 of [15, Aubin, Bayen & Saint-Pierre] for more details.

2. Viability is one of the properties required in the framework of an *inverse approach*[11–21]: we are interested by the *construction* and the *computation* of *regulation maps* $(t,x) \rightsquigarrow R(t,x)$ and of the aperture $\Omega \geq 0$ of the temporal window $[T-\Omega, T]$ over which the evolution governed by $x'(t) \in R(t, x(t))$ guaranteeing at least the viability property (2.1), p. 32 and other ones, such as the ones listed below.

2.2 Restoring Viability

Before mentioning them, just few obvious observations are in order. The evolutions governed by (2.2), p. 32 are affine functions

$$\forall\, t \in \mathbb{R}, \quad x(t) \;=\; x - y(T-t) \tag{2.5}$$

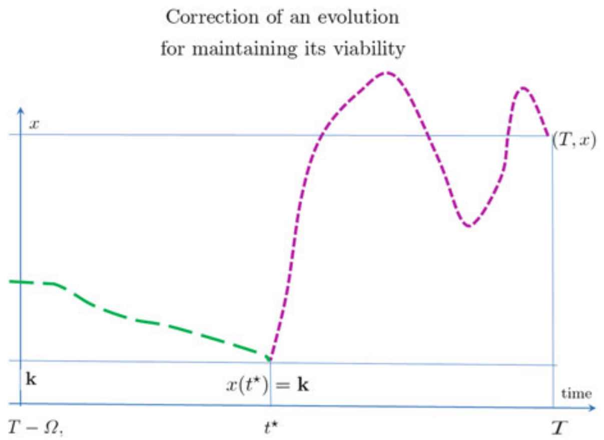
where y is a constant endowment (velocity) and x the terminal condition at T. It is viable on the temporal window of aperture $\Omega := \dfrac{(x-\mathbf{k})}{y}$.

Either the endowment $y = x'(T-\Omega^\flat) < 0$ is negative, and thus the scarcity requirement is violated, or $y^\star = x'(T-\Omega^\flat) \geq 0$, and the evolution remains viable at least for some time. If $y = x'(T-\Omega^\flat) = 0$, then $x(t) = \mathbf{k}$ for all $t \in [T-\Omega^\flat, T]$, and fails to satisfy the terminal condition except if $x = \mathbf{k}$. If $y > 0$, then the solution reaches x at time T if $y = \dfrac{x-\mathbf{k}}{\Omega^\flat}$ where $\Omega^\flat := \Omega^\flat(T,x,y) := \dfrac{x-\mathbf{k}}{y}$ is the aperture of the temporal window $[T-\Omega^\flat(T,x,y), T]$ on which x can be reached at time T by the evolution starting from $\mathbf{k}$ at time $T-\Omega^\flat(T,x,y)$.

Hence, as soon as an evolution hits the viability threshold at time $T-\Omega^\flat(T,x,y) := \frac{x-\mathbf{k}}{y}$, the terminal state can be reached with the endowment y satisfying $\dfrac{x-\mathbf{k}}{\Omega^\flat(T,x,y)}$.

Observe that *starting from any initial state* $x_{T-\Omega} > \mathbf{k}$ *at the beginning* $T-\Omega$ *of a temporal window* $[T-\Omega, T]$, *any evolution is viable* either *on* $[T-\Omega, T]$ *until it reaches* x *at time* T or *there exists some aperture* $\Omega^\flat \in [0, \Omega]$ *when* $x(T-\Omega^\flat) = \mathbf{k}$ *reaches the viability threshold.*

This is the time when the viability question arises for providing a new (positive) endowment or/and a new commodity (strictly larger than the viability threshold) for allowing the evolution to restart and reach x at time T:



14 ***[Restoring Viability]*** *Whenever viability is at stakes at the time $t^\sharp = T - \Omega^\flat$ in the sense that $x(t^\sharp) = \mathbf{k}$, how can we choose an endowment $y^\star = x'(t^\sharp)$ or/and a commodity $x^\star > \mathbf{k}$ such that the solution $x(\cdot)$ governed by the system (2.2), p. 32 starting at $(t^\sharp, \mathbf{k})$ with a velocity $y^\star$ is viable on $[T - \Omega^\flat, T]$ and reaches x at time T?*

Any requirement added to answer this question will define an adequate regulation map R allowing to restore the viability by evolutions governed by differential inclusion $x'(t) \in R(t, x(t))$ on the temporal window $[T - \Omega^\flat, T]$.

We provide a short list of examples of additional requirements.

1. **Required Aperture.** The aperture Ω is given and the viability is required on the whole temporal window $[T - \Omega, T]$, i.e., $\omega^\star(T, x, y) = \Omega$. Then the regulation map R_Ω is obviously defined by

$$y \;=\; R_\Omega(T, x) \;:=\; \frac{x - \mathbf{k}}{\Omega}$$

so that we have computed this feedback in this simple case, as well as the initial commodity $x(T - \Omega) = \Omega R_\Omega(T, x)$.
2. **Viability Threshold Endowment Condition.** When the evolution $x(\cdot)$ hits the viability threshold $\mathbf{k}$ at time $t^\sharp$, the endowment is fixed as a function $\Delta(t^\sharp)$. The function $t \mapsto \Delta(t)$ is called a *viability threshold condition*. This viability threshold condition being given, we construct a regulation map $R_\Delta(T, x)$ (which may be set-valued when the viability threshold condition is not monotone) providing the set of endowments provided when scarcity is at stakes for reaching x at time T (see Sect. 2.3, p. 37).
3. **Viability Threshold Impulse Condition.** Instead of imposing the endowment through a viability threshold condition when the evolution $x(\cdot)$ hits the viability

threshold $\mathbf{k}$ at time $t^\sharp$, we allow the evolution of the commodity to be discontinuous and to "jump" at a new level $x^+ = \Phi(t^\sharp)$ with an *impulse* (infinite velocity). Starting from this new condition $x(t^\sharp) = \Phi(t^\sharp)$, we build a regulation map R_Φ such that any velocity $y \in R_\Phi(T, x)$ will allow to reach x at time T. Such a system is an example of *impulse dynamical systems*. See more details in Sect. 12.2 of [15, Aubin, Bayen & Saint-Pierre] and in the recent book [74, *Hybrid Dynamical Systems*] by *Rafal Goebel* et al.

We can combine the viability threshold condition and impulse requirements by associating with time $t^\sharp$ the new state $x(t^\sharp) = \Phi(t^\sharp)$ and the velocity $y(t^\sharp) = \Delta(t^\sharp)$ (see Sect. 2.5, p. 41).

4. **Governing the Evolution of Endowments.** For the time, we imposed that the endowments were constant parameters, governed by the differential equation $y'(t) = 0$. We may require that the velocity $y'(t)$ of the endowments depend linearly on the velocities $x'(t) = y(t)$, i.e., that the evolution of the endowments is from now on governed by a differential equation

$$y'(t) \;=\; \mathbf{m}(t)y(t) + \mathbf{l}(t) \tag{2.6}$$

reaching $\mathbf{C}$ at time T.

We also allow the viability threshold $\mathbf{k}(t)$ of the endowment to depend on time (see Chap. 3, p. 47).

5. **Choosing Initial Endowments in a Set.** Imposing a unique endowment y when the commodity reaches the viability threshold may be too strong a requirement. We may bring a little more flexibility by requiring only that the endowment y can be chosen and evolve in a given interval $[0, c]$. In other words, we impose only a constraint on the endowment.

This flexibility is translated as (simple) differential inclusion

$$\begin{cases} (i) \;\; |x'(t)| = y(t) \\ (ii) \;\; y'(t) = 0 \end{cases} \tag{2.7}$$

See Chap. 4, p. 57.

6. **Endowment Depending on the Commodities and Prices.** Until now, we assumed that the endowment financed the transaction of one commodity under the assumption that the price was constant (and equal to 1). If not, prices $t \mapsto p(t)$ evolve and their velocities are regarded as *price fluctuations*.

We denote by

$$\begin{cases} U(x(t), p(t)) \;:=\; p(t)x(t) \text{ the } \textit{patrimonial value} \\ E(x(t), p(t)) \;=\; p(t)x'(t) + p'(t)x(t) \text{ its } \textit{impetus} \end{cases}$$

We associate with an economic pair (x, p) and an endowment y the set $\mathscr{U}(x, p; y)$ of transaction-fluctuations (υ, π) defined by

$$\mathscr{U}(x, p; y) \;:=\; \{(\upsilon, \pi) \text{ such that } p\upsilon + \pi x \;\leq\; y\}$$

Hence the coevolution of commodities, prices, and endowment is governed by the *differential inclusion*

$$\forall\, t \in [T-\Omega, T], \quad (x'(t), p'(t)) \in \mathscr{U}(x(t), p(t), y(t)) \text{ and } y'(t) = 0 \qquad (2.8)$$

Therefore, concerning the financial aspect, the retroactions laws associating endowments $y \in R(t, \upsilon)$ with time t and patrimonial value υ that have been mentioned can be used to define the regulation maps $(x'(t), p'(t)) \in \mathscr{U}(x(t), p(t), R(t, p(t)x(t)))$.

Naturally, the situation is much more complex, because commodities $x(t) \in \mathbb{R}^{\ell}$ and prices $p(t) \in \mathbb{R}^{\ell^\star}$ range over ℓ-dimensional spaces, or even, in the case of n economic agents, commodity allocations $x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^{n\ell}$ over a $n\ell$ dimensional one. Additional economic constraints of the form $(x(t), p(t)) \in K(t)$ may be added to the financial ones bearing on the impetuses

$$E(x(t), p(t)) \;:=\; \sum_{i=1}^{n} \left\langle p(t), x_i'(t)\right\rangle + \left\langle p'(t), x_i(t)\right\rangle \;=\; \frac{d}{dt}U(x(t), p(t))$$

of the patrimonial value

$$U(x(t), p(t)) \;:=\; \sum_{i=1}^{n} \langle p(t), x_i(t)\rangle$$

These issues are the topics studied in Chap. 7, p. 105.

We investigate in the framework of system (2.2), p. 32, when the endowment provider acts when the commodity reaches the *viability threshold* for financing the economic agent by providing him either some endowment or some amount of commodity above the threshold or both.

We begin by the case when it provides only an endowment and next, endowment and commodity.

2.3 Providing Endowment

Proposition 2.3.1. ***[The Threshold Endowment Regulation Map]*** *The viability threshold endowment function Δ being given, the associated regulation map R_Δ provides the aperture $\Omega^\flat \geq 0$ and the amount y of the endowment necessary to reach x at time T in a viable way. The apertures $\Omega^\flat$ are the fixed points of*

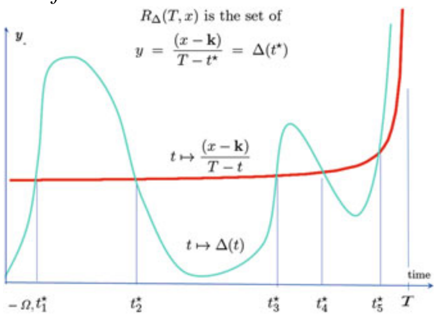
$$\Omega^\flat \Delta(T - \Omega^\flat) \;=\; x - \mathbf{k} \qquad (2.9)$$

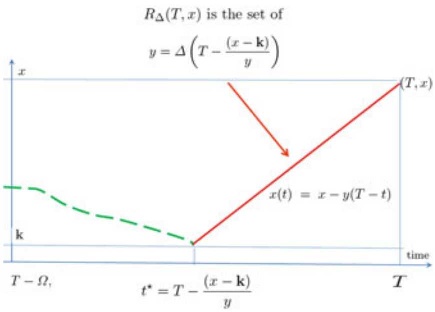
and the amounts $y = \Delta(T - \Omega^\flat)$ of endowments is the subset $R_\Delta(T, x)$ of fixed points y of

$$y \; = \; \Delta\left(T - \frac{(x-\mathbf{k})}{y}\right) \tag{2.10}$$

The regulation map R_{Δ} is single valued if and only if the viability threshold endowment condition Δ is strictly decreasing.
Otherwise, whenever a pair of initial instants t_i satisfies $u(t_1, t_2) := \frac{t_1 - t_2}{\Delta(t^1) - \Delta(t_2)} \geq 0$, then there exists a shock at time $t^\star = t_1 + u(t_1, t_2)\Delta(t_2) = t_2 + u(t_1, t_2)\Delta(t_1)$ and $x^\star = \mathbf{k} + u(t_1, t_2)y_1 = \mathbf{k} + u(t_1, t_2)y_2$.

Proof.



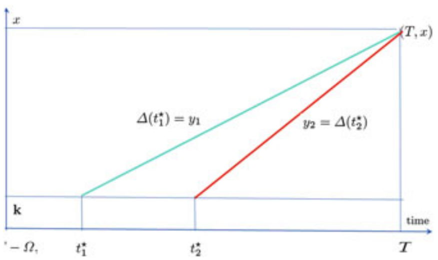


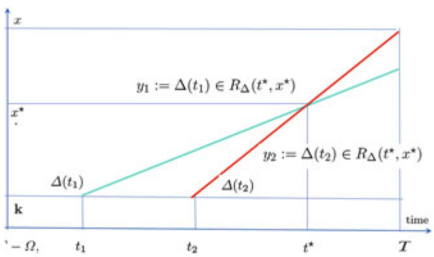
Let (T, x) be fixed and consider the evolution $x(t) := x - y(T - t)$ associated with the endowment y reaching the target $x(t^\sharp) = \mathbf{k}$ at time $t^\sharp := T - \Omega^\flat$ when

$$\Omega^\flat \; = \; \frac{(x-\mathbf{k})}{y} \geq 0 \text{ or } y \; = \; \frac{(x-\mathbf{k})}{\Omega^\flat}$$

Since the threshold condition requires that the endowment $y = \Delta(T - \Omega^\flat)$ is equal to $\Delta(T - \Omega^\flat)$, we infer that the aperture is a solution to the equation $\Omega^\flat \Delta(T - \Omega^\flat) = x - \mathbf{k}$ or, equivalently, that the endowment is a fixed point $y = \Delta\left(T - \frac{(x-\mathbf{k})}{y}\right)$.

Hence the regulation map $R_{\Delta}(T, x)$ is a subset, which can be empty (no solution), reduced to one element $\{r_{\Delta}(T, x)\}$ or the subset of values $y = \Delta(t^\sharp)$ such that $\frac{(x-\mathbf{k})}{T - t^\sharp} = \Delta(t^\sharp)$ containing more than one element.





Assume that there exist two endowments $y_i \in R_\Delta(T,x)$, $i = 1,2$, such $y_1 < y_2$ and $t_i^\sharp := T - \dfrac{(x-\mathbf{k})}{y_i}$. Then

$$t_1^\sharp \; < \; t_2^\sharp \text{ and } \Delta(t_1^\sharp) = y_1 < y_2 = \Delta(t_2^\sharp)$$

1. If Δ is strictly decreasing, we would deduce that $\Delta(t_2^\sharp) \le \Delta(t_1^\sharp) < \Delta(t_2^\sharp)$, which is impossible. In this case, the regulation map $R_\Delta(T,x) = \{r_\Delta(T,x)\}$ is single-valued (there is no shock).
2. Conversely, assume that Δ is "not strictly decreasing" in the sense than there exist $t_i \le T$, $i = 1,2$, such that

$$t_1 \; < \; t_2 \text{ and } \Delta(t_1) \; \le \; \Delta(t_2)$$

Let us consider the "characteristics" $x_i(t) := \mathbf{k} + \Delta(t_i)(t - t_i)$ starting at $\mathbf{k}$ at time t_i. They intersect at time $t^\star$ satisfying

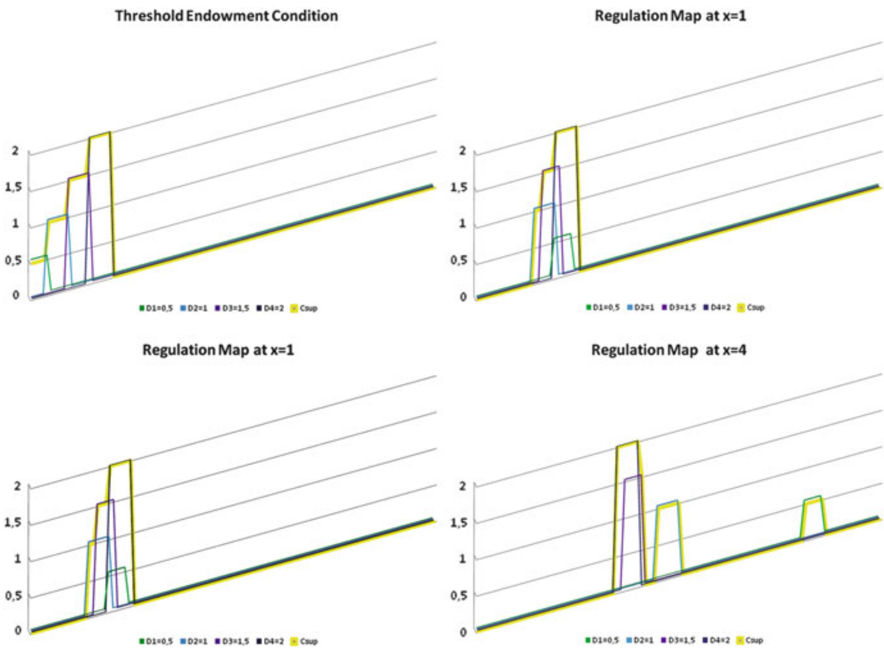
$$t^\star \; = \; t_2 + y_1 \frac{t_2 - t_1}{y_2 - y_1} \; \ge \; t_2 \; \ge \; t_1$$

Denoting by $x^\star = x_1(t^\star) = x_2(t^\star)$ the common value of the characteristics, we infer that both $y_i := \Delta(t_i)$ belong to $R_\Delta(t^\star, x^\star)$, i.e., that a shock happens at time $t^\star \ge t_2$ and $x^\star$. ■

In the extent that set-valued maps are rejected for lack of economic (or physical) sense, we can provide selections of the map R_Δ, which are single-valued, but discontinuous. Among them, we single-out upper $r_\Delta^\sharp(t,x)$ and the lower selections $r_\Delta^\flat(t,x)$ defined by

$$\begin{cases} r_\Delta^\sharp(t,x) =: \max R_\Delta(t,x) \\ r_\Delta^\flat(t,x) =: \min R_\Delta(t,x) \end{cases} \tag{2.11}$$

Hence $R_\Delta(t,x) \subset [r_\Delta^\sharp(t,x), r_\Delta^\flat(t,x)]$. We deduce that the apertures satisfy inequality $\Omega^\flat(T,x,r_\Delta^\sharp(t,x)) \le \Omega^\flat(T,x,r_\Delta^\flat(t,x))$.



15 ***Regulation Map with Shocks.*** *Consider a viability threshold endowment function Δ which is a not strictly increasing* step *(or* staircase, piecewise constant*) functions with four steps, displayed in the left figure of the first row.*
For a fixed commodity x, the figure in the right displays the graph of $t \rightsquigarrow R_\Delta(t, x)$ which is set-valued map (having shocks*) as well as the graphs of its upper selection $r_\Delta^\sharp(t, x)$. Since the endowment condition is increasing, shocks may happen. Indeed, the three other figures display the graphs of the maps $t \mapsto R_\Delta(t, x)$ for three other values of the commodity, where shocks are produced.*

2.4 Viability Property of the Regulation Map

Lemma 2.4.1. *[**Viability Property**] Let $y \in R_\Delta(T, x) = R_\Delta(T, x(T))$. Then, for any $t \in [T - \Omega^\flat(T, x, y), T]$, y belongs to $R_\Delta(t, x(t))$. In other words, for every $t \in [T - \Omega^\flat(T, x, y), T]$, $(t, x(t), y) \in Graph(R_\Delta)$ is viable, or, equivalently, the graph of R_Δ is viable under the characteristic system (2.2), p. 32.*

Proof. This is obvious, since the evolution $x(t) = x - y(T - t)$ starting at $t^\sharp := T - \Omega^\flat$ reaches $x(t)$ at time t and starts from $(t^\sharp, \mathbf{k})$, so that $y \in R_\Delta(t, x(t))$. ■

This viability property implies formally that, whenever $(t, x) \rightsquigarrow R_{\Delta}(t, x) = \{r(t, x)\}$ is single-valued and differentiable, it is the solution to the *Burgers partial differential equation*

$$\frac{\partial r(t,x)}{\partial t} + r(t,x)\frac{\partial r(t,x)}{\partial x} \;=\; 0 \tag{2.12}$$

satisfying the Dirichlet condition

$$\forall\, t \in [t^{\sharp}, T],\quad r(t, \mathbf{k}) \;=\; \Delta(t) \tag{2.13}$$

so that the endowment threshold function $t \mapsto \Delta(t)$ plays the role of a *Dirichlet boundary condition* in the language of partial differential equations.

Indeed, differentiating the viability relation $r(t, x(t)) = y$ with respect to t, we obtain,

$$\forall\, t \in [t^{\sharp}, T],\quad \frac{\partial r(t,x(t))}{\partial t} + r(t,x(t))\frac{\partial r(t,x(t))}{\partial x} \;=\; 0$$

The Burgers partial differential equation with Dirichlet condition is the cornerstone of the *theory of shock waves*, at the origin of a myriad investigations. We refer only at [64, *Partial Differential Equations*] by *Craig Evans* for a mathematical exposition and to [96, *History of Shock Waves, Explosions and Impact*] by *Peter Krehl* for a physicist one.

Set-valued analysis defining “graphical derivatives” of set-valued maps can be used to give a rigorous mathematical meaning to the concept of a “derivative of a set-valued map,” and consequently, to the concept of “set-valued solution” to the Burgers equation. We thus derive that the set-valued map R_{Δ} is the *unique solution with closed graph* satisfying the Burgers equation with Dirichlet condition.

These partial differential equations are just mentioned for those familiar with this theory. Chapter 16 of [15, Aubin, Bayen & Saint-Pierre] provides details on the viability approach for solving conservation laws that it is not useful to reproduce here.

2.5 Providing Commodity

In this case, we assume that the endowment provider furnishes not only an endowment $\Delta(t^{\sharp})$ when the commodity reaches its threshold level $\mathbf{k}$, but also an amount $\Xi(t^{\sharp}) > \mathbf{k}$ through a *threshold commodity function* Ξ.

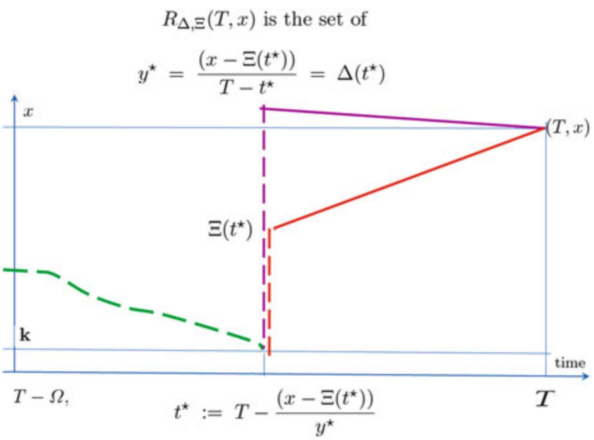
If the amount $\Xi(t^{\sharp}) > x$ is larger than the objective x at time T, then the agent must refund the provider with some (negative) endowment. If we accept this situation, we have to assume that the endowment function takes also negative values.

Proposition 2.5.1. ***[The Threshold Endowment and Commodity Regulation Map]*** *The viability threshold endowment function Δ and the commodity function Ξ being given, the associated regulation map R_{Δ} providing the endowment and the commodity necessary to reach x at time T in a viable way is the subset $R_{\Delta,\Xi}(T,x)$ of endowments $y = \Delta(T-\Omega^{\flat})$ such that the aperture $\Omega^{\flat}$ is a solution to the equation*

$$\Xi(T-\Omega^{\flat}) + \Omega^{\flat}\Delta(T-\Omega^{\flat}) \; = \; x \tag{2.14}$$

so that the endowments y are given by

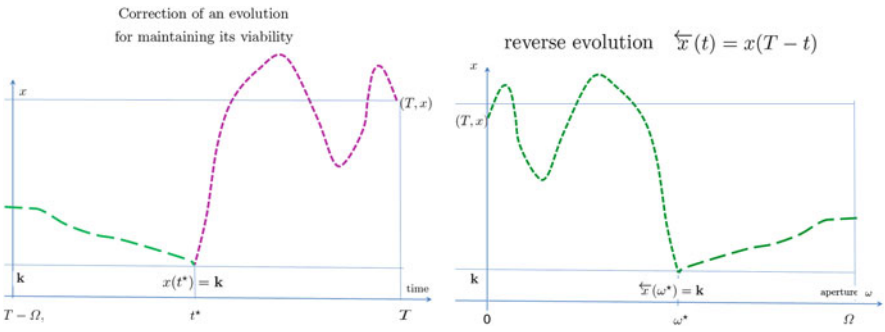
$$y \;\in\; \Delta(\Xi^{-1}(x-\Omega^{\flat}y)) \tag{2.15}$$



Proof. The proof if a simple adaptation of Proposition 2.3.1, p. 37, where we replace $\mathbf{k}$ by $\Xi(t^\star)$ in the formula $\Delta\left(T - \frac{(x-\Xi(t^\star))}{y^\star}\right)$ providing the endowment for this new threshold where $t^\star = T - \frac{(x-\Xi(t^\star))}{y^\star}$. ■

2.6 Towards Viability Solutions

We reformulate the proof in a reverse way for showing that $(T,x) \rightsquigarrow R_{\Delta}(T,x)$ is the viability solution in a sense that we describe now.



16 ***[Correction of a Evolution To Remain Viable]*** *Left, the temporal graph of an evolution $x(\cdot)$ corrected when it bumps the threshold. Right, its reverse $\overline{x}(\cdot)$.*

Let us introduce the reverse functions $\tau(t) := T - t$, $\overleftarrow{x}(t) := x(T-t) = x - yt$ and $\overleftarrow{y}(t) := y(T-t) = y$ (see Fig. 16, p. 43). We observe that this evolution is governed by the reverse characteristic system

$$\begin{cases} (i) & \tau'(t) = -1 \\ (ii) & \overleftarrow{x}'(t) = -\overleftarrow{y}(t) \\ (iii) & \overleftarrow{y}'(t) = 0 \end{cases} \tag{2.16}$$

starts at (T, x, y) at initial time 0.

It satisfies the constraint $T - t \geq 0$ and $x - yt \geq \mathbf{k}$, i.e., that it is viable in the environment $\mathscr{K} := \mathbb{R}_+ \times [\mathbf{k}, +\infty[\times \mathbb{R}_+$. To say that $y \in \Delta(T - \Omega^\flat)$ whenever $x(T - \Omega^\flat) = \mathbf{k}$ amounts to saying that $(T - \Omega^\flat, x - y(T - \Omega^\flat), y) = (0, \mathbf{k}, y)$ and $y = \Delta(T - \Omega^\flat)$.

Therefore, we introduce

1. the *environment* $\mathscr{K} := \mathbb{R}_+ \times [\mathbf{k}, +\infty[\times \mathbb{R}_+$;
2. the *target* $\mathscr{C} := \{(t, \mathbf{k}, \Delta(t))_{t \geq 0}\}$,

to say that $y \in R_\Delta(T, x)$ amounts to saying that (T, x, y) *is the initial condition of the solution* $t \mapsto (T - t, x - yt, y)$ *governed by the reverse characteristic system (2.16), p. 43,* viable *in the environment* $\mathscr{K}$ *until it* captures *the target* $\mathscr{C}$ *at time* $\Omega^\flat \in [0, \Omega]$.

Why this little awkward hard work while we already solved explicitly the problem in Proposition 2.5.1, p. 42?

The answer is provided in terms of *viable capture basin*[38–124], since we have proved that y belongs to $R_\Delta(T, x)$ if and only if (T, x, y) belongs to the capture basin $\mathrm{Capt}_{(2.16)}(\mathscr{K}, \mathscr{C})$. This is in this sense that R_Δ is a (graphical) viability solution:

$$\mathrm{Graph}(R_\Delta) = \mathrm{Capt}_{(2.16)}(\mathscr{K}, \mathscr{C}) \tag{2.17}$$

2.7 The Viabilist Strategy

The point is that, except in this simplest example, the extension of Proposition 2.5.1, p. 42 to more general problems is difficult: even in this simplest case, the computation of the solution amounts to find the set of fixed points of the nonlinear equation $y = \Delta\left(T - \frac{(x - \mathbf{k})}{y}\right)$, which transfers the difficulties to the (big) ones concerning *the resolution of nonlinear equations or inclusions*.

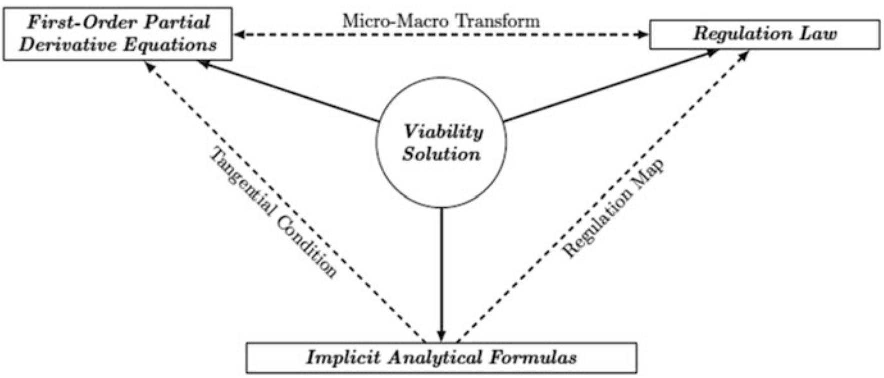
However, by now, many properties of viable capture basin are known (many of them are gathered in [15, Aubin, Bayen & Saint-Pierre]). Among them are viability kernel algorithms, examples of algorithms computing the viable capture basins. Viability kernel algorithms are part of "set-valued numerical analysis" handling the computations of subsets instead of plain vectors at each step of the algorithm. They provide the numerical solutions, whatever is the function ρ in the characteristic system (2.2), p. 32, the time-dependent viability threshold and the threshold endowment condition. This is how we computed the solution of the example (Fig. 15, p. 40), instead of using algorithms for solving partial differential equations.

This set-valued approach is used pervasively: maps, single-valued or set-valued, are characterized by their graphs and numerical functions by their epigraphs. Closedness of viable capture basins, graphs, and epigraphs is the minimal "continuity property" we require.

Now that this example has been carried over, we shall solve all other problems by defining viability solutions for adequate characteristic systems, constraints, and targets.

Whenever possible, we provide (implicit) analytical formulas characterizing the viability solution and derive the regulation maps. Not because they help us to compute the solutions, but since they tell tales about the portfolio, unfortunately in an esoteric mathematical language.

These viability solutions are also solutions of partial differential equations or inclusions. We shall not insist on this side of the story for lack of (apparent) economic relevance (just the opposite approach of the physicists who use these partial differential inclusions as the starting point of their study and working much harder than we will do for studying their solutions).



17 *[**Viability Solution: The Mother of Solutions to Three Problems**] This diagram describes the three problems under investigation: the underlying invariance property, the microscopic version dealing with the regulation laws and the macroscopic approach through first-order partial differential equation (conservation laws and Hamilton–Jacobi equations). The tools of viability theory allow us to show that the viability solutions solve these three other problems at once and, often, under weaker assumptions.*

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