

Chapter 2

Equilibrium Laser Pulse Heating and Thermal Stress Analysis

Abstract When the heating duration becomes greater than the thermalization time of the substrate material, equilibrium heating takes place in the laser irradiated region. In this case, the classical Fourier heating law governs the energy transport. Although the heating process is complicated, some useful assumptions enable to obtain the closed form solution for temperature and stress fields. Since the analytical solution provides the functional relation between the dependent variable and the independent parameters, it provides better physical insight into the heating problem than that of the numerical analysis. In this chapter, equilibrium heating of solid surfaces heated by a laser beam is considered. The closed form solution for the resulting temperature and stress fields are presented for various heating situations. The study also covers the phase change taking place at the irradiated region during the laser treatment process.

2.1 Introduction

Laser pulse heating of metallic surfaces can be modeled using the Fourier heating law for the pulse lengths greater than the thermalization time of the substrate material. Laser pulses can be considered as a step input pulse type or a time exponentially decaying pulses. In addition, the boundary conditions are important and the solution changes for different boundary conditions resembling the actual physical configurations. Therefore, the analytical solution for the laser heating problem depends on the pulse type and the boundary conditions incorporated in relation to the physical situations. During the laser heating process, thermal strain is developed because of the temperature gradient generated in the solid substrate. Depending on the thermal expansion coefficient of the substrate material and the temperature gradient, thermal strain results in excessive thermal stress levels in the substrate material. Therefore, thermal stress developed because of step input laser and time exponentially varying pulses for various boundary conditions are considered, and the closed form solutions for temperature and stress fields are presented in line with the previous studies [1–8].

The laser spot diameter at the workpiece surface is small and heat transfer in the radial direction is considerably smaller than that corresponding to the axial direction. Therefore, one-dimensional modeling of the laser heating process predicts reasonably accurate temperature distribution at the surface and inside the substrate material. In addition, the absorption depth of laser irradiated radiation is considerably small in the surface as compared to the thickness of the substrate material; consequently, the substrate material can be assumed as a semi-infinite body.

2.2 Step Input Laser Pulse Heating

The boundary conditions influence the thermal stress distribution in the irradiated region. In, general two-boundary conditions can be considered associated with the practical laser heating applications. The first boundary condition involves with stress free surface where the surface free to expand during the heating cycle. The second boundary condition is stress continuity at the surface because of the presence of the film at the surface. The closed form solutions for laser heating and thermal stress development are given below the appropriate sub headings in line with the previous studies [1–3].

2.2.1 Stress Free Boundary at the Surface

The laser heating pulse can be considered as a step input pulse. In the actual laser pulse, there exists the pulse rise and fall times and the actual laser pulse deviates slightly from the step input pulse due to small pulse rise and fall times. However, it is convenient to use a step input pulse in the analysis for the mathematical simplicity. In order to construct a single step intensity pulse, it is necessary to subtract two unit step functions with a time shift between them. The first step function will start at $t = 0$ and the other will start at $t + \Delta t$. The step intensity pulse is therefore:

$$SP(t) = 1(t) - 1(t - \Delta t) \quad (2.1)$$

where:

$$1(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \quad (2.2)$$

and

$$1(t - \Delta t) = \begin{cases} 1, & t > \Delta t \\ 0, & t < \Delta t \end{cases} \quad (2.3)$$

The mathematical arrangement of temperature rise in the solid substrate due to a single step intensity pulse is given here; however, the analyses related to two

successive pulses are similar to the single step intensity pulse and, therefore, it is avoided herein. The Fourier heat transfer equation for a laser heating pulse can be written as:

$$\frac{\partial^2 T}{\partial x^2} + \frac{I_1 \delta}{k} (C_1 * SP(t)) e^{-\delta x} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (2.4)$$

where:

$$I_1 = (1 - r_f) I_0 \quad (2.5)$$

Initially, the substrate material is assumed to at a constant uniform temperature; therefore, the initial condition is:

$$\text{At } t = 0 \Rightarrow T(x, 0) = T_0 \quad (2.6)$$

Since the heating duration is short and no assist gas cooling is assumed; therefore, the radiation and convective losses from the surface are negligible. Consequently, the corresponding boundary condition is:

$$\text{At } x = 0 \Rightarrow \left[\frac{\partial T}{\partial x} \right]_{x=0} = 0 \quad (2.7)$$

Since the depth of the irradiated region is limited with the depth of absorption, the size of the absorption depth is considerably smaller than the depth of the workpiece; therefore a semi-infinite workpiece is considered. This assumption leads to a boundary condition of constant temperature at an infinitely depth below the surface, i.e. the influence of the laser heating pulse is negligible at a depth infinitely long from the surface. Therefore,

$$\text{At } x = \infty \Rightarrow T(\infty, t) = 0 \quad (2.8)$$

The Laplace transform of Eq. 2.4 with respect to t , results:

$$\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{I_1 \delta}{k} (C_1 * SP(s)) e^{-\delta x} = \frac{1}{\alpha} [s \bar{T}(x, s) - T(x, 0)] \quad (2.9)$$

where:

$$SP(s) = \frac{1}{s} - \frac{e^{-(\Delta t)s}}{s} \quad (2.10)$$

Introducing the initial condition and rearranging Eq. 2.9 yields:

$$\frac{\partial^2 \bar{T}}{\partial x^2} - \lambda^2 \bar{T} = -\frac{I_1 \delta}{k} (C_1 * SP(s)) e^{-\delta x} - \frac{T_0}{\alpha} \quad (2.11)$$

where $\lambda^2 = s/\alpha$ and s is the transform variable. Equation 2.11 has the solution:

$$T(x, s) = A e^{\lambda x} + B e^{-\lambda x} - \frac{I_1 \delta}{k} \frac{(C_1 * SP(s))}{(\delta^2 - s/\alpha)} e^{-\delta x} + \frac{T_0}{s} \quad (2.12)$$

where A and B are constants. Introducing the boundary conditions determines the constants A and B , i.e.:

$$B = 0, \text{ and } A = \frac{I_1 \delta^2 (C_1 * SP(s))}{k\lambda(\delta^2 - s/\alpha)}.$$

Substitution of A and B in Eq. 2.12 yields:

$$\bar{T}(x, s) = \frac{I_1 \delta^2 \sqrt{\alpha} (C_1 * SP(s)) e^{-\lambda x}}{k\sqrt{s}(\delta^2 - \lambda^2)} - \frac{I_1 \delta (C_1 * SP(s)) e^{-\delta x}}{k(\delta^2 - \lambda^2)} + \frac{T_0}{s} \quad (2.13)$$

which gives the thermal solution in the Laplace transform domain.

The inverse Laplace transform of Eq. 2.13 gives the temperature distribution inside the substrate material in non-dimensional form. This is possible by defining dimensionless quantities, which are:

$$x^* = x\delta : t^* = \alpha\delta^2 t : T^* = \frac{Tk\delta}{I_1} \quad (2.14)$$

The dimensionless temperature distribution becomes:

$$T^* = T_0^* + C_1 \left\{ e^{-x^*} (e^{t^*} - U[1]) - \frac{e^{t^*}}{2} \left[e^{-x^*} \operatorname{erfc} \left(-\sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}} \right) - e^{-x^*} \operatorname{erfc} \left(\sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}} \right) \right] + \left[\frac{2\sqrt{t^*}}{\sqrt{\pi}} e^{-\frac{x^{*2}}{4t^*}} - x^* \operatorname{erfc} \left(\frac{x^*}{2\sqrt{t^*}} \right) \right] \right\} - C_1 \left\{ e^{-x^*} (e^{t^* - \Delta t^*} - U[1 - \Delta t^*]) - \frac{e^{t^* - \Delta t^*}}{2} \left[e^{-x^*} \operatorname{erfc} \left(-\sqrt{t^* - \Delta t^*} + \frac{x^*}{2\sqrt{t^* - \Delta t^*}} \right) - e^{-x^*} \operatorname{erfc} \left(\sqrt{t^* - \Delta t^*} + \frac{x^*}{2\sqrt{t^* - \Delta t^*}} \right) \right] + \left[\frac{2\sqrt{t^* - \Delta t^*}}{\sqrt{\pi}} e^{-\frac{x^{*2}}{4(t^* - \Delta t^*)}} - x^* \operatorname{erfc} \left(\frac{x^*}{2\sqrt{t^* - \Delta t^*}} \right) \right] \right\} \quad (2.15)$$

where erfc is the complementary error function, which is:

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) \text{ and } \operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

Equation 2.15 is the closed form solution for temperature distribution.

To solve for the stress distribution inside the substrate material, equation governing the momentum in a one-dimensional solid for a linear elastic case can be considered [9], i.e.:

$$\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{1}{c_1^2} \frac{\partial^2 \sigma_x}{\partial t^2} = c_2 \frac{\partial^2 T}{\partial t^2} \quad (2.16)$$

where c_1 is the wave speed in the solid

$$c_1 = \sqrt{\frac{E}{\rho}} \text{ and } c_2 = \frac{1+\nu}{1-\nu} \rho \alpha_T$$

where ν is Poisson's ratio, ρ is the density of the solid and α_T is the thermal expansion coefficient of the solid.

Initial and boundary conditions for the temperature field in Eq. 2.16 are similar to those given for Eq. 2.4. Initially substrate material is considered as free from the stress. In addition, as time extends to infinity, the stress field vanishes in the substrate material. Therefore, the initial and final conditions for the stress field become:

$$\text{At } t = 0 \Rightarrow \sigma_x = 0 \quad (2.17)$$

and

$$\text{At } t = \infty \Rightarrow \sigma_x = 0 \quad (2.18)$$

The consideration of no mechanical force at the surface prior to laser irradiation pulse leads to stress free boundary conditions at the surface. In addition, the effective depth of laser irradiation is considerably smaller than the workpiece thickness. Therefore, the assumption of semi-infinite body holds in the stress analysis. In this case, the thermal strain disappears at a depth infinitely long from the surface. Consequently, the corresponding boundary conditions yield:

$$\text{At } x = 0 \Rightarrow \sigma_x = 0 \quad (2.19)$$

and

$$\text{At } x = \infty \Rightarrow \sigma_x = 0 \quad (2.20)$$

Taking the Laplace transform of Eq. 2.16 with respect to time yields:

$$\begin{aligned} \frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{1}{c_1^2} [s^2 \bar{\sigma}_x(x, s) - s \sigma_x(x, 0) - \dot{\sigma}_x(x, 0)] \\ = c_2 [s^2 \bar{T}(x, s) - s T(x, 0) - \dot{T}(x, 0)] \end{aligned} \quad (2.21)$$

where $\bar{\sigma}_x(x, s)$ and $\bar{T}(x, s)$ are the Laplace transforms of thermal stress and temperature, respectively in the x and s domains.

By substituting the initial conditions, Eq. 2.21 reduces to:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = c_2 s^2 \bar{T}(x, s) - c_2 s T_0 \quad (2.22)$$

Considering the temperature distribution for a pulse with exponential temporal variation, Eq. 2.13, and substituting into Eq. 2.22 and solving for the stress field, yields:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x = c_2 s^2 \left[\frac{I_1 \delta^2 \sqrt{\alpha} (C_1 * SP(s)) e^{-\lambda x}}{k \sqrt{s} (\delta^2 - \lambda^2)} - \frac{I_1 \delta (C_1 * SP(s)) e^{-\delta x}}{k (\delta^2 - \lambda^2)} + \frac{T_0}{s} \right] - c_2 s T_0 \quad (2.23)$$

The complementary and the particular solutions for Eq. 2.23 are:

$$(\bar{\sigma}_x)_c = D_1 e^{\frac{sx}{c_1}} + D_2 e^{\frac{-sx}{c_1}} \quad (2.24)$$

While the particular solution has two parts, the first part is:

$$(\bar{\sigma}_x)_{p_1} = D_3 e^{-\sqrt{\frac{s}{\alpha}} x} \quad (2.25)$$

The second part of the particular solution is:

$$(\bar{\sigma}_x)_{p_2} = D_4 e^{-\delta x} \quad (2.26)$$

Solving for the particular solutions yields:

$$D_3 = \frac{c_2 s^2 I_1 \delta^2 (C_1 * SP(s)) \sqrt{\alpha}}{k \sqrt{s} (\delta^2 - \lambda^2) (\lambda^2 - s^2/c_1^2)} \quad (2.27)$$

and

$$D_4 = \frac{c_2 s^2 I_1 \delta (C_1 * SP(s))}{k (\delta^2 - \lambda^2) (s^2/c_1^2 - \delta^2)} \quad (2.28)$$

So, the general solution for the stress field becomes:

$$(\bar{\sigma}_x)_g = D_1 e^{\frac{sx}{c_1}} + D_2 e^{\frac{-sx}{c_1}} + D_3 e^{-\sqrt{\frac{s}{\alpha}} x} + D_4 e^{-\delta x} \quad (2.29)$$

Form the boundary condition ($x = \infty \Rightarrow \sigma_x = 0$), it yields $D_1 = 0$:

Then,

$$(\bar{\sigma}_x)_g = D_2 e^{\frac{-sx}{c_1}} + D_3 e^{-\sqrt{\frac{s}{\alpha}} x} + D_4 e^{-\delta x} \quad (2.30)$$

Consider the boundary condition at the surface, where at $x = 0 \Rightarrow \bar{\sigma}_x = 0$, the constant in Eq. 2.30 becomes:

$$D_2 = -D_3 - D_4 \quad (2.31)$$

Then:

$$\bar{\sigma}_x(x, s) = D_3 e^{-\sqrt{\frac{s}{\alpha}}x} - D_3 e^{\frac{-sx}{c_1}} + D_4 e^{-\delta x} - D_4 e^{\frac{-sx}{c_1}} \quad (2.32)$$

Finding the solution for σ_x in the x and t domain, we should take the inverse Laplace transform for each term in Eq. 2.32. To accomplish this, the following definitions are introduced:

$$\begin{aligned} \text{Term1} &= D_3 e^{-\sqrt{\frac{s}{\alpha}}x} & \text{Term2} &= -D_3 e^{\frac{-sx}{c_1}} \\ \text{Term3} &= D_4 e^{-\delta x} & \text{Term4} &= -D_4 e^{\frac{-sx}{c_1}} \end{aligned} \quad (2.33)$$

Consequently, the solution for the stress distribution becomes the summation of the inverse Laplace transforms of the above terms.

Therefore, the inverse of Laplace transform of Terms (Term1, Term2, Term3, Term4) are:

$$\text{Term1} = \frac{I_1 \delta^2 C_1 c_2 \sqrt{\alpha}}{k} \left[\frac{e^{-\sqrt{\frac{s}{\alpha}}x}}{\sqrt{s}(\delta^2 - s/\alpha)(1/\alpha - s/c_1^2)} - \frac{e^{-\sqrt{\frac{s}{\alpha}}x} e^{-(\Delta t)s}}{\sqrt{s}(\delta^2 - s/\alpha)(1/\alpha - s/c_1^2)} \right] \quad (2.34)$$

After partial fraction of the above equation:

Term1

$$\begin{aligned} &= \frac{I_1 \delta^2 C_1 c_2 \sqrt{\alpha}}{k} \left[\frac{\alpha e^{-\sqrt{\frac{s}{\alpha}}x}}{\sqrt{s}\delta^2} - \frac{\sqrt{s}\alpha^4 e^{-\sqrt{\frac{s}{\alpha}}x}}{(\alpha^2\delta^2 - c_1^2)(s\alpha - c_1^2)} - \frac{c_1^2\alpha\sqrt{s}e^{-\sqrt{\frac{s}{\alpha}}x}}{\delta^2(\alpha^2\delta^2 - c_1^2)(\alpha\delta^2 - s)} \right. \\ &\quad \left. - \frac{\alpha e^{-\sqrt{\frac{s}{\alpha}}x} e^{-(\Delta t)s}}{\sqrt{s}\delta^2} + \frac{\sqrt{s}\alpha^4 e^{-\sqrt{\frac{s}{\alpha}}x} e^{-(\Delta t)s}}{(\alpha^2\delta^2 - c_1^2)(s\alpha - c_1^2)} + \frac{c_1^2\alpha\sqrt{s}e^{-\sqrt{\frac{s}{\alpha}}x} e^{-(\Delta t)s}}{\delta^2(\alpha^2\delta^2 - c_1^2)(\alpha\delta^2 - s)} \right] \end{aligned} \quad (2.35)$$

let $G_1 = \frac{I_1 \delta^2 C_1 c_2 \sqrt{\alpha}}{k}$, then:

$$\text{Term1} = [G_1 \text{Term11}] + [G_1 \text{Term21}] + [G_1 \text{Term31}] \quad (2.36)$$

where

$$\begin{aligned} \text{Term11} &= \frac{\alpha e^{-\sqrt{\frac{s}{\alpha}}x} [1 - e^{-\Delta t s}]}{\sqrt{s}\delta^2}, \quad \text{Term21} = \frac{\sqrt{s}\alpha^4 e^{-\sqrt{\frac{s}{\alpha}}x} [1 - e^{-\Delta t s}]}{(\alpha^2\delta^2 - c_1^2)(s\alpha - c_1^2)} \text{ and} \\ \text{Term31} &= \frac{c_1^2\alpha\sqrt{s}e^{-\sqrt{\frac{s}{\alpha}}x} [1 - e^{-\Delta t s}]}{s^2(\alpha^2\delta^2 - c_1^2)(\alpha\delta^2 - s)} \end{aligned} \quad (2.37)$$

The inverse of Laplace transform of Term1 is $\mathcal{L}^{-1}[\text{Term1}]$, i.e.:

$$\mathcal{L}^{-1}[\text{Term1}] = \mathcal{L}^{-1}[\text{Term11}] + \mathcal{L}^{-1}[\text{Term21}] + \mathcal{L}^{-1}[\text{Term31}] \quad (2.38)$$

The inverse of Laplace transform of terms becomes:

$$\mathfrak{L}^{-1}[\text{G}_1 \text{Term11}] = \text{G}_1 \left[\frac{\alpha}{\delta^2} \left(\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{1}{\sqrt{\pi(t-\Delta t)}} \exp\left(-\frac{x^2}{4\alpha(t-\Delta t)}\right) \right) \right] \quad (2.39)$$

$$\begin{aligned} & \mathfrak{L}^{-1}[\text{G}_1 \text{Term21}] \\ &= \text{G}_1 \frac{\alpha^3}{2(\alpha^2 \delta^2 - c_1^2)} \left[\begin{aligned} & -\frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4}\right) - \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} + \frac{c_1 x}{\alpha}} \operatorname{erfc}\left(-c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \\ & + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} + \frac{c_1 x}{\alpha}} \operatorname{erfc}\left(c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) + \frac{2}{\sqrt{\pi(t-\Delta t)}} \exp\left(\frac{-x^2}{4\alpha(t-\Delta t)}\right) \\ & + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2(t-\Delta t)}{\alpha} + \frac{c_1 x}{\alpha}} \operatorname{erfc}\left(-c_1 \sqrt{\frac{(t-\Delta t)}{\alpha}} + \frac{x}{2\sqrt{\alpha(t-\Delta t)}}\right) \\ & - \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2(t-\Delta t)}{\alpha} + \frac{c_1 x}{\alpha}} \operatorname{erfc}\left(c_1 \sqrt{\frac{(t-\Delta t)}{\alpha}} + \frac{x}{2\sqrt{\alpha(t-\Delta t)}}\right) \end{aligned} \right] \quad (2.40) \end{aligned}$$

$$\begin{aligned} & \mathfrak{L}^{-1}[\text{G}_1 \text{Term31}] \\ &= \text{G}_1 \frac{c_1^2 \alpha}{2\delta^2(c_1^2 - \alpha^2 \delta^2)} \left[\begin{aligned} & -\frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4\alpha t}\right) - \sqrt{\alpha} e^{\alpha \delta^2 t - \delta x} \operatorname{erfc}\left(-\delta \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \\ & + \sqrt{\alpha} \delta e^{\alpha \delta^2 t + \delta x} \operatorname{erfc}\left(\delta \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) + \frac{2}{\sqrt{\pi(t-\Delta t)}} \exp\left(\frac{-x^2}{4\alpha(t-\Delta t)}\right) \\ & + \sqrt{\alpha} \delta e^{\alpha \delta^2(t-\Delta t) - \delta x} \operatorname{erfc}\left(-\delta \sqrt{\alpha(t-\Delta t)} + \frac{x}{2\sqrt{\alpha(t-\Delta t)}}\right) \\ & - \sqrt{\alpha} \delta e^{\alpha \delta^2(t-\Delta t) + \delta x} \operatorname{erfc}\left(\delta \sqrt{\alpha(t-\Delta t)} + \frac{x}{2\sqrt{\alpha(t-\Delta t)}}\right) \end{aligned} \right] \quad (2.41) \end{aligned}$$

However, the Term 2 is:

$$\text{Term2} = -\frac{c_2 \delta^2 I_1 C_1 \sqrt{\alpha}}{k} \left[\frac{e^{-\frac{s}{c_1} x}}{\sqrt{s}(\delta^2 - s/\alpha)(1/\alpha - s/c_1^2)} - \frac{e^{-\frac{s}{c_1} x} e^{-(\Delta t)s}}{\sqrt{s}(\delta^2 - s/\alpha)(1/\alpha - s/c_1^2)} \right] \quad (2.42)$$

Let:

$$\text{G}_2 = -\frac{c_2 \delta^2 I_1 C_1 \sqrt{\alpha}}{k} \quad (2.43)$$

After the partial fraction, Term2 can be written as:

$$\text{Term2} = \text{G}_2 [\text{Term12} + \text{Term22} + \text{Term32}] \quad (2.44)$$

where

$$\text{Term12} = \frac{\alpha}{\delta^2} \frac{e^{-\frac{s}{c_1}x}}{\sqrt{s}} : \quad \text{Term22} = \frac{c_1^2 \sqrt{s} \alpha e^{-\frac{s}{c_1}x}}{\delta^2 (c_1^2 - \alpha^2 \delta^2) (\alpha \delta^2 - s)}$$

and

$$: \text{Term32} = \frac{\sqrt{s} \alpha^4 e^{-\frac{s}{c_1}x}}{\delta^2 (\alpha^2 \delta^2 - c_1^2) (\alpha s - c_1^2)} \quad (2.45)$$

The inverse of Laplace transform of Term 2 comes out to be:

$$\mathcal{L}^{-1}[\text{Term2}] = \mathcal{L}^{-1}[\text{G}_2 \text{Term12}] + \mathcal{L}^{-1}[\text{G}_2 \text{Term22}] + \mathcal{L}^{-1}[\text{G}_2 \text{Term32}] \quad (2.46)$$

or

$$\begin{aligned} \mathcal{L}^{-1}[\text{G}_2 \text{Term12}] = \text{G}_2 \frac{\alpha}{\delta^2} & \left[\frac{1}{\sqrt{\pi(t - x/c_1)}} \cdot 1\left(t - \frac{x}{c_1}\right) \right. \\ & \left. - \frac{1}{\sqrt{\pi(t - \Delta t - x/c_1)}} \cdot 1\left(t - \Delta t - \frac{x}{c_1}\right) \right] \end{aligned} \quad (2.47)$$

and

$$\begin{aligned} \mathcal{L}^{-1}[\text{G}_2 \text{Term22}] \\ = \text{G}_2 \frac{c_1^2 \alpha}{\delta^2 (\alpha^2 \delta^2 - c_1^2)} & \left[\left(\frac{1}{\sqrt{\pi(t - x/c_1)}} + \sqrt{\alpha} \delta e^{\alpha \delta^2 t} \text{erf}\left(\sqrt{\alpha} \delta \sqrt{t - \frac{x}{c_1}}\right) \right) \cdot 1\left(t - \frac{x}{c_1}\right) \right. \\ & \left. - \left(\frac{1}{\sqrt{\pi(t - \Delta t - x/c_1)}} + \sqrt{\alpha} \delta e^{\alpha \delta^2 (t - \Delta t)} \text{erf}\left(\sqrt{\alpha} \delta \sqrt{t - \Delta t - \frac{x}{c_1}}\right) \right) \cdot 1\left(t - \Delta t - \frac{x}{c_1}\right) \right] \end{aligned} \quad (2.48)$$

and

$$\begin{aligned} \mathcal{L}^{-1}[\text{G}_2 \text{Term32}] \\ = \text{G}_2 \frac{\alpha^3}{(\alpha^2 \delta^2 - c_1^2)} & \left[- \left(\frac{1}{\sqrt{\pi(t - x/c_1)}} + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2}{\alpha} t} \text{erf}\left(\frac{c_1}{\sqrt{\alpha}} \sqrt{t - \frac{x}{c_1}}\right) \right) \cdot 1\left(t - \frac{x}{c_1}\right) \right. \\ & \left. + \left(\frac{1}{\sqrt{\pi(t - \Delta t - x/c_1)}} + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2}{\alpha} (t - \Delta t)} \text{erf}\left(\frac{c_1}{\sqrt{\alpha}} \sqrt{t - \Delta t - \frac{x}{c_1}}\right) \right) \cdot 1\left(t - \Delta t - \frac{x}{c_1}\right) \right] \end{aligned} \quad (2.49)$$

where erf (y) is the error function of the variable y.

However, Term3 is:

$$\text{Term3} = \frac{c_2 I_1 \delta C_1}{k} \left[\frac{s e^{-\delta x}}{(s^2/c_1^2 - \delta^2)(\delta^2 - s/\alpha)} - \frac{s e^{-\delta x} e^{-(\Delta t)s}}{(s^2/c_1^2 - \delta^2)(\delta^2 - s/\alpha)} \right] \quad (2.50)$$

Let:

$$G_3 = \frac{c_2 I_1 \delta C_1}{k} \quad (2.51)$$

The Term3 can be written as:

$$\text{Term3} = [G_3 \text{Term13}] + [G_3 \text{Term23}] + [G_3 \text{Term33}] \quad (2.52)$$

After the partial fraction, the Term 3 becomes:

$$\text{Term13} = \frac{c_1^2 \alpha e^{-\delta x}}{2\delta(s - c_1 \delta)(c_1 - \alpha \delta)} : \quad \text{Term23} = \frac{c_1^2 \alpha e^{-\delta x}}{2\delta(s + c_1 \delta)(c_1 + \alpha \delta)}$$

and

$$\text{Term13} = \frac{c_1^2 \alpha^2 e^{-\delta x}}{(c_1^2 - \alpha^2 \delta^2)(s - \alpha \delta^2)} \quad (2.53)$$

The inverse of Laplace transform of Term3 comes out to be:

$$\mathcal{L}^{-1}[\text{Term3}] = \mathcal{L}^{-1}[G_3 \text{Term13}] + \mathcal{L}^{-1}[G_3 \text{Term23}] + \mathcal{L}^{-1}[G_3 \text{Term33}] \quad (2.54)$$

where

$$\mathcal{L}^{-1}[G_3 \text{Term13}] = G_3 \frac{c_1^2 \alpha e^{-\delta x}}{2\delta(c_1 - \alpha \delta)} \left[e^{c_1 \delta(t - \Delta t)} - e^{c_1 \delta t} \right] \quad (2.55)$$

and

$$\mathcal{L}^{-1}[G_3 \text{Term23}] = G_3 \frac{c_1^2 \alpha e^{-\delta x}}{2\delta(c_1 + \alpha \delta)} \left[e^{c_1 \delta t} - e^{c_1 \delta(t - \Delta t)} \right] \quad (2.56)$$

and

$$\mathcal{L}^{-1}[G_3 \text{Term33}] = G_3 \frac{c_1^2 \alpha^2 e^{-\delta x}}{(c_1^2 - \alpha^2 \delta^2)} \left[e^{\alpha \delta^2 t} - e^{\alpha \delta^2(t - \Delta t)} \right] \quad (2.57)$$

However, Term4 is:

$$\text{Term4} = -\frac{c_2 I_1 \delta C_1}{k} \left[\frac{se^{-\frac{s}{c_1}x}}{(s^2/c_1^2 - \delta^2)(\delta^2 - s/\alpha)} - \frac{se^{-\frac{s}{c_1}x}e^{-(\Delta t)s}}{(s^2/c_1^2 - \delta^2)(\delta^2 - s/\alpha)} \right] \quad (2.58)$$

Let:

$$G_4 = -\frac{c_2 I_1 \delta C_1}{k} \quad (2.59)$$

Term4 can be written as:

$$\text{Term4} = [\text{G}_4\text{Term14}] + [\text{G}_4\text{Term24}] + [\text{G}_4\text{Term34}] \quad (2.60)$$

After partial fraction of Term4, it yields:

$$\text{Term14} = \frac{c_1^2 \alpha e^{-\frac{s}{c_1}x}}{2\delta(s - c_1\delta)(c_1 - \alpha\delta)} : \quad \text{Term24} = \frac{c_1^2 \alpha e^{-\frac{s}{c_1}x}}{2\delta(s + c_1\delta)(c_1 + \alpha\delta)}$$

and

$$\text{Term34} = \frac{c_1^2 \alpha^2 e^{-\frac{s}{c_1}x}}{(c_1^2 - \alpha^2 \delta^2)(s - \alpha\delta^2)} \quad (2.61)$$

The inversion of Term 4 is:

$$\mathcal{L}^{-1}[\text{Term4}] = \mathcal{L}^{-1}[\text{G}_4\text{Term14}] + \mathcal{L}^{-1}[\text{G}_4\text{Term24}] + \mathcal{L}^{-1}[\text{G}_4\text{Term34}] \quad (2.62)$$

where

$$\begin{aligned} & \mathcal{L}^{-1}[\text{G}_4\text{Term14}] \\ &= \text{G}_4 \frac{c_1^2 \alpha}{2\delta(c_1 - \alpha\delta)} \left[e^{c_1\delta\left(t - \Delta t - \frac{x}{c_1}\right)} \cdot 1\left(t - \Delta t - \frac{x}{c_1}\right) - e^{c_1\delta\left(t - \frac{x}{c_1}\right)} \cdot 1\left(t - \frac{x}{c_1}\right) \right] \end{aligned} \quad (2.63)$$

and

$$\begin{aligned} & \mathcal{L}^{-1}[\text{G}_4\text{Term24}] \\ &= \text{G}_4 \frac{c_1^2 \alpha}{2\delta(c_1 + \alpha\delta)} \left[e^{-c_1\delta\left(t - \frac{x}{c_1}\right)} \cdot 1\left(t - \frac{x}{c_1}\right) - e^{-c_1\delta\left(t - \Delta t - \frac{x}{c_1}\right)} \cdot 1\left(t - \Delta t - \frac{x}{c_1}\right) \right] \end{aligned} \quad (2.64)$$

and

$$\begin{aligned} & \mathcal{L}^{-1}[\text{G}_4\text{Term34}] \\ &= \text{G}_4 \frac{c_1^2 \alpha^2}{(c_1^2 - \alpha^2 \delta^2)} \left[e^{\alpha\delta^2\left(t - \frac{x}{c_1}\right)} \cdot 1\left(t - \frac{x}{c_1}\right) - e^{\alpha\delta^2\left(t - \Delta t - \frac{x}{c_1}\right)} \cdot 1\left(t - \Delta t - \frac{x}{c_1}\right) \right] \end{aligned} \quad (2.65)$$

where $1\left(t - \frac{x}{c_1}\right)$ is a unit step function.

The closed form solution of stress distribution can be written as:

$$\sigma_x(x, t) = \mathcal{L}^{-1}[\text{Term1}] + \mathcal{L}^{-1}[\text{Term2}] + \mathcal{L}^{-1}[\text{Term3}] + \mathcal{L}^{-1}[\text{Term4}] \quad (2.66)$$

where $\mathcal{L}^{-1}$ represents the inverse sign of Laplace transform.

Presenting the stress distribution in dimensionless form, the additional dimensionless quantities are defined, i.e.:

$$c_1 = \frac{c_1}{\alpha \delta} \text{ and } \sigma_x = \frac{k \sigma_x}{I_1 \delta \alpha^2 c_2 C_1} \quad (2.67)$$

$U[1]$ is the unit step function, which is $U[1] = \left(t - \frac{x}{c_1}\right)$, and $U[1 - \Delta t]$ is the unit step function for the time shift, i.e. $U[1 - \Delta t] = 1 \left(t - \Delta t - \frac{x}{c_1}\right)$

Therefore, for the dimensionless stress distribution, the following results are obtained:

$$(\sigma_x)_1 = (\sigma_x)_{11} + (\sigma_x)_{21} + (\sigma_x)_{31} \quad (2.68)$$

where

$$(\sigma_x)_{11} = \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) - \frac{1}{\sqrt{\pi(t - \Delta t)}} \exp\left(-\frac{x^2}{4(t - \Delta t)}\right) \quad (2.69)$$

and

$$(\sigma_x)_{21} = \frac{1}{2(1 - c_1^2)} \left[\begin{aligned} & -\frac{2}{\sqrt{\pi t^*}} \exp\left(\frac{-x^{*2}}{4t^*}\right) - c_1^* e^{c_1^{*2} t^* - c_1^* x^*} \operatorname{erfc}\left(-c_1^* \sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}}\right) \\ & + c_1^* e^{c_1^{*2} t^* + c_1^* x^*} \operatorname{erfc}\left(c_1^* \sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}}\right) \\ & + \frac{2}{\sqrt{\pi(t^* - \Delta t^*)}} \exp\left(\frac{-x^{*2}}{4(t^* - \Delta t^*)}\right) \\ & + c_1^* e^{c_1^{*2}(t^* - \Delta t^*) - c_1^* x^*} \operatorname{erfc}\left(-c_1^* \sqrt{(t^* - \Delta t^*)} + \frac{x^*}{2\sqrt{(t^* - \Delta t^*)}}\right) \\ & - c_1^* e^{c_1^{*2}(t^* - \Delta t^*) + c_1^* x^*} \operatorname{erfc}\left(c_1^* \sqrt{(t^* - \Delta t^*)} + \frac{x^*}{2\sqrt{(t^* - \Delta t^*)}}\right) \end{aligned} \right] \quad (2.70)$$

and

$$(\sigma_x)_{31} = -\frac{c_1^{*2}}{2(1 - c_1^{*2})} \left[\begin{aligned} & -\frac{2}{\sqrt{\pi t^*}} \exp\left(\frac{-x^{*2}}{4t^*}\right) - e^{t^* - x^*} \operatorname{erfc}\left(-\sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}}\right) + e^{t^* + x^*} \operatorname{erfc}\left(\sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}}\right) \\ & + \frac{2}{\sqrt{\pi(t^* - \Delta t^*)}} \exp\left(\frac{-x^{*2}}{4(t^* - \Delta t^*)}\right) + e^{(t^* - \Delta t^*) - x^*} \operatorname{erfc}\left(-\sqrt{(t^* - \Delta t^*)} + \frac{x^*}{2\sqrt{(t^* - \Delta t^*)}}\right) \\ & - e^{(t^* - \Delta t^*) + x^*} \operatorname{erfc}\left(\sqrt{(t^* - \Delta t^*)} + \frac{x^*}{2\sqrt{(t^* - \Delta t^*)}}\right) \end{aligned} \right] \quad (2.71)$$

However, $(\sigma_x^\bullet)_2$ is:

$$(\sigma_x^\bullet)_2 = (\sigma_x^\bullet)_{12} + (\sigma_x^\bullet)_{22} + (\sigma_x^\bullet)_{32} \quad (2.72)$$

where

$$(\sigma_x^\bullet)_{12} = -\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} \cdot U[1] + \frac{1}{\sqrt{\pi(t^\bullet - \Delta t^\bullet - x^\bullet/c_1^\bullet)}} \cdot U[1 - \Delta t^\bullet] \quad (2.73)$$

and

$$(\sigma_x^\bullet)_{22} = -\frac{c_1^{\bullet 2}}{(1 - c_1^{\bullet 2})} \left[\left(\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + e^{t^\bullet} \operatorname{erf} \left(\sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}} \right) \right) \cdot U[1] \right. \\ \left. - \left(\frac{1}{\sqrt{\pi(t^\bullet - \Delta t^\bullet - x^\bullet/c_1^\bullet)}} + e^{(t^\bullet - \Delta t^\bullet)} \operatorname{erf} \left(\sqrt{t^\bullet - \Delta t^\bullet - \frac{x^\bullet}{c_1^\bullet}} \right) \right) \cdot U[1 - \Delta t^\bullet] \right] \quad (2.74)$$

and

$$(\sigma_x^\bullet)_{32} = -\frac{1}{(1 - c_1^{\bullet 2})} \left[-\left(\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + c_1^\bullet e^{c_1^{\bullet 2} t^\bullet} \operatorname{erf} \left(c_1^\bullet \sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}} \right) \right) \cdot U[1] \right. \\ \left. + \left(\frac{1}{\sqrt{\pi(t^\bullet - \Delta t^\bullet - x^\bullet/c_1^\bullet)}} + c_1^\bullet e^{c_1^{\bullet 2} (t^\bullet - \Delta t^\bullet)} \operatorname{erf} \left(c_1^\bullet \sqrt{t^\bullet - \Delta t^\bullet - \frac{x^\bullet}{c_1^\bullet}} \right) \right) \cdot U[1 - \Delta t^\bullet] \right] \quad (2.75)$$

The term $(\sigma_x^\bullet)_3$ is:

$$(\sigma_x^\bullet)_3 = (\sigma_x^\bullet)_{13} + (\sigma_x^\bullet)_{23} + (\sigma_x^\bullet)_{33} \quad (2.76)$$

where

$$(\sigma_x^\bullet)_{13} = \frac{c_1^{\bullet 2} e^{-x^\bullet}}{2(c_1^\bullet - 1)} \left[e^{c_1^\bullet(t^\bullet - \Delta t^\bullet)} - e^{c_1^\bullet t^\bullet} \right] \quad (2.77)$$

and

$$(\sigma_x^\bullet)_{23} = \frac{c_1^{\bullet 2} e^{-x^\bullet}}{2(c_1^\bullet + 1)} \left[e^{-c_1^\bullet t^\bullet} - e^{-c_1^\bullet(t^\bullet - \Delta t^\bullet)} \right] \quad (2.78)$$

and

$$(\sigma_x^\bullet)_{33} = \frac{c_1^{\bullet 2} e^{-x^\bullet}}{(c_1^{\bullet 2} - 1)} \left[e^{t^\bullet} - e^{(t^\bullet - \Delta t^\bullet)} \right] \quad (2.79)$$

The term $(\sigma_x^*)_4$ is:

$$(\sigma_x^*)_4 = (\sigma_x^*)_{14} + (\sigma_x^*)_{24} + (\sigma_x^*)_{34} \quad (2.80)$$

where

$$(\sigma_x^*)_{14} = \frac{c_1^{*2}}{2(c_1^* - 1)} \left[e^{c_1^* \left(t^* - \Delta t^* - \frac{t^*}{c_1^*} \right)} \cdot U[1 - \Delta t^*] - e^{c_1^* \left(t^* - \frac{t^*}{c_1^*} \right)} \cdot U[1] \right] \quad (2.81)$$

and

$$(\sigma_x^*)_{24} = \frac{c_1^{*2}}{2(c_1^* + 1)} \left[e^{-c_1^* \left(t^* - \frac{t^*}{c_1^*} \right)} \cdot U[1] - e^{-c_1^* \left(t^* - \Delta t^* - \frac{t^*}{c_1^*} \right)} \cdot U[1 - \Delta t^*] \right] \quad (2.82)$$

and

$$(\sigma_x^*)_{34} = \frac{c_1^{*2}}{(c_1^{*2} - 1)} \left[e^{\left(t^* - \frac{t^*}{c_1^*} \right)} \cdot U[1] - e^{\left(t^* - \Delta t^* - \frac{t^*}{c_1^*} \right)} \cdot U[1 - \Delta t^*] \right] \quad (2.83)$$

Consequently, the dimensionless form of stress equation becomes:

$$\sigma_x^* = (\sigma_x^*)_1 + (\sigma_x^*)_2 + (\sigma_x^*)_3 + (\sigma_x^*)_4 \quad (2.84)$$

The solution of Eqs. 2.15 and 2.84 provides data for temperature and stress fields. The mathematical arguments for the stress equation due to two successive pulse heating situation are not given inhere due to the length arguments. However, the closed form solution for the stress field (Eq. 2.84) can be used for heating situation due to two successive pulses, provided that the following conditions are satisfied:

$$\sigma_x^* = (\sigma_x^*)_{\text{first pulse}} + (\sigma_x^*)_{\text{second pulse}} \quad (2.85)$$

$(\sigma_x^*)_{\text{first pulse}}$ can be obtained after setting $\Delta t^* = t_1^* + dt_1^* + \theta_1$ in Eq. 2.84, where dt_1^* is the pulse length of the first pulse and θ_1 is the cooling period after the first pulse. It should be noted that the duration of the cooling period of the first pulse is the same of the first pulse's length ($\theta_1 = dt_1^*$). This requires that $0 \leq t^* \leq t_1^* + dt_1^* + \theta_1$ (covering the heating and the cooling periods; the second pulse initiates after the cooling period of the first pulse. $(\sigma_x^*)_{\text{f secondpulse}}$ can be obtained when setting $\Delta t^* = t_2^* + dt_2^* + \theta_2$ in Eq. 2.84, where dt_2^* is the pulse length of the second pulse and θ_2 is the cooling period after the second pulse. This requires that $t_1^* + dt_1^* + \theta_1 \leq t^* \leq t_2^* + dt_2^* + \theta_2$. In the present simulations the pulse lengths and cooling periods of the first and second pulses of the two successive pulses are set the same ($dt_1^* = dt_2^*$ and $\theta_1 = \theta_2$).

2.2.2 Stress Continuity Boundary at the Surface

The zero stress gradient at the surface represents the case occurring during laser heating of the coated surfaces. In this case, the surface of the substrate material is coated by a high absorbent material, such as paint, to improve the absorptivity at the surface. However, the absorbent material has different mechanical properties, but the stress continuity across the substrate material and thin layer of the coat occurs. This results in zero stress gradient at the surface of the substrate material.

The heat transfer equation for a laser step input heating pulse can be written similar to Eq. 2.5, which is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{I_1 \delta}{k} e^{-\delta x} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (2.86)$$

where, $I_1 = (1 - r_f)I_o$ (similar to Eq. 2.5), x is the distance, t is the time, k is the thermal conductivity, α is the thermal diffusivity, δ is the absorption coefficient r_f is the reflection coefficient, and I_o is the peak power intensity. Since the solution of Eq. 2.86 is similar to Eq. 2.4, the boundary conditions and the solution of Eq. 2.86 is given lightly.

The initial and boundary conditions are:

$$\text{At time } t = 0 \rightarrow T(x, 0) = 0 \quad (2.87)$$

and

$$\text{At the surface } x = 0 \rightarrow \left[\frac{\partial T}{\partial x} \right]_{x=0} = 0 \quad (2.88)$$

$$\text{and at } x = \infty \rightarrow T(t, \infty) = 0$$

The solution of Eq. 2.86 can be obtained possibly through Laplace transformation method, i.e., with respect to t , the Laplace transformation of Eq. 2.86 yields:

$$\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{I_1 \delta}{ks} e^{-\delta x} = \frac{1}{\alpha} [s\bar{T} - T(x, 0)] \quad (2.89)$$

where $\bar{T} = T(x, s)$. Using the initial condition, $T(x, 0) = 0$, Eq. 2.87 yields:

$$\frac{\partial^2 \bar{T}}{\partial x^2} - q^2 \bar{T} = -\frac{I_1 \delta}{ks} e^{-\delta x} \quad (2.90)$$

where $q^2 = \frac{s}{\alpha}$. Equation 2.89 has a solution:

$$\bar{T} = Ae^{qx} + Be^{-qx} + \frac{I_1 \alpha \delta}{ks(s - \alpha \delta^2)} e^{-\delta x} \quad (2.91)$$

where A and B are the constants and they are calculated through the boundary conditions. Substituting boundary condition, $\frac{\partial \bar{T}}{\partial x} = 0$ at the surface ($x = 0$), it gives:

$$A = B + \frac{I_1 \alpha \delta^2}{qks(s - \alpha \delta^2)} \quad (2.92)$$

The boundary condition, $\bar{T} = 0$ at $x = \infty$, results $A = 0$ in Eq. 2.91. Therefore, B yields:

$$B = -\frac{I_1 \delta^2}{qks(q^2 - \delta^2)} \quad (2.93)$$

Hence, Eq. 2.91 becomes:

$$\bar{T} = \frac{I_1 \alpha \delta}{ks(s - \alpha \delta^2)} e^{-\delta x} - \frac{I_1 \delta^2}{qks(q^2 - \delta^2)} e^{-qx} \quad (2.94)$$

Rearrangement of Eq. 2.94 yields:

$$\bar{T} = \frac{I_1 \alpha \delta}{ks(s - \alpha \delta^2)} e^{-\delta x} - \frac{I_1 \delta}{2kqs} \left[\frac{e^{-qx}}{q - \delta} - \frac{e^{-qx}}{q + \delta} \right] \quad (2.95)$$

The Inverse Laplace Transformation of Eq. 2.95 results:

$$\begin{aligned} T(x, t) = & \frac{I_1}{k} \left[2\sqrt{\alpha t} \text{ierfc} \left(\frac{x}{2\sqrt{\alpha t}} \right) - \frac{1}{2\delta} e^{-\delta x} \right. \\ & + \frac{1}{2\delta} e^{x\delta^2 t + \delta x} \text{erfc} \left(\delta\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}} \right) \\ & \left. + \frac{1}{2\delta} e^{x\delta^2 t - \delta x} \text{erfc} \left(\delta\sqrt{\alpha t} - \frac{x}{2\sqrt{\alpha t}} \right) \right] \end{aligned} \quad (2.96)$$

where erf is the error function, erfc is the complementary error function, and ierfc is the integral of complementary error function, which are:

$$\begin{aligned} \text{erf}(\chi) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-\chi^2} d\chi \\ \text{erfc}(\chi) &= 1 - \text{erf}(\chi) \\ \text{ierfc}(\chi) &= \frac{1}{\sqrt{\pi}} e^{-\chi^2} - \chi \text{erfc}(\chi) \end{aligned} \quad (2.97)$$

Introducing dimensionless quantities as:

$$\tau = \alpha \delta^2 t : x' = x\delta : T' = \frac{k\delta}{I_1} T \quad (2.98)$$

Substituting the dimensionless quantities in Eq. 2.96, it yields:

$$\begin{aligned}
T'(x', \tau) = & \left[2\sqrt{\tau} \operatorname{erfc}\left(\frac{x'}{2\sqrt{\tau}}\right) - \frac{1}{2}e^{-x'} \right. \\
& + \frac{1}{2}e^{\tau+x'} \operatorname{erfc}\left(\sqrt{\tau} + \frac{x'}{2\sqrt{\tau}}\right) \\
& \left. + \frac{1}{2}e^{\tau-x'} \operatorname{erfc}\left(\sqrt{\tau} - \frac{x'}{2\sqrt{\tau}}\right) \right]
\end{aligned} \quad (2.99)$$

Equation 2.99 is used to compute the dimensionless temperature profiles inside the substrate material.

The equation governing the momentum in one-dimensional solid for a linear elastic case can be used to formulate the thermal stress field in one-dimensional semi-infinite solid. Therefore, Eq. 2.16 is used to formulate the momentum equation.

The new initial conditions for Eq. 2.16 are:

$$\begin{aligned}
& \text{At } t = 0 \rightarrow, \text{ for the stress equation, } \sigma_x = 0 \\
& \text{and at } t = \infty \rightarrow, \text{ for the stress equation, } \sigma_x = 0 \\
& \text{At } t = 0 \rightarrow, \text{ for the heat transfer equation, } T = 0 \\
& \text{and at } t = \infty \rightarrow, \text{ for the heat transfer equation, } T = 0
\end{aligned} \quad (2.100)$$

In the case of zero stress gradient at the surface, the relevant boundary conditions are:

$$\begin{aligned}
& \text{At } x = 0 \rightarrow, \text{ for the stress equation, } \frac{\partial \sigma_x}{\partial x} = 0 \text{ and } \frac{\partial \sigma_x}{\partial t} = 0 \\
& \text{and at } x = 0 \rightarrow, \text{ for the heat transfer equation, } T = 0: \frac{\partial T}{\partial t} = 0 \\
& \text{At } x = \infty \rightarrow, \text{ for the stress equation, } \sigma_x = 0 \\
& \text{and at } x = \infty \rightarrow, \text{ for the heat transfer equation, } T = 0
\end{aligned} \quad (2.101)$$

The solution of thermal stress equation (Eq. 2.16) is possible using the Laplace transformation method, i.e. Laplace transformation of Eq. 2.16 with respect to t yields:

$$\begin{aligned}
\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{1}{c_1^2} [s^2 \bar{\sigma}_x(x, s) - s\sigma_x(x, 0) - \dot{\sigma}_x(x, 0)] \\
= c_2 [s^2 \bar{T}(x, s) - sT(x, 0) - \dot{T}(x, 0)]
\end{aligned} \quad (2.102)$$

where $\bar{\sigma}_x(x, s)$ and $\bar{T}(x, s)$ are the Laplace transform of thermal stress and temperature, respectively. Introducing the initial condition, Eq. 2.102 yields:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = c_2 s^2 \bar{T}(x, s) \quad (2.103)$$

However, it is noted from Eq. 2.94 that $\bar{T}$ is:

$$\bar{T} = \gamma_1 e^{-\sqrt{\frac{s}{\alpha}}x} + \gamma_2 e^{-\delta x} \quad (2.104)$$

where:

$$\begin{aligned} \gamma_1 &= \frac{I_1 \delta}{k} \frac{\delta \sqrt{\alpha}}{s \sqrt{s} (\delta^2 - \frac{s}{\alpha})} \text{ and} \\ \gamma_2 &= -\frac{I_1 \delta}{k} \frac{1}{s (\delta^2 - \frac{s}{\alpha})} \end{aligned} \quad (2.105)$$

Substitution of Eq. 2.104 into Eq. 2.103 yields:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = c_2 \frac{I_1 \delta}{k} \left[\frac{\delta \sqrt{\alpha} \sqrt{s} e^{-\sqrt{\frac{s}{\alpha}}x}}{(\delta^2 - \frac{s}{\alpha})} - \frac{s e^{-\delta x}}{(\delta^2 - \frac{s}{\alpha})} \right] \quad (2.106)$$

If we define M_1 and M_2 as:

$$\begin{aligned} M_1(s) &= c_2 \frac{I_1 \delta}{k} \left[\frac{\delta \sqrt{\alpha} \sqrt{s}}{(\delta^2 - \frac{s}{\alpha})} \right] \\ M_2(s) &= -c_2 \frac{I_1 \delta}{k} \left[\frac{s}{(\delta^2 - \frac{s}{\alpha})} \right] \end{aligned} \quad (2.107)$$

then Eq. 2.106 becomes:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = M_1(s) e^{-\sqrt{\frac{s}{\alpha}}x} + M_2(s) e^{-\delta x} \quad (2.108)$$

Equation 2.108 has homogenous $(\bar{\sigma}_x)_h$ and particular $(\bar{\sigma}_x)_p$ solutions, i.e.:

$$\bar{\sigma}_x = (\bar{\sigma}_x)_h + (\bar{\sigma}_x)_p \quad (2.109)$$

The homogenous solution is:

$$(\bar{\sigma}_x)_h = D_1 e^{\frac{s}{c_1^2}x} + D_2 e^{-\frac{s}{c_1^2}x} \quad (2.110)$$

and the particular solution has two parts. The first part is:

$$(\bar{\sigma}_x)_{p1} = G_1 e^{-\sqrt{\frac{s}{\alpha}}x} \quad (2.111)$$

Substituting Eq. 2.111 into Eq. 2.103 yields:

$$G_1 = \frac{M_1(s)}{\frac{s}{\alpha} - \frac{s^2}{c_1^2}} \quad (2.112)$$

The second part of the particular solution is:

$$(\bar{\sigma}_x)_{p1} = G_2 e^{-\delta x} \quad (2.113)$$

Substituting Eq. 2.113 into Eq. 2.108 yields:

$$G_2 = \frac{M_2(s)}{\delta^2 - \frac{s^2}{c_1^2}} \quad (2.114)$$

Therefore, Eq. 2.109 becomes:

$$\bar{\sigma}_x = (\bar{\sigma}_x)_h + (\bar{\sigma}_x)_{p1} + (\bar{\sigma}_x)_{p2}$$

or

$$\bar{\sigma}_x = D_1 e^{\frac{s}{c_1}x} + D_2 e^{-\frac{s}{c_1}x} + \frac{M_1(s)}{\frac{s}{\alpha} - \frac{s^2}{c_1^2}} e^{-\sqrt{\frac{s}{\alpha}}x} + \frac{M_2(s)}{\delta^2 - \frac{s^2}{c_1^2}} e^{-\delta x} \quad (2.115)$$

where D_1 and D_2 are constants and they will be calculated through boundary conditions. The coefficient D_1 must be zero, since $c_1 > 0$ then σ_x can be finite. Equation 2.115 becomes:

$$\bar{\sigma}_x = D_2 e^{-\frac{s}{c_1}x} + f_1(s) e^{-\sqrt{\frac{s}{\alpha}}x} + f_2(s) e^{-\delta x} \quad (2.116)$$

where

$$f_1(s) = c_2 \frac{I_1 \delta}{k} \left[\frac{\delta \sqrt{\alpha} \sqrt{s}}{s(\delta^2 - \frac{s}{\alpha})(\frac{1}{\alpha} - \frac{s}{c_1^2})} \right] \quad (2.117)$$

or

$$f_1(s) = c_2 c_1^2 \frac{I_1 \delta^2}{k} \alpha \sqrt{\alpha} \left[\frac{1}{\sqrt{s}(s - \alpha \delta^2)(s - \frac{c_1^2}{\alpha})} \right] \quad (2.118)$$

and

$$f_2(s) = -c_2 c_1^2 \frac{I_1 \delta}{k} \alpha \left[\frac{s}{(s - \alpha \delta^2)(s^2 - c_1^2 \delta^2)} \right] \quad (2.119)$$

Let us define the constants C_{10} and C_{20} as:

$$C_{10} = c_2 c_1^2 \frac{I_1 \delta^2}{k} \alpha \sqrt{\alpha} \quad (2.120)$$

and

$$C_{20} = -c_2 c_1^2 \frac{I_1 \delta}{k} \alpha \quad (2.121)$$

D_2 can be found through taking the derivative of Eq. 2.116 and setting to zero and knowing that D_1 is zero, as explained before, it yields:

$$\frac{s}{c_1} D_2 - \sqrt{\frac{s}{\alpha}} f_1(s) + \delta f_2(s) = 0 \quad (2.122)$$

Therefore, D_2 becomes:

$$D_2 = \frac{c_1}{\sqrt{s\alpha}} f_1(s) - \frac{c_1}{s} \delta f_2(s) \quad (2.123)$$

Substituting D_2 and setting D_I as zero into Eq. 2.108, it yields:

$$\begin{aligned} \bar{\sigma}_x = & -c_2 c_1^3 \frac{I_1 \delta \alpha}{k} e^{-\frac{s}{c_1} x} \left[\frac{1}{s(s - \alpha \delta^2) \left(\left(s - \frac{c_1^2}{\alpha} \right) \right)} - \frac{1}{(s - \alpha \delta^2)(s - c_1 \delta)(s + c_1 \delta)} \right] \\ & + c_2 c_1^2 \frac{I_1 \delta^2}{k} \alpha \sqrt{\alpha} \left[\frac{e^{-\sqrt{\frac{s}{\alpha}} x}}{\sqrt{s}(s - \alpha \delta^2) \left(s - \frac{c_1^2}{\alpha} \right)} - \frac{e^{-\frac{s}{c_1} x}}{(s - \alpha \delta^2)(s - c_1 \delta)(s + c_1 \delta)} \right] \end{aligned} \quad (2.124)$$

Let us introduce the followings:

$$\begin{aligned} Term1 &= - \frac{e^{-\frac{s}{c_1} x}}{s(s - \alpha \delta^2) \left(s - \frac{c_1^2}{\alpha} \right)} \\ Term2 &= \frac{e^{-\frac{s}{c_1} x}}{(s - \alpha \delta^2)(s - c_1 \delta)(s + c_1 \delta)} \\ Term3 &= \frac{\sqrt{\alpha} \delta e^{-\sqrt{\frac{s}{\alpha}} x}}{\sqrt{s}(s - \alpha \delta^2) \left(s - \frac{c_1^2}{\alpha} \right)} \\ Term4 &= - \frac{e^{-\frac{s}{c_1} x}}{(s - \alpha \delta^2)(s - c_1 \delta)(s + c_1 \delta)} \end{aligned} \quad (2.125)$$

Using partial fraction expansion, the following relation can be obtained for $Term1$:

$$Term1 = - \frac{1}{c_1 \delta} \frac{e^{-\frac{s}{c_1} x}}{s} - \frac{c_1 \delta}{(\alpha^2 \delta^4 - c_1^2 \delta^2)} \frac{e^{-\frac{s}{c_1} x}}{(s - \alpha \delta^2)} - \frac{c_1 \delta}{\left(\frac{c_1^4}{\alpha^2} - c_1^2 \delta^2 \right)} \frac{e^{-\frac{s}{c_1} x}}{\left(s - \frac{c_1^2}{\alpha} \right)} \quad (2.126)$$

The inversion of Laplace transformation of $Term1$ yields:

$$\mathcal{L}^{-1} Term1 = \left[1 \left(t - \frac{x}{c_1} \right) \right] \left[\begin{aligned} & - \frac{1}{c_1 \delta} - \frac{c_1 \delta}{(\alpha^2 \delta^4 - c_1^2 \delta^2)} e^{\alpha \delta^2 \left(t - \frac{x}{c_1} \right)} \\ & - \frac{c_1 \delta}{\left(\frac{c_1^4}{\alpha^2} - c_1^2 \delta^2 \right)} e^{\frac{c_1^2}{\alpha} \left(t - \frac{x}{c_1} \right)} \end{aligned} \right] \quad (2.127)$$

Similarly, the inversion of Laplace transformation of $Term2$ yields:

$$\begin{aligned} \mathcal{L}^{-1} Term2 &= \left[1 \left(t - \frac{x}{c_1} \right) \right] \left[\frac{c_1 \delta}{(\alpha \delta^2 - c_1 \delta)(\alpha \delta^2 + c_1 \delta)} e^{\alpha \delta^2 \left(t - \frac{x}{c_1} \right)} \right. \\ & \quad \left. + \frac{1}{2(c_1 \delta - \alpha \delta^2)} e^{c_1 \delta \left(t - \frac{x}{c_1} \right)} + \frac{1}{2(c_1 \delta + \alpha \delta^2)} e^{-c_1 \delta \left(t - \frac{x}{c_1} \right)} \right] \end{aligned} \quad (2.128)$$

Using partial fraction expansion, the following relation can be obtained for *Term3*:

$$Term3 = \frac{\delta\sqrt{\alpha}}{\sqrt{s}} e^{-\sqrt{\frac{s}{\alpha}}x} \left[\frac{1}{\left(\frac{c_1^2}{\alpha} - \alpha\delta^2\right)\left(s - \frac{c_1^2}{\alpha}\right)} + \frac{1}{\left(\alpha\delta^2 - \frac{c_1^2}{\alpha}\right)\left(s - \alpha\delta^2\right)} \right] \quad (2.129)$$

or

$$Term3 = \frac{\delta\alpha\sqrt{\alpha}}{(\alpha\delta^2 - c_1^2)} \left[-\frac{e^{-\sqrt{\frac{s}{\alpha}}x}}{\sqrt{s}\left(s - \frac{c_1^2}{\alpha}\right)} + \frac{e^{-\sqrt{\frac{s}{\alpha}}x}}{\sqrt{s}\left(s - \alpha\delta^2\right)} \right] \quad (2.130)$$

The inversion of Laplace transformation of *Term3* yields:

$$\mathcal{L}^{-1}Term3 = \frac{\delta\alpha\sqrt{\alpha}}{(\alpha\delta^2 - c_1^2)} \left\{ \begin{aligned} &\frac{e^{\alpha\delta^2 t}}{2\delta\sqrt{\alpha}} \left[\begin{aligned} &e^{-\delta x} \operatorname{erfc}\left(-\delta\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \\ &-e^{\delta x} \operatorname{erfc}\left(\delta\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \end{aligned} \right] \\ &-\frac{e^{\frac{c_1^2}{\alpha} t}}{2c_1\sqrt{\alpha}} \left[\begin{aligned} &e^{-\frac{xc_1}{\alpha}} \operatorname{erfc}\left(-c_1\sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \\ &-e^{\frac{xc_1}{\alpha}} \operatorname{erfc}\left(c_1\sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \end{aligned} \right] \end{aligned} \right\} \quad (2.131)$$

The Laplace inversion of *Term4* yields:

$$\mathcal{L}^{-1}Term3 = \frac{\alpha\delta^2}{(\alpha^2\delta^4 - c_1^2\delta^2)} e^{\alpha\delta^2 t} + \frac{1}{2(c_1\delta - \alpha\delta^2)} [e^{c_1\delta t} - e^{-c_1\delta t}] e^{\delta x} \quad (2.132)$$

Using the dimensionless quantities and summing the terms *Term1*, *Term2*, *Term3*, and *Term4* gives the Laplace inversion of dimensionless stress, i.e.:

$$\begin{aligned} \sigma'_x = &\frac{I_1\delta^2}{k} c_2(c_1)^2 \left\{ \left[\frac{e^{c_1'\tau - \frac{x'}{c_1'}}}{2(c_1' - 1)} + \frac{e^{-c_1'\tau + \frac{x'}{c_1'}}}{2(c_1' + 1)} - \frac{e^{(c_1')^2\tau - c_1'x'}}{c_1'(c_1' + 1)} - 1 \right] u \right. \\ &+ \frac{e^\tau}{2(1 - (c_1')^2)} \left[e^{-x'} \operatorname{erfc}\left(-\sqrt{\tau} + \frac{x'}{2\sqrt{\tau}}\right) - e^{x'} \operatorname{erfc}\left(\sqrt{\tau} + \frac{x'}{2\sqrt{\tau}}\right) \right] \\ &- \frac{e^{(c_1')^2\tau}}{2c_1'(1 - (c_1')^2)} \left[e^{-c_1'x'} \operatorname{erfc}\left(-c_1'\sqrt{\tau} + \frac{x'}{2\sqrt{\tau}}\right) - e^{c_1'x'} \operatorname{erfc}\left(c_1'\sqrt{\tau} + \frac{x'}{2\sqrt{\tau}}\right) \right] \\ &\left. - e^{-x'} \left[\frac{e^\tau}{1 - (c_1')^2} + \frac{1}{2(c_1' - 1)} (e^{c_1'\tau} - e^{-c_1'\tau}) \right] \right\} \end{aligned} \quad (2.133)$$

As indicated earlier, when $\tau < \frac{\lambda'}{c_1}$ (or $t < \frac{x}{c_1}$), the step function is zero ($u = 0$) and it has a value of one ($u = 1$) when $\tau > \frac{\lambda'}{c_1}$. Therefore, during the time interval $\tau < \frac{\lambda'}{c_1}$ the step function in Eq. 2.133 is set to zero ($u = 0$) or else it is set to 1.

Equation 2.133 is used to compute the dimensionless stress distribution for zero stress gradient at the surface.

2.3 Time Exponentially Varying Laser Pulse Heating

Since the laser power intensity varies with time, temporal variation of the pulse intensity is incorporated in the stress analysis. In addition, the laser pulse intensity distribution can be resembled through employing a time exponentially varying pulse intensity profile. The boundary conditions are important formulating the stress states in the irradiated material. Consequently, the closed form solutions for the thermal stress generated inside the substrate material are presented due to different boundary conditions for the time exponentially decaying laser pulse under the following sub-headings in line with the previous studies [3–5].

2.3.1 Stress Free Boundary at the Surface

The Fourier heat transfer equation due to time exponentially decaying laser heating pulse can be written as:

$$\frac{\partial^2 T}{\partial x^2} + \frac{I_1 \delta}{k} (e^{-\beta t}) e^{-\delta x} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (2.134)$$

where $I_1 = (1 - r_f) I_0$.

It can be assumed that the convection losses from the surface are neglected during the heating pulse. This implies the insulated boundary at the substrate surface. In addition, since the substrate material is assumed to be semi-infinite body, temperature at a depth approaching to infinity becomes the same as the initial temperature of the substrate material. Therefore, the corresponding boundary conditions are:

The insulated boundary condition at the surface yields:

At the free surface:

$$x = 0 \Rightarrow \left[\frac{\partial T}{\partial x} \right]_{x=0} = 0 \quad (2.135)$$

Temperature can be considered as approaching an initial temperature at a depth infinitely below the surface, i.e.:

$$\text{At } x = \infty \Rightarrow T(\infty, t) = 0 \quad (2.136)$$

Due to the simplicity the initial temperature of the substrate material is set to zero and temperature is assumed to be uniform inside the solid substrate at time equal to zero (before heating initiates). Therefore, the initial condition becomes:

$$\text{At } t = 0 \Rightarrow T(x, 0) = 0 \quad (2.137)$$

The closed form solution of Eq. 2.134 can be obtained through a Laplace Transformation method after employing the appropriate boundary and initial conditions. In this case, the Laplace Transformation of Eq. 2.134 with respect to t , results:

$$\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{I_1 \delta}{k} \frac{e^{-\delta x}}{(s + \beta)} = \frac{1}{\alpha} [s \bar{T}(x, s) - T(x, 0)] \quad (2.138)$$

Introducing the initial condition and rearranging Eq. 2.138 yields:

$$\frac{\partial^2 \bar{T}}{\partial x^2} - h^2 \bar{T} = -\frac{I_1 \delta}{k} \frac{e^{-\delta x}}{(s + \beta)} \quad (2.139)$$

where $h^2 = s/\alpha$ and s is the transform variable. Equation 2.139 has the solution:

$$T(x, s) = A e^{hx} + B e^{-hx} - \frac{I_1 \delta e^{-\delta x}}{k(s + \beta)(\delta^2 - h^2)} \quad (2.140)$$

where A and B are constants. Introducing the boundary conditions will allow determining the constants A and B , i.e.:

$$A = 0, \text{ and: } B = \frac{I_1 \delta^2}{kh(s + \beta)(\delta^2 - h^2)} \quad (2.141)$$

After substituting the values of A and B in Eq. 2.140, it yields:

$$\bar{T}(x, s) = -\frac{I_1 \delta}{k(s + \beta)} \left[\frac{\delta \exp(-hx)}{h(h^2 - \delta^2)} - \frac{\exp(-\delta x)}{(h^2 - \delta^2)} \right] \quad (2.142)$$

which gives the solution for temperature in Laplace domain.

The inverse Laplace Transform of Eq. 2.142 gives the temperature distribution inside the substrate material in space (x) and time (t) domain as follows [3–5]:

$T(x, t)$

$$= \frac{I_1 \delta}{2k} \left(\frac{\alpha}{\beta + \alpha \delta^2} \right) \left\{ i \delta \sqrt{\frac{\alpha}{\beta}} \exp(-\beta t) \cdot \begin{bmatrix} \exp\left(ix \sqrt{\frac{\beta}{\alpha}}\right) \cdot \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} + i\sqrt{\beta t}\right) \\ - \exp\left(-ix \sqrt{\frac{\beta}{\alpha}}\right) \cdot \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} - i\sqrt{\beta t}\right) \end{bmatrix} \right. \\ \left. + \exp(\alpha \delta^2 t) \begin{bmatrix} \exp() \cdot \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \delta\sqrt{\alpha t}\right) - \exp(-\delta x) \cdot \text{Erfc}\left(\delta\sqrt{\alpha t} - \frac{x}{2\sqrt{\alpha t}}\right) \\ - 2\exp(-(\beta t + \delta x)) \end{bmatrix} \right\} \quad (2.143)$$

where Erfc is the complementary error function. Equation 2.143 is the closed form solution for temperature distribution. The temperature distribution in non-dimensional form is possible by defining dimensionless quantities and substituting in Eq. 2.143. The dimensionless quantities are:

$$x^* = x\delta; t^* = \alpha \delta^2 t; T^* = \frac{Tk\delta}{I_1}; \beta^* = \frac{\beta}{\alpha \delta^2} \quad (2.144)$$

The dimensionless temperature distribution becomes:

$$T^*(x^*, t^*) = \frac{1}{2} \left(\frac{t^*}{\beta^* + t^*} \right) \left\{ i \sqrt{\frac{t^*}{\beta^*}} \exp(-\beta^*) \cdot \begin{bmatrix} \exp\left(ix^* \sqrt{\frac{\beta^*}{t^*}}\right) \left(\frac{x^*}{2\sqrt{t^*}} + i\sqrt{\beta^*}\right) \\ - \exp\left(-ix^* \sqrt{\frac{\beta^*}{t^*}}\right) \cdot \text{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - i\sqrt{\beta^*}\right) \end{bmatrix} \right. \\ \left. + \exp(t^*) \begin{bmatrix} \exp(x^*) \cdot \text{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} + \sqrt{t^*}\right) - \exp(-x^*) \cdot \text{Erfc}\left(\sqrt{t^*} - \frac{x^*}{2\sqrt{t^*}}\right) \\ - 2\exp(-(\beta^* + x^*)) \end{bmatrix} \right\} \quad (2.145)$$

To obtain the closed form solution for the stress distribution inside the semi-infinite one-dimensional substrate material, equation governing the momentum in a one-dimensional solid for a linear elastic case can be considered. In this case, the momentum equation becomes the same as Eq. 2.16.

Since the substrate surface is free to expand during the heating process, a free stress condition is adopted at the surface of the substrate material. Moreover, as the depth below the free surface increases to infinity, the thermal stress approaches the initial stress conditions, which is the stress free condition, i.e. it is assumed that the substrate material is initially free from any stresses. Therefore, the corresponding boundary conditions are:

$$\text{At } x = 0 \Rightarrow \sigma_x = 0 \quad (2.146)$$

and

$$\text{At } x = \infty \Rightarrow \sigma_x = 0 \quad (2.147)$$

For the temperature, the boundary conditions is:

$$\text{At } x = 0 \Rightarrow \frac{\partial T}{\partial x} = 0 \quad (2.148)$$

and

$$\text{At } x = \infty \Rightarrow T = 0 \quad (2.149)$$

Since the substrate material is assumed to be initially stress free and the thermal stress becomes the same as the initial stress condition as time approaches infinity, the initial and final conditions for the stress field yield:

$$\text{At } t = 0 \Rightarrow \sigma_x = 0 \quad (2.150)$$

and

$$\text{At } t = \infty \Rightarrow \sigma_x = 0 \quad (2.151)$$

The initial condition for temperature is:

$$\text{At } t = 0 \Rightarrow T(x, 0) = 0 \quad (2.152)$$

Since the laser pulse decays exponentially with time, the final and the initial temperature of the substrate material becomes the same as the time approaches infinity, i.e.:

$$\text{At } t = \infty \Rightarrow T(x, \infty) = 0 \quad (2.153)$$

Taking the Laplace Transformation of Eq. 2.16 with respect to time yields:

$$\begin{aligned} \frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{1}{c_1^2} [s^2 \bar{\sigma}_x(x, s) - s\sigma_x(x, 0) - \dot{\sigma}_x(x, 0)] \\ = c_2 [s^2 \bar{T}(x, s) - sT(x, 0) - \dot{T}(x, 0)] \end{aligned} \quad (2.154)$$

where $\bar{\sigma}_x(x, s)$ and $\bar{T}(x, s)$ are the Laplace Transforms of thermal stress and temperature respectively in the x and s domains.

By substituting the initial conditions, Eq. 2.154 reduces to:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = c_2 s^2 \bar{T}(x, s) \quad (2.155)$$

Considering the temperature distribution for time exponentially varying pulse and substituting into Eq. 2.155, and solving for the stress field, yields:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = -c_2 s^2 \frac{I_1 \delta}{k(s + \beta)} \left[\frac{\delta \exp(-hx)}{h(h^2 - \delta^2)} - \frac{\exp(-\delta x)}{(h^2 - \delta^2)} \right] \quad (2.156)$$

which can be arranged further:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = \frac{-I_1 \delta^2 c_2 s^2}{k(s + \beta)h(h^2 - \delta^2)} e^{-hx} + \frac{I_1 \delta c_2 s^2}{k(s + \beta)(h^2 - \delta^2)} e^{-\delta x} \quad (2.157)$$

Now let W_1 and W_2 are defined as:

$$W_1 = \frac{-I_1 \delta^2 c_2 s^2}{k(s + \beta)h(h^2 - \delta^2)} \quad (2.158)$$

and

$$W_2 = \frac{I_1 \delta c_2 s^2}{k(s + \beta)(h^2 - \delta^2)} \quad (2.159)$$

Then Eq. 2.157 becomes:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = W_1 e^{-hx} + W_2 e^{-\delta x} \quad (2.160)$$

The homogeneous and the particular solutions for Eq. 2.160 are:

$$(\bar{\sigma}_x)_h = C_1 e^{\frac{sx}{c_1}} + C_2 e^{\frac{-sx}{c_1}} \quad (2.161)$$

while the particular solution has two parts, the first part is:

$$(\bar{\sigma}_x)_{p_1} = Q_1 e^{-hx} \quad (2.162)$$

Substituting Eq. 2.162 in Eq. 2.160 yields:

$$Q_1 = \frac{W_1}{h^2 - \frac{s^2}{c_1^2}} \quad (2.163)$$

The second part of the particular solution is:

$$(\bar{\sigma}_x)_{p_2} = Q_2 e^{-\delta x} \quad (2.164)$$

Substituting Eq. 2.164 in Eq. 2.160 yields:

$$Q_2 = \frac{W_2}{\delta^2 - \frac{s^2}{c_1^2}} \quad (2.165)$$

Therefore, the general solution for the stress field becomes:

$$(\bar{\sigma}_x)_g = C_1 e^{\frac{sx}{c_1}} + C_2 e^{\frac{-sx}{c_1}} + Q_1 e^{-hx} + Q_2 e^{-\delta x} \quad (2.166)$$

Form the boundary condition ($x = \infty \Rightarrow \sigma_x = 0$), it yields $C_I = 0$:
Then,

$$(\bar{\sigma}_x)_g = C_2 e^{\frac{-sx}{c_1}} + \frac{W_1}{h^2 - \frac{s^2}{c_1^2}} e^{-hx} + \frac{W_2}{\delta^2 - \frac{s^2}{c_1^2}} e^{-\delta x} \quad (2.167)$$

Substituting for W_1 and W_2 in Eq. 2.167 results:

$$(\bar{\sigma}_x)_g = C_2 e^{\frac{-sx}{c_1}} + g_1(s) e^{-hx} + g_2(s) e^{-\delta x} \quad (2.168)$$

where:

$$g_1(s) = \frac{-I_1 \delta^2 c_2 s^2}{k(s + \beta) h (h^2 - \delta^2) \left(h^2 - \frac{s^2}{c_1^2} \right)} \quad (2.169)$$

and

$$g_2(s) = \frac{I_1 \delta c_2 s^2}{k(s + \beta) (h^2 - \delta^2) \left(\delta^2 - \frac{s^2}{c_1^2} \right)} \quad (2.170)$$

Now substituting for $h = s/\alpha$ and simplifying the expressions for $g_1(s)$ and $g_2(s)$ yields:

$$g_1(s) = \frac{I_1 \delta^2 c_2 \sqrt{\alpha} \alpha c_1^2}{k(s + \beta) h (s - \alpha \delta^2) \left(s - \frac{c_1^2}{\alpha} \right)} \quad (2.171)$$

and

$$g_2(s) = \frac{-I_1 \delta c_2 s^2 c_1^2}{k(s + \beta) (s - \alpha \delta^2) (s^2 - c_1^2 \delta^2)} \quad (2.172)$$

Rearranging the two expressions as:

$$g_1(s) = C_3 \left[\frac{\sqrt{s}}{(s + \beta) h (s - \alpha \delta^2) (s - c_1^2/\alpha)} \right] \quad (2.173)$$

and

$$g_2(s) = C_4 \left[\frac{s^2}{(s + \beta) (s - \alpha \delta^2) (s - c_1 \delta) (s + c_1 \delta)} \right] \quad (2.174)$$

where:

$$C_3 = \frac{I_1 \delta^2 c_2 \sqrt{\alpha} \alpha c_1^2}{k} \quad (2.175)$$

and

$$C_4 = \frac{-I_1 c_2 c_1^2 \delta \alpha}{k} \quad (2.176)$$

Using partial fraction, the following relations can be obtained:

$$g_1(s) = C_3 \left[\frac{\frac{\sqrt{s}}{(\alpha \delta^2 + \beta)(\alpha \delta^2 - c_1^2/\alpha)(s - \alpha \delta^2)} + \frac{\alpha^2 \sqrt{s}}{(c_1^2 + \alpha \beta)(c_1^2 - \alpha^2 \delta^2)(s - c_1^2/\alpha)}}{\frac{\sqrt{s}}{(\beta + \alpha \delta^2)(\beta + c_1^2/\alpha)(s + \beta)}} \right] \quad (2.177)$$

and

$$g_2(s) = C_4 \left[\frac{\frac{\alpha \delta^2}{(\alpha \delta^2 + \beta)(\alpha^2 \delta^2 - c_1^2)(s - \alpha \delta^2)} + \frac{c_1}{2(c_1 \delta + \beta)(c_1 - \alpha \delta)(s - c_1 \delta)}}{\frac{c_1}{2(c_1 \delta - \beta)(c_1 + \alpha \delta)(s + c_1 \delta)} + \frac{\beta^2}{(-\beta - \alpha \delta^2)(\beta^2 - c_1^2 \delta^2)(s + \beta)}} \right] \quad (2.178)$$

Consider the boundary condition at the surface in the stress field, where at $x = 0 \Rightarrow \bar{\sigma}_x = 0$, the constant in Eq. 2.167 becomes:

$$C_2 = -g_1(s) - g_2(s) \quad (2.179)$$

Then:

$$\bar{\sigma}_x(x, s) = g_1(s)e^{-\sqrt{\frac{s}{\alpha}}x} - g_1(s)e^{\frac{-\alpha x}{c_1}} + g_2(s)e^{-\delta x} - g_2(s)e^{\frac{-\alpha x}{c_1}} \quad (2.180)$$

Finding the solution for σ_x in the x and t domain, we should take the inverse Laplace Transform for each term in 2.180. To perform this, the following designations are introduced:

$$\begin{aligned} \text{Term1} &= g_1(s)e^{-\sqrt{\frac{s}{\alpha}}x} & \text{Term2} &= g_1(s)e^{\frac{-\alpha x}{c_1}} \\ \text{Term3} &= g_2(s)e^{-\delta x} & \text{Term4} &= g_2(s)e^{\frac{-\alpha x}{c_1}} \end{aligned} \quad (2.181)$$

Consequently, the solution for stress distribution becomes the summation of the inverse Laplace Transforms of the above terms. Therefore, the Laplace inversion of Terms (Term1, Term2, Term3, Term4) are:

$\mathfrak{F}^{-1}[\text{Term1}]$

$$= C_3 \left\{ \begin{aligned} & \frac{1}{2(\alpha\delta^2 + \beta)(\alpha\delta^2 - c_1^2/\alpha)} \left[\frac{2}{\sqrt{\pi t}} \cdot e^{\frac{-x^2}{4\alpha t}} + \delta\sqrt{\alpha} \cdot e^{\delta^2 \alpha t} \begin{pmatrix} e^{-\delta x} \cdot \text{Erfc}\left(-\delta\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \\ -e^{\delta x} \cdot \text{Erfc}\left(\delta\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \end{pmatrix} \right] \\ & + \frac{\alpha^2}{2(c_1^2 + \alpha\beta)(c_1^2 - \alpha\delta^2)} \left[\frac{2}{\sqrt{\pi t}} \cdot e^{\frac{-x^2}{4\alpha t}} + \frac{c_1}{\sqrt{\alpha}} \cdot e^{\frac{c_1^2 t}{\alpha}} \begin{pmatrix} e^{\frac{-c_1 x}{\alpha}} \cdot \text{Erfc}\left(-c_1\sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \\ -e^{\frac{c_1 x}{\alpha}} \left(c_1\sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \end{pmatrix} \right] \\ & + \frac{1}{2(\beta + \alpha\delta^2)(\beta + c_1^2/\alpha)} \left[\frac{2}{\sqrt{\pi t}} \cdot e^{\frac{-x^2}{4\alpha t}} + i\sqrt{\beta} \cdot e^{-\beta t} \begin{pmatrix} e^{-xi\sqrt{\frac{\beta}{\alpha}}} \cdot \text{Erfc}\left(-i\sqrt{\beta t} + \frac{x}{2\sqrt{\alpha t}}\right) \\ -e^{xi\sqrt{\frac{\beta}{\alpha}}} \cdot \text{Erfc}\left(i\sqrt{\beta t} + \frac{x}{2\sqrt{\alpha t}}\right) \end{pmatrix} \right] \end{aligned} \right\} \quad (2.182)$$

and

$$\mathfrak{F}^{-1}[\text{Term2}] = C_3 \left\{ \begin{aligned} & \frac{\frac{1}{\sqrt{\pi(t-x/c_1)}} + \sqrt{\alpha\delta^2} e^{\alpha\delta^2(t-x/c_1)} \text{Erf}\left[\sqrt{\alpha\delta^2(t-x/c_1)}\right]}{(\alpha\delta^2 - c_1^2/\alpha)(\alpha\delta^2 + \beta)} \\ & + \alpha^2 \left(\frac{\frac{1}{\sqrt{\pi(t-x/c_1)}} + \sqrt{\frac{c_1^2}{\alpha}} e^{\frac{-c_1^2}{\alpha}(t-x/c_1)} \text{Erf}\left[\sqrt{\frac{c_1^2}{\alpha}(t-x/c_1)}\right]}{(c_1^2 + \alpha\beta)(c_1^2 - \alpha\delta^2)} \right) \\ & + \frac{\frac{1}{\sqrt{\pi(t-x/c_1)}} + i\sqrt{\beta} e^{-\beta(t-x/c_1)} \text{Erf}\left[i\sqrt{\beta(t-x/c_1)}\right]}{(\beta + c_1^2/\alpha)(\alpha\delta^2 + \beta)} \end{aligned} \right\} \cdot 1\left(t - \frac{x}{c_1}\right) \quad (2.183)$$

and

$$\mathfrak{F}^{-1}[\text{Term3}] = C_4 e^{-\delta x} \left\{ \begin{aligned} & \frac{\alpha^2 \delta^2 e^{\alpha\delta^2 t}}{(\alpha\delta^2 + \beta)(\alpha\delta^2 - c_1^2)} + \frac{c_1 e^{c_1 \delta t}}{2(c_1 \delta + \beta)(c_1 - \alpha\delta)} \\ & + \frac{c_1 e^{-c_1 \delta t}}{2(c_1 \delta - \beta)(c_1 + \alpha\delta)} + \frac{\beta^2 e^{-\beta t}}{(-\beta - \alpha\delta^2)(\beta^2 - c_1^2 \delta^2)} \end{aligned} \right\} \quad (2.184)$$

and

$$\begin{aligned} \mathcal{L}^{-1}[\text{Term4}] = C_4 & \left[\frac{\alpha^2 \delta^2 e^{\alpha \delta^2 (t-x/c_1)}}{(\alpha \delta^2 + \beta)(\alpha^2 \delta^2 - c_1^2)} + \frac{c_1 e^{c_1 \delta (t-x/c_1)}}{2(c_1 \delta + \beta)(c_1 - \alpha \delta)} \right. \\ & \left. + \frac{c_1 e^{-c_1 \delta (t-x/c_1)}}{2(c_1 \delta - \beta)(c_1 + \alpha \delta)} + \frac{\beta^2 e^{-\beta (t-x/c_1)}}{(-\beta - \alpha \delta^2)((\beta^2 - c_1^2 \delta^2))} \right] \\ & \cdot 1 \left(t - \frac{x}{c_1} \right) \end{aligned} \quad (2.185)$$

where $1(t - \frac{x}{c_1})$ is a unit step function and Erf (y) is the error function of the variable y. It should be noted that the value of a unit step function takes one or zero, i.e.:

$$t \leq \frac{x}{c_1} \Rightarrow 1 \left(t - \frac{x}{c_1} \right) = 0 \quad \text{and} \quad t > \frac{x}{c_1} \Rightarrow 1 \left(t - \frac{x}{c_1} \right) = 1 \quad (2.186)$$

The closed form solution of stress distribution can be written as:

$$\sigma_x(x, t) = \mathcal{L}^{-1}[\text{Term1}] + \mathcal{L}^{-1}[\text{Term2}] + \mathcal{L}^{-1}[\text{Term3}] + \mathcal{L}^{-1}[\text{Term4}] \quad (2.187)$$

where $\mathcal{L}^{-1}$ represents the inverse sign of Laplace Transformation.

Presenting the stress distribution in dimensionless form, the additional dimensionless quantities are defined, i.e.:

$$c_1^* = \frac{c_1}{\alpha \delta} \quad (2.188)$$

and

$$\sigma_x^* = \frac{k \sigma_x}{I_1 \delta \alpha^2 c_2 (c_1^*)^2} \quad (2.189)$$

$U [1]$ is the unit step function, which is $U[1] = \left(t^* - \frac{x^*}{c_1^*} \right)$. The values of unit step function are:

$$t^* \leq \frac{x^*}{c_1^*} \Rightarrow U[1] \left(t^* - \frac{x^*}{c_1^*} \right) = 0 \quad \text{and} \quad t^* > \frac{x^*}{c_1^*} \Rightarrow U[1] \left(t^* - \frac{x^*}{c_1^*} \right) = 1 \quad (2.190)$$

Therefore, for the dimensionless stress distribution, the followings are resulted:

$$(\sigma_x^*)_1 = \left\{ \begin{aligned} & \frac{1}{(t^* + \beta^*)(1 - (c_1^*)^2)} \left[\frac{\sqrt{t^*} e^{-\frac{(x^*)^2}{4t^*}}}{\sqrt{\pi}} + \frac{t^*}{2} \left(\frac{e^{-x^*} \left(-\sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}} \right)}{-e^{x^*} \cdot \operatorname{erfc} \left(\sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}} \right)} \right) \right] \\ & + \frac{1}{((c_1^*)^2 - 1)((c_1^*)^2 t^* + \beta^*)} \left[\frac{\sqrt{t^*} e^{-\frac{(x^*)^2}{4t^*}}}{\sqrt{\pi}} + \frac{c_1^* t^* e^{-(c_1^*)^2 t^*}}{2} \left(\frac{e^{-c_1^* x^*} \cdot \operatorname{erfc} \left(-c_1^* \sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}} \right)}{-e^{c_1^* x^*} \cdot \operatorname{erfc} \left(c_1^* \sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}} \right)} \right) \right] \\ & + \frac{(t^*)^{3/2}}{(\beta^* + t^*)(\beta^* + (c_1^*)^2 t^*)} \left[\frac{e^{-\frac{(x^*)^2}{4t^*}}}{\sqrt{\pi}} + \frac{i\sqrt{\beta^*} e^{-\beta^*}}{2} \left(\frac{e^{-ix^*} \sqrt{\frac{\beta^*}{t^*}} \cdot \operatorname{erfc} \left(-i\sqrt{\beta^*} + \frac{x^*}{2\sqrt{t^*}} \right)}{-e^{ix^*} \sqrt{\frac{\beta^*}{t^*}} \cdot \operatorname{erfc} \left(i\sqrt{\beta^*} + \frac{x^*}{2\sqrt{t^*}} \right)} \right) \right] \end{aligned} \right\} \quad (2.191)$$

$$(\sigma_x^*)_2 = \left\{ \begin{aligned} & \frac{1}{(t^* + \beta^*)(1 - (c_1^*)^2)} \left[\frac{t^*}{\sqrt{\pi} \sqrt{(t^* - x^*/c_1^*)}} + t^* e^{(t^* - x^*/c_1^*)} \left(\sqrt{(t^* - x^*/c_1^*)} \right) \right] \\ & \frac{1}{((c_1^*)^2 - 1)((c_1^*)^2 + \beta^*/t^*)} \left[\frac{1}{\sqrt{\pi} \sqrt{(t^* - x^*/c_1^*)}} + c_1^* e^{-((c_1^*)^2 t^* - c_1^* x^*)} \cdot \operatorname{Erf} \left(\sqrt{(c_1^*)^2 t^* - c_1^* x^*} \right) \right] \\ & + \frac{(t^*)^{3/2}}{(t^* + \beta^*)(\beta^* + (c_1^*)^2 t^*)} \left[\frac{\sqrt{t^*}}{\sqrt{\pi} \sqrt{(t^* - x^*/c_1^*)}} + i\sqrt{\beta^*} e^{-\beta^* (1 - x^*/c_1^* t^*)} \cdot \operatorname{erf} \left(i\sqrt{\beta^* (1 - x^*/c_1^* t^*)} \right) \right] \end{aligned} \right\} \cdot U[1] \quad (2.192)$$

$$(\sigma_x^*)_3 = \left\{ \begin{aligned} & \frac{-t^* e^{t^* - x^*}}{(1 - (c_1^*)^2)(t^* + \beta^*)} - \frac{t^* c_1^* e^{c_1^* t^* - x^*}}{2(c_1^* t^* + \beta^*)(c_1^* - 1)} \\ & - \frac{t^* c_1^* e^{-c_1^* t^* - x^*}}{2(c_1^* t^* - \beta^*)(c_1^* + 1)} \\ & - \frac{(\beta^*)^2 t^* e^{-\beta^* - x^*}}{(-\beta^* - t^*)(\beta^* - (c_1^* t^*)^2)} \end{aligned} \right\} \quad (2.193)$$

$$(\sigma_x^*)_4 = \left\{ \begin{aligned} & \frac{-t^* e^{(t^* - x^*/c_1^*)}}{(1 - (c_1^*)^2)(t^* + \beta^*)} - \frac{t^* c_1^* e^{c_1^* t^* - x^*}}{2(c_1^* t^* + \beta^*)(c_1^* - 1)} - \frac{t^* c_1^* e^{-(c_1^* t^* - x^*)}}{2(c_1^* t^* - \beta^*)(c_1^* + 1)} \\ & - \frac{(\beta^*)^2 t^* e^{-\beta^* (1 - x^*/c_1^* t^*)}}{(-\beta^* - t^*)(\beta^* - (c_1^* t^*)^2)} \end{aligned} \right\} \cdot U[1] \quad (2.194)$$

Consequently, the dimensionless form of stress equation (Eq. 2.187) is:

$$\sigma_x^* = (\sigma_x^*)_1 + (\sigma_x^*)_2 + (\sigma_x^*)_3 + (\sigma_x^*)_4 \quad (2.195)$$

Equations 2.145 and 2.195 are used to compute the dimensionless temperature and stress distributions inside the substrate material.

2.3.2 Stress Free Boundary and Convection at the Surface

In practical applications, an inert assisting gas is used to protect the surface from high temperature exothermic reactions. This intern results in surface cooling due to convection effect of the assisting gas. Therefore, the convective boundary condition needs to be incorporated for laser gas assisted heating situations.

The Fourier heat transfer equation for a laser pulse decaying exponentially with time can be written similar to Eq. 2.134. Therefore, the boundary conditions and the closed form solution of the heat equation is given lightly.

The substrate material is considered as a semi-infinite body and heated by a laser beam on the surface. The convective boundary condition is assumed be on the substrate surface. In addition, as the depth is considered to extend to infinity and the temperature to go down to zero. Heating occurs in the surface region during the laser pulse. Therefore, the corresponding boundary conditions are:

$$\text{At } x = 0 \Rightarrow \left[\frac{\partial T}{\partial x} \right]_{x=0} = \frac{h}{k} (T(0, t) - T_0) \quad (2.196)$$

and

$$\text{At } x = \infty \Rightarrow T(\infty, t) = 0 \quad (2.197)$$

Initially substrate material is assumed to be at uniform temperature. Therefore, the initial condition is:

$$\text{At } t = 0 \Rightarrow T(x, 0) = 0 \quad (2.198)$$

The Laplace transformation of Eq. 2.16 with respect to t , results in:

$$\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{I_1 \delta}{k} \frac{e^{-\delta x}}{(s + \beta)} = \frac{1}{\alpha} [s \bar{T}(x, s) - T(x, 0)] \quad (2.199)$$

Introducing the initial condition and rearranging the Eq. 2.199, it yields:

$$\frac{\partial^2 \bar{T}}{\partial x^2} - g^2 \bar{T} = -\frac{I_1 \delta}{k} \frac{e^{-\delta x}}{(s + \beta)} \quad (2.200)$$

where $g^2 = s/\alpha$ and s is the Laplace transform variable. Equation 2.203 has the solution:

$$\bar{T}(x, s) = A_1 e^{gx} + A_2 e^{-gx} - \frac{I_1 \delta e^{-\delta x}}{k(s + \beta)(\delta^2 - g^2)} \quad (2.201)$$

where A and B are constants. Introducing the boundary conditions will enable calculation of the constants A_1 and A_2 , i.e.:

$A_1 = 0$, and:

$$A_2 = \frac{I_1 \delta (h + \delta k)}{k(s + \beta)(\delta^2 - g^2)(h + kg)} + \frac{hT_0}{s(h + kg)} \quad (2.202)$$

After substituting the values of A_1 and A_2 in Eq. 2.201, it yields:

$$\bar{T}(x, s) = \frac{I_1 \delta (h + \delta k) e^{-gx}}{k(s + \beta)(\delta^2 - g^2)(h + kg)} + \frac{hT_0 e^{-gx}}{s(h + kg)} - \frac{I_1 \delta e^{-\delta x}}{k(\delta^2 - g^2)(s + \beta)} \quad (2.203)$$

which gives the solution for the temperature distribution in the Laplace domain.

The inverse Laplace Transform of Eq. 2.206 provides the temperature distribution within the substrate material in space x and time t . The mathematical arrangements of the Laplace inversion of Eq. 2.203 are given in the previous study [3–5]. Therefore, the equation after the Laplace inversion is given below as:

$$T(x, t) = \frac{I_1 \delta x^{3/2} (h + k\delta)}{k^2} \left\{ \begin{aligned} & - \frac{e^{(\alpha\delta^2 t - \delta x)} \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} - \delta\sqrt{\alpha t}\right)}{2(\beta + \alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k} + \delta\sqrt{\alpha}\right)} - \frac{e^{(\alpha\delta^2 t + \delta x)} \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} - \delta\sqrt{\alpha t}\right)}{2(\beta + \alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k} + \delta\sqrt{\alpha}\right)} \\ & + \frac{e^{-\beta t} e^{\sqrt{(\beta/\alpha)x}} \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \sqrt{\beta t}\right)}{2(\beta + \alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k} - \sqrt{\beta t}\right)} + \frac{e^{-\beta t} e^{-\sqrt{(\beta/\alpha)x}} \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} - \sqrt{\beta t}\right)}{2(\beta + \alpha\delta^2)\left(\frac{h\sqrt{\alpha}}{k} + \sqrt{\beta t}\right)} \\ & + \frac{h\sqrt{\alpha} e^{hx/k} e^{(h^2/k^2)t} \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h}{k}\sqrt{\alpha t}\right)}{k\left(\frac{h^2}{k^2} + \beta\right)\left(\frac{h^2}{k^2} - \alpha\delta^2\right)} - \frac{ke^{-\delta x} (e^{-\alpha\delta^2 t} - e^{-\beta t})}{\sqrt{\alpha}(\beta + \alpha\delta^2)(h + k\delta)} \end{aligned} \right\} \\ + T_0 \left[\text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - e^{hx/k} e^{(h^2/k^2)t} \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h}{k}\sqrt{\alpha t}\right) \right] \quad (2.204)$$

where Erfc is the complementary error function. Equation 2.204 is the closed form solution for temperature distribution. The temperature distribution can be expressed in a non-dimensional form by introducing dimensionless quantities and substituting in Eq. 2.204. The dimensionless quantities are:

$$x^* = x\delta; t^* = \alpha\delta^2 t; T^* = \frac{Tk\delta}{I_1}; h^* = \frac{h}{\delta k}; \beta^* = \beta t$$

The dimensionless temperature distribution for a full pulse is then:

$$T^*(x^*, t^*) = (h^* + 1) \left\{ \begin{aligned} & \left[\frac{e^{-x^*} e^{t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{t^*}\right)}{2(\beta^* + 1)(h^* + 1)} - \frac{e^{-x^*} e^{t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{t^*}\right)}{2(\beta^* + 1)(h^* - 1)} + \frac{e^{-\beta^*} e^{\sqrt{\beta^*} x^* i} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{\beta^* t^*} i\right)}{2(\beta^* + 1)(h^* - \sqrt{\beta^*} i)} \right. \\ & \left. + \frac{e^{-\beta^*} e^{t^*} e^{-\sqrt{\beta^*} x^* i} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{\beta^* t^*} i\right)}{2(\beta^* + 1)(h^* + \sqrt{\beta^*} i)} + \frac{h^* e^{h^* x^*} e^{(h^*)^2 t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - h^* \sqrt{t^*}\right)}{2((h^*)^2 + \beta^*)((h^*)^2 - 1)} - \frac{e^{-x^*} (e^{-t^*} - e^{-\beta^* t^*})}{(\beta^* + 1)(h^* + 1)} \right] \\ & - \left[\frac{e^{-x^*} e^{t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{t^*}\right)}{2(\gamma^* + 1)(h^* + 1)} - \frac{e^{-x^*} e^{t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{t^*}\right)}{2(\gamma^* + 1)(h^* - 1)} + \frac{e^{-\gamma^*} e^{\sqrt{\gamma^*} x^* i} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{\gamma^* t^*} i\right)}{2(\gamma^* + 1)(h^* - \sqrt{\gamma^*} i)} \right. \\ & \left. + \frac{e^{-\gamma^*} e^{t^*} e^{-\sqrt{\gamma^*} x^* i} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{\gamma^* t^*} i\right)}{2(\gamma^* + 1)(h^* + \sqrt{\gamma^*} i)} + \frac{h^* e^{h^* x^*} e^{(h^*)^2 t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - h^* \sqrt{t^*}\right)}{2((h^*)^2 + \gamma^*)((h^*)^2 - 1)} - \frac{e^{-x^*} (e^{-t^*} - e^{-\gamma^* t^*})}{(\gamma^* + 1)(h^* + 1)} \right] \end{aligned} \right\} \quad (2.205)$$

In order to solve for the stress distribution within the substrate it is possible to consider the equation governing the momentum in a one-dimensional solid for the linear elastic case. Therefore, Eq. 2.16 is used to formulate thermal stress distribution in the substrate material.

In order to solve the momentum equation (Eq. 2.16) it is necessary to establish the initial conditions for stress and temperature fields. In this case, the substrate material is assumed to be free from stresses initially (at time = 0) and as the time extends to infinity, the stress free state must apply in the substrate. The same initial condition for the temperature is applied as in Eq. 2.198 provided that as time approaches infinity, the temperature in the substrate material reduces to zero. This is due to the fact that the laser pulse decays exponentially with time; therefore, as time approaches infinity, the laser pulse intensity becomes zero. Therefore, the initial and boundary conditions for the stress field are:

$$\text{At } t = 0 \Rightarrow \sigma_x = 0 \quad (2.206)$$

and

$$\text{At } t = \infty \Rightarrow \sigma_x = 0 \quad (2.207)$$

and

$$\text{At } x = 0 \Rightarrow \sigma_x = 0 \quad (2.208)$$

and

$$\text{At } x = \infty \Rightarrow \sigma_x = 0 \quad (2.209)$$

Taking the Laplace Transformation of Eq. 2.16 with respect to time yields:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{1}{c_1^2} \left[s^2 \bar{\sigma}_x(x, s) - s \sigma_x(x, 0) - \dot{\sigma}_x(x, 0) \right] = c_2 \left[s^2 \bar{T}(x, s) - s T(x, 0) - \dot{T}(x, 0) \right] \quad (2.210)$$

where $\bar{\sigma}_x(x, s)$ and $\bar{T}(x, s)$ are the Laplace transforms of thermal stress and temperature respectively in the x and s domains.

By substituting the initial conditions, Eq. 2.210 reduces to:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = c_2 s^2 \bar{T}(x, s) \quad (2.211)$$

Considering the temperature distribution in a Laplace domain for an exponentially decaying pulse with time, equation (Eq. 2.200), and substituting it into Eq. 2.211, and solving for the stress field, yields:

$$\begin{aligned} \frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = -c_2 s^2 \left[\frac{I_1 \delta (h + \delta k) e^{-gx}}{k(s + \beta)(\delta^2 - g^2)(h + kg)} \right. \\ \left. + \frac{hT_0 e^{-gx}}{s(h + kg)} - \frac{I_1 \delta e^{-\delta x}}{k(\delta^2 - g^2)(s + \beta)} \right] \end{aligned} \quad (2.212)$$

Now let M_1 and M_2 be defined as:

$$M_1 = \frac{I_1 \delta (h + \delta k) c_2 s^2}{k(s + \beta)(\delta^2 - g^2)(h + kg)} + \frac{hT_0 c_2 s}{(h + kg)} \quad (2.213)$$

and

$$M_2 = \frac{I_1 \delta c_2 s^2}{k(\delta^2 - g^2)(s + \beta)} \quad (2.214)$$

Then, Eq. 2.212 becomes:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = M_1 e^{-gx} + M_2 e^{-\delta x} \quad (2.215)$$

The complementary and the particular solutions of Eq. 2.215 are:

$$(\bar{\sigma}_x)_h = A_3 e^{\frac{sx}{c_1}} + A_4 e^{\frac{-sx}{c_1}} \quad (2.216)$$

While the particular solution has two parts, the first part is:

$$(\bar{\sigma}_x)_{p1} = G_1 e^{-\sqrt{\frac{s}{2}}x} \quad (2.217)$$

Substituting Eq. 2.217 in Eq. 2.215 yields:

$$G_1 = \frac{M_1}{g^2 - \frac{s^2}{c_1^2}} \quad (2.218)$$

The second part of the particular solution is:

$$(\bar{\sigma}_x)_{p_2} = G_2 e^{-\delta x} \quad (2.219)$$

Substituting Eq. 2.219 in Eq. 2.215 yields:

$$G_2 = \frac{M_2}{\delta^2 - \frac{s^2}{c_1^2}} \quad (2.220)$$

Hence, the general solution for the stress field becomes:

$$(\bar{\sigma}_x)_g = A_3 e^{\frac{sx}{c_1}} + A_4 e^{\frac{-sx}{c_1}} + G_1 e^{-gx} + G_2 e^{-\delta x} \quad (2.221)$$

From the boundary condition ($x = \infty \Rightarrow \sigma_x = 0$), this yields $A_3 = 0$.

Then, Eq. 2.21 reduces to:

$$(\bar{\sigma}_x)_g = A_4 e^{\frac{-sx}{c_1}} + G_1 e^{-\sqrt{\frac{s}{\alpha}}x} + G_2 e^{-\delta x} \quad (2.222)$$

Consider the boundary condition of the stress field at the surface, where at $x = 0 \Rightarrow \partial \bar{\sigma}_x = 0$, the constant in Eq. 2.222 becomes:

$$A_4 = -\frac{c_1}{s} \left[\sqrt{\frac{s}{\alpha}} G_1(s) + \delta G_2(s) \right]$$

Therefore, Eq. 2.222 becomes:

$$\begin{aligned} \bar{\sigma}_x(x, s) = & G_1(s) e^{-\sqrt{\frac{s}{\alpha}}x} - G_1(s) \frac{c_1}{\sqrt{s\alpha}} e^{\frac{-sx}{c_1}} \\ & + G_2(s) e^{-\delta x} - G_2(s) \frac{c_1 \delta}{s} e^{\frac{-sx}{c_1}} \end{aligned} \quad (2.223)$$

Finding the solution for σ_x in the x and t domain, one should take the inverse Laplace Transform for each term in Eq. 2.223. To do this, the following terms are introduced:

$$\begin{aligned} \text{Term1} &= G_1(s) e^{-\sqrt{\frac{s}{\alpha}}x} & \text{Term2} &= -G_1(s) \frac{c_1}{\sqrt{s\alpha}} e^{\frac{-sx}{c_1}} \\ \text{Term3} &= G_2(s) e^{-\delta x} & \text{Term4} &= -G_2(s) \frac{c_1 \delta}{s} e^{\frac{-sx}{c_1}} \end{aligned} \quad (2.224)$$

Consequently, the solution for the stress distribution is the summation of the inverse Laplace Transforms of the above terms. Therefore, the Laplace inversion of Terms (Term1, Term2, Term3, Term4) can be stated as follows:

Term1 is composed of the terms:

$$\text{Term1} = \text{Term11} + \text{Term 21} \quad (2.225)$$

where

$$\text{Term11} = \frac{I_1 \delta(h + \delta k) c_2}{k} \left[\frac{s e^{-\sqrt{\frac{s}{\alpha}} x}}{(\delta^2 - s/\alpha)(s + \beta) \left(h + k\sqrt{s/\alpha}\right) (1/\alpha - s/c_1^2)} \right] \quad (2.226)$$

and

$$\text{Term21} = h T_0 c_2 \left[\frac{e^{-\sqrt{\frac{s}{\alpha}} x}}{\left(h + k\sqrt{s/\alpha}\right) (1/\alpha - s/c_1^2)} \right] \quad (2.227)$$

$$\text{Let } C_{10} = \frac{I_1 \delta(h + \delta k) C_2}{k} \quad (2.228)$$

and

$$C_{20} = h T_0 c_2 \quad (2.229)$$

Therefore, the Laplace transformation of Term1 can be written as:

$$\mathfrak{L}^{-1}[\text{Term1}] = \mathfrak{L}^{-1}[\text{Term11}] + \mathfrak{L}^{-1}[\text{Term21}] \quad (2.230)$$

where

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term11}] &= \mathfrak{L}^{-1}[\text{Term111}] + \mathfrak{L}^{-1}[\text{Term211}] + \mathfrak{L}^{-1}[\text{Term311}] \\ &+ \mathfrak{L}^{-1}[\text{Term411}] + \mathfrak{L}^{-1}[\text{Term511}] \\ &+ \mathfrak{L}^{-1}[\text{Term611}] + \mathfrak{L}^{-1}[\text{Term711}] + \mathfrak{L}^{-1}[\text{Term811}] \end{aligned} \quad (2.231)$$

The Laplace transformations of the terms are:

$$\mathfrak{L}^{-1}[\text{Term111}] = -\frac{c_1^2 h^2 k^3 \alpha^2 \sqrt{\alpha} C_{10}}{2(\alpha^2 h^2 - c_1^2 k^2)(h^2 \alpha + k^2 \beta)(h^2 - k^2 \delta^2)} \left[\frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} - \frac{h x}{k}} \text{Erfc}\left(-\frac{h\sqrt{\alpha} t}{k} + \frac{x}{2\sqrt{\alpha} t}\right) \right. \\ \left. + \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} + \frac{h x}{k}} \text{Erfc}\left(\frac{h\sqrt{\alpha} t}{k} + \frac{x}{2\sqrt{\alpha} t}\right) \right] \quad (2.232)$$

and

$$\mathfrak{L}^{-1}[\text{Term211}] = -\frac{c_1^2 h \alpha^3 \sqrt{\beta} C_{10}}{2(\beta + \alpha \delta^2)(h^2 \alpha + k^2 \beta)(c_1^2 + \alpha \beta)} \left[\sqrt{\beta} e^{\beta t - \sqrt{\frac{\beta}{\alpha}} x} \text{Erfc}\left(-\sqrt{\beta} t + \frac{x}{2\sqrt{\alpha} t}\right) \right. \\ \left. + \sqrt{\beta} e^{\sqrt{\frac{\beta}{\alpha}} x + \beta t} \text{Erfc}\left(\sqrt{\beta} t + \frac{x}{2\sqrt{\alpha} t}\right) \right] \quad (2.233)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term311}] = & -\frac{c_1^3 h \alpha^4 \sqrt{\alpha} C_{10}}{2(\alpha^2 h^2 - c_1^2 k^2)(c_1^2 + \alpha\beta)(\delta^2 \alpha^2 - c_1^2)} \\ & \times \left[\frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} - \frac{c_1 x}{\alpha}} \text{Erfc}\left(-c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \right. \\ & \left. + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} + \frac{c_1 x}{\alpha}} \text{Erfc}\left(c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \right] \end{aligned} \quad (2.234)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term411}] = & \frac{c_1^2 h \alpha^2 \sqrt{\alpha} \delta C_{10}}{2(\beta + \alpha\delta^2)(\delta^2 \alpha^2 - c_1^2)(h^2 - k^2 \delta^2)} \\ & \times \left[\sqrt{\alpha} \delta e^{\alpha\delta^2 t - \delta x} \text{Erfc}\left(-\delta \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \right. \\ & \left. + \sqrt{\alpha} e^{\alpha\delta^2 t + \delta x} \text{Erfc}\left(\delta \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \right] \end{aligned} \quad (2.235)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term511}] = & \frac{c_1^2 h^2 k^3 \alpha^2 \sqrt{\alpha} C_{10}}{2(\alpha^2 h^2 - c_1^2 k^2)(h^2 \alpha + k^2 \beta)(h^2 - k^2 \delta^2)} \\ & \times \left[\frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4\alpha t}\right) + \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} - \frac{hx}{k}} \text{Erfc}\left(-\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) \right. \\ & \left. - \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} + \frac{hx}{k}} \text{Erfc}\left(\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) \right] \end{aligned} \quad (2.236)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term611}] = & \frac{c_1^2 k \alpha^2 \sqrt{\alpha} \beta C_{10}}{2(c_1^2 + \alpha\beta)(h^2 \alpha + k^2 \beta)(\beta + \alpha\delta^2)} \\ & \times \left[\frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4\alpha t}\right) + \sqrt{\beta t - x} \sqrt{\frac{\beta}{\alpha}} \text{Erfc}\left(-\sqrt{\beta t} + \frac{x}{2\sqrt{\alpha t}}\right) \right. \\ & \left. - \sqrt{\beta} e^{\beta t + x} \sqrt{\frac{\beta}{\alpha}} \text{Erfc}\left(\sqrt{\beta t} + \frac{x}{2\sqrt{\alpha t}}\right) \right] \end{aligned} \quad (2.237)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term711}] &= \frac{c_1^2 k \alpha^3 \sqrt{\alpha} C_{10}}{2(\alpha^2 h^2 - c_1^2 k^2)(c_1^2 + \alpha\beta)(\delta^2 \alpha^2 - c_1^2)} \\ &\times \left[\frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4\alpha t}\right) + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} - \frac{c_1 x}{\alpha}} \text{Erfc}\left(-c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \right. \\ &\quad \left. - \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} + \frac{c_1 x}{\alpha}} \text{Erfc}\left(c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \right] \end{aligned} \quad (2.238)$$

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term811}] &= \frac{c_1^2 k \alpha^2 \sqrt{\alpha} \delta^2 C_{10}}{2(\beta + \alpha \delta^2)(\delta^2 \alpha^2 - c_1^2)(k^2 \delta^2 - h^2)} \\ &\times \left[\frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4\alpha t}\right) + \sqrt{\alpha} \delta e^{\alpha \delta^2 t - \delta x} \text{Erfc}\left(-\delta \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \right. \\ &\quad \left. - \sqrt{\alpha} \delta e^{\alpha \delta^2 t + \delta x} \text{Erfc}\left(\delta \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \right] \end{aligned} \quad (2.239)$$

The Laplace inversion of Term21 can be written as:

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term21}] &= \mathfrak{L}^{-1}[\text{Term121}] + \mathfrak{L}^{-1}[\text{Term221}] \\ &\quad + \mathfrak{L}^{-1}[\text{Term321}] + \mathfrak{L}^{-1}[\text{Term421}] \end{aligned} \quad (2.240)$$

Therefore, the Laplace inversions of composing terms are:

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term121}] &= \frac{c_1^2 h k \alpha \sqrt{\alpha} C_{20}}{2h(\alpha^2 h^2 - c_1^2 k^2)} \left[\frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} - \frac{hx}{k}} \text{Erfc}\left(-\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) \right. \\ &\quad \left. + \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} + \frac{hx}{k}} \text{Erfc}\left(\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) \right] \end{aligned} \quad (2.241)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term221}] &= -\frac{c_1 h \alpha^2 \sqrt{\alpha} C_{20}}{2(\alpha^2 h^2 - c_1^2 k^2)} \left[\frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} - \frac{c_1 x}{\alpha}} \text{Erfc}\left(-c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \right. \\ &\quad \left. + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} + \frac{c_1 x}{\alpha}} \text{Erfc}\left(c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \right] \end{aligned} \quad (2.242)$$

and

$$\mathfrak{L}^{-1}[\text{Term321}] = -\frac{c_1^2 k \alpha \sqrt{\alpha} C_{20}}{2(\alpha^2 h^2 - c_1^2 k^2)} \left[\begin{aligned} &\frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4}\right) + \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2}{k^2} - \frac{hx}{k}} \text{Erfc}\left(-\frac{h\sqrt{\alpha}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) \\ &- \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2}{k^2} + \frac{hx}{k}} \text{Erfc}\left(\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) \end{aligned} \right] \quad (2.243)$$

and

$$\mathfrak{L}^{-1}[\text{Term421}] = \frac{c_1^2 k \alpha \sqrt{\alpha} C_{20}}{2(\alpha^2 h^2 - c_1^2 k^2)} \left[\begin{aligned} &\frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4}\right) + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} - \frac{c_1 x}{\alpha}} \text{Erfc}\left(-c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \\ &- \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} + \frac{c_1 x}{\alpha}} \text{Erfc}\left(c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \end{aligned} \right] \quad (2.244)$$

The Laplace transform of Term2 can be written as:

$$\mathfrak{L}^{-1}[\text{Term2}] = \mathfrak{L}^{-1}[\text{Term12}] + \mathfrak{L}^{-1}[\text{Term22}] \quad (2.245)$$

Therefore, the Laplace inversions of composing terms are:

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term12}] &= \mathfrak{L}^{-1}[\text{Term112}] + \mathfrak{L}^{-1}[\text{Term212}] + \mathfrak{L}^{-1}[\text{Term312}] \\ &+ \mathfrak{L}^{-1}[\text{Term412}] + \mathfrak{L}^{-1}[\text{Term512}] \\ &+ \mathfrak{L}^{-1}[\text{Term612}] + \mathfrak{L}^{-1}[\text{Term712}] + \mathfrak{L}^{-1}[\text{Term812}] \end{aligned} \quad (2.246)$$

Knowing that:

$$C_{30} = -\frac{I_1 \delta(h + \delta k) c_2 c_1}{\sqrt{\alpha} k} \quad (2.247)$$

and

$$C_{40} = -\frac{h T_0 c_2 c_1}{\sqrt{\alpha}} \quad (2.248)$$

Hence:

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term112}] &= -\frac{c_1^2 h k^6 \alpha^2 C_{30}}{k^2 (c_1^2 k^2 - h^2 \alpha^2) (h^2 \alpha + k^2 \beta) (h^2 - k^2 \delta^2)} \\ &\times \left[\begin{aligned} &\frac{1}{\sqrt{\pi(t - x/c_1)}} + \\ &\frac{h\sqrt{\alpha}}{k} e^{\frac{h^2}{k^2} - \frac{hx}{k}} \text{Erf}\left(\frac{h\sqrt{\alpha}}{k} \sqrt{t - x/c_1}\right) \end{aligned} \right] \cdot 1\left(t - \frac{x}{c_1}\right) \end{aligned} \quad (2.249)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term212}] &= \frac{c_1^2 h \alpha^3 C_{30}}{(\beta + \alpha \delta^2)(h^2 \alpha + k^2 \beta)(c_1^2 + \alpha \beta)} \\ &\times \left[\frac{1}{\sqrt{\pi(t - x/c_1)}} + \frac{1}{\sqrt{-\beta} e^{-\beta t} \text{Erf}\left(\sqrt{-\beta(t - x/c_1)}\right)} \right] \cdot 1\left(t - \frac{x}{c_1}\right) \end{aligned} \quad (2.250)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term312}] &= -\frac{c_1^2 h \alpha^5 C_{30}}{(\alpha^2 h^2 - c_1^2 k^2)(c_1^2 + \alpha \beta)(\delta^2 \alpha^2 - c_1^2)} \\ &\times \left[\frac{1}{\sqrt{\pi(t - x/c_1)}} + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2}{\alpha} t} \text{Erf}\left(\frac{c_1}{\sqrt{\alpha}} \sqrt{t - x/c_1}\right) \right] \cdot 1\left(t - \frac{x}{c_1}\right) \end{aligned} \quad (2.251)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term412}] &= \frac{c_1^2 h \alpha^2 C_{30}}{(\beta + \alpha \delta^2)(\delta^2 \alpha^2 - c_1^2)(h^2 - k^2 \delta^2)} \\ &\times \left[\frac{1}{\sqrt{\pi(t - x/c_1)}} + \frac{1}{\sqrt{\alpha} \delta e^{\alpha \delta^2 t} \text{Erf}\left(\sqrt{\alpha} \delta \sqrt{t - x/c_1}\right)} \right] \cdot 1\left(t - \frac{x}{c_1}\right) \end{aligned} \quad (2.252)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term512}] &= \frac{c_1^2 k^5 \alpha^2 C_{30}}{(\alpha^2 h^2 - c_1^2 k^2)(h^2 \alpha + k^2 \beta)(h^2 - k^2 \delta^2)} \\ &\times \left[D(t - x/c_1) + \frac{h^2 \alpha}{k^2} e^{\frac{h^2 \alpha}{k^2}(t - x/c_1)} \right] \cdot 1\left(t - \frac{x}{c_1}\right) \end{aligned} \quad (2.253)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term612}] &= -\frac{c_1^2 k \alpha^2 \sqrt{\alpha} C_{30}}{(c_1^2 + \alpha \beta)(h^2 \alpha + k^2 \beta)(\beta + \alpha \delta^2)} \left[D(t - x/c_1) - \beta e^{-\beta(t - x/c_1)} \right] \\ &\cdot 1\left(t - \frac{x}{c_1}\right) \end{aligned} \quad (2.254)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term712}] = & -\frac{c_1^2 k \alpha^4 \sqrt{\alpha} C_{30}}{(\alpha^2 h^2 - c_1^2 k^2)(c_1^2 + \alpha \beta)(\delta^2 \alpha^2 - c_1^2)} \\ & \times \left[D(t - x/c_1) + \frac{c_1^2}{\alpha} e^{\frac{c_1^2}{\alpha}(t-x/c_1)} \right] \cdot 1\left(t - \frac{x}{c_1}\right) \end{aligned} \quad (2.255)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term812}] = & \frac{c_1^2 k \alpha \sqrt{\alpha} C_{30}}{(\beta + \alpha \delta^2)(\delta^2 \alpha^2 - c_1^2)(k^2 \delta^2 - h^2)} \\ & \times \left[D(t - x/c_1) + \alpha \delta^2 e^{\alpha \delta^2(t-x/c_1)} \right] \cdot 1\left(t - \frac{x}{c_1}\right) \end{aligned} \quad (2.256)$$

The Laplace transformation of Term22 can be written as:

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term22}] = & \mathfrak{L}^{-1}[\text{Term122}] + \mathfrak{L}^{-1}[\text{Term222}] \\ & + \mathfrak{L}^{-1}[\text{Term322}] + \mathfrak{L}^{-1}[\text{Term422}] \end{aligned} \quad (2.257)$$

Therefore, the Laplace transformations of the composing terms are:

$$\mathfrak{L}^{-1}[\text{Term122}] = \frac{\alpha C_{40}}{h \sqrt{\pi(t - x/c_1)}} \cdot 1\left(t - \frac{x}{c_1}\right) \quad (2.258)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term222}] = & \frac{c_1^2 k^2 \alpha C_{40}}{h(\alpha^2 h^2 - c_1^2 k^2)} \\ & \times \left[\frac{1}{\sqrt{\pi(t - x/c_1)}} + \frac{h \sqrt{\alpha}}{k} e^{\frac{h^2 \alpha}{k^2} t} \text{Erf}\left(\frac{h \sqrt{\alpha}}{k} \sqrt{t - \frac{x}{c_1}}\right) \right] \cdot 1\left(t - \frac{x}{c_1}\right) \end{aligned} \quad (2.259)$$

and

$$\begin{aligned} \mathfrak{L}^{-1}[\text{Term322}] = & -\frac{h \alpha^3 C_{40}}{(\alpha^2 h^2 - c_1^2 k^2)} \\ & \times \left[\frac{1}{\sqrt{\pi(t - x/c)}} + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2}{\alpha} t} \text{Erf}\left(\frac{c_1}{\sqrt{\alpha}} \sqrt{t - \frac{x}{c_1}}\right) \right] \cdot 1\left(t - \frac{x}{c_1}\right) \end{aligned} \quad (2.260)$$

and

$$\mathfrak{L}^{-1}[\text{Term422}] = -\frac{k\sqrt{\alpha}C_{40}}{h^2}D(t-x/c_1) \cdot 1\left(t-\frac{x}{c_1}\right) \quad (2.261)$$

and

$$\mathfrak{L}^{-1}[\text{Term522}] = \frac{c_1^2 k^3 \sqrt{\alpha} C_{40}}{h^2 (c_1^2 k^2 - h^2 \alpha^2)} \left[D(t-x/c_1) + \frac{h^2 \alpha}{k^2} e^{\frac{h^2 \alpha}{k^2} (t-x/c_1)} \right] \cdot 1\left(t-\frac{x}{c_1}\right) \quad (2.262)$$

and

$$\mathfrak{L}^{-1}[\text{Term622}] = -\frac{k\alpha^2 \sqrt{\alpha} C_{40}}{(c_1^2 k^2 - h^2 \alpha^2)} \left[D(t-x/c_1) + \frac{c_1^2}{\alpha} e^{\frac{c_1^2}{\alpha} (t-x/c_1)} \right] \cdot 1\left(t-\frac{x}{c_1}\right) \quad (2.263)$$

and

$$\text{Term3} = \frac{-I_1 \delta c_2}{k} \left[\frac{s^2 e^{-\delta x}}{(\delta^2 - s/\alpha)(s + \beta)(\delta^2 - s^2/c_1^2)} \right] \quad (2.264)$$

and

$$\mathfrak{L}^{-1}[\text{Term3}] = \frac{-I_1 \delta c_2}{k} \left\{ \begin{aligned} & \frac{c_1^2 c_1 \delta t - \delta x}{2(\beta + c_1 \delta)(c_1 - \alpha \delta)} - \frac{c_1^3 \alpha e^{-(c_1 \delta t + \delta x)}}{2(c_1 \delta - \beta)(c_1 + \alpha \delta)} - \frac{c_1^2 \alpha \beta^2 e^{-(+\delta x)}}{(\beta^2 - c_1^2 \delta^2)(\beta + \alpha \delta^2)} \\ & + \frac{c_1^2 \alpha^3 \delta^2 e^{\alpha \delta^2 t - \delta x}}{(\alpha^2 \delta^2 - c_1^2)(\beta + \alpha \delta^2)} \end{aligned} \right\} \quad (2.265)$$

and

$$\text{Term4} = \frac{I_1 \delta^2 c_2 c_1}{k} \left[\frac{s e^{-\frac{s}{c_1} x}}{(\delta^2 - s/\alpha)(s + \beta)(\delta^2 - s^2/c_1^2)} \right] \quad (2.266)$$

and

$$\mathfrak{L}^{-1}[\text{Term4}] = \frac{I_1 \delta^2 c_2 c_1}{k} \left\{ \begin{aligned} & \frac{c_1^2 \alpha e^{c_1 \delta (t-x/c_1)}}{2\delta(\beta + c_1 \delta)(c_1 - \alpha \delta)} + \frac{c_1^2 \alpha e^{-c_1 \delta (t-x/c_1)}}{2\delta(c_1 \delta - \beta)(c_1 + \alpha \delta)} \\ & - \frac{c_1^2 \alpha \beta e^{-\beta(t-x/c_1)}}{(c_1^2 \delta^2 - \beta^2)(\beta + \alpha \delta^2)} - \frac{c_1^2 \alpha^2 e^{\alpha \delta^2 (t-x/c_1)}}{(c_1^2 - \alpha^2 \delta^2)(\beta + \alpha \delta^2)} \end{aligned} \right\} \cdot 1\left(t-\frac{x}{c_1}\right) \quad (2.267)$$

where $1(t - \frac{x}{c_1})$ is a unit step function, $\text{Erf}(y)$ is the error function of the variable y and $D(t - x/c_1)$ is the Dirac delta function. The unit step function has the values of 0 for $t \leq \frac{x}{c_1}$ and 1 for $t > \frac{x}{c_1}$.

The closed form solution of the stress distribution can be written as:

$$\sigma_x(x, t) = \mathcal{L}^{-1}[\text{Term1}] + \mathcal{L}^{-1}[\text{Term2}] + \mathcal{L}^{-1}[\text{Term3}] + \mathcal{L}^{-1}[\text{Term4}] \quad (2.268)$$

where $\mathcal{L}^{-1}$ represents the inverse sign of the Laplace transformation.

The additional dimensionless quantities are defined to present the stress distribution in the dimensionless form, i.e.:

$$c_1^* = \frac{c_1}{\alpha \delta} : \sigma_x^* = \frac{k \sigma_x}{I_1 \delta \alpha^2 c_2} \text{ and } U[1] = \left(t^* - \frac{x^*}{c_1^*} \right) \quad (2.269)$$

where $U(1)$ is the dimensionless unit step function. Therefore, for the dimensionless stress distribution, the followings are resulted:

$$(\sigma_x^*)_1 = (\sigma_x^*)_{11} + (\sigma_x^*)_{21} \quad (2.270)$$

where

$$(\sigma_x^*)_{11} = (\sigma_x^*)_{111} + (\sigma_x^*)_{211} + (\sigma_x^*)_{311} + (\sigma_x^*)_{411} + (\sigma_x^*)_{511} + (\sigma_x^*)_{611} + (\sigma_x^*)_{711} + (\sigma_x^*)_{811} \quad (2.271)$$

In the dimensionless form:

$$(\sigma_x^*)_{111} = - \frac{(h^* + 1)h^{*2}c_1^{*2}}{2(h^{*2} - 1)(h^{*2} - c_1^{*2})(h^{*2} + \beta^*/t^*)} \left[\begin{aligned} & h^* e^{h^{*2}t^* - h^*x^*} \text{Erfc} \left(-h^* \sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}} \right) \\ & + h^* e^{h^{*2}t^* + h^*x^*} \text{Erfc} \left(h^* \sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}} \right) \end{aligned} \right] \quad (2.272)$$

and

$$(\sigma_x^*)_{211} = - \frac{(h^* + 1)h^*c_1^{*2}\beta^*}{2(\beta^*/t^* + 1)(h^{*2} + \beta^*/t^*)(c_1^{*2} + \beta^*/t^*)} \left[\begin{aligned} & e^{\beta^* - \sqrt{\frac{\beta^*}{t^*}}x^*} \text{Erfc} \left(-\sqrt{\beta^*} + \frac{x^*}{2\sqrt{t^*}} \right) \\ & + e^{\beta^* + \sqrt{\frac{\beta^*}{t^*}}x^*} \text{Erfc} \left(\sqrt{\beta^*} + \frac{x^*}{2\sqrt{t^*}} \right) \end{aligned} \right] \quad (2.273)$$

and

$$(\sigma_x^\bullet)_{311} = -\frac{(h^\bullet + 1)h^\bullet c_1^{\bullet 3}}{2(h^{\bullet 2} - c_1^{\bullet 2})(1 - c_1^{\bullet 2})(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[\begin{aligned} &c_1^{\bullet 2} e^{c_1^{\bullet 2} t^\bullet - c_1^\bullet x^\bullet} \text{Erfc} \left(-c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}} \right) \\ &+ c_1^{\bullet 2} e^{c_1^{\bullet 2} t^\bullet + c_1^\bullet x^\bullet} \text{Erfc} \left(c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}} \right) \end{aligned} \right] \quad (2.274)$$

and

$$(\sigma_x^\bullet)_{411} = \frac{(h^\bullet + 1)h^\bullet c_1^{\bullet 2}}{2(h^{\bullet 2} - 1)(1 - c_1^{\bullet 2})(1 + \beta^\bullet/t^\bullet)} \left[\begin{aligned} &e^{t^\bullet - x^\bullet} \text{Erfc} \left(-\sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}} \right) \\ &+ e^{t^\bullet + x^\bullet} \text{Erfc} \left(\sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}} \right) \end{aligned} \right] \quad (2.275)$$

and

$$(\sigma_x^\bullet)_{511} = \frac{(h^\bullet + 1)h^{\bullet 2} c_1^{\bullet 2}}{2(h^{\bullet 2} - 1)(h^{\bullet 2} - c_1^{\bullet 2})(h^{\bullet 2} + \beta^\bullet/t^\bullet)} \times \left[\begin{aligned} &\frac{2}{\sqrt{\pi t^\bullet}} \exp \left(\frac{-x^{\bullet 2}}{4t^\bullet} \right) + h^\bullet e^{h^{\bullet 2} t^\bullet - h^\bullet x^\bullet} \text{Erfc} \left(-h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}} \right) \\ &- h^\bullet e^{h^{\bullet 2} t^\bullet + h^\bullet x^\bullet} \text{Erfc} \left(h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}} \right) \end{aligned} \right] \quad (2.276)$$

and

$$(\sigma_x^*)_{611} = \frac{(h^* + 1)c_1^{*2} \beta^*}{2t^*(\beta^*/t^* + 1)(h^{*2} + \beta^*/t^*)(c_1^{*2} + \beta^*/t^*)} \times \left[\begin{aligned} &\frac{2}{\sqrt{\pi t^*}} \exp \left(\frac{-x^{*2}}{4t^*} \right) \\ &+ \sqrt{\frac{\beta^*}{t^*}} e^{\beta^* - \sqrt{\frac{\beta^*}{t^*}} x^*} \text{Erfc} \left(-\sqrt{\beta^*} + \frac{x^*}{2\sqrt{t^*}} \right) \\ &- \sqrt{\frac{\beta^*}{t^*}} e^{\beta^* + \sqrt{\frac{\beta^*}{t^*}} x^*} \text{Erfc} \left(\sqrt{\beta^*} + \frac{x^*}{2\sqrt{t^*}} \right) \end{aligned} \right] \quad (2.277)$$

and

$$\begin{aligned}
 (\sigma_x^\bullet)_{711} = & \frac{(h^\bullet + 1)c_1^{\bullet 4}}{2(h^{\bullet 2} - c_1^{\bullet 2})(1 - c_1^{\bullet 2})(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \\
 & \times \left[\frac{2}{\sqrt{\pi t^\bullet}} \exp\left(\frac{-x^{\bullet 2}}{4t^\bullet}\right) + c_1^\bullet e^{c_1^{\bullet 2}t^\bullet - c_1^\bullet x^\bullet} \operatorname{Erfc}\left(-c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \right] \\
 & \left[-c_1^\bullet e^{c_1^{\bullet 2}t^\bullet + c_1^\bullet x^\bullet} \operatorname{Erfc}\left(c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \right]
 \end{aligned} \tag{2.278}$$

and

$$\begin{aligned}
 (\sigma_x^\bullet)_{811} = & \frac{(h^\bullet + 1)c_1^{\bullet 2}}{2(1 - h^{\bullet 2})(1 - c_1^{\bullet 2})(1 + \beta^\bullet/t^\bullet)} \\
 & \times \left[\frac{2}{\sqrt{\pi t^\bullet}} \exp\left(\frac{-x^{\bullet 2}}{4t^\bullet}\right) + e^{t^\bullet - x^\bullet} \operatorname{Erfc}\left(-\sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \right] \\
 & \left[-e^{t^\bullet + x^\bullet} \operatorname{Erfc}\left(\sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \right]
 \end{aligned} \tag{2.279}$$

The dimensionless form of $(\sigma_x^\bullet)_{21}$ is:

$$(\sigma_x^\bullet)_{21} = (\sigma_x^\bullet)_{121} + (\sigma_x^\bullet)_{221} + (\sigma_x^\bullet)_{321} + (\sigma_x^\bullet)_{421} \tag{2.280}$$

where

$$(\sigma_x^\bullet)_{121} = \frac{T_0^\bullet h^\bullet c_1^{\bullet 2}}{2(h^{\bullet 2} - c_1^{\bullet 2})} \left[\begin{aligned} & h^\bullet e^{h^{\bullet 2}t^\bullet - h^\bullet x^\bullet} \operatorname{Erfc}\left(-h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \\ & + h^\bullet e^{h^{\bullet 2}t^\bullet + h^\bullet x^\bullet} \operatorname{Erfc}\left(h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \tag{2.281}$$

and

$$(\sigma_x^\bullet)_{221} = -\frac{T_0^\bullet h^{\bullet 2} c_1^\bullet}{2(h^{\bullet 2} - c_1^{\bullet 2})} \left[\begin{aligned} & c_1^\bullet e^{c_1^{\bullet 2}t^\bullet - c_1^\bullet x^\bullet} \operatorname{Erfc}\left(-c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \\ & + c_1^\bullet e^{c_1^{\bullet 2}t^\bullet + c_1^\bullet x^\bullet} \operatorname{Erfc}\left(c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \tag{2.282}$$

and

$$\begin{aligned}
 (\sigma_x^\bullet)_{321} = & -\frac{T_0^\bullet h^\bullet c_1^{\bullet 2}}{2(h^{\bullet 2} - c_1^{\bullet 2})} \\
 & \times \left[\frac{2}{\sqrt{\pi t^\bullet}} \exp\left(\frac{-x^{\bullet 2}}{4t^\bullet}\right) + h^\bullet e^{h^{\bullet 2} t^\bullet - h^\bullet x^\bullet} \operatorname{Erfc}\left(-h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \right] \\
 & - h^\bullet e^{h^{\bullet 2} t^\bullet + h^\bullet x^\bullet} \operatorname{Erfc}\left(h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \quad (2.283)
 \end{aligned}$$

and

$$\begin{aligned}
 (\sigma_x^\bullet)_{421} = & \frac{T_0^\bullet c_1^{\bullet 2} h^\bullet}{2(h^{\bullet 2} - c_1^{\bullet 2})} \left[\frac{2}{\sqrt{\pi t^\bullet}} \exp\left(\frac{-x^{\bullet 2}}{4t^\bullet}\right) + c_1^\bullet e^{c_1^{\bullet 2} t^\bullet - c_1^\bullet x^\bullet} \operatorname{Erfc}\left(-c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \right] \\
 & - c_1^\bullet e^{c_1^{\bullet 2} t^\bullet + c_1^\bullet x^\bullet} \operatorname{Erfc}\left(c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \quad (2.284)
 \end{aligned}$$

The dimensionless form of $(\sigma_x^\bullet)_2$ is:

$$(\sigma_x^\bullet)_2 = (\sigma_x^\bullet)_{12} + (\sigma_x^\bullet)_{22} \quad (2.285)$$

where

$$\begin{aligned}
 (\sigma_x^\bullet)_{12} = & (\sigma_x^\bullet)_{112} + (\sigma_x^\bullet)_{212} + (\sigma_x^\bullet)_{312} + (\sigma_x^\bullet)_{412} \\
 & + (\sigma_x^\bullet)_{512} + (\sigma_x^\bullet)_{612} + (\sigma_x^\bullet)_{712} + (\sigma_x^\bullet)_{812} \quad (2.286)
 \end{aligned}$$

and

$$\begin{aligned}
 (\sigma_x^\bullet)_{112} = & \frac{(h^\bullet + 1)h^\bullet c_1^{\bullet 3}}{(h^{\bullet 2} - 1)(h^{\bullet 2} - c_1^{\bullet 2})(h^{\bullet 2} + \beta^\bullet/t^\bullet)} \\
 & \times \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + h^\bullet e^{h^{\bullet 2} t^\bullet} \operatorname{Erf}\left(h^\bullet \sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}}\right) \right] \cdot U[1] \quad (2.287)
 \end{aligned}$$

and

$$\begin{aligned}
 (\sigma_x^\bullet)_{212} = & -\frac{(h^\bullet + 1)c_1^{\bullet 3}h^\bullet}{(\beta^\bullet/t^\bullet + 1)(h^{\bullet 2} + \beta^\bullet/t^\bullet)(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \\
 & \times \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + \sqrt{-\frac{\beta^\bullet}{t^\bullet}} e^{-\beta^\bullet} \text{Erf}\left(\sqrt{-\beta^\bullet + \frac{\beta^\bullet x^\bullet}{c_1^\bullet t^\bullet}}\right) \right] \cdot U[1]
 \end{aligned} \tag{2.288}$$

and

$$\begin{aligned}
 (\sigma_x^\bullet)_{312} = & \frac{(h^\bullet + 1)c_1^{\bullet 3}h^\bullet}{(h^{\bullet 2} - c_1^{\bullet 2})(1 - c_1^{\bullet 2})(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \\
 & \times \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + c_1^\bullet e^{c_1^{\bullet 2}t^\bullet} \text{Erf}\left(c_1^\bullet \sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}}\right) \right] \cdot U[1]
 \end{aligned} \tag{2.289}$$

and

$$\begin{aligned}
 (\sigma_x^\bullet)_{412} = & \frac{(h^\bullet + 1)c_1^{\bullet 3}h^\bullet}{(1 - h^{\bullet 2})(1 - c_1^{\bullet 2})(1 + \beta^\bullet/t^\bullet)} \\
 & \times \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + e^{t^\bullet} \text{Erf}\left(\sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}}\right) \right] \cdot U[1]
 \end{aligned} \tag{2.290}$$

and

$$\begin{aligned}
 (\sigma_x^\bullet)_{512} = & -\frac{(h^\bullet + 1)c_1^{\bullet 3}}{(h^{\bullet 2} - 1)(h^{\bullet 2} - c_1^{\bullet 2})(h^{\bullet 2} + \beta^\bullet/t^\bullet)} \\
 & \times \left[D(t^\bullet - x^\bullet/c_1^\bullet) + h^{\bullet 2} e^{h^{\bullet 2}t^\bullet - \frac{h^{\bullet 2}x^\bullet}{c_1^\bullet}} \right] \cdot U[1]
 \end{aligned} \tag{2.291}$$

and

$$\begin{aligned}
 (\sigma_x^\bullet)_{612} = & \frac{(h^\bullet + 1)c_1^{\bullet 3}}{(\beta^\bullet/t^\bullet + 1)(h^{\bullet 2} + \beta^\bullet/t^\bullet)(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \\
 & \times \left[D(t^\bullet - x^\bullet/c_1^\bullet) - \frac{\beta^\bullet}{t^\bullet} e^{-\beta^\bullet + \frac{\beta^\bullet x^\bullet}{c_1^\bullet}} \right] \cdot U[1]
 \end{aligned} \tag{2.292}$$

and

$$(\sigma_x^\bullet)_{712} = \frac{(h^\bullet + 1)c_1^{\bullet 3}}{(h^{\bullet 2} - c_1^{\bullet 2})(1 - c_1^{\bullet 2})(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \times \left[D(t^\bullet - x^\bullet/c_1^\bullet) + c_1^{\bullet 2} e^{c_1^{\bullet 2} t^\bullet - c_1^\bullet x^\bullet} \right] \cdot U[1] \quad (2.293)$$

and

$$(\sigma_x^\bullet)_{812} = -\frac{(h^\bullet + 1)c_1^{\bullet 3}}{(1 - h^{\bullet 2})(1 - c_1^{\bullet 2})(1 + \beta^\bullet/t^\bullet)} \left[D(t^\bullet - x^\bullet/c_1^\bullet) + e^{t^\bullet - \frac{x^\bullet}{c_1^\bullet}} \right] \cdot U[1] \quad (2.294)$$

The dimensionless for of $(\sigma_x^\bullet)_{22}$ is:

$$(\sigma_x^\bullet)_{22} = (\sigma_x^\bullet)_{122} + (\sigma_x^\bullet)_{222} + (\sigma_x^\bullet)_{322} + (\sigma_x^\bullet)_{422} \quad (2.295)$$

where

$$(\sigma_x^\bullet)_{122} = -\frac{T_0^\bullet c_1^{\bullet 3}}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} \cdot U[1] \quad (2.296)$$

and

$$(\sigma_x^\bullet)_{222} = -\frac{T_0^\bullet c_1^{\bullet 3}}{(h^{\bullet 2} - c_1^{\bullet 2})} \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + h^\bullet e^{h^{\bullet 2} t^\bullet} \text{Erf} \left(h^\bullet \sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}} \right) \right] \cdot U[1] \quad (2.297)$$

and

$$(\sigma_x^\bullet)_{322} = \frac{T_0^\bullet c_1^\bullet h^{\bullet 2}}{(h^{\bullet 2} - c_1^{\bullet 2})} \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + c_1^\bullet e^{c_1^{\bullet 2} t^\bullet} \text{Erf} \left(c_1^\bullet \sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}} \right) \right] \cdot U[1] \quad (2.298)$$

and

$$(\sigma_x^\bullet)_{422} = \frac{T_0^\bullet c_1^\bullet}{h^\bullet} D(t^\bullet - x^\bullet/c_1^\bullet) \cdot U[1] \quad (2.299)$$

and

$$(\sigma_x^\bullet)_{522} = \frac{T_0^\bullet c_1^{\bullet 3}}{h^\bullet (h^{\bullet 2} - c_1^{\bullet 2})} \left[D(t^\bullet - x^\bullet/c_1^\bullet) + h^{\bullet 2} e^{h^{\bullet 2} t^\bullet - \frac{h^{\bullet 2} x^\bullet}{c_1^\bullet}} \right] \cdot U[1] \quad (2.300)$$

and

$$(\sigma_x^*)_{622} = -\frac{T_0^* c_1^* h^*}{(h^{*2} - c_1^{*2})} \left[D(t^* - x^*/c_1^*) + c_1^{*2} e^{c_1^{*2} t^* - c_1^* x^*} \right] \cdot U[1] \quad (2.301)$$

and

$$(\sigma_x^*)_3 = -\frac{c_1^{*3} t^* e^{c_1^* t^* - x^*}}{2(\beta^* + c_1^* t^*)(c_1^* - 1)} + \frac{c_1^{*3} t^* e^{-c_1^* t^* - x^*}}{2(c_1^* t^* - \beta^*)(c_1^* + 1)} \quad (2.302)$$

$$+ \frac{c_1^{*2} t^* \beta^{*2} e^{-\beta^* - x^*}}{(\beta^* + t^*)(\beta^{*2} - c_1^{*2} t^{*2})} - \frac{c_1^{*2} t^* e^{t^* - x^*}}{(\beta^* + t^*)(1 - c_1^{*2})}$$

and

$$(\sigma_x^*)_4 = \left[\frac{c_1^{*3} t^* e^{c_1^* t^* - x^*}}{2(\beta^* + c_1^* t^*)(c_1^* - 1)} + \frac{c_1^{*3} t^* e^{-c_1^* t^* - x^*}}{2(c_1^* t^* - \beta^*)(c_1^* + 1)} \right] \cdot U[1] \quad (2.303)$$

$$- \frac{c_1^{*3} t^* \beta^{*2} e^{-\beta^* + \frac{\beta^* x^*}{c_1^* t^*}}}{(\beta^* + t^*)(c_1^{*2} t^{*2} - \beta^{*2})} - \frac{c_1^{*3} t^* e^{t^* - \frac{x^*}{c_1^*}}}{(\beta^* + t^*)(c_1^{*2} - 1)}$$

Consequently, the dimensionless form of the stress equation is:

$$\sigma_x^* = (\sigma_x^*)_1 + (\sigma_x^*)_2 + (\sigma_x^*)_3 + (\sigma_x^*)_4 \quad (2.304)$$

The dimensionless temperature (Eq. 2.205) and stress distributions (Eq. 2.304) are computed during the heating pulse.

2.3.3 Stress Boundary at the Surface

The Fourier heat transfer equation due to time exponentially decaying laser pulse can be written similar to Eq. 2.134. Therefore, the boundary conditions and the solution of the heat equation is given lightly.

In the analysis, no heat convection is considered from the free surface of the substrate material. The depth well below the surface ($x \cong \infty$), temperature remains the same. Therefore, the corresponding boundary conditions are:

At the surface:

$$\text{At } x = 0 \Rightarrow \left. \frac{\partial T}{\partial x} \right|_{x=0} = 0 \quad (2.305)$$

and

At depth infinity:

$$\text{At } x = \infty \Rightarrow T(\infty, t) = 0 \quad (2.306)$$

Initially, substrate material is considered at uniform temperature. Hence, the initial condition is:

Initially:

$$\text{At } t = 0 \Rightarrow T(x, 0) = 0 \quad (2.307)$$

The Laplace Transformation of Eq. 2.16 with respect to t , results:

$$\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{I_1 \delta}{k} \frac{e^{-\delta x}}{(s + \beta)} = \frac{1}{\alpha} [s \bar{T}(x, s) - T(x, 0)] \quad (2.308)$$

Introducing the initial condition and rearranging Eq. 2.308 yields:

$$\frac{\partial^2 \bar{T}}{\partial x^2} - h^2 \bar{T} = -\frac{I_1 \delta}{k} \frac{e^{-\delta x}}{(s + \beta)} \quad (2.309)$$

where $h^2 = s/\alpha$ and s is the transform variable. Equation 2.309 has the solution:

$$T(x, s) = A e^{hx} + B e^{-hx} - \frac{I_1 \delta e^{-\delta x}}{k(s + \beta)(\delta^2 - h^2)} \quad (2.310)$$

where A and B are constants. Introducing the boundary conditions will allow determining the constants A and B , i.e.:

$$\bar{T}(x, s) = -\frac{I_1 \delta}{k(s + \beta)} \left[\frac{\delta \exp(-hx)}{h(h^2 - \delta^2)} - \frac{\exp(-\delta x)}{(h^2 - \delta^2)} \right] \quad (2.311)$$

which gives the solution for temperature in Laplace domain.

The inverse Laplace Transform of Eq. 2.311 gives the temperature distribution inside the substrate material in space x and time t domain as follows:

$$T(x, t) = \frac{I_1 \delta}{2k} \left(\frac{\alpha}{\beta + \alpha \delta^2} \right) \times \left\{ i \delta \sqrt{\frac{\alpha}{\beta}} \exp(-\beta t) \cdot \left[\exp\left(ix \sqrt{\frac{\beta}{\alpha}} \right) \cdot \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} + i\sqrt{\beta t} \right) - \exp\left(-ix \sqrt{\frac{\beta}{\alpha}} \right) \cdot \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} - i\sqrt{\beta t} \right) \right] + \exp(\alpha \delta^2 t) \left[\exp(\delta x) \cdot \text{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \delta \sqrt{\alpha t} \right) - \exp(-) \cdot \text{Erfc}\left(\delta \sqrt{\alpha t} - \frac{x}{2\sqrt{\alpha t}} \right) \right] - 2 \exp(-(\beta t + \delta x)) \right\} \quad (2.312)$$

where Erfc is the complementary error function. Equation 2.312 is the closed form solution for temperature distribution. The temperature distribution in non-dimensional form is possible by defining dimensionless quantities and substituting in Eq. 2.312. The dimensionless quantities are:

$$x^* = x\delta: t^* = \alpha\delta^2 t: T^* = \frac{Tk\delta}{I_1}: \beta^* = \frac{\beta}{\alpha\delta^2} \quad (2.313)$$

The dimensionless temperature distribution becomes:

$$T^*(x^*, t^*) = \frac{1}{2} \left(\frac{t^*}{\beta^* + t^*} \right) \times \left\{ i \sqrt{\frac{t^*}{\beta^*}} \exp(-\beta^*) \cdot \begin{bmatrix} \exp\left(ix^* \sqrt{\frac{\beta^*}{t^*}}\right) \cdot \text{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} + i\sqrt{\beta^*}\right) \\ - \exp\left(-ix^* \sqrt{\frac{\beta^*}{t^*}}\right) \left(\frac{x^*}{2\sqrt{t^*}} - i\sqrt{\beta^*}\right) \end{bmatrix} \right. \\ \left. + \exp(t^*) \begin{bmatrix} \exp(x^*) \cdot \text{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} + \sqrt{t^*}\right) - \exp(-x^*) \cdot \text{Erfc}\left(\sqrt{t^*} - \frac{x^*}{2\sqrt{t^*}}\right) \\ - 2 \exp(-(\beta^* + x^*)) \end{bmatrix} \right\} \quad (2.314)$$

To solve for the stress distribution inside the substrate material, equation governing the momentum in a one-dimensional solid for a linear elastic case can be considered and Eq. 2.16 can be incorporated in the analysis.

To solve the stress equation (Eq. 2.16), two boundary conditions for the stress fields should be defined. The first boundary condition is time exponentially decaying stress at the surface due to recoil pressure developed. It should be noted that the recoil pressure generated at surface decays almost exponentially with time and it acts as a stress at the surface as soon as the recoil pressure is generated. Moreover, as the distance below the surface increases further (extends almost to infinity), the temperature gradient diminishes; in which case, thermal stress approaches zero in this region. Therefore, the corresponding boundary conditions are:

Stress at the surface due to recoil pressure:

$$\text{At } x = 0 \Rightarrow \sigma_x|_{x=0} = \sigma_o(e^{-bt} - e^{-at}) \quad (2.315)$$

Stress free as depth approaches infinity:

$$\text{At } x = \infty \Rightarrow \sigma_x|_{x=\infty} = 0 \quad (2.316)$$

Initially, the substrate material is assumed to be free from thermal stress and as the time approaches infinity, heating diminishes, the substrate material also becomes free from thermal stress. In this case, initial and final conditions for the stress field are:

Initially stress free substrate material:

$$\text{At time } t = 0 \Rightarrow \sigma_x|_{t=0} = 0 \quad (2.317)$$

Finally stress free substrate material:

$$\text{At time } t = \infty \Rightarrow \sigma_x|_{t=\infty} = 0 \quad (2.318)$$

In the case of temperature term in Eq. 2.16, initially substrate material is assumed at a zero uniform temperature as similar to initial condition for Eq. 2.311. As time approaches infinity, temperature becomes zero, since heating diminishes. It should be noted that the laser pulse intensity decays exponentially with time. Therefore, initial and final conditions for temperature term in Eq. 2.16 are:

Initially:

$$\text{At } t = 0 \Rightarrow T(x, 0) = 0 \quad (2.319)$$

and

Finally:

$$\text{At } t = \infty \Rightarrow T(x, \infty) = 0 \quad (2.320)$$

Taking the Laplace Transformation of Eq. 2.16 with respect to time yields:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{1}{c_1^2} [s^2 \bar{\sigma}_x(x, s) - s\sigma_x(x, 0) - \dot{\sigma}_x(x, 0)] = c_2 [s^2 \bar{T}(x, s) - sT(x, 0) - \dot{T}(x, 0)] \quad (2.321)$$

where $\bar{\sigma}_x(x, s)$ and $\bar{T}(x, s)$ are the Laplace Transforms of thermal stress and temperature respectively in the x and s domains.

By substituting the initial conditions, Eq. 2.16 reduces to:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = c_2 s^2 \bar{T}(x, s) \quad (2.322)$$

Considering the temperature distribution for time exponentially varying pulse, Eq. 2.318, and substituting into Eq. 2.322, and solving for the stress field, yields:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = -c_2 s^2 \frac{I_1 \delta}{k(s + \beta)} \left[\frac{\delta \exp(-hx)}{h(h^2 - \delta^2)} - \frac{\exp(-\delta x)}{(h^2 - \delta^2)} \right] \quad (2.323)$$

which can be arranged further:

$$\begin{aligned} \frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) &= \frac{-I_1 \delta^2 c_2 s^2}{k(s + \beta) h (h^2 - \delta^2)} e^{-hx} \\ &+ \frac{I_1 \delta c_2 s^2}{k(s + \beta) (h^2 - \delta^2)} e^{-\delta x} \end{aligned} \quad (2.324)$$

Now let N_1 and N_2 are defined as:

$$N_1 = \frac{-I_1 \delta^2 c_2 s^2}{k(s + \beta)h(h^2 - \delta^2)} \quad (2.325)$$

and

$$N_2 = \frac{I_1 \delta c_2 s^2}{k(s + \beta)(h^2 - \delta^2)} \quad (2.326)$$

Equation 2.16 becomes:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = N_1 e^{-hx} + N_2 e^{-\delta x} \quad (2.327)$$

The homogeneous and the particular solutions for 2.328 are:

$$(\bar{\sigma}_x)_h = C_1 e^{\frac{sx}{c_1}} + C_2 e^{\frac{-sx}{c_1}} \quad (2.328)$$

while the particular solution has two parts, the first part is:

$$(\bar{\sigma}_x)_{p_1} = Q_1 e^{-hx} \quad (2.329)$$

Substituting Eq. 2.329 in Eq. 2.327 yields:

$$Q_1 = \frac{N_1}{h^2 - \frac{s^2}{c_1^2}} \quad (2.330)$$

The second part of the particular solution is:

$$(\bar{\sigma}_x)_{p_2} = Q_2 e^{-\delta x} \quad (2.331)$$

Substituting Eq. 2.331 in Eq. 2.327 yields:

$$Q_2 = \frac{N_2}{\delta^2 - \frac{s^2}{c_1^2}} \quad (2.332)$$

therefore, the general solution for the stress field becomes:

$$(\bar{\sigma}_x)_g = C_1 e^{\frac{sx}{c_1}} + C_2 e^{\frac{-sx}{c_1}} + Q_1 e^{-hx} + Q_2 e^{-\delta x} \quad (2.333)$$

Form the boundary condition ($x = \infty \Rightarrow \sigma_x = 0$), it yields $C_1 = 0$:

Then,

$$(\bar{\sigma}_x)_g = C_2 e^{\frac{-sx}{c_1}} + \frac{N_1}{h^2 - \frac{s^2}{c_1^2}} e^{-hx} + \frac{N_2}{\delta^2 - \frac{s^2}{c_1^2}} e^{-\delta x} \quad (2.334)$$

Substituting for N_1 and N_2 in Eq. 2.334 results:

$$(\bar{\sigma}_x)_g = C_2 e^{\frac{-\alpha x}{c_1}} + g_1(s) e^{-hx} + g_2(s) e^{-\delta x} \quad (2.335)$$

where:

$$g_1(s) = \frac{-I_1 \delta^2 c_2 s^2}{k(s + \beta) h (h^2 - \delta^2) \left(h^2 - \frac{s^2}{c_1^2} \right)} \quad (2.336)$$

and

$$g_2(s) = \frac{I_1 \delta c_2 s^2}{k(s + \beta) (h^2 - \delta^2) \left(\delta^2 - \frac{s^2}{c_1^2} \right)} \quad (2.337)$$

Now substituting for $h = s/\alpha$ and simplifying the expressions for $g_1(s)$ and $g_2(s)$ yields:

$$g_1(s) = \frac{I_1 \delta^2 c_2 \sqrt{s \alpha} c_1^2}{k(s + \beta) h (s - \alpha \delta^2) \left(s - \frac{c_1^2}{\alpha} \right)} \quad (2.338)$$

and

$$g_2(s) = \frac{-I_1 \delta c_2 s^2 c_1^2}{k(s + \beta) (s - \alpha \delta^2) (s^2 - c_1^2 \delta^2)} \quad (2.339)$$

Rearranging the two expressions as:

$$g_1(s) = C_3 \left[\frac{\sqrt{s}}{(s + \beta) h (s - \alpha \delta^2) (s - c_1^2/\alpha)} \right] \quad (2.340)$$

and

$$g_2(s) = C_4 \left[\frac{s^2}{(s + \beta) (s - \alpha \delta^2) (s - c_1 \delta) (s + c_1 \delta)} \right] \quad (2.341)$$

where:

$$C_3 = \frac{I_1 \delta^2 c_2 \sqrt{\alpha} \alpha c_1^2}{k} \quad (2.342)$$

and

$$C_4 = \frac{-I_1 c_2 c_1^2 \delta \alpha}{k} \quad (2.343)$$

Using partial fraction, the following relations can be obtained:

$$g_1(s) = C_3 \left[\frac{\sqrt{s}}{(\alpha\delta^2 + \beta)(\alpha\delta^2 - c_1^2/\alpha)(s - \alpha\delta^2)} + \frac{\alpha^2\sqrt{s}}{(c_1^2 + \alpha\beta)(c_1^2 - \alpha^2\delta^2)(s - c_1^2/\alpha)} \right. \\ \left. + \frac{\sqrt{s}}{(\beta + \alpha\delta^2)(\beta + c_1^2/\alpha)(s + \beta)} \right] \quad (2.344)$$

and

$$g_2(s) = C_4 \left[\frac{\alpha\delta^2}{(\alpha\delta^2 + \beta)(\alpha^2\delta^2 - c_1^2)(s - \alpha\delta^2)} + \frac{c_1}{2(c_1\delta + \beta)(c_1 - \alpha\delta)(s - c_1\delta)} \right. \\ \left. + \frac{c_1}{2(c_1\delta - \beta)(c_1 + \alpha\delta)(s + c_1\delta)} + \frac{\beta^2}{(-\beta - \alpha\delta^2)(\beta^2 - c_1^2\delta^2)(s + \beta)} \right] \quad (2.345)$$

Consider the boundary condition at the surface, where:

$$\text{At } x = 0 \Rightarrow \sigma_x = \sigma_o(e^{-bt} - e^{-at}) \quad (2.346)$$

Laplace transformation of the boundary condition gives:

$$\bar{\sigma}_s = \sigma_o \left[\frac{1}{s + b} - \frac{1}{s + a} \right] \quad (2.347)$$

Then, the constant in Eq. 2.334 becomes:

$$C_2 = \sigma_o \left[\frac{1}{s + b} - \frac{1}{s + a} \right] - g_1(s) - g_2(s) \quad (2.348)$$

Then:

$$\bar{\sigma}_x(x, s) = g_1(s)e^{-\sqrt{\frac{s}{\alpha}}x} - g_1(s)e^{\frac{-sx}{c_1}} \\ + g_2(s)e^{-\delta x} - g_2(s)e^{\frac{-sx}{c_1}} + \sigma_o \left[\frac{1}{s + b} - \frac{1}{s + a} \right] e^{\frac{-sx}{c_1}} \quad (2.349)$$

Finding the solution for σ_x in the x and t domain, we should take the inverse Laplace Transform for each term in Eq. 2.349. To perform this, the following designations are introduced:

$$\begin{aligned} \text{Term1} &= g_1(s)e^{-\sqrt{\frac{s}{\alpha}}x} & \text{Term2} &= g_1(s)e^{\frac{-sx}{c_1}} \\ \text{Term3} &= g_2(s)e^{-\delta x} & \text{Term4} &= g_2(s)e^{\frac{-sx}{c_1}} \\ \text{Term5} &= \sigma_o \left[\frac{1}{s + b} - \frac{1}{s + a} \right] e^{\frac{-sx}{c_1}} \end{aligned} \quad (2.350)$$

Consequently, the solution for stress distribution becomes the summation of the inverse Laplace Transforms of the above terms.

The details of Laplace Transformation of the terms are given in Appendix 1.

Therefore, the Laplace inversions of Terms (Term1, Term2, Term3, Term4, Term5) are:

$$\mathfrak{L}^{-1}[\text{Term1}] = C_3 \left\{ \begin{aligned} & \frac{1}{2(\alpha\delta^2 + \beta)(\alpha\delta^2 - c_1^2/\alpha)} \left[\frac{2}{\sqrt{\pi t}} \cdot e^{\frac{-x^2}{4\alpha t}} + \delta\sqrt{\alpha} \cdot e^{\delta^2 \alpha t} \begin{pmatrix} e^{-\delta x} \cdot \text{Erfc}\left(-\delta\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \\ -e^{\delta x} \cdot \text{Erfc}\left(\delta\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \end{pmatrix} \right] \\ & + \frac{\alpha^2}{2(c_1^2 + \alpha\beta)(c_1^2 - \alpha^2\delta^2)} \left[\frac{2}{\sqrt{\pi t}} \cdot e^{\frac{-x^2}{4\alpha t}} + \frac{c_1}{\sqrt{\alpha}} \cdot e^{\frac{c_1^2 t}{\alpha}} \begin{pmatrix} e^{\frac{-c_1 x}{\alpha}} \cdot \text{Erfc}\left(-c_1\sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \\ -e^{\frac{c_1 x}{\alpha}} \cdot \text{Erfc}\left(c_1\sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \end{pmatrix} \right] \\ & + \frac{1}{2(\beta + \alpha\delta^2)(\beta + c_1^2/\alpha)} \left[\frac{2}{\sqrt{\pi t}} \cdot e^{\frac{-x^2}{4\alpha t}} + i\sqrt{\beta} \cdot e^{-\beta t} \begin{pmatrix} e^{-xi\sqrt{\frac{\beta}{\alpha}}} \cdot \text{Erfc}\left(-i\sqrt{\beta t} + \frac{x}{2\sqrt{\alpha t}}\right) \\ -e^{xi\sqrt{\frac{\beta}{\alpha}}} \cdot \text{Erfc}\left(i\sqrt{\beta t} + \frac{x}{2\sqrt{\alpha t}}\right) \end{pmatrix} \right] \end{aligned} \right\} \quad (2.351)$$

and

$$\mathfrak{L}^{-1}[\text{Term2}] = C_3 \left\{ \begin{aligned} & \frac{\frac{1}{\sqrt{\pi(t-x/c_1)}} + \sqrt{\alpha\delta^2} e^{\alpha\delta^2(t-x/c_1)} \text{Erf}\left[\sqrt{\alpha\delta^2(t-x/c_1)}\right]}{(\alpha\delta^2 - c_1^2/\alpha)(\alpha\delta^2 + \beta)} \\ & + \alpha^2 \left(\frac{\frac{1}{\sqrt{\pi(t-x/c_1)}} + \sqrt{\frac{c_1^2}{\alpha}} e^{\frac{-c_1^2}{\alpha}(t-x/c_1)} \text{Erf}\left[\sqrt{\frac{c_1^2}{\alpha}(t-x/c_1)}\right]}{(c_1^2 + \alpha\beta)(c_1^2 - \alpha^2\delta^2)} \right) \\ & + \frac{\frac{1}{\sqrt{\pi(t-x/c_1)}} + i\sqrt{\beta} e^{-\beta(t-x/c_1)} \text{Erf}\left[i\sqrt{\beta(t-x/c_1)}\right]}{(\beta + c_1^2/\alpha)} \end{aligned} \right\} \cdot 1\left(t - \frac{x}{c_1}\right) \quad (2.352)$$

$$\mathfrak{L}^{-1}[\text{Term3}] = C_4 e^{-\delta x} \left\{ \begin{aligned} & \frac{\alpha^2 \delta^2 e^{\alpha\delta^2 t}}{(\alpha\delta^2 + \beta)(\alpha^2 \delta^2 - c_1^2)} + \frac{c_1 e^{c_1 \delta t}}{2(c_1 \delta + \beta)(c_1 - \alpha\delta)} \\ & + \frac{c_1 e^{-c_1 \delta t}}{2(c_1 \delta - \beta)(c_1 + \alpha\delta)} + \frac{\beta^2 e^{-\beta t}}{(-\beta - \alpha\delta^2)(\beta^2 - c_1^2 \delta^2)} \end{aligned} \right\} \quad (2.353)$$

and

$$\mathfrak{L}^{-1}[\text{Term4}] = C_4 \left[\frac{\alpha^2 \delta^2 e^{\alpha \delta^2 (t-x/c_1)}}{(\alpha \delta^2 + \beta)(\alpha^2 \delta^2 - c_1^2)} + \frac{c_1 e^{c_1 \delta (t-x/c_1)}}{2(c_1 \delta + \beta)(c_1 - \alpha \delta)} \right] \cdot 1 \left(t - \frac{x}{c_1} \right) + \frac{c_1 e^{-c_1 \delta (t-x/c_1)}}{2(c_1 \delta - \beta)(c_1 + \alpha \delta)} + \frac{\beta^2 e^{-\beta (t-x/c_1)}}{(-\beta - \alpha \delta^2)(\beta^2 - c_1^2 \delta^2)} \cdot 1 \left(t - \frac{x}{c_1} \right) \quad (2.354)$$

and

$$\mathfrak{L}^{-1}[\text{Term5}] = \sigma_o \left(e^{-b \left(t - \frac{x}{c_1} \right)} - e^{-a \left(t - \frac{x}{c_1} \right)} \right) \cdot 1 \left(t - \frac{x}{c_1} \right) \quad (2.355)$$

where $1 \left(t - \frac{x}{c_1} \right)$ is a unit step function and Erf (y) is the error function of the variable y.

The closed form solution of stress distribution can be written as:

$$\sigma_x(x, t) = \mathfrak{L}^{-1}[\text{Term1}] + \mathfrak{L}^{-1}[\text{Term2}] + \mathfrak{L}^{-1}[\text{Term3}] + \mathfrak{L}^{-1}[\text{Term4}] + \mathfrak{L}^{-1}[\text{Term5}] \quad (2.356)$$

where $\mathfrak{L}^{-1}$ represents the inverse sign of Laplace Transformation.

Presenting the stress distribution in dimensionless form, the additional dimensionless quantities are defined, i.e.:

$$c_1^* = \frac{c_1}{\alpha \delta} \quad (2.357)$$

and

$$\sigma_x^* = \frac{k \sigma_x}{I_1 \delta \alpha^2 c_2 (c_1^*)^2} \quad (2.358)$$

and

$$\sigma_o^* = \frac{k \sigma_o}{I_1 \delta \alpha^2 c_2 (c_1^*)^2} \quad (2.359)$$

U [1] is the unit step function, which is $U[1] = \left(t * -\frac{x^*}{c_1^*} \right)$

Therefore, for the dimensionless stress distribution, the followings are resulted:

$$(\sigma_x^*)_1 = \left\{ \begin{aligned} & \frac{1}{(t^* + \beta^*)(1 - (c_1^*)^2)} \left[\frac{\sqrt{t^*} e^{-\frac{(x^*)^2}{4t^*}}}{\sqrt{\pi}} + \frac{t^*}{2} \begin{pmatrix} e^{-x^*} \cdot \text{erfc}\left(-\sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}}\right) \\ -e^{x^*} \cdot \text{erfc}\left(\sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}}\right) \end{pmatrix} \right] \\ & + \frac{1}{((c_1^*)^2 - 1)((c_1^*)^2 t^* + \beta^*)} \left[\frac{\sqrt{t^*} e^{-\frac{(x^*)^2}{4t^*}}}{\sqrt{\pi}} + \frac{c_1^* t^* e^{-(c_1^*)^2 t^*}}{2} \begin{pmatrix} e^{-c_1^* x^*} \cdot \text{erfc}\left(-c_1^* \sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}}\right) \\ -e^{c_1^* x^*} \cdot \text{erfc}\left(c_1^* \sqrt{t^*} + \frac{x^*}{2\sqrt{t^*}}\right) \end{pmatrix} \right] \\ & + \frac{(t^*)^{3/2}}{(\beta^* + t^*)(\beta^* + (c_1^*)^2 t^*)} \left[\frac{e^{-\frac{(x^*)^2}{4t^*}}}{\sqrt{\pi}} + \frac{i\sqrt{\beta^*} e^{-\beta^*}}{2} \begin{pmatrix} e^{-ix^* \sqrt{\frac{\beta^*}{t^*}}} \cdot \text{erfc}\left(-i\sqrt{\beta^*} + \frac{x^*}{2\sqrt{t^*}}\right) \\ -e^{ix^* \sqrt{\frac{\beta^*}{t^*}}} \cdot \text{erfc}\left(i\sqrt{\beta^*} + \frac{x^*}{2\sqrt{t^*}}\right) \end{pmatrix} \right] \end{aligned} \right\} \quad (2.360)$$

and

$$(\sigma_x^*)_2 = \left\{ \begin{aligned} & \frac{1}{(t^* + \beta^*)(1 - (c_1^*)^2)} \left[\frac{t^*}{\sqrt{\pi} \sqrt{(t^* - x^*/c_1^*)}} + t^* e^{(t^* - x^*/c_1^*)} \cdot \text{Erf}\left(\sqrt{(t^* - x^*/c_1^*)}\right) \right] \\ & \frac{1}{((c_1^*)^2 - 1)((c_1^*)^2 + \beta^*/t^*)} \left[\frac{1}{\sqrt{\pi} \sqrt{(t^* - x^*/c_1^*)}} + c_1^* e^{-((c_1^*)^2 t^* - c_1^* x^*)} \left(\sqrt{(c_1^*)^2 t^* - c_1^* x^*} \right) \right] \\ & + \frac{(t^*)^{3/2}}{(t^* + \beta^*)(\beta^* + (c_1^*)^2 t^*)} \left[\frac{\sqrt{t^*}}{\sqrt{\pi} \sqrt{(t^* - x^*/c_1^*)}} + i\sqrt{\beta^*} e^{-\beta^* (1 - x^*/c_1^* t^*)} \cdot \text{erf}\left(i\sqrt{\beta^* (1 - x^*/c_1^* t^*)}\right) \right] \end{aligned} \right\} \cdot U[1] \quad (2.361)$$

and

$$(\sigma_x^*)_3 = \left\{ \begin{aligned} & \frac{-t^* e^{t^* - x^*}}{(1 - (c_1^*)^2)(t^* + \beta^*)} - \frac{t^* c_1^* e^{c_1^* t^* - x^*}}{2(c_1^* t^* + \beta^*)(c_1^* - 1)} - \frac{t^* c_1^* e^{-c_1^* t^* - x^*}}{2(c_1^* t^* - \beta^*)(c_1^* + 1)} \\ & - \frac{(\beta^*)^2 t^* e^{-\beta^* - x^*}}{(-\beta^* - t^*)((\beta^*)^2 - ((c_1^* t^*)^2))} \end{aligned} \right\} \quad (2.362)$$

and

$$(\sigma_x^*)_4 = \left\{ \begin{aligned} & \frac{-t^* e^{(t^*-x^*/c_1^*)}}{(1-(c_1^*)^2)(t^*+\beta^*)} - \frac{t^* c_1^* e^{c_1^* t^*-x^*}}{2(c_1^* t^* + \beta^*)(c_1^* - 1)} - \frac{t^* c_1^* e^{-(c_1^* t^*-x^*)}}{2(c_1^* t^* - \beta^*)(c_1^* + 1)} \\ & - \frac{(\beta^*)^2 t^* e^{-\beta^*(1-x^*/c_1^* t^*)}}{(-\beta^* - t^*)(\beta^*)^2 - (c_1^* t^*)^2} \end{aligned} \right\} \cdot U[1] \quad (2.363)$$

and

$$(\sigma_x^*)_5 = \sigma_o^* \left(e^{-b \left(t^* - \frac{x^*}{c_1^*} \right)} - e^{-a \left(t^* - \frac{x^*}{c_1^*} \right)} \right) \cdot 1 \left(t^* - \frac{x^*}{c_1^*} \right) \quad (2.364)$$

Consequently, the dimensionless form of stress equation (Eq. 2.356) becomes:

$$\sigma_x^* = (\sigma_x^*)_1 + (\sigma_x^*)_2 + (\sigma_x^*)_3 + (\sigma_x^*)_4 + (\sigma_x^*)_5 \quad (2.365)$$

Equations 2.314 and 2.365 are used to compute the dimensionless temperature and stress distributions inside the substrate material.

2.4 Entropy Analysis Due to Thermal Stress Field

Entropy analysis provides information on thermodynamic irreversibility in the thermal system. It also aids to quantify the loss work in the system because of the heat transfer. Although the formulation of the entropy generation rate for laser pulse heating is the same for different pulse types, the amount of entropy generation rate becomes different for different pulses. Consequently, the general formulation of entropy generation rate is given below in line with the previous studies [6–8].

Entropy generation in the solid substrate can be written as (2.10):

$$\frac{DS}{Dt} = \nabla \cdot \left(\frac{\bar{q}}{T} \right) \quad (2.366)$$

or

$$\frac{DS}{Dt} = \frac{1}{T} \nabla \cdot \bar{q} - \frac{\bar{q}}{T^2} \nabla T \quad (2.367)$$

Consider the relation:

$$h = h(T, e) \quad (2.368)$$

or

$$dh = \left(\frac{\partial h}{\partial t} \right)_e dT + \left(\frac{\partial h}{\partial e} \right)_T de \quad (2.369)$$

However, the term $\left(\frac{\partial h}{\partial e} \right)_T$ can be written as [11]:

$$\rho \left(\frac{\partial h}{\partial e} \right)_T = C_k(T) \cong \left(\frac{\partial C_k}{\partial T} \right)_{T_o} (T - T_o) + \dots \quad (2.370)$$

where C_k is the heat capacity and its value at reference temperature is assumed to be zero. If $(T - T_o)$ is the almost the same order of T_o , then:

$$C_k(T) = \left(\frac{\partial C_k}{\partial T} \right)_{T_o} T_o \quad (2.371)$$

It is known that:

$$\left(\frac{\partial h}{\partial T} \right)_e \equiv C_p \quad (2.372)$$

Therefore:

$$\rho dh = \rho C_p dT + \left(\frac{\partial C_k}{\partial T} \right)_{T_o} T_o de \quad (2.373)$$

Furthermore, it was shown that [11]:

$$\left(\frac{\partial C_k}{\partial T} \right)_{T_o} = \frac{3E}{1 - 2\nu} \alpha_T \quad (2.374)$$

where E is the Young's modulus and α_T is the thermal expansion coefficient. Therefore, Eq. 2.369 can be rearranged as:

$$\rho \frac{dh}{dt} = \rho C_p \frac{dT}{dt} + \frac{3E}{1 - 2\nu} \alpha_T T_o \frac{de}{dt} \quad (2.375)$$

It was also shown that [11]:

$$\nabla \cdot \vec{q} = \rho \frac{dh}{dt} \quad (2.376)$$

Therefore Eq. 2.375 becomes:

$$\nabla \cdot \vec{q} = \rho C_p \frac{dT}{dt} + \frac{3E}{1 - 2\nu} \alpha_T T_o \frac{de}{dt} \quad (2.377)$$

Inserting Eq. 2.377 into Eq. 2.367 and knowing that $q = -k \frac{\partial T}{\partial x}$, it yields:

$$\frac{DS}{Dt} = \frac{1}{T} \left[\frac{k}{T} (\nabla T)^2 + \rho C_p \frac{dT}{dt} \right] + \frac{3E}{1-2\nu} \alpha_T \frac{T_o}{T} \frac{de}{dt} \quad (2.378)$$

However, the strain can be written in terms of displacement (U), i.e.: $e = \frac{\partial U}{\partial x}$. Therefore Eq. 2.378 becomes:

$$\frac{DS}{Dt} = \frac{1}{T} \left[\frac{k}{T} (\nabla T)^2 + \rho C_p \frac{dT}{dt} \right] + \frac{3E}{1-2\nu} \alpha_T \frac{T_o}{T} \frac{d}{dt} \left(\frac{\partial U}{\partial x} \right) \quad (2.379)$$

In the non-dimensional form:

$$\frac{DS^*}{Dt^*} = \frac{1}{T^*} \left[\frac{1}{T^*} (\nabla T^*)^2 + \frac{dT^*}{dt^*} \right] + \frac{3E\alpha_T}{\rho C_p (1-2\nu)} \frac{T_o^*}{T^*} \frac{d}{dt^*} \left(\frac{\partial U^*}{\partial x^*} \right) \quad (2.380)$$

where

$$U^* = U\delta \text{ and } S^* = \frac{S}{\rho C_p \alpha \delta^2} \quad (2.381)$$

The first term on the right hand side of Eq. 2.380 represents the entropy generation due to temperature field while the second term is entropy generation due to stress field.

Equation 2.381 is used to compute the entropy generation due to temperature and stress fields.

The lost work can be evaluated using a Gouy-Stodola theorem [10]. In this case, the lost work can be written as:

$$W_{lost}^* = \int T_o^* dS^* \quad (2.382)$$

where W_{lost}^* is the dimensionless lost work and T_o^* is the reference temperature. Equation 2.383 is used to compute the lost work.

In order to formulate the displacement, the relation between the stress field and the displacement needs to be established. However, the thermal strain along the x-axis can be written as [11]:

$$\varepsilon_x = \frac{(1+\nu)(1-2\nu)}{(1-\nu)E} \sigma_x + \frac{1+\nu}{1-\nu} \alpha_T \Delta T \quad (2.383)$$

where ν , E , and α_T are Poisson's ratio, elastic module, and thermal expansion coefficient of a solid material, respectively. The equation of motion in a solid substrate in the one-dimensional form can be written as:

$$\frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial V}{\partial t} \quad (2.384)$$

where V is the velocity, which can be written in terms of displacement (U) as:

$$V = \frac{\partial U}{\partial t} \quad (2.385)$$

The equation of motion becomes:

$$\frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial^2 U}{\partial t^2} \quad (2.386)$$

Therefore, Eq. 2.386 gives the relation between the thermal stress and the displacement, which is needed for the entropy calculations.

2.5 Findings and Discussions

The findings of the thermal stress developed in the laser irradiated region is presented below under the appropriate sub-headings in line with the previous studies [1–8].

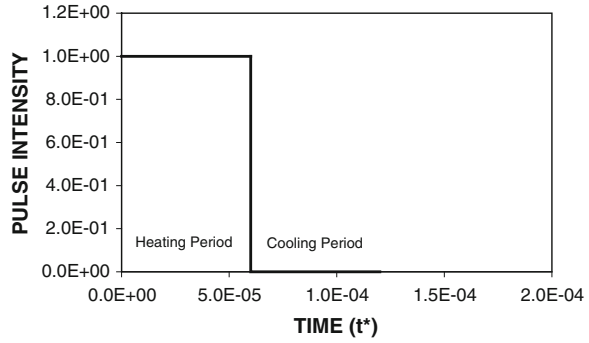
2.5.1 Step Input Pulse Heating

2.5.1.1 Stress Free Boundary Condition at the Surface

A closed form solution for the stress field is obtained using the Laplace transform method for a single and two successive step input pulses. However, the analytical solution obtained is limited with the free stress condition at the surface, which is the most common situation in laser machining applications. The stress field in the solid phase of the substrate material is modeled although the liquid and gas phases are developed during the laser high intensity irradiation. This is because of the fact that the thermal strain generated in these phases is insignificant due to the free molecular activity in these phases. The elastic stress field is considered because of minimizing the complications associated with the non-linear equation of motion when the stress level exceeds the elastic limit of the substrate material. The non-dimensional closed form solution for the stress field can be applicable to any material, provided that the laser pulse shape is in the step input form. In line with the previous study [1–3], the two pulse modes are considered in the analysis, which are a single step input pulse and two successive step input pulses as shown in Figs. (2.1) and (2.2) while Table 2.1 gives the laser properties used in the simulations.

Figures (2.3) and (2.4) show non-dimensional temperature profile inside the substrate material for three different heating periods for single and two successive step input pulses. The differences in temperature profiles appear in the surface

Fig. 2.1 Laser step input pulse used in the analysis and in the simulations



region where, the magnitude of temperature is high and the temperature gradient is small. The small temperature gradient in the surface region results from the energy gain in the substrate material from the irradiated field. It should be noted that the amount of the absorbed laser power varies exponentially with the depth inside the substrate material (Beer Lambert's law), i.e. it is maximum at the surface and decays exponentially with increasing depth. Consequently, the energy gain from the irradiated field dominates the diffusional energy transport from the surface vicinity to the solid bulk. It should be noted that the diffusional energy transport enhances as the temperature gradient increases. Moreover, as the depth below the surface increases, the temperature gradient becomes large due to a sharp change in temperature in this region and smaller amount of absorbed energy from the irradiated field with increasing depth. Therefore, diffusional energy transport dominates internal energy gain of the substrate material through absorption in this region. Since the time domain selected is close to each other, the temperature variation along the depth is almost similar, particularly at some depth below the surface. It should be noted that the selection of a close time domain is due to examination of stress wave propagation in the substrate material, i.e. a long time domain obscures the appearance of the stress wave within the surface region, since it has a considerably high speed. In the case of the heating situation for two successive pulses, some variation in temperature profiles in the surface region occurs at different heating durations. As the heating duration increases ($t^* = 0.003$), the influence of the second pulse on the temperature rise is more pronounced in the surface region. This, in turn, elevates the temperature rise at the surface.

Figure (2.5) shows non-dimensional stress inside the substrate material for three heating periods while Fig. (2.6) shows non-dimensional stress distribution in the region close to the surface. Thermal stress is compressive in the region of the surface and becomes tensile with increasing depth from the surface. This situation can be seen from Fig. (2.6). As the depth increases beyond the absorption depth ($x^* = 1$), it decays sharply. However, the location at which the thermal stress reaches its first peak varies with the heating period. This is because of the stress wave propagation inside the substrate material. The depth at which the wave

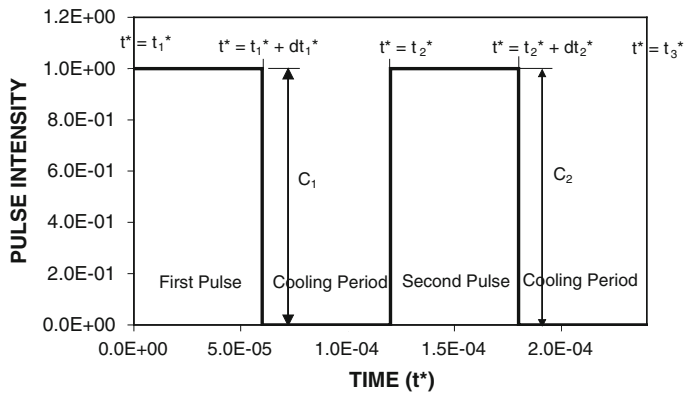
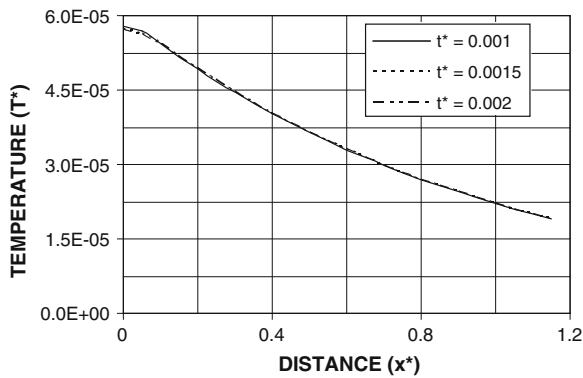


Fig. 2.2 Two successive dimensionless laser step input pulses used in the analysis and in the simulations ($dt_1^* = dt_2^*$)

Table 2.1 Laser pulse properties used in the simulations

	First pulse	Second pulse
Dimensionless pulse length ($\tau\alpha\delta^2$, τ is the pulse length)	0.00006	0.00006
Dimensionless intensity ($\frac{I_o}{k\delta}$, I_o laser peak power intensity)	1	1
Dimensionless cooling period between pulses (θ_1 and θ_2)	0.00006	0.00006

Fig. 2.3 Dimensionless temperature profiles inside the substrate material obtained at three different dimensionless heating periods for one step input pulse



reaches its full amplitude depends on the wave speed and the duration of the wave that travels. Consequently, longer the heating period results in the deeper the location at which wave reaches its maximum amplitude.

Fig. 2.4 Dimensionless temperature profiles inside the substrate material obtained at three different dimensionless heating periods for two successive pulses

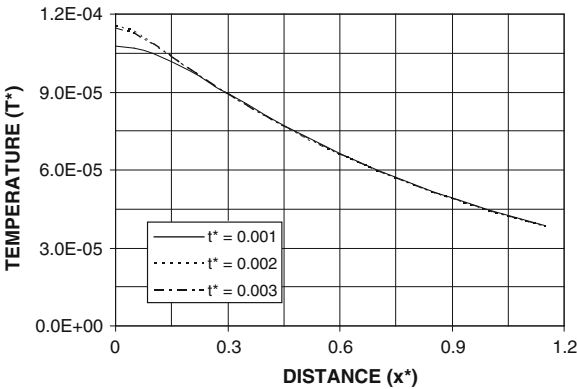


Fig. 2.5 Dimensionless thermal stress inside the substrate material for different dimensionless heating periods for single pulse

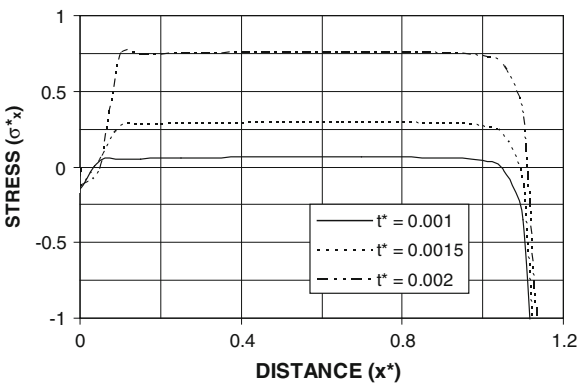
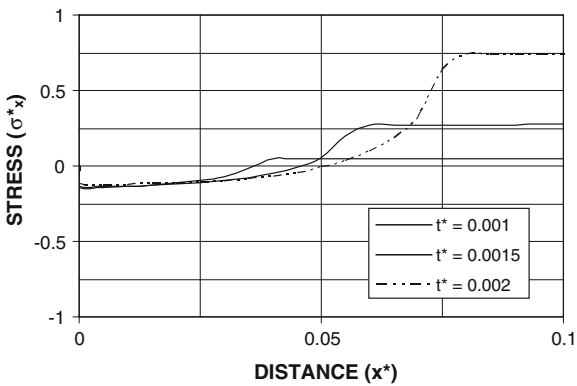


Fig. 2.6 Dimensionless thermal stress in the surface vicinity of the substrate material for different dimensionless heating periods for single pulse



2.5.1.2 Stress Continuity Boundary at the Surface

Figure (2.7) shows dimensionless stress levels inside the substrate material at different heating periods. Stress field inside substrate material is compressive. This is due to the stress boundary condition introduced at the surface ($(\partial T^*/\partial x^*)|_{x^*=0} = 0$). The stress level is high in the surface vicinity and as the distance increases it reduces sharply. Moreover, the stress level reduces to zero at some distance below the surface. The depth, where the zero stress occurs, increases with progressing time. This indicates that the stress wave propagates into the substrate material with a wave speed c_I .

Figure (2.8) shows the temporal variation of dimensionless stress field at different depths inside the substrate material. The stress level is considerably high at the surface and as the depth increases, the stress level reduces. Moreover, the magnitude of stress level inside the substrate material rises after certain period from the pulse beginning. This indicates that the stress wave propagates into the substrate material. However, the intensity and shape of the stress wave is modified by the stress boundary condition at the surface, i.e. stress wave is compressive and its magnitude increases with progressing time.

2.5.2 Time Exponentially Varying Laser Pulse Heating

2.5.2.1 Stress Free Boundary at the Surface

Figure (2.9) shows temporal variation of dimensionless temperature at different locations inside the substrate material. Temperature rises rapidly in the early heating period and the rate of temperature rise reduces as the heating progresses. This occurs because of the energy gain and energy transport mechanisms inside the

Fig. 2.7 Dimensionless stress distribution inside the substrate material for different heating periods

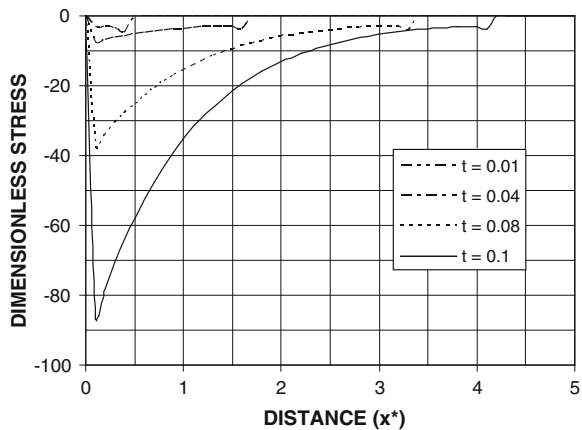


Fig. 2.8 Temporal variation of dimensionless stress at different locations inside the substrate material

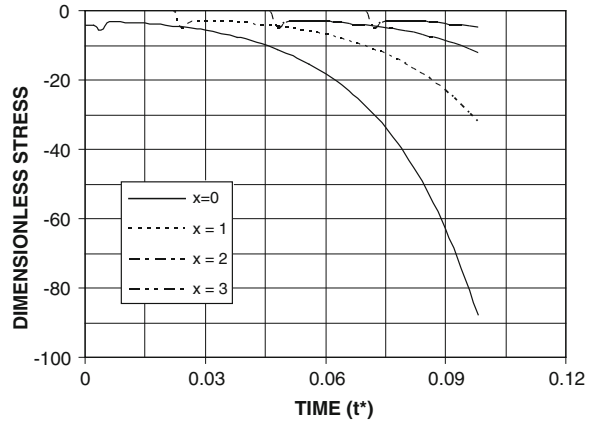
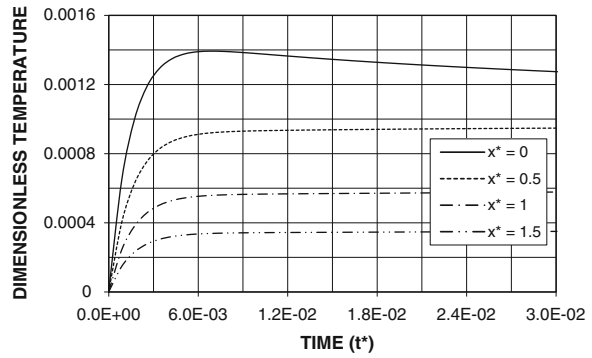
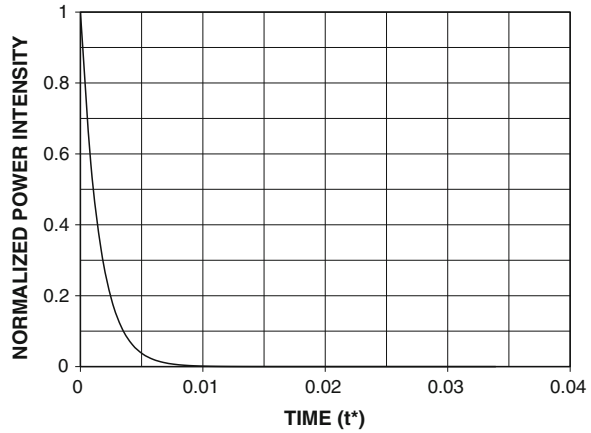


Fig. 2.9 Temporal variation of dimensionless temperature at different locations inside the substrate material



substrate material. In this case the internal energy gain by the substrate material from the irradiated field dominates over the conduction losses in the early heating period. As the heating period progresses internal energy gain and temperature gradient in the surface region increase. Consequently, a stage is reached when the conduction losses from the surface region to the solid bulk becomes important due to high temperature gradient developed in this region with progressing time. Since the pulse intensity decreases with time as shown in Fig. (2.10), temperature also decreases with further progressing heating time. Therefore, temperature curve decays after reaching its maximum value. The rise of temperature at some depth below the surface is not as high as that corresponding to the surface. This occurs because of the energy absorbed by the substrate material at some depth below the surface, i.e. energy absorbed decreases exponentially with increasing depth (Lambert's law). As the heating progresses further ($t^* > 6 \times 10^{-3}$) temperature in the surface region ($x^* < 1$) reduces while at some depth below the surface ($x^* \geq 1$) increases gradually with heating period. This occurs because of the diffusional heating of the substrate material at some depth below the surface due to high temperature gradient in this region. In the region limited by the absorption

Fig. 2.10 Normalized power intensity distribution

depth ($x^* < 1$) energy gain by the substrate material reduces considerably with further progressing time because of time exponentially decaying pulse intensity. However, high temperature gradient enhances the energy loss from this region to solid bulk, i.e. conduction losses dominates over the internal energy gain of the substrate material. This, in turn, results in reducing temperature in this region with further progressing time.

Figure (2.11) shows temporal variation of dimensionless stress at different locations inside substrate material. The occurrence of peak stress at different periods indicates the propagation of the wave inside the substrate material, i.e. location $x^* = 0.1$, the peak stress occurs at about $t^* = 0.3 \times 10^{-2}$ while at $x^* = 1.5$ it occurs at about $t^* = 2.4 \times 10^{-2}$. The magnitude of maximum stress is high at $x^* = 0.5$. This is because of the development of the maximum temperature gradient at this location at $t^* \cong 1.25 \times 10^{-2}$. The stress level decreases with increasing time and as the time progresses further it decays gradually.

2.5.2.2 Stress Free Boundary and Convection at the Surface

The closed form solution for the stress distribution due to an exponentially decaying pulse with time and the convective boundary condition at the surface have been derived. Steel was employed to simulate the temperature and stress fields. The laser pulse parameter (β) and substrate material properties are given Table 2.2. Figure (2.12) shows the laser pulse used in the simulations.

Figure (2.13) shows the dimensionless temperature distribution within the substrate material for various dimensionless heating periods. The influence of the heat transfer coefficient on the temperature distribution becomes significant when the dimensionless heat transfer coefficient reaches $h' \geq 2.02 \times 10^{-2}$ ($\approx 10^8 \text{ W/m}^2\text{K}$). In this case, the temperature and its gradient in the surface region are reduced. This can also be seen from Fig. (2.14), in which the temperature

Fig. 2.11 Temporal variation of dimensionless stress distribution at different locations inside the substrate material

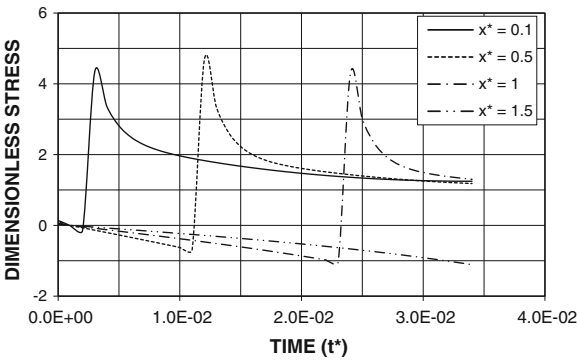


Table 2.2 Material properties and laser pulse parameter used in the simulations

ρ (kg/m ³)	δ (1/m)	α (m ² /s)	α_T (1/K)	ν	E (Pa)	Cp (J/kgK)	k (W/mK)	β (1/s)
7930	6.16×10^6	3.7×10^{-5}	1.6×10^{-5}	0.29	2.10×10^{11}	510	150	1.53×10^{11}

gradient is shown. The temperature gradient in the surface region is reduced to its minimum. At the point of minimum temperature gradient, the internal energy gain by the substrate from the irradiated area is balanced by the diffusional energy transport from the substrate to the solid bulk. In this case, the depth beyond the point of minimum temperature gradient diffusional energy transport dominates over the internal energy gain of the substrate material due to absorption of irradiated field. The point of minimum temperature gradient changes with the heat transfer coefficient, which is more pronounced for the heating period of 0.021. Moreover, the sharp decay in the temperature gradient in the surface region ($x' \leq 0.1$) is because of: (i) the absorption process, i.e. the absorbed energy decreases exponentially with increasing depth (Beer Lambert’s law), and (ii) the internal energy gain in the surface region is high and diffusional energy transport due to the temperature gradient from this region to the solid bulk is low, i.e. the increase in temperature due to diffusional energy transport in the neighboring region is low; therefore, the temperature profile is governed by the internal energy gain in this region.

Figure (2.15) shows the dimensionless stress distribution within the substrate material for different dimensionless heat transfer coefficients and times. The thermal stress is zero at the surface as a result of the surface boundary condition used in the analysis and it increases sharply close to the surface. The thermal stress is tensile in this region due to expansion of the surface. As the depth increases ($x' > 0.06$), the stress becomes compressive, as a result of the thermal strain developed in this region, i.e., at this depth and beyond the material contracts resulting in a compressive thermal stress field. The influence of the heat transfer coefficient on the stress development is considerable, as illustrated by

Fig. 2.12 Laser pulse used in the simulations

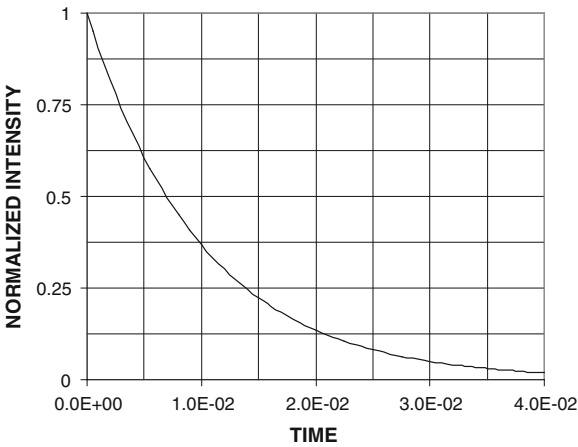


Fig. 2.13 Dimensionless temperature distributions within the substrate material

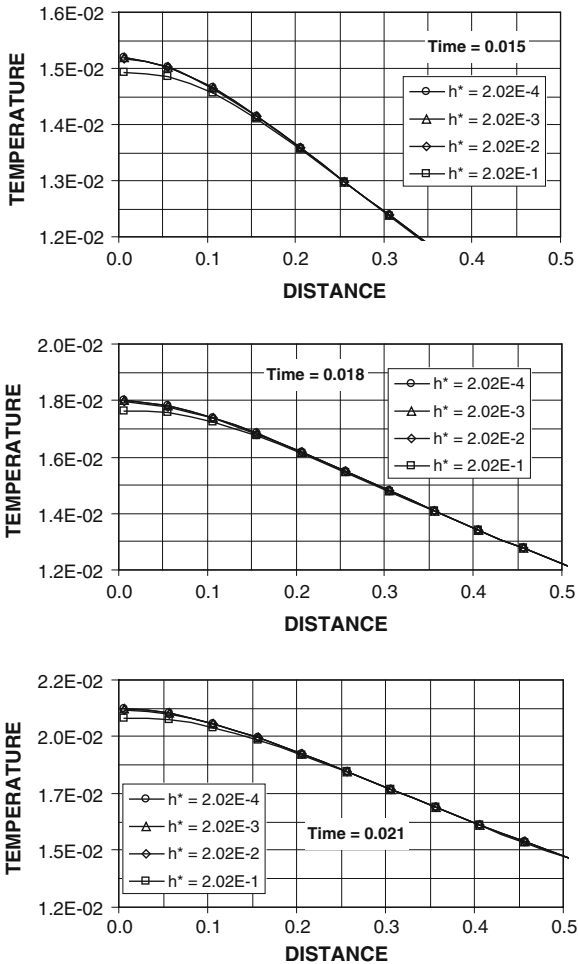
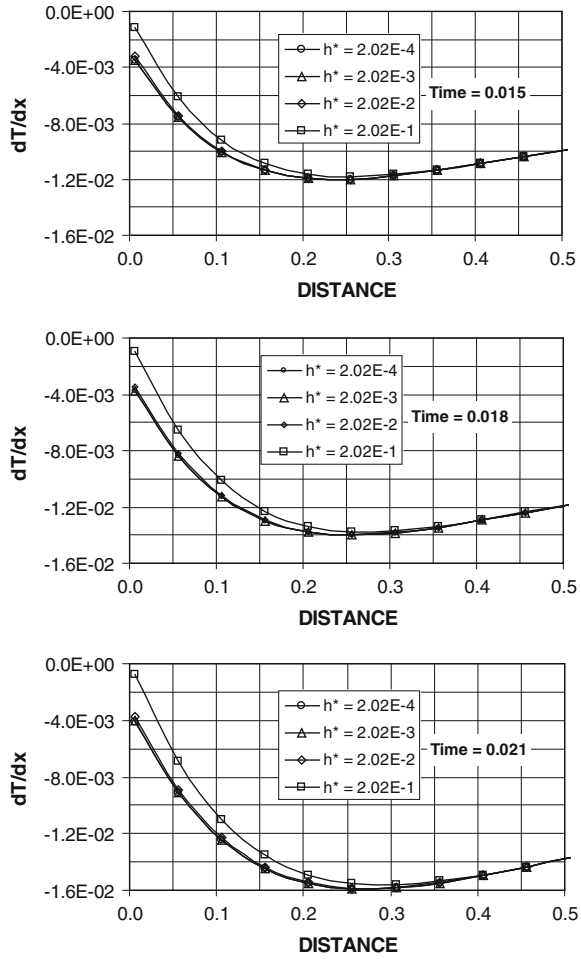


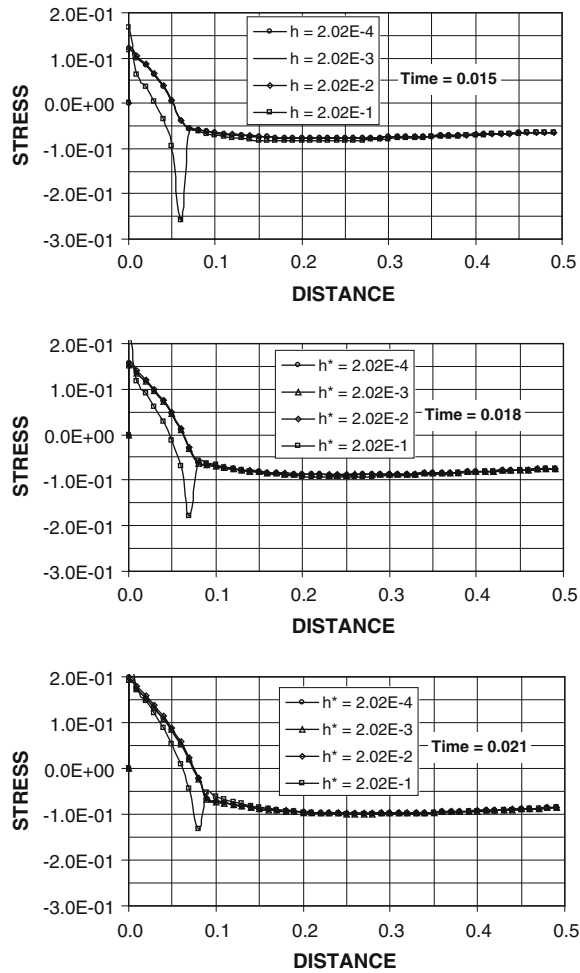
Fig. 2.14 Dimensionless temperature gradient within the substrate material



$h = 0.0202(10^9 \text{ W/m}^2\text{K})$. In this case, the stress developed is compressive and with a high magnitude in the vicinity of the surface and decays sharply as the depth increases. However, the compressive stress wave is developed at some point below the surface. The magnitude of the stress wave is lower at this point as time progresses. In addition, the magnitude of the thermal stress levels, corresponding to a heat transfer coefficient other than $h = 0.0202$, increases with an increase in time, provided that this increase is less than 10 %.

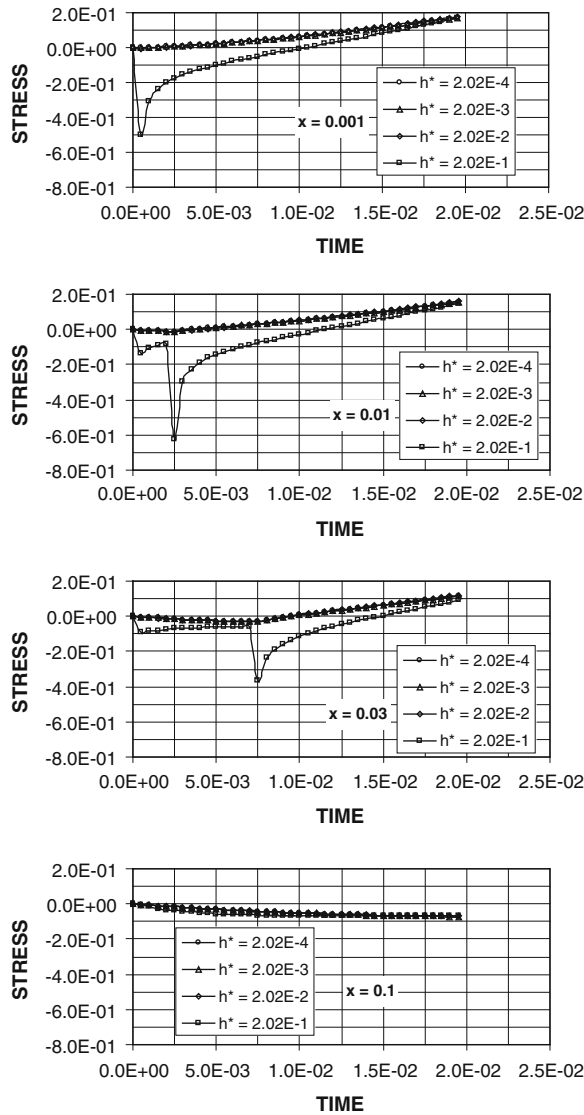
Figure (2.16) shows the change in thermal stress for different dimensionless heat transfer coefficients with time. Since the substrate material is considered initially to be stress free, i.e. the magnitude of the stress level is zero at time $t = 0$. The stress level is tensile in the vicinity of the surface ($x' = 0.001$). This is because of the free expansion of the surface, in which case the strain becomes

Fig. 2.15 Dimensionless stress distribution within the substrate material



positive. However, the thermal stress becomes compressive as the depth increases beyond $x = 0.01$. This occurs as a result of the compression of the substrate material. The effect of the heat transfer coefficient on the stress distribution behavior with time is significant for a heat transfer coefficient of $h = 0.0202$. In this case, the temperature gradient in the surface region differs significantly from those corresponding to the other heat transfer coefficients. This, in turn, generates a compressive stress wave propagating into the material. The magnitude of the stress wave is reduced as it propagates into the substrate material. Moreover, the stress wave behavior disappears at a depth of 0.1, i.e. the stress behavior beyond this depth changes significantly. This shows that the stress wave does not attain a depth of $x = 0.1$ during the time interval considered in the present study.

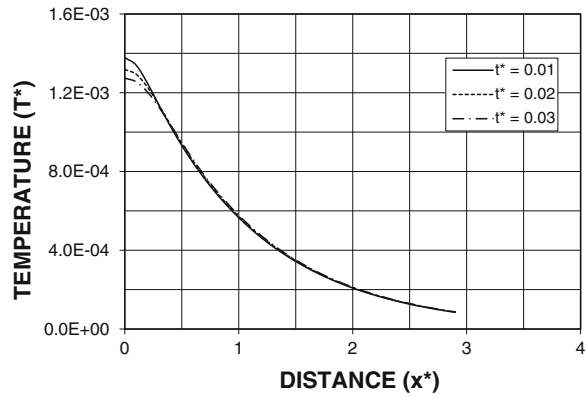
Fig. 2.16 Change in of dimensionless stress distribution with time



2.5.2.3 Stress Boundary at the Surface

Figure (2.17) shows dimensionless temperature distribution inside the substrate material at different heating periods. The decay of temperature in the surface region is slower as compared to some depth below the surface. This occurs because of the insulated boundary condition at the surface as well as the internal energy gain from the irradiated field in this region. In this case, the amount of energy absorbed from the irradiated field in the surface region is higher than that

Fig. 2.17 Dimensionless temperature distributions inside the substrate material for different time



corresponding to some depth below the surface due to the Lambert's law of absorption, i.e., energy absorbed by the substrate reduces exponentially with distance away from the surface. Consequently, energy absorption enhances the internal energy gain in the surface region, which dominates over the conduction energy transfer from this region to the solid bulk. However, as the depth below the surface increases further, amount of energy absorbed reduces and conduction losses dominates the internal energy gain of the substrate material. Hence, the temperature gradient changes considerably in this region.

Figure (2.18) shows stress levels inside the substrate material at different periods. It is evident that the stress wave propagates into the substrate material with a wave speed. The magnitude of peak stress reduces as the depth below the surface increases towards the solid bulk. This indicates that material dumps gradually the magnitude of stress level as the wave propagates towards the solid bulk. Moreover, the time occurrence of peak stress inside the substrate material differs for the stress free condition and stress condition at the substrate surface. This is because of the temporal variation of the stress distribution at the surface and the temporal variation of laser pulse intensity distribution, which differ, i.e. pulse intensity decays with a progressing time while stress distribution at the substrate increases to reach its peak, then decreases with increasing time as shown in Fig. (2.18).

2.5.3 Entropy Analysis Due to Thermal Stress Field

2.5.3.1 Step Input Pulse

Figure (2.19) shows dimensionless entropy generation due to temperature field at different heating periods. Entropy generation is low in the surface region and it increases steadily with increasing depth from the surface. The increase in entropy

Fig. 2.18 Dimensionless stress distribution inside the substrate material at different times for stress free and stress conditions at the surface

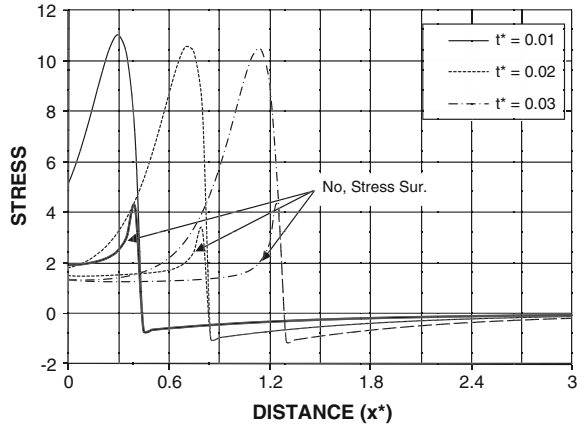
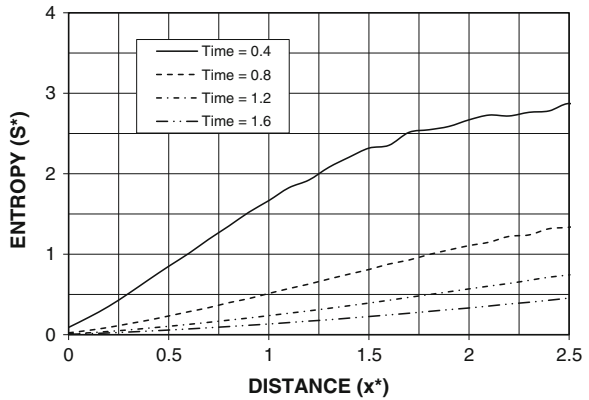


Fig. 2.19 Dimensionless entropy generation profiles due to temperature field inside the substrate material at different heating periods



can be explained in terms of entropy equation. In this case, term $(\nabla T^*)^2$ increases in magnitude in the region next to the surface vicinity (Fig. (2.20)). However, temperature is less in this region next to the surface vicinity. Consequently, increase in entropy due to term $(\nabla T^*)^2$ is enhanced by the term $(\frac{1}{T^2})$, i.e. $\frac{1}{T^2}(\nabla T^*)^2$ increases with depth. Moreover, in the region further away from the surface $(\nabla T^*)^2$ does not attain high values as much as in the surface region, but term $(\frac{1}{T^2})$ increases considerably due to low temperature in this region. Similarly, the term $(\frac{1}{T^2} \frac{dT^*}{dt^*})$ increases with increasing depth, since $(\frac{1}{T^2})$ attains higher values as compared to term $(\frac{1}{T^2} \frac{dT^*}{dt^*})$. Therefore, entropy generation due to temperature field increases with depth. Moreover, entropy generation in the early heating period is high. In this case, rapid rise of temperature in the surface region results in high rate of entropy generation in the early heating period.

Fig. 2.20 Dimensionless temperature gradient inside the substrate material at different dimensionless heating periods

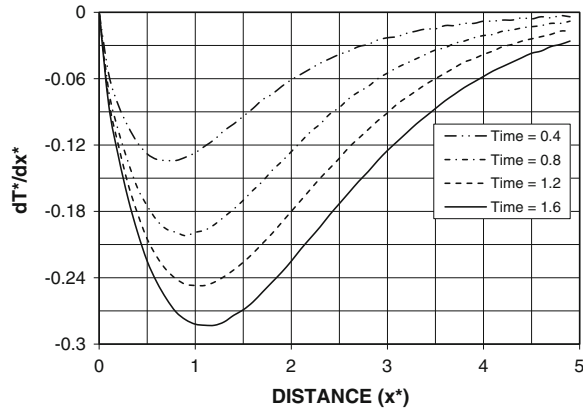
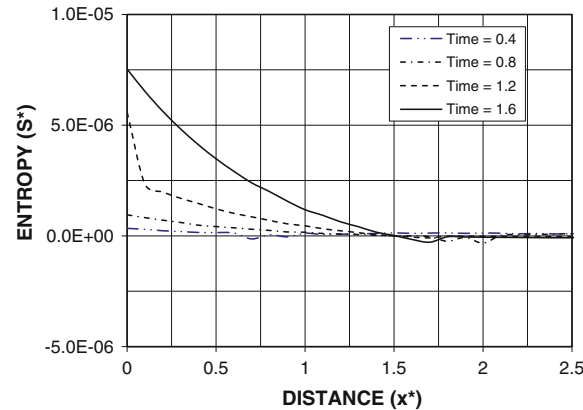


Figure (2.21) shows dimensionless entropy generation due to stress field at different heating periods. Entropy generation in the surface region is high and as the distance increases from the surface it reduces, particularly sharply for the long heating period ($t^* = 1.6$). Attainment of high entropy generation in the surface region is because of high magnitude of surface displacement in this region. Entropy contribution of displacement is presented by the term $(\frac{d}{dr}(\frac{\partial U^*}{\partial x^*}))$ in entropy equation. High entropy generation in the surface region is high values of the term $(\frac{d}{dr}(\frac{\partial U^*}{\partial x^*}))$ in this region. When comparing the magnitude of entropy generation due to stress field with its counterpart corresponding to temperature field, it is evident that entropy generation due to temperature field is considerably high, i.e. the ratio of entropy is in the order of 10^{-5} . This is because of the small displacement of the surface during the heating process.

Fig. 2.21 Dimensionless entropy generation profiles due to stress field inside the substrate material at different heating periods



2.5.3.2 Time Exponentially Decaying Pulse

Figure (2.22) shows the dimensionless volumetric entropy generation inside the substrate material due to temperature field. Entropy generation is high in the surface region and reduces sharply as the depth increases from the surface. The sharp decay of the entropy generation in the region next to the surface vicinity is because of the temperature gradient, which varies significantly in this region. Moreover, entropy generation due to temperature field is influenced considerably by the energy diffusion $\left(\frac{k}{T^*}(\nabla T^*)^2\right)$ as well as by the energy storage $\left(\frac{\partial T^*}{\partial t^*}\right)$. Consequently, entropy profiles do not exactly follow the square of temperature gradient, i.e. the influence of energy storage on entropy generation is none zero. It was shown that at some depth below the surface energy balance attains among the energy absorbed from the irradiated field, internal energy gain of the substrate material and the diffusional energy transport from the surface region to the solid bulk. Consequently, entropy generation reduces to minimum in the region close to the location of the equilibrium point, i.e. an equilibrium state is reached. As the distance increases from the point of the equilibrium state, entropy generation increases sharply. In this case, the diffusional energy transport dominates the internal energy gain due to the energy absorbed from the irradiated field. Consequently, the decay of temperature gradient changes in this region. As the depth increases further towards to solid bulk, entropy generation attains a steady value, i.e. a steady state is reached. As the heating period progresses, the magnitude of entropy generation reduces; in which case, the laser power intensity reduces with progressing time. Moreover, the location of minimum entropy generation moves away from the surface as heating period progresses.

Figure (2.23) shows the dimensionless volumetric entropy generation inside the substrate material due to stress field at different heating periods. Although the entropy generation does not follow the stress distribution (Fig. 2.24), the location of peak entropy generation coincides with the location of the peak stress inside the substrate material. The negative sign of the entropy generation rate at some depth below the surface is because of the negative magnitude of stress field (compressive

Fig. 2.22 Dimensionless entropy generation inside the material due to temperature field at different heating periods

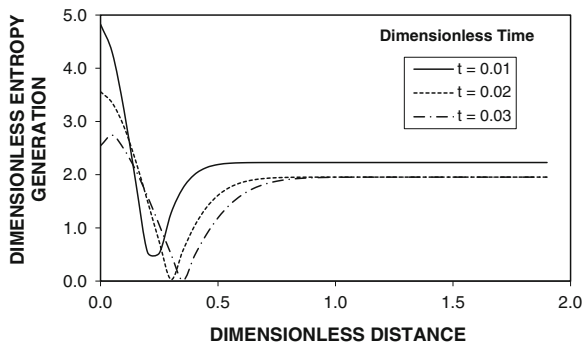


Fig. 2.23 Dimensionless entropy generation inside the material due to stress field at different heating periods

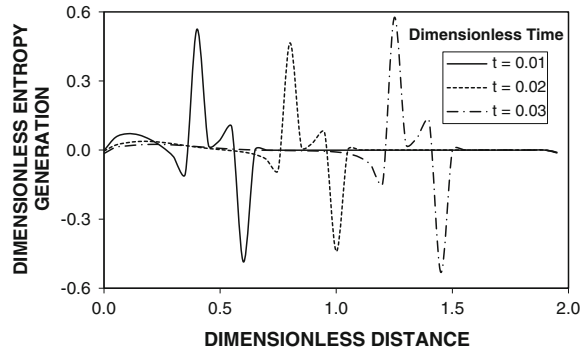
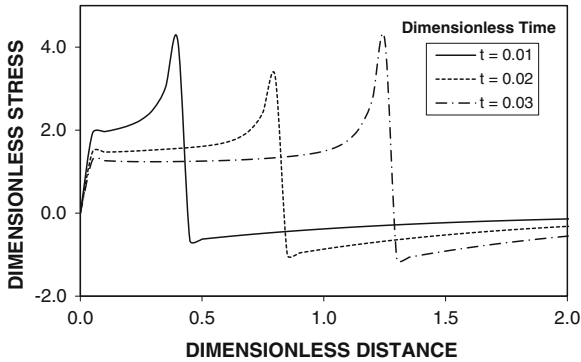


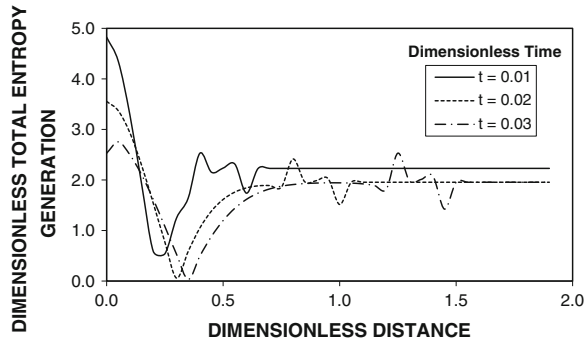
Fig. 2.24 Dimensionless thermal stress inside the material at different heating periods



stress field), which is due to the propagation of the stress wave and, in all cases, the entropy generation is positive. It should be noted that the negative sign in the entropy generation curve is the indication of direction of the stress field and, in all cases, the entropy generation is positive. The entropy generation inside the substrate material appears like a cyclic with time. This is because of the stress behavior, which occurs in the wave form. It should be noted that since the substrate material is assumed to be stress free at initial state and elastic in nature, once the thermal field diminishes it returns the initial state. Consequently, the final net balance of entropy generation should be zero. This situation can be seen from Fig. (2.23), i.e. the final net balance of entropy generation is zero.

Figure (2.25) shows the dimensionless total entropy generation inside the substrate material, due to temperature and stress fields, at different heating periods. Entropy profile in the surface region follows the entropy profile corresponding to the temperature field. This indicates that entropy generation in this region is dominated by the temperature field and the entropy generation due to stress field is negligibly small. As the depth increases further towards the solid bulk, entropy generation due to stress fields become important. In this case, the magnitude of

Fig. 2.25 Dimensionless total entropy generation inside the material at different heating periods



entropy generation due to temperature field becomes less than the peak values of the entropy generation due to stress field. This, in turn, results in cyclic appearance on the entropy curve in the region at some depth below the surface.

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Analysis

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