Chapter 2
Hypergeometric Series with Applications

Hypergeometric function arises in connection with solutions of the second order ordinary differential equations with several regular singular points. Although Gamma function does not satisfy any differential equation with algebraic coefficients, a detailed discussion would give us an opportunity to introduce all necessary notation as well as a number of very useful tools.

2.1 The Gamma Function

2.1.1 Definition and Recurrence Relation

One way to define the Gamma function is via the integral representation also referred to as *Euler integral of the second kind*:

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}. \tag{2.1}$$

By elementary integration we immediately find that

$$\Gamma(1) = \int_0^\infty dt\, e^{-t} = 1.$$

We observe that integral (2.1) is absolutely convergent for $\mathrm{Re} z > 0$ and therefore defines an analytic function in this domain.[1] For $\mathrm{Re} z < 0$, on the other hand, the

[1] Indeed the integral

$$\frac{d\Gamma(z)}{dz} = \int_0^\infty dt\ (\ln t) t^{z-1} e^{-t}$$

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integral diverges at its lower limit. So we need to analytically continue the Gamma function to Re$z < 0$ region from Re$z > 0$ region. To this end, we integrate by parts,

$$\Gamma(z) = \frac{1}{z} \int_0^\infty d(t^z) e^{-t} = \frac{\Gamma(z+1)}{z},$$

to establish the *recurrence relation*:

$$\Gamma(z+1) = z\Gamma(z). \tag{2.2}$$

Obviously, for any integer n we then have a useful identity

$$\Gamma(n+1) = n!. \tag{2.3}$$

Therefore the Gamma function can be considered as a generalization of the factorial for non-integer arguments.

2.1.2 Analytic Properties

The identity (2.2) can be used to accomplish the analytic continuation. Indeed, the relation

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

defines $\Gamma(z)$ in the region $-1 < \text{Re}z < 0$ where Eq. (2.1) can still be used on the rhs of the above equation. Applying the recurrence relation repeatedly ($n+1$ times), one obtains

$$\begin{aligned} \Gamma(z) &= \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{(z+1)z} = \dots \\ &= \frac{\Gamma(z+n+1)}{(z+n)(z+n-1)\dots(z+1)z}, \end{aligned} \tag{2.4}$$

which defines $\Gamma(z)$ in $-n-1 < \text{Re}z < -n$. As the above process can be continued indefinitely, we have the definition of $\Gamma(z)$ for all Re$z < 0$. In addition, we see that the Gamma function is analytic for all finite values of z except non-positive integers. To investigate the nature of the singularities of the Gamma function near $z = -n$ ($n = 0, 1, 2, \dots$), put

$$z = -n + \delta$$

(Footnote 1 continued)
is clearly convergent in Re$z > 0$ as are all higher derivatives.

in the above formula and expand in $\delta \to 0$:

$$\Gamma(-n+\delta) = \frac{\Gamma(1+\delta)}{\delta(1-\delta)\dots(n-\delta-1)(n-\delta)(-1)^n} = \frac{(-1)^n}{n!\,\delta} + O(\delta^0).$$

We thus discover that points $z = -n$ are simple poles of the Gamma function with residues

$$\mathrm{Res}\,\Gamma(z)|_{z=-n} = \frac{(-1)^n}{n!}.$$

These are the only singularities of the Gamma function for finite z.

Let us consider the integral

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds A(s)\Gamma(-s)(-z)^s,$$

where the integration contour is along the imaginary axis but is slightly deformed around the origin so as to leave the $s = 0$ pole on the right. Next assume that the function $A(s)$ is analytic in $\mathrm{Re}\, s > 0$ and is not too divergent as $\mathrm{Re}\, s \to +\infty$. Then for $|z| < 1$ we can shift the integration contour all the way to the right leaving zero result plus the contribution from the string of poles of the Gamma function in the integrand:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds A(s)\Gamma(-s)(-z)^s = \sum_{n=0}^{\infty} \frac{A(n)}{n!} z^n. \tag{2.5}$$

If the function $A(s)$ is chosen in such a way that the values $A(n)$ coincide with the coefficients of the Taylor expansion of a (different) function $f(z)$, that means $A(n) = f^{(n)}(0)$, then Eq. (2.5) provides a contour integral representation for the function $f(z)$ inside the unit circle.[2]

This provides a useful tool for the analytic continuation of Taylor series outside the unit circle as, depending on the properties of the function $A(s)$, it may happen that integral (2.5) still converges for $|z| > 1$ and can be computed by shifting the contour of integration to the left.

2.1.3 Complement Formula

For our subsequent studies of the hypergeometric series, we shall need another identity satisfied by the Gamma function for all complex z:

[2] In general, the Gamma function is not the only function that can be used to recover the Taylor series and other functions with similar pole structure, for example $1/\sin(\pi s)$ could be used as well see Sect. 1.3.6.

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{2.6}$$

also referred to as *complement* or *reflection formula*. Indeed, from the pole structure of the Gamma function, it is clear that the expression $\sin(\pi z)\,\Gamma(z)\Gamma(1-z)$ has no poles at all on the real axis and is, in fact, analytic for all finite z and tends to π for $z \to 0$. We do not know, however, whether it is bounded and so the Liouville theorem cannot really be used here. We shall therefore prove identity (2.6) from the defining integral representation, Eq. (2.1), which we rewrite as ($t = x^2$)

$$\Gamma(z) = 2\int_0^\infty dx\, x^{2z-1} e^{-x^2}.$$

We shall for the moment restrict ourselves to the real values $z = \lambda$ belonging to the interval $\lambda \in (0, 1)$. One then finds

$$\Gamma(\lambda)\Gamma(1-\lambda) = 4\int_0^\infty\int_0^\infty dx dy x^{2\lambda-1} y^{-2\lambda+1} e^{-x^2-y^2} = 4\int_0^{\pi/2} d\varphi (\cot\varphi)^{2\lambda-1}\int_0^\infty d\rho\rho e^{-\rho^2},$$

where the polar co-ordinates $x = \rho\cos\varphi, \quad y = \rho\sin\varphi$ have been used.[3] The ρ–integral in the above is elementary and equal to $1/2$. To calculate the φ–integral substitute $x = \cot\varphi$:

$$\int_0^{\pi/2} d\varphi (\cot\varphi)^{2\lambda-1} = \int_0^\infty \frac{x^{2\lambda-1}dx}{x^2+1}.$$

This is the same type of branch–cut integral studied in Sect. 1.3.2 and so given by the formula (1.14) with $a = 2\lambda$:

$$\int_0^\infty \frac{x^{2\lambda-1}dx}{x^2+1} = \frac{\pi}{\sin(2\pi\lambda)}\sum_{\pm}\operatorname{Res}\frac{(-z)^{2\lambda-1}}{z^2+1}\Big|_{z=\pm i} = \frac{\pi}{\sin(2\pi\lambda)}\left[\frac{(-i)^{2\lambda-1}}{2i} + \frac{(i)^{2\lambda-1}}{-2i}\right]$$
$$= \frac{\pi}{2i\sin(2\pi\lambda)}\left(e^{-i\pi\lambda+i\pi/2} - e^{i\pi\lambda-i\pi/2}\right) = \frac{\pi\cos\pi\lambda}{\sin(2\pi\lambda)} = \frac{\pi}{2\sin\pi\lambda}.$$

We thus conclude that the equality

$$\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin\pi\lambda}$$

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[3] This is a generally useful trick in mathematical physics, employed to obtain various product identities and for instance for the evaluation of the Gaussian integral.

holds on the interval $\lambda \in (0, 1)$. From the analytic continuation theorem of Sect. 1.3.6 it follows that Eq. (2.6) is valid for all finite z in the complex plane.

The value

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{2.7}$$

immediately follows from (2.6). Setting $z \to \frac{1}{2} + z$, another complement identity follows from (2.6):

$$\Gamma\left(\frac{1}{2} + z\right)\Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\cos \pi z}. \tag{2.8}$$

2.1.4 Stirling's Formula

In the following sections we shall need to estimate various integrals involving Gamma functions. Therefore the $|z| \to \infty$ asymptotic form is given here for future reference:

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-1/2} e^{-z} \left[1 + \frac{1}{12z} + O\left(\frac{1}{z^2}\right)\right]. \tag{2.9}$$

This expression is known as the *Stirling's formula* and is valid for $|\arg z| < \pi - \delta$ (δ an infinitesimal). The complete proof of (2.9) is somewhat lengthy (the expansion in brackets [...] is in terms of Bernoulli numbers) and can be looked up in [3]. Equation (2.9) follows from (2.2) and the fact that $\Gamma(1) = 1$ and $\Gamma(n) = (n-1)!$. For positive integers $z = n$ Eq. (2.9) reduces to the Stirling's formula for $n!$ familiar from the elementary analysis.

2.1.5 Euler Integral of the First Kind

Assuming Re$z > 0$ and Re$w > 0$, we define the Beta function as

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt. \tag{2.10}$$

From the symmetry $t \leftrightarrow 1 - t$ it follows that $B(z, w) = B(w, z)$.

Next we prove that the following relation to Gamma functions holds:

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z + w)}. \tag{2.11}$$

Proceed by reducing the double integral as in Sect. 2.1.3

$$\Gamma(z)\Gamma(w) = 4\int_0^\infty\int_0^\infty dx dy\, x^{2z-1}y^{2w-1}e^{-x^2-y^2} = 4\int_0^\infty d\rho\,\rho^{2(z+w)-1}e^{-\rho^2}$$
$$\times\int_0^{\pi/2} d\varphi\,(\cos\varphi)^{2z-1}(\sin\varphi)^{2w-1}.$$

The ρ–integral is now equal to $\Gamma(z+w)/2$. In the φ–integral substitute $\sin^2\varphi = t$:

$$(\cos\varphi)^{2z-1}(\sin\varphi)^{2w-1}d\varphi = \frac{1}{2}(1-t)^{z-1}t^{w-1}dt,$$

and thus obtain the definition (2.10). Hence

$$\Gamma(z)\Gamma(w) = \Gamma(z+w)B(w,z) = \Gamma(z+w)B(z,w)$$

indeed holds.

2.1.6 Duplication Formula

One practical usage of the Euler integral is for the derivation of the duplication formula for the Gamma function. To see this set $z = w$ in the Euler integral

$$B(z,z) = \int_0^1 t^{z-1}(1-t)^{z-1}dt = \int_0^1 [t(1-t)]^{z-1}dt.$$

The integrand in the above is evidently symmetric about the point $t = 1/2$ (i.e. under the substitution $t \leftrightarrow 1-t$) and therefore

$$B(z,z) = 2\int_0^{1/2} [t(1-t)]^{z-1}dt.$$

Now substitute $t = (1-\sqrt{\tau})/2$. We have $dt = -d\tau/(4\sqrt{\tau})$, $t(1-t) = (1-\sqrt{\tau})(1+\sqrt{\tau})/4$ and therefore

$$B(z,z) = \frac{2}{4^z}\int_0^1 \tau^{-1/2}(1-\tau)^{z-1}d\tau = 2^{1-2z}B\left(\frac{1}{2},z\right).$$

This expression is known as *Legendre duplication formula*. Recalling that $\Gamma(1/2) = \sqrt{\pi}$ [see (2.7)] as well as using (2.11), we deduce the identity for the Gamma function

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{2.12}$$

which is also known as the *duplication formula*. While the Euler integral only converges for $\mathrm{Re}z > 0$, formula (2.12) is actually valid for all complex z in the spirit of the analytic continuation.

2.1.7 Infinite Product Representation

It is not difficult to show that for any b and c the following remarkable statement holds (see for example Sect. 1.1 of Ref. [7] for the proof),

$$\lim_{u\to\infty} \frac{u^b\,\Gamma(u+c)}{u^c\,\Gamma(u+b)} = 1. \tag{2.13}$$

One way to verify it is to use the Stirling formula (2.9). Let us now set $b = z + 1$ and $c = 1$ and restrict u to integers. Then

$$\lim_{n\to\infty} \frac{n^z\,\Gamma(n+1)}{\Gamma(z+n+1)} = 1.$$

Using (2.3) and the fact that $\Gamma(1) = 1$ we obtain

$$\lim_{n\to\infty} \frac{n^z\,n!}{z(z+1)\ldots(z+n)\,\Gamma(z)} = 1,$$

which immediately leads to the infinite product representation:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z\,n!}{z\,(z+1)\cdots(z+n)}. \tag{2.14}$$

Obviously, this representation is valid for all complex numbers z, except the poles $z = 0, -1, -2, \ldots$. With its help some of the important relations can be recovered in a very convenient way. For example for the recurrence relation we have

$$\begin{aligned}\Gamma(z+1) &= \lim_{n\to\infty} \frac{n^{z+1}\,n!}{(z+1)\cdots(z+n+1)}\\ &= \lim_{n\to\infty}\frac{nz}{z+n+1}\lim_{n\to\infty}\frac{n^z\,n!}{z(z+1)\cdots(z+n)} = z\Gamma(z).\end{aligned}$$

There is also another way to show the compatibility of the representations (2.14) and (2.1). For this purpose consider the integral

$$G_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt,$$

for Re $z > 0$. As is known from the elementary analysis, the factor $(1 - t/n)^n$ in the integrand limits to the function e^{-t} at $n \to \infty$. One may expect therefore that at infinite growth of n the function $G_n(z)$ limits to the Gamma function (2.1). To prove it, put $t = n(1 - e^{-v})$ so that the above integral rewrites as

$$G_n(z) = n^z \int_0^\infty v^{z-1} e^{-nv} f(v) dv, \;\; f(v) = \left(\frac{1 - e^{-v}}{v}\right)^{z-1} e^{-v},$$

where $f(v)$ is a continuous function of v and $f(0) = 1$. Hence as $n \to \infty$

$$G_n(z) = \Gamma(z)\Big[1 + O(n^{-1})\Big].$$

On the other hand we can verify that the function $G_n(z)$ satisfies (2.14). To do that we integrate by parts taking into account that n are positive integers. After the substitution $\tau = t/n$ one then finds

$$\begin{aligned} G_n(z) &= n^z \int_0^1 (1-\tau)^n \tau^{z-1} d\tau = n^z \Big[\underbrace{\frac{1}{z}\tau^z (1-\tau)^n \Big|_0^1}_{=0} \\ &+ \frac{n}{z} \int_0^1 (1-\tau)^{n-1} \tau^z d\tau \Big] = n^z \, \frac{n(n-1)}{z(z+1)} \int_0^1 (1-\tau)^{n-2} \tau^{z+1} d\tau = \ldots \\ &= \frac{1 \cdot 2 \ldots n}{z(z+1) \ldots (z+n)} \, n^z . \end{aligned}$$

Taking the limit $n \to \infty$ we then recover (2.14).

Representation (2.14) is often used to establish new properties of $\Gamma(z)$. In particular, after some algebra[4] one obtains the Gamma function in the form of the Weierstrass infinite product

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \Big[\left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \Big]. \tag{2.15}$$

[4] The rhs of (2.14) rewrites as

Here γ is the Euler constant, which is defined by

$$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] = 0.5772157\ldots$$

Clearly, from the Weierstrass formula (2.15) it follows that only negative integers or zero are the poles of $\Gamma(z)$.

It could also be shown that the complement formula (2.6) follows directly from (2.14):

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -\frac{z}{\Gamma(z)\Gamma(1-z)} = -z^2\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right) = -\frac{z\sin\pi z}{\pi}.$$

The same applies to the duplication formula (2.12).

Furthermore, the infinite product (2.14) allows to prove the following general formula (known as the Gauss multiplication formula):

$$\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\ldots\,\Gamma\left(z+\frac{m-1}{m}\right) = (2\pi)^{\frac{1}{2}(m-1)}m^{\frac{1}{2}-mz}\Gamma(mz), \tag{2.16}$$

where $m = 2, 3\ldots$. To prove this multiplication rule, notice that $\Gamma(mz)$, by a slight modification of definition (2.14), is given by

$$\Gamma(mz) = \lim_{n\to\infty}\frac{(mn)!(mn)^{mz}}{\prod_{k=0}^{mn}(mz+k)}. \tag{2.17}$$

Similarly, for a product of the Gamma functions in the lhs of (2.16), by using the same identity (2.14), we have

$$\left[\prod_{k=0}^{m-1}\Gamma\left(z+\frac{k}{m}\right)\right]^{-1} = \lim_{n\to\infty}\frac{1}{(n!)^m n^{mz+(m-1)/2}m^{mn+m}}\prod_{k=0}^{nm+m-1}(mz+k).$$

(Footnote 4 continued)

$$\lim_{n\to\infty}\frac{n!n^z}{z(z+1)\ldots(z+n)} = \lim_{n\to\infty}\frac{e^{z\ln n}}{z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)^{-1}$$

$$= \frac{e^{\overbrace{-\lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right)z}^{-\gamma z}}}{z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{-1}e^{z/k}.$$

Note that although it seems that here the Euler's constant γ is introduced artificially, we will see shortly that it is deeply related to the Gamma function.

Combining this expression with (2.17) we obtain,

$$\frac{(m)^{-mz}\Gamma(mz)}{\prod_{k=0}^{m-1}\Gamma(z+\frac{k}{m})} = \lim_{n\to\infty}\frac{(mn)!\,n^{(m-1)/2}}{(n!)^m\,m^{mn+m}}\prod_{k=nm+1}^{nm+m-1}[(mz+k)/n].$$

Now we see that in fact the rhs of the above relation is independent of z and is equal to[5]

$$m^{-\frac{1}{2}}(2\pi)^{-\frac{1}{2}(m-1)}.$$

Hence finally we have

$$\frac{\Gamma(mz)}{\prod_{k=0}^{m-1}\Gamma(z+\frac{k}{m})} = (2\pi)^{-\frac{1}{2}(m-1)}m^{-\frac{1}{2}+mz},$$

as required.

Another interesting identity is derived from the multiplication formula (2.16) by setting $z = 1/m$,

$$\Gamma\Big(\frac{1}{m}\Big)\Gamma\Big(\frac{2}{m}\Big)\ldots\,\Gamma\Big(\frac{m-1}{m}\Big) = \frac{(2\pi)^{\frac{1}{2}(m-1)}}{\sqrt{m}},\ m = 2, 3, \ldots .$$

2.2 Mellin Integral Transform

Let us get back to the definition (2.1) and rescale the integration variable by a parameter p, $t \to pt$, then we obtain

$$\int_0^\infty dt\, t^{z-1}\, e^{-pt} = \Gamma(z)/p^z. \tag{2.19}$$

[5] Indeed, at infinite growth of n the involved product reduces to

$$\lim_{n\to\infty}\prod_{k=nm+1}^{nm+m-1}[(mz+k)/n] = m^{m-1}$$

and the pre–factor, by using the Stirling's formula for the factorial

$$n! \sim \sqrt{2\pi n}\,(n/e)^n, \tag{2.18}$$

is given by

$$\lim_{n\to\infty}\frac{(mn)!n^{(m-1)/2}}{(n!)^m m^{mn+m}} = (2\pi)^{-\frac{1}{2}(m-1)}m^{\frac{1}{2}-m}.$$

In this result we immediately recognize the Laplace transform of the power function $f(t) = t^{z-1}$. One can define another integral transform assuming t^{z-1} to be its kernel instead of the exponential e^{-pt}. It is referred to as *Mellin transform* and is formally defined as

$$F_M(s) = \int_0^\infty dx\, f(x)\, x^{s-1}, \tag{2.20}$$

as long as the integral is convergent. In general, the integral does exist for complex s in some strip depending on the function $f(x)$.

Let $f(x)$ be continuous on the positive real semiaxis. Similar to the exponential bound indices we can introduce power law bound indices α_0 and α_∞, defined as

$$\lim_{x\to 0} |f(x)| \le C_0\, x^{\alpha_0} \quad \text{and} \quad \lim_{x\to\infty} |f(x)| \le C_\infty\, x^{\alpha_\infty},$$

where $C_{0,\infty}$ are some positive constants. Then we can perform the following estimation

$$\begin{aligned}\left|\int_0^\infty dx\, f(x)x^{s-1}\right| &\le \int_0^1 dx\, |f(x)|x^{\mathrm{Re}(s)-1} + \int_1^\infty dx\, |f(x)|x^{\mathrm{Re}(s)-1} \\ &\le C_0 \int_0^1 dx\, x^{\mathrm{Re}(s)+\alpha_0-1} + C_\infty \int_1^\infty dx\, x^{\mathrm{Re}(s)+\alpha_\infty-1}.\end{aligned}$$

The last two integrals exist for $\mathrm{Re}(s) > -\alpha_0$ and $\mathrm{Re}(s) < -\alpha_\infty$, respectively. Thus the Mellin transform of $f(x)$ exists for any complex s in the *fundamental strip* $-\alpha_0 < \mathrm{Re}(s) < -\alpha_\infty$. For any polynomial $\alpha_\infty > \alpha_0$ and therefore the Mellin transform does not exist. For the exponential function the fundamental strip is $(0, \infty)$.

The Mellin transform is closely related to the Gamma function. This is obvious as Mellin transform of the exponential function e^{-t} $(t > 0)$ is the Gamma function $\Gamma(s)$. Another examples are:

- for the function $f(t) = e^{-1/t}/t$ we obtain

$$F_M(s) = \int_0^\infty dt\, \frac{e^{-1/t}}{t}\, t^{s-1} \underbrace{\Rightarrow}_{1/t=x} \int_0^\infty dx x^{-s} e^{-x} = \Gamma(1-s),$$

- for the function $f(t) = (1+t)^{-a}$, setting $t = x(1-x)^{-1}$ we get $(1+t)^{-1} = 1-x$, $dt = (1-x)^{-2}dx$ and thus:

$$\begin{aligned}F_M(s) &= \int_0^\infty t^{s-1}(1+t)^{-a}dt = \int_0^1 x^{s-1}(1-x)^{a-s-1}dx = B(s, a-s) \\ &= \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)},\end{aligned}$$

where in the last step we have used Eq. (2.10) for the Beta-function.

Mellin transforms have interesting properties, a number of which we shall use in the subsequent chapters. Probably the most important of them is the transform of a function with a scaled argument, $f(x) = g(ax)$, then

$$F_M(s) = \int_0^\infty dx\, g(ax)\, x^{s-1} = a^{-s} \int_0^\infty dx\, g(x)\, x^{s-1} = a^{-s} G_M(s). \tag{2.21}$$

It is particularly useful in solving functional equations of the type

$$f(x) = g(x) + f(ax), \tag{2.22}$$

which, after applying the Mellin transform take a form of a simple algebraic relationship

$$F_M(s) = G_M(s)/(1 - a^{-s}).$$

This property can be very useful when deriving interesting functional relations. The inversion formula for the Mellin transform is given by

$$f(x) = \frac{1}{2\pi i} \int\limits_{a-i\infty}^{a+i\infty} F_M(s) x^{-s} ds, \tag{2.23}$$

where the integration is along a vertical line through $\mathrm{Re}s = a$. To get this formula, we use definition (2.20) and introduce the new variables by the substitution $x = e^{-\xi}$ and $s = a + 2\pi i \beta$. The Mellin transform writes then as

$$F_M(s) = \int\limits_{-\infty}^{\infty} d\xi\, f(e^{-\xi}) e^{-a\xi} e^{-2\pi i \beta \xi}.$$

Hence for a given value $\mathrm{Re}s = a$ belonging to the fundamental strip, the Mellin transform of a function is expressed as a Fourier transform. Using Fourier's inversion theorem the original is:

$$f(e^{-\xi}) e^{-a\xi} = \int\limits_{-\infty}^{\infty} F_M(s) e^{2\pi i \beta \xi} d\beta.$$

Going now back to variables x and s, we obtain the result (2.23):

$$f(x) = x^{-a} \int\limits_{-\infty}^{\infty} F_M(s) x^{-2\pi i \beta} d\beta = \frac{1}{2\pi i} \int\limits_{a-i\infty}^{a+i\infty} F_M(s) x^{-s} ds.$$

Example 2.1
Let us consider the Jacobi's *elliptic* ϑ_3*-function*, which is defined as

$$\vartheta_3(0, e^{-\pi x}) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi x n^2} = 1 + 2f(x).$$

We would like to show that

$$\sqrt{y}\,\vartheta_3(0, y) = \vartheta_3(0, 1/y) \quad \text{or} \quad 1 + 2f(x) = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}}\, f(1/x). \tag{2.24}$$

The Mellin transform of $f(x)$ can be found to be

$$F_M(s) = \frac{1}{\pi^s}\,\Gamma(s)\,\zeta(2s),$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2.25}$$

is the Riemann *Zeta function*. The fundamental strip for the function $f(x)$ is $\mathrm{Re} s > 0$, so that the inverse transformation is

$$f(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} ds\, x^{-s}\, \frac{1}{\pi^s}\,\Gamma(s)\,\zeta(2s),$$

with the integration contour P being a straight line parallel to the imaginary axis, see Fig. 2.1. The integrand $F_M(s)$ has two poles at $s = 0$ and $s = 1/2$, the respective residua at which can be read off the following expansions,

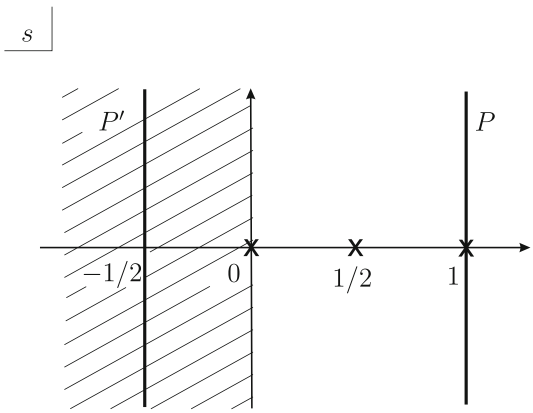

Fig. 2.1 Contour shift used in Example 2.1

$$F(s)\Big|_{s\to 0} = -\frac{1}{2s} + \dots, \quad F(s)\Big|_{s\to 1/2} = -\frac{1}{2(s-1/2)} + \dots$$

By shifting the integration contour to P' (which, in fact, lies in the 'forbidden' region) we have to take out the poles. This procedure yields

$$f(x) = -\frac{1}{2} + \frac{1}{2\sqrt{x}} + \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} ds\, x^{-s}\, \frac{1}{\pi^s}\,\Gamma(s)\,\zeta(2s).$$

We now perform the substitution $s \to 1/2 - s$ which returns our integration contour to the original P and obtain

$$= -\frac{1}{2} + \frac{1}{2\sqrt{x}} - \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} ds\, x^{-1/2+s}\, \frac{1}{\pi^{1/2-s}}\,\Gamma(1/2-s)\,\zeta(1-2s).$$

In the next step we use the identity[6]

$$\Gamma(1/2-s)\,\zeta(1-2s) = \frac{1}{\pi^{2s-1/2}}\Gamma(s)\,\zeta(2s) \tag{2.26}$$

and arrive at

$$\begin{aligned} f(x) &= -\frac{1}{2} + \frac{1}{2\sqrt{x}} + \frac{1}{2\pi i\,\sqrt{x}}\int_{1-i\infty}^{1+i\infty} ds\, x^{s}\, \frac{1}{\pi^s}\,\Gamma(s)\,\zeta(2s) \\ &= -\frac{1}{2} + \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{x}} f(1/x), \end{aligned}$$

which proves the relation (2.24).

2.3 Series Expansion

For numerical evaluation of the Gamma function near some point $z \in \mathbb{C}$, a series expansion around it could be useful. Before doing this we introduce Psi or Digamma function related to the logarithmic derivative of $\Gamma(z)$, that is:

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz}\ln\Gamma(z). \tag{2.27}$$

Below we investigate the main properties of the Psi function which, evidently, are closely related to the properties of the Gamma function.

[6] This identity can be derived using the Hurvitz formula given in e.g. Sect. 13.15 of [3].

Taking the logarithm of both sides of the Weierstrass formula (2.15) and then differentiating, we obtain the series representation of the Psi function:

$$\Psi(z) = -\gamma + \sum_{k=1}^{\infty}\Big(\frac{1}{k} - \frac{1}{z+k-1}\Big). \tag{2.28}$$

We observe that the rhs of the above sum is absolutely convergent for $\mathrm{Re}z > 0$ and hence the function $\Psi(z)$, just like the Gamma function, is an analytic function in this domain.[7] The structure of the series expansion (2.28) leads to the following recurrence relation

$$\Psi(z) = \Psi(z+1) - \sum_{k=1}^{\infty}\Big(\frac{1}{z+k-1} - \frac{1}{z+k}\Big) = \Psi(z+1) - \frac{1}{z}, \tag{2.30}$$

which ensures the analytic continuation of the Psi function to the region $\mathrm{Re}z < 0$ from the region $\mathrm{Re}z > 0$. Indeed, applying the recurrence relation repeatedly, one finds

$$\begin{aligned}\Psi(z) &= \Psi(z+1) - \frac{1}{z} = \Psi(z+2) - \frac{1}{z} - \frac{1}{z+1} = \cdots \\ &= \Psi(z+n) - \frac{1}{z} - \frac{1}{z+1} - \cdots - \frac{1}{z+n-1},\end{aligned}$$

the formula that defines $\Psi(z)$ in the region $-n < \mathrm{Re}z < -n+1$. Continuing this process we obtain $\Psi(z)$ for all $\mathrm{Re}z < 0$, except negative integers $z = -n$ ($n = 0, 1, 2, \cdots$), which are simple poles of the Psi function.

Obviously, zeros of the Psi function are the extrema of the Gamma function. Since the values

$$\Psi(1) = -\gamma \qquad \text{and} \qquad \Psi(2) = 1 - \gamma$$

immediately follow from the series expansion (2.28) and the recurrence relation (2.30), respectively, $\Psi(z)$ has one zero $\mathrm{Re}z = x_0 \in [1, 2]$ on the positive semiaxis. On the negative semiaxis the Psi function has a single zero between each consecutive negative integers, which are the poles of the Gamma function.

For $\mathrm{Re}z > 0$, the Psi function is represented by the integral

---

[7] If we go on differentiating the relation (2.28) several times, we obtain

$$\Psi^{(n)}(z) = \frac{d^n}{dz^n}\Psi(z) = (-1)^{n+1} n! \sum_{k=1}^{\infty} \frac{1}{(k+z-1)^{n+1}}. \tag{2.29}$$

Clearly, the first derivative $\Psi^{(1)} = \sum_{k=1}^{\infty}(k+z-1)^{-2}$ is convergent in $\mathrm{Re}z > 0$ domain as are all higher derivatives.

$$\Psi(z) = \int_0^\infty \Big[e^{-\xi} - (1+\xi)^{-z}\Big]\frac{d\xi}{\xi}. \tag{2.31}$$

To establish this result, we start with the double integral

$$I(t) = \int_0^\infty \Big(\int_1^t e^{-sz} ds\Big) dz = \int_0^\infty \frac{e^{-z} - e^{-tz}}{z}\, dz.$$

On the other hand, calculation of the integrals in the reversed order yields

$$I(t) = \int_1^t \Big(\int_0^\infty e^{-sz} dz\Big) ds = \int_1^t \frac{dt}{t} = \ln t.$$

Hence we conclude that

$$\int_0^\infty \frac{e^{-z} - e^{-tz}}{z}\, dz = \ln t. \tag{2.32}$$

Next we observe that the first derivative of the Gamma function is

$$\Gamma'(z) = \int_0^\infty e^{-t} t^{z-1} \ln t \; dt.$$

Replacing $\ln t$ in this expression by (2.32) we obtain

$$\begin{aligned}\Gamma'(z) &= \int_0^\infty \Big(e^{-\xi} \underbrace{\int_0^\infty t^{z-1} e^{-t} dt}_{\Gamma(z)} - \underbrace{\int_0^\infty t^{z-1} e^{-t(1+\xi)} dt}_{(1+\xi)^{-z}\Gamma(z)}\Big)\frac{d\xi}{\xi} \\ &= \Gamma(z) \int_0^\infty \Big[e^{-\xi} - (1+\xi)^{-z}\Big]\frac{d\xi}{\xi},\end{aligned}$$

which immediately leads to (2.31).

By logarithmic differentiation of the complement (2.6) and duplication (2.12) formulae for the Gamma function we find the corresponding identities for the Psi function:

$$\Psi(1-z) = \Psi(z) + \pi \cot \pi z,$$
$$2\Psi(2z) = \Psi(z) + \Psi\Big(z + \frac{1}{2}\Big) + 2\ln 2.$$

Last formula being evaluated at $z = 1/2$ yields the value

$$\Psi\Big(\frac{1}{2}\Big) = -\gamma - 2\ln 2.$$

Next we investigate the Psi and Gamma functions near the point $z = 1$. We begin with derivatives of the Psi function given by the relation (2.29). At $z = 1$ we have

$$\Psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1),$$

where the Zeta function $\zeta(k)$ [see the definition (2.25)] is convergent for $k > 1$. Then for the Taylor series expansion we obtain

$$\Psi(1+z) = -\gamma + \sum_{k=2}^{\infty} (-1)^k \zeta(k) z^{k-1} \quad \text{for} \quad |z| < 1.$$

Using term by term integration of the both sides of this relation, one obtains the Taylor expansion of the logarithm of the Gamma function in the form

$$\ln\Big[\Gamma(1+z)\Big] = -\gamma z + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} z^k, \qquad |z| < 1.$$

From this series expansion and with a help of the recurrence relation (2.2), the behaviour of $\Gamma(z)$ near a the point $z = 0$ is

$$\Gamma(z) = e^{-\gamma z}\Big[\frac{1}{z} + \frac{\zeta(2)}{2} z + O(z^2)\Big], \qquad \text{as } |z| \ll 1,$$

which is consistent with $\Gamma(z)$ having a simple pole at $z = 0$.

2.3.1 Hankel Contour Integral Representation

Let us consider the following integral

$$I_\gamma(z) = \int_\gamma d\xi \, (-\xi)^{z-1} \, e^{-\xi},$$

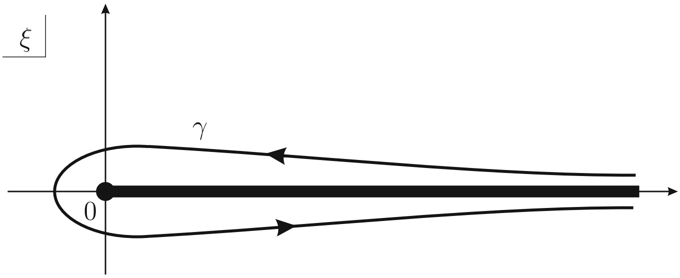

Fig. 2.2 Contour γ used in the Hankel integral representation

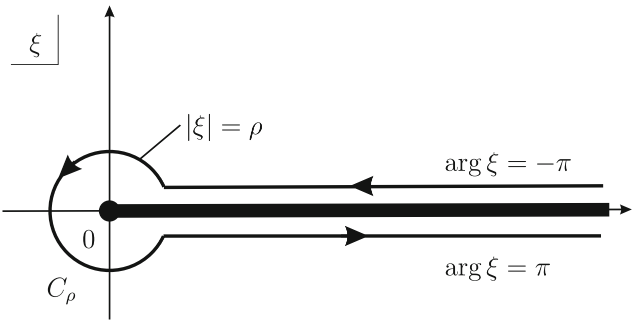

Fig. 2.3 The deformed contour of Fig. 2.2

along the contour γ, which starts at the point $x + i0^+$, $x > 0$ just above the real axis, encircles the coordinate origin counterclockwise and returns to the point $x - i0^+$ just below the real axis, see Fig. 2.2. We fix the branch of the multivalued function $(-\xi)^{z-1}$ by the convention that in $(-\xi)^{z-1} = \exp[(z-1)\ln(-\xi)]$, where the part $\ln(-\xi)$ is purely real on the negative real axis. Thus on γ we have $-\pi \le \arg(-\xi) \le \pi$ and the complex plane $\mathbb{C}$ is cut along the positive real semiaxis.

As the next step we deform the contour in such a way that it now consists of two line segments $C_\pm$ along the upper/lower edge of the cut and a circle C_ρ with radius ρ around the coordinate origin as is shown in Fig. 2.3. Then $\arg(-\xi) = \mp\pi$, so that $(-t)^{z-1} = e^{\mp i(z-1)}\,\xi^{z-1}$ on the respective contour parts $C_\pm$. We choose the parametrisation on C_ρ to be $(-\xi) = \rho\, e^{i\theta}$, then

$$
\begin{aligned}
I_\gamma(z) = \int_{C_+} + \int_{C_\rho} + \int_{C_-} &= \int_x^\rho d\xi\, e^{-i\pi(z-1)}\,\xi^{z-1}\,e^{-\xi} \\
&+ \int_{-\pi}^{\pi} \rho e^{i\theta}\, i d\theta\,(\rho\, e^{i\theta})^{z-1}\, e^{-\rho(\cos\theta + i\sin\theta)} + \int_\rho^x d\xi\, e^{i\pi(z-1)}\,\xi^{z-1}\,e^{-\xi} \\
&= -2i\,\sin(\pi z)\int_\rho^x d\xi\,\xi^{z-1}\,e^{-\xi} + i\rho^z \int_{-\pi}^{\pi} d\theta\, e^{iz\theta - \rho(\cos\theta + i\sin\theta)}.
\end{aligned}
$$

Taking the limit $\rho \to 0$ we infer that for $\mathrm{Re}z > 0$

$$I_\gamma(z) = -2i\,\sin(\pi z)\int_0^x d\xi\,\xi^{z-1}\,e^{-\xi}.$$

In the last step we send $x \to \infty$ and obtain

$$\Gamma(z) = -\frac{1}{2i\;\sin(\pi z)}\int_\gamma d\xi\,(-\xi)^{z-1}\,e^{-\xi}, \tag{2.33}$$

which is a well known contour integral representation of the Gamma function due to Hankel. It is important that the contour γ does not pass through the point $\xi = 0$ and therefore the integral in (2.33) is the analytic function of z in the whole complex plane. Hence the Hankel formula (2.33) is valid for all z, except $z = 0, \pm 1, \pm 2, \cdots$, where $\sin(\pi z) = 0$.

2.4 Hypergeometric Series: Basic Properties

2.4.1 Definition and Convergence

The following functional series in the variable z

$$F(a, b; c; z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{2.34}$$

are called *hypergeometric series*.[8] Here $(a)_0 = 1$, $(a)_1 = a$, $(a)_2 = a(a+1)$, and generally

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{2.35}$$

is the Pochhammer symbol, $(1)_n = n!$.[9] Similar definitions, $(b)_n = \Gamma(b+n)/\Gamma(b)$ and $(c)_n = \Gamma(c+n)/\Gamma(c)$, hold for $(b)_n$ and $(c)_n$ and a, b, and c are, in general, complex parameters. Obviously (2.34) is symmetric with respect to the exchange $a \leftrightarrow b$.

For later convenience we define the general term

$$A_n = \frac{(a)_n(b)_n}{(c)_n n!},$$

[8] In the more recent literature it is sometimes called ${}_2F_1(a, b; c; z)$ in order to indicate the number and arrangement of parameters. We believe, however, that this convention unnecessarily complicates notation.

[9] Remember that $\Gamma(z+1) = z\Gamma(z)$.

so that the hypergeometric series is represented as $F(a, b; c; z) = \sum_{n=0}^{\infty} A_n z^n$. From the standard ratio test for the series convergence

$$\lim_{n\to\infty}\left|\frac{A_{n+1}z^{n+1}}{A_n z^n}\right| = \left[\lim_{n\to\infty}\frac{(n+a)(n+b)}{(n+c)(n+1)}\right]|z| = |z|$$

it follows that the hypergeometric series is absolutely convergent inside the unit circle $|z| < 1$, where it defines an analytic function: the hypergeometric function. As with the Gamma function it is possible to analytically continue the hypergeometric function outside the analyticity domain to which the original definition is limited. We shall do so in later sections.

2.4.2 Special Cases

All the elementary functions and many special functions including all orthogonal polynomials can be obtained as special cases of the hypergeometric function. Here we consider some particularly simple and important examples.
(i) When $b = c$ we recognise the elementary series[10]

$$F(a, b; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}. \tag{2.36}$$

Interestingly, the above expression does not depend on b any more. This expansion naturally converges in the unit circle $|z| < 1$, but can clearly be extended to the whole complex plane. Then it has one singular point at $z = 1$ and for non-integer values of a a branch cut on $z \in [1, \infty)$. We shall see later that these properties still hold in the general case. In particular, for $a = 1$ one obtains the elementary geometric series:

$$F(1, b; b; z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}. \tag{2.37}$$

(ii) For $b = a + 1/2$ and $c = 3/2$ the following identity

[10] Alternatively one can compute the Taylor coefficients

$$\frac{d^n}{dz^n}(1-z)^{-a} = a\frac{d^{n-1}}{dz^{n-1}}(1-z)^{-a-1} = \ldots = a(a+1)\ldots(a+n-1)(1-z)^{-a-n}$$

and find that

$$\left.\frac{d^n}{dz^n}(1-z)^{-a}\right|_{z=0} = (a)_n.$$

$$F\left(a, \frac{1}{2}+a; \frac{3}{2}; z^2\right) = \frac{1}{2z(1-2a)}\Big[(1+z)^{1-2a} - (1-z)^{1-2a}\Big] \tag{2.38}$$

holds. In particular, for $a = 1$ and then $b = c$ it reduces to the above case, $F\left(1, 3/2; 3/2; z^2\right) = (1-z^2)^{-1}$. Otherwise using the duplication formula (2.12), the generic coefficient of the series is equal to

$$A_n = \frac{\Gamma(a+n)\Gamma\left(a+\frac{1}{2}+n\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}+n\right)n!} = \frac{\Gamma(2a+2n)}{\Gamma(2a)(2n+1)!},$$

leading to

$$F\left(a, \frac{1}{2}+a; \frac{3}{2}; z^2\right) = \frac{1}{z(2a-1)}\sum_{n=0}^{\infty}\frac{\Gamma(2a-1+2n+1)}{\Gamma(2a-1)(2n+1)!}\,z^{2n+1}.$$

We observe now that only odd powers of z contribute to the above sum over n. Adding and subtracting sums of even powers of z we obtain

$$\begin{aligned} F(\ldots) = {} & \frac{1}{2z(2a-1)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(2a-1+2n+1)}{\Gamma(2a-1)(2n+1)!}z^{2n+1} + \sum_{n=0}^{\infty}\frac{\Gamma(2a-1+2n)}{\Gamma(2a-1)(2n)!}z^{2n}\right] \\ & + \frac{1}{2z(2a-1)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(2a-1+2n+1)}{\Gamma(2a-1)(2n+1)!}z^{2n+1} - \sum_{n=0}^{\infty}\frac{\Gamma(2a-1+2n)}{\Gamma(2a-1)(2n)!}z^{2n}\right], \end{aligned}$$

which, in turn, can be rewritten as

$$F(\ldots) = \frac{1}{2z(2a-1)}\Bigg[\underbrace{\sum_{k=0}^{\infty}\frac{\Gamma(2a-1+k)}{\Gamma(2a-1)k!}z^k}_{(1-z)^{-2a+1}} - \underbrace{\sum_{k=0}^{\infty}\frac{\Gamma(2a-1+k)}{\Gamma(2a-1)k!}(-z)^k}_{(1+z)^{-2a+1}}\Bigg],$$

where in the last step we have used the above identity (2.36). Interestingly, for $a = (1-n)/2$, $z = \sqrt{5}$ one obtains a relation to the famous *Fibonacci numbers*,[11]

[11] Fibonacci sequence is the series of integer numbers F_n defined by the linear recurrence equation $F_n = F_{n-1} + F_{n-2}$. By definition, the first two numbers are 0 and 1 and each next number is found by adding up the two numbers before it: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, $\cdots$. The closed form for this sequence is given by

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = \frac{n}{2^{n-1}}\, F\left(\frac{1-n}{2}, \frac{2-n}{2}; \frac{3}{2}; 5\right). \tag{2.39}$$

Since one of the numbers $(1-n)/2$, $(2-n)/2$ is always a negative integer (or zero) for $n \geq 1$, (2.39) is in fact a finite sum.
(iii) For $a = -b$ and $c = 1/2$ one can show that

$$F\left(a, -a; \frac{1}{2}; -z^2\right) = \frac{1}{2}\left[(\sqrt{1+z^2}+z)^{2a} + (\sqrt{1+z^2}-z)^{2a}\right]. \tag{2.40}$$

In order to prove it we compare the series expansions in variable z^2 for rhs and lhs separately. For the rhs the Taylor expansion leads to the following series

$$1 + \sum_{k=0}^{\infty} 2^{2k+2} \frac{a^2(a^2-1)(a^2-2^2)\dots(a^2-k^2)}{(2k+2)!} z^{2k+2}. \tag{2.41}$$

The lhs is given by

$$\sum_{n=0}^{\infty} (-1)^n (A)_n z^{2n},$$

where $(A)_0 = 1$ and

$$(A)_n = \frac{(a)_n(-a)_n}{n!(\frac{1}{2})_n} = 2^n \frac{a(a+1)\dots(a+n-1)\overbrace{[-a(1-a)\dots(n-1-a)]}^{(-1)^n a(a-1)\dots(a-n+1)}}{n!\underbrace{(2n-1)!!}_{(2^{-n})(2n)!/n!}},$$

which simplifies to

$$1 + \sum_{n=1}^{\infty} 2^{2n} \frac{a^2(a^2-1)(a^2-2^2)\dots[a^2-(n-1)^2]}{(2n)!} z^{2n}.$$

In this last expression we replace the summation index as $n = k+1$ and obtain the Taylor expansion (2.41) of the rhs.
(iv) The identity

$$F\left(a, -a; \frac{1}{2}; \sin^2 z\right) = \cos(2az) \tag{2.42}$$

follows from (iii) after the substitution $z \to i \sin z$.
(v) In the case $a = b = 1$ and $c = 2$, the Pochhammer symbols are: $(c)_n = (1+n)(a)_n$, $(a)_n = (b)_n = n!$, and $A_n = 1/(1+n)$. One obtains then an interesting relation to the logarithm function,

$$F(1,1;2;z) = \sum_{n=0}^{\infty} \frac{z^n}{1+n} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n} = -\frac{1}{z}\ln(1-z). \tag{2.43}$$

(vi) When $a = b = 1/2$ and $c = 3/2$

$$F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right) = \frac{\arcsin\sqrt{z}}{\sqrt{z}}. \tag{2.44}$$

In order to verify that we calculate the generic coefficient in the hypergeometric series:

$$A_n = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2n+1)n!}.$$

Using the duplication formula

$$\Gamma\left(n+\frac{1}{2}\right)\Gamma(n) = 2^{1-2n}\sqrt{\pi}\,\Gamma(2n)$$

one obtains

$$A_n = \frac{(2n)!}{2^{2n}(n!)^2}\,\frac{1}{2n+1},$$

so that for $0 < z < 1$

$$\begin{aligned} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}\,\frac{z^n}{2n+1} &= \frac{1}{\sqrt{z}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}\,\frac{(\sqrt{z})^{2n+1}}{2n+1} \\ &= \frac{\arcsin\sqrt{z}}{\sqrt{z}}. \end{aligned} \tag{2.45}$$

In particular,

$$F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1\right) = \frac{\pi}{2}. \tag{2.46}$$

(vii) When $a = 1/2,\ b = 1$, and $c = 3/2$ one obtains:

$$F\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right) = \frac{1}{2z}\ln\frac{1+z}{1-z}, \tag{2.47}$$

since because of the coefficient $A_n = \frac{1}{2n+1}$ we have

$$F\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right) = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n+1} = \frac{1}{2z}\ \underbrace{2\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}}_{\ln\frac{1+z}{1-z}}.$$

(viii) By an analytic continuation $F(\dots, z^2) \to F(\dots, -z^2)$ of (2.47) one can verify that

$$F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{1}{z} \arctan z. \tag{2.48}$$

For other examples we refer to [8].

2.4.3 Pochhammer Integral and the Gauss Summation Theorem

Let us consider the integral

$$I = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,$$

which is convergent for $|z| < 1$, $\mathrm{Re}b > 0$, and $\mathrm{Re}(c-b) > 0$. Inserting the elementary series, Eq. (2.36),

$$(1-zt)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} t^n z^n,$$

one obtains:

$$I = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} \int_0^1 t^{b+n-1}(1-t)^{c-b-1}dt.$$

We observe that the n^{th} term in this sum is nothing but Euler integral (2.10) with $z = b + n$ and $w = c - b$ and so

$$I = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} = \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}.$$

In this way we obtain the following integral representation for the hypergeometric function due to Pochhammer

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt. \tag{2.49}$$

Using it we can obtain interesting information about the hypergeometric function near the point $z = 1$. As one can see from Eq. (2.36) this point is a singular point of F at least for some values of the parameters a, b, and c. (We shall see later that it nearly always is a singular point, except when $c-a-b$ is a positive integer.) This does not mean that the limit $\lim\limits_{z\to 1} F(a,b;c;z)$ is necessarily infinite.[12] We defer the investigation of the exact nature of the singularity of F at $z=1$ to the later section and focus here on the conditions for the existence of the limit $\lim\limits_{z\to 1} F(a,b;c;z)$. These can be studied with the help of the above Pochhammer formula. Indeed, substituting $z=1$ in Eq. (2.49) leads to the integral of the type (2.10)

$$\int_0^1 t^{b-1}(1-t)^{c-a-b-1}dt = \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)},$$

which is convergent for $\mathrm{Re}b > 0$ and $\mathrm{Re}(c-a-b) > 0$. It follows that under this condition the limit in question exists and is equal to

$$\lim_{z\to 1} F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.50}$$

The condition $\mathrm{Re}b > 0$ can be removed, because we can just replace Eq. (2.49) by the version with a and b interchanged, $a \leftrightarrow b$, but the other condition $\mathrm{Re}(c-a-b) > 0$ will remain essential. This result is also known as the *Gauss summation theorem*.

We also would like to remark that the general symmetry $a \leftrightarrow b$ of the hypergeometric function is, of course, intact in Eq. (2.50).

2.5 Differential Equations

2.5.1 Hypergeometric Equation: Solutions Around $z=0$

The hypergeometric series (2.34) satisfy the second order differential equation

$$z(1-z)F'' + [c-(a+b+1)z]F' - abF = 0 \tag{2.51}$$

known as the hypergeometric equation. Indeed, substituting $F = \sum\limits_{n=0}^{\infty} A_n z^n$ into this equation, equating powers of z and collecting terms results in a two-term recurrence relation:

$$(n+c)(n+1)A_{n+1} - (n+a)(n+b)A_n = 0.$$

This relation is obviously solved by

[12] Just think about the following example: $z=1$ is a singular, branch point of the function $f(z) = z+\sqrt{1-z}$, yet the limit is well defined and equal to $\lim\limits_{z\to 1} f(z) = 1$.

$$A_n = \frac{(a)_n (b)_n}{(c)_n n!} = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)n!},$$

which is the general term of the hypergeometric series. Conversely, up to an arbitrary multiplicative constant the hypergeometric function provides a solution to the hypergeometric equation regular at $z \to 0$.

According to the elementary analysis, the hypergeometric equation, being an equation of second order, has two linearly independent solutions. So we must find the second solution, which is linearly independent of $F(a, b; c; z)$. To this end we substitute $F \to z^\alpha F$. Using $F' \to z^\alpha F' + \alpha z^{\alpha-1} F$ and $F'' \to z^\alpha F'' + 2\alpha z^{\alpha-1} F'$ $+ \alpha(\alpha-1) z^{\alpha-2} F$, one easily finds

$$z(1-z)F'' + [c + 2\alpha - (a + b + 2\alpha + 1)z]F' \\ + \left[\frac{\alpha(\alpha+c-1)}{z} - \alpha(a+b+\alpha) - ab\right] F = 0.$$

If we choose $\alpha = 1 - c$ then the $1/z$ term vanishes and the above equation takes the same form as the original Eq. (2.51) but with a new set of parameters which we denote as $(\bar{a}, \bar{b}, \bar{c})$.[13] The parameter $\bar{c}$ is readily seen to be equal to $\bar{c} = 2 - c$. The two remaining parameters satisfy the system of equations

$$\bar{a}+\bar{b} = a+b-2c+2, \qquad \bar{a}\bar{b} = (1-c)(a+b-c+1)+ab = (a-c+1)(b-c+1),$$

which is clearly resolved by

$$\bar{a} = a - c + 1, \qquad \bar{b} = b - c + 1.$$

It follows that the second, linearly independent solution of (2.51) is of the form

$$z^{1-c} F(a-c+1, b-c+1; 2-c; z).$$

We see that the second solution is singular in the limit $z \to 0$.

The general solution of the hypergeometric equation, applicable inside the unit circle $|z| < 1$ is then obtained as a linear combination of the two above solutions

$$A_1 F(a, b; c; z) + B_1 z^{1-c} F(a-c+1, b-c+1; 2-c; z) \tag{2.52}$$

with arbitrary coefficients A_1 and B_1.

[13] This notation should not be confused with complex conjugate.

2.5.2 *Solutions Around* $z = 1$ *and their Relation to Solutions Around* $z = 0$

To investigate the hypergeometric equation around the point $z = 1$ we substitute $z \to 1 - z$, which immediately results in

$$z(1-z)F'' - [a+b+1-c-(a+b+1)z]F' - abF = 0.$$

The equation is again of the same form as Eq. (2.51) with the new set of parameters $\bar{a} = a$, $\bar{b} = b$, and $\bar{c} = a + b + 1 - c$. The general solution, convergent inside the circle $|z - 1| < 1$, can therefore be read off formula (2.52):

$$A_2 F(a, b; a+b+1-c; 1-z) + B_2 (1-z)^{c-a-b} F(c-a, c-b; c+1-a-b; 1-z), \tag{2.53}$$

where A_2 and B_2 are again arbitrary coefficients, which are, of course, different from those in Eq. (2.52). We see that, like around $z = 0$ there are two linearly independent solutions around $z = 1$, one regular and one singular.

The circles $|z| < 1$ and $|z - 1| < 1$ obviously overlap. In the overlap region, the two solutions (2.52) and (2.53) represent one and the same function. This means that each of the two particular solutions in Eq. (2.52) should be expressible as a linear combination of the two solutions in Eq. (2.53) and vice versa. In particular, one should have

$$\begin{aligned} F(a, b; c; z) = {} & AF(a, b; a+b+1-c; 1-z) \\ & + B(1-z)^{c-a-b} F(c-a, c-b; c+1-a-b; 1-z), \end{aligned}$$

with this time definite coefficients A and B. To begin with let us assume $\mathrm{Re}(c - a - b) > 0$ and set $z = 1$ in the above. Using the Gauss summation theorem, Eq. (2.50), for the lhs one finds

$$A = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Next we set $z = 0$

$$1 = AF(a, b; a+b+1-c; 1) + BF(c-a, c-b; c+1-a-b; 1).$$

Using again the Gauss theorem and identity (2.6), we simplify the first term on the rhs to

$$\begin{aligned} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} = \frac{\sin[\pi(c-a)]\sin[\pi(c-b)]}{\sin(\pi c)\sin[\pi(c-a-b)]} \\ & = \frac{\cos[\pi(a-b)] - \cos[\pi(2c-a-b)]}{\cos[\pi(a+b)] - \cos[\pi(2c-a-b)]} = 1 + \frac{\cos[\pi(a-b)] - \cos[\pi(a+b)]}{\cos[\pi(a+b)] - \cos[\pi(2c-a-b)]} \end{aligned}$$

$$= 1 - \frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi c)\sin[\pi(c-a-b)]}.$$

It follows that

$$B = \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(1-c)\Gamma(c+1-a-b)}\frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi c)\sin[\pi(c-a-b)]} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}.$$

Collecting the results, we obtain the following relation between the regular solution around $z = 0$ and the two solutions around $z = 1$:

$$\begin{aligned} F(a,b;c;z) = &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b+1-c;1-z) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} \\ &\times F(c-a,c-b;c+1-a-b;1-z). \end{aligned} \tag{2.54}$$

On the grounds of the analytic continuation theorem of Sect. 1.3.6, one can now argue that the restriction $\mathrm{Re}(c-a-b) > 0$ can be removed so that the above formula is valid for all a, b, and c and for $\mathrm{Re}(a+b-c) > 0$ reveals the nature of the divergency of the function $F(a,b;c;z)$ as z approaches 1 from inside the unit circle:

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\frac{1}{(1-z)^{a+b-c}} + O\left[(1-z)^{1+c-a-b}\right].$$

The remaining three formulas relating solutions in Eqs. (2.52) and (2.53) can be obtained in a similar manner or read off Eq. (2.54) by suitable substitutions of the parameters and the variables z and $1-z$.

2.5.3 Barnes' Integral

We have so far found linearly independent solutions to hypergeometric Eq. (2.51) in terms of hypergeometric series convergent inside the circles $|z| < 1$ and $|z-1| < 1$ and obtained the relations between these solutions by using the Gauss summation theorem. To find solutions outside these circles and in particular for $z \to \infty$, we could, in principle, continue along these lines by studying Eq. (2.51) under the substitution $z \to 1/z$.

It is, however, advantageous to employ a different method here based on the integral representation ideas of Sect. 2.1.2. Clearly, if we set in Eq. (2.5)

$$A(s) = \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+s)}$$

then we recover the correct Taylor coefficients $A(n) = (a)_n(b)_n/(c)_n$. This gives rise to the integral

$$I = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} \Gamma(-s)(-z)^s ds,$$

where the integration contour is mainly along the imaginary axis but is deformed in such a way as to avoid the three strings of poles $s = n$, $s = -a - n$, and $s = -b - n$ ($n = 0, 1, 2, \ldots$). This integral is known as *Barnes' integral*. The convergence properties of Barnes' integral are such that it is well defined for all z and the integration contour can be shifted all the way to the right for $|z| < 1$ and all the way to the left for $|z| > 1$ (see [3] for details).

So, for $|z| < 1$ the poles $s = n$ contribute and, in accordance with formula (2.5), we have

$$I = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b; c; z).$$

For $|z| > 1$, the poles at $s = -a - n$ and $s = -b - n$ contribute, so

$$I = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(b-a-n)}{\Gamma(c-a-n)} \Gamma(a+n)(-z)^{-a-n} + (a \leftrightarrow b).$$

Now we use identity (2.6) in order to have n and not $-n$ in the arguments of the Gamma functions:

$$I = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\sin[\pi(c-a-n)]}{\sin[\pi(b-a-n)]} \frac{\Gamma(1-c+a+n)\Gamma(a+n)}{\Gamma(1-a+b+n)} (-z)^{-a-n} + (a \leftrightarrow b).$$

Next we re-write this as

$$I = \frac{\sin[\pi(c-a)]}{\sin[\pi(b-a)]} \frac{\Gamma(a)\Gamma(1-c+a)}{\Gamma(1-a+b)} (-z)^{-a} \sum_{n=0}^{\infty} \frac{(a)_n(1-c+a)_n}{(1-a+b)_n n!} \left(\frac{1}{z}\right)^n + (a \leftrightarrow b)$$

and use identity (2.6) again to obtain

$$I = \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(c-a)} (-z)^{-a} F\left(a, 1-c+a; 1-a+b; \frac{1}{z}\right) + (a \leftrightarrow b).$$

The two ways of calculating Barnes' integral thus result in the analytic continuation formula from $|z| < 1$ to $|z| > 1$ for the hypergeometric function:

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} F\left(a, a+1-c; a+1-b; \frac{1}{z}\right)$$
$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} F\left(b, b+1-c; b+1-a; \frac{1}{z}\right). \quad (2.55)$$

2.6 Confluent Hypergeometric Series

Given the very appealing form of the hypergeometric series coefficients it might appear surprising that in Sect. 2.4.2 we did not recover the exponential function as a special case. In fact, e^x can be represented by a distinct class of confluent hypergeometric series, which we would like to discuss next.

2.6.1 Definition and Differential Equation

Consider the following functional series in the variable z

$$F(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!(c)_n} z^n, \quad (2.56)$$

where coefficients $(a)_n$ and $(c)_n$ are the Pochhammer symbols introduced in (2.35). According to the standard ratio test we obtain

$$\lim_{n\to\infty} \left| \frac{(a)_{n+1}(c)_n}{(a)_n(c)_{n+1}} \frac{z^{n+1}}{z^n} \frac{n!}{(n+1)!} \right| = \left\{ \lim_{n\to\infty} \left| \frac{(a+n)}{(c+n)(n+1)} \right| \right\} |z| \underbrace{\rightarrow}_{\text{finite } z} 0.$$

Therefore the above series defines an analytic function for all finite z. This series, called the *confluent hypergeometric function*,[14] is closely connected to the hypergeometric function and is obtained as a limit of $F(a,b;c;z/b)$ at $b \to \infty$:

$$F(a,c;z) = \lim_{b\to\infty} F\left(a, b; c; \frac{z}{b}\right).$$

Substituting $z \to z/b$ in the hypergeometric equation (2.51) and then taking the limit $b \to \infty$, we find that the confluent function is a solution of the following differential equation

$$zF''(a,c;z) + (c-z)F'(a,c;z) - aF(a,c;z) = 0. \quad (2.57)$$

[14] Sometimes it is also referred to as *Kummer series* or *Kummer function* with the common notation $M(a,c,z)$.

Indeed, substitute the series $F = \sum_{n=0}^{\infty} B_n z^n$ as well as its derivatives

$$F' = \sum_{n=1}^{\infty} n B_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) B_{n+1} z^n \quad \text{and} \quad F'' = \sum_{n=1}^{\infty} (n+1) n B_{n+1} z^{n-1}$$

into (2.57). Then equating powers of z, for coefficients B_n we obtain the recurrence relation

$$(c+n)(n+1)B_{n+1} - (a+n)B_n = 0,$$

which is obviously solved by $B_n = (a)_n/n!(c)_n$ as prescribed by (2.56).

Differential Eq. (2.57) for the confluent hypergeometric function has only two singularities: one simple pole at the point $z = 0$ and an irregular singularity at infinity. It differs from the case of the hypergeometric Eq. (2.51) which has an additional pole at the point $z = 1$. For the Eq. (2.57), this pole therefore merges into one of the above enumerated singularities, namely into the irregular one.[15] The term 'confluent' for the function $F(a, c; z)$ displays precisely this circumstance.

In order to find the second solution of Eq. (2.57) we substitute $F \to z^{1-c}F$. The respective derivatives are $F' \to (1-c)z^{-c}F + z^{1-c}F'$ and $F'' \to z^{1-c}F'' + 2(1-c)z^{-c}F' - c(1-c)z^{-c-1}F$, and one easily obtains the equation

$$zF'' + (2-c-z)F' - (a-c+1)F = 0,$$

[15] Both the hypergeometric Eq. (2.51) and its degenerate form (2.57) is, in fact, a special case of the Riemann's differential equation (see, for instance [3]). The latter allows three simple poles to be situated anywhere in the complex plane z including infinity. In a particular case of two poles fixed at $z = 0$ and $z = \infty$ as well as a third one at z_3, respectively, the Riemann's hypergeometric equation has the following form

$$u'' + \Big(\frac{1-\alpha-\alpha'}{z} + \frac{1-\gamma-\gamma'}{z-z_3}\Big)u' + \Big[-\frac{z_3\,\alpha\alpha'}{z^2(z-z_3)} + \frac{\beta\beta'}{z(z-z_3)} + \frac{z_3\,\gamma\gamma'}{z(z-z_3)^2}\Big]u = 0.$$

The parameters $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ (which are referred to as exponents) satisfy the condition

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' - 1 = 0.$$

Setting $z_3 = 1$ and $\alpha = \gamma = 0$ (observe $\alpha, \beta, \gamma \leftrightarrow \alpha', \beta', \gamma'$ symmetry), the Riemann's equation immediately becomes similar to the hypergeometric Eq. (2.51). Considering another set of exponents, namely setting $\beta = 0$, $\beta' = -z_3$, and $\alpha + \alpha' = 1$ and moving the singular point z_3 to infinity $z_3 \to \infty$ the Riemann's equation transforms to the following one

$$u'' + u' + \Big[\frac{\alpha(1-\alpha)}{z^2} + \frac{\gamma}{z}\Big]u = 0,$$

which is known as the *Whittaker's differential equation*. Further substitution $u(z) = F(z)\, e^{-z} z^{\alpha}$ reduces this equation to the confluent form

$$zF'' + (2\alpha - z)F' - (\alpha - \gamma)F = 0.$$

which has the same form as the original Eq. (2.57), but with a new set of parameters, namely, $a \to a - c + 1$ and $c \to 2 - c$. Hence the second, linearly independent solution takes the form

$$z^{1-c} F(a - c + 1, 2 - c; z).$$

This solution is singular in the limit $z \to 0$.

2.6.2 Integral Representation

For $\mathrm{Re} c > \mathrm{Re} a > 0$, the following integral representation,

$$F(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt}\, t^{a-1} (1-t)^{c-a-1} dt \tag{2.58}$$

is valid. In order to prove it we use the integral representation of the Beta function (Euler integral of the first kind), Eq. (2.10). The ratio of the Pochhammer symbols entering the definition (2.56) is expressed as follows

$$\begin{aligned} \frac{(a)_n}{(c)_n} &= \frac{\Gamma(c)}{\Gamma(a)} \frac{\Gamma(a+n)}{\Gamma(c+n)} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} B(a+n, c-a) \\ &= \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a+n-1} (1-t)^{c-a-1} dt, \end{aligned}$$

where we have used (2.11) to write $B(a+n, c-a) = \Gamma(a+n)\Gamma(c-a)/\Gamma(c+n)$. Inserting this result into the series (2.56) and changing the order of the summation and integration, we finally obtain

$$F(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} \underbrace{\sum_{n=0}^{\infty} \frac{(zt)^n}{n!}}_{e^{zt}}\, dt.$$

With the help of Eq. (2.58), we can derive the following functional relation:

$$F(a, c; z) = e^z F(c - a, c; -z). \tag{2.59}$$

To that end we need only to substitute $t \to 1 - t$ in the integrand of (2.58). Then we have

$$F(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{-z(t-1)} t^{c-a-1} (1-t)^{a-1} dt$$
$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \frac{\Gamma(c-a)\Gamma(a)}{\Gamma(c)} e^z F(c-a, c; -z) = e^z F(c-a, c; -z).$$

The following recurrence relation

$$aF(a+1, c+1; z) = (a-c)F(a, c+1; z) + cF(a, c; z) \tag{2.60}$$

holds as well. Indeed, using the integral representation Eq. (2.58), we obtain

$$aF(a+1, c+1; z) - cF(a, c; z) = \frac{a\ \Gamma(c+1)}{\Gamma(a+1)\Gamma(c-a)} \int_0^1 e^{zt} t^a (1-t)^{c-a-1} dt$$
$$- \frac{c\ \Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1} dt = - \frac{c\ \Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a} dt$$
$$= - \underbrace{\frac{c\ \Gamma(c)}{\Gamma(a)\Gamma(c-a)} \frac{\Gamma(a)\Gamma(c-a+1)}{\Gamma(c+1)}}_{=(c-a)} F(a, c+1; z) = (a-c)F(a, c+1; z).$$

2.6.3 Special Cases

Just as for the ordinary hypergeometric function, some of the elementary and special functions can be expressed as special cases of the confluent hypergeometric function.
(i) In a particular case of $a = b$ we recognize the exponential function

$$F(a, a; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z, \tag{2.61}$$

which is obviously independent on a. The above functional relation (2.59) is, evidently, satisfied: $F(a, a; z) = e^z F(0, a; -z) \equiv e^z$.
(ii) The error function, which is widely used in probability and statistics,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{2.62}$$

is expressed as

$$\text{erf}(x) = \frac{2x}{\sqrt{\pi}}\, F\Big(\frac{1}{2}, \frac{3}{2}; -x^2\Big). \tag{2.63}$$

It immediately follows from the integral representation (2.58):

$$F\Big(\frac{1}{2}, \frac{3}{2}; -x^2\Big) = \underbrace{\frac{\Gamma(3/2)}{\Gamma(1/2)\Gamma(1)}}_{=1/2} \int_0^1 \frac{e^{-x^2 t}}{\sqrt{t}} dt = \frac{1}{x} \int_0^x e^{-t^2} dt.$$

(iii) When $a = 1$ and $b = 2$, the following identity holds:

$$F(1, 2; 2z) = \underbrace{\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}}_{=1} \int_0^1 e^{2zt} dt = \frac{1}{2z}(e^{2z} - 1) = e^z\, \frac{\sinh z}{z}. \tag{2.64}$$

(iv) In Example 1.2 on page 11 we have encountered Bessel function of integer order. It can be generalized to non-integer case, for instance the *modified Bessel function* of the first kind can be represented by the following series[16]

[16] The Bessel functions are solutions $y(z)$ of the differential equation [3]

$$\frac{d^2 y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \Big(1 - \frac{\nu^2}{z^2}\Big) y = 0.$$

The Bessel function of the first kind, which is regular around $z = 0$, is defined by its Taylor series expansion:

$$J_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k!\Gamma(\nu + k + 1)}, \quad |\arg z| < \pi.$$

For non-integer ν, the functions $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent, so that $J_{-\nu}(z)$ may serve as a second solution, which has a singularity at $z = 0$. Bessel functions of the second kind, denoted as $N_\nu(z)$ and sometimes called the *Neumann functions*, are related to $J_{-\nu}(z)$ by

$$N_\nu(z) = \frac{1}{\sin \nu\pi}\Big[\cos \nu\pi\, J_\nu(z) - J_{-\nu}(z)\Big], \quad |\arg z| < \pi.$$

For integer $\nu = n$, this solution is defined by taking the $\lim_{\nu\to n} N_\nu(z)$. For a special case of a purely imaginary argument, the Bessel equation reduces to the form

$$\frac{d^2 y}{dz^2} + \frac{1}{z}\frac{dy}{dz} - \Big(\frac{\nu^2}{z^2} + 1\Big) y = 0.$$

The respective solutions, the modified Bessel functions of the first and second kind, are defined by

$$I_\nu(z) = i^{-\nu} J_\nu(iz) \;\text{ and }\; K_\nu(z) = \frac{\pi}{2 \sin \nu\pi}[I_{-\nu}(z) - I_\nu(z)], \tag{2.65}$$

respectively. The $K_\nu(z)$ function is known also as the *Macdonald function*. As functions of a real argument at large values of $|z|$, $I_\nu(z) \sim z^{-1/2}\, e^{|z|}$ and $K_\nu(z) \sim z^{-1/2}\, e^{-|z|}$ are exponentially increasing and decreasing functions, respectively. For more details see for instance [9].

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k} k! \Gamma(\nu + k + 1)}. \tag{2.66}$$

It arises in many problems of applied physics and for $\mathrm{Re}(\nu + 1/2) > 0$ can be related to the confluent hypergeometric function via

$$F\left(\frac{1}{2} + \nu, 1 + 2\nu; 2z\right) = \Gamma(\nu + 1) e^z \left(\frac{2}{z}\right)^\nu I_\nu(z). \tag{2.67}$$

To establish this result, we consider the integral

$$\mathcal{I} = \int_{-1}^{1} dt\, e^{zt} (1 - t^2)^{\nu - \frac{1}{2}}.$$

We observe that the integral representation (2.58), by means of a substitution $t \to (t + 1)/2$, can be rewritten as

$$F(a, c; z) = \frac{2^{1-c} e^{\frac{z}{2}}}{B(a, c - a)} \int_{-1}^{1} e^{\frac{z}{2}t} (1 - t)^{c-a-1} (1 + t)^{a-1} dt,$$

whence at $c = 2a = 2\nu + 1$ and $z \to 2z$

$$\mathcal{I} = \frac{\sqrt{\pi}\, \Gamma(\nu + \frac{1}{2})\, e^{-z}}{\Gamma(\nu + 1)}\, F\left(\frac{1}{2} + \nu, 1 + 2\nu; 2z\right). \tag{2.68}$$

On the other hand, at $\mathrm{Re}(\nu + \frac{1}{2}) > 0$ the integral $\mathcal{I}$ may be expressed as

$$\mathcal{I} = \left(\frac{2}{z}\right)^\nu \sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)\, I_\nu(z). \tag{2.69}$$

This result is easily proven by an expansion of the exponential in the integrand of $\mathcal{I}$ (notice that the integral $\mathcal{I}$ is an even function of z) in powers of z and a subsequent re-arrangement of the integration and the summation order:

$$\mathcal{I} = 2 \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \sum_{k=0}^{\infty} \frac{(zt)^{2k}}{(2k)!}\, dt = \sum_{k=0}^{\infty} \frac{(z)^{2k}}{(2k)!}\, B\left(k + \frac{1}{2}, \nu + \frac{1}{2}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(z)^{2k}}{(2k)!} \frac{\overbrace{\Gamma\left(k + \frac{1}{2}\right)}^{\sqrt{\pi}(2k)!/2^{2k}k!} \Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu + k + 1)} = \sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right) \underbrace{\sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k} k! \Gamma(k + \nu + 1)}}_{I_\nu(z)\, (2/z)^\nu}.$$

Equating the rhs of (2.68) and (2.69), we obtain the identity (2.67). In particular, for $\nu = 1/2$ from (2.67) and (2.64) we obtain a special case for the modified Bessel function

$$I_{\frac{1}{2}}(z) = \sqrt{\frac{z}{2}}\,\frac{e^{-z}}{\underbrace{\Gamma\left(1+\frac{1}{2}\right)}_{\sqrt{\pi}/2}}\,\underbrace{F(1,2;2z)}_{e^{z}\sinh z/z} = \sqrt{\frac{2}{\pi z}}\sinh z. \tag{2.70}$$

Further important special cases will be given in Sect. 4.10.

2.7 Generalized Hypergeometric Series

The hypergeometric series (2.34) can be generalized to the case of p parameters of a and b-type and q parameters of c-type

$$_pF_q[(a);(b);z] = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}, \tag{2.71}$$

where $(a) = a_1,\ldots,a_p$, $(a_i)_n = a_i(a_i+1)\cdots(a_i+n)$ and $(b_i)_n = b_i(b_i+1)\cdots(b_i+n)$ are the Pochhammer symbols. The series (2.71) converges for all finite z if $p \le q$, for all $|z| < 1$ at $p = q+1$ and diverges for all $z \ne 0$ if $p > q+1$. It can be easily shown that these series satisfy the following differential equation:

$$\begin{aligned}\Delta(\Delta+b_1-1)(\Delta+b_2-1)\cdots(\Delta+b_q-1)F\\ = z(\Delta+a_1)(\Delta+a_2)\cdots(\Delta+a_p)F,\end{aligned} \tag{2.72}$$

where

$$\Delta = z\frac{d}{dz}.$$

The most widespread class of generalized hypergeometric functions is the one with $p = q+1$, a typical representative of which is the 'conventional' series $F(\ldots) = {}_2F_1(\ldots)$ considered above. An analytic continuation beyond the singularity at $z = 1$ can be found with the help of the Barnes' integral (2.5). To that end we define in Eq. (2.5)

$$A(s) = \frac{\prod_{i=1}^{p}\Gamma(a_i+s)}{\prod_{j=1}^{q}\Gamma(b_j+s)}.$$

For $|z| < 1$ we then shift the integration contour to ∞[17] and obtain

[17] Or alternatively close the contour by an semi circle to the right of the integration line of the Barnes' integral.

$$I = \frac{\prod_{i=1}^{q} \Gamma(a_j)}{\prod_{j=1}^{p} \Gamma(b_i)} \, {}_pF_q[(a); (b); z].$$

For $|z| > 1$ we move the integration line to $-\infty$ instead. Then we have to take care of the poles at $s = -a_k - n$, so

$$I = \sum_{k=1}^{p} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{\prime p} \Gamma(a_i - a_k - n)}{\prod_{j=1}^{q} \Gamma(b_j - a_k - n)} \Gamma(a_k + n) \frac{(-1)^n}{n!} (-z)^{-a_k - n},$$

where the primed product does not contain the term $i = k$. Now we would like to use the identity (2.6) in the form

$$\Gamma(c - n) = \pi(-1)^n / [\sin(\pi c)\, \Gamma(1 - c + n)],$$

in order to rewrite the above expression as

$$I = \sum_{k=1}^{p} (-z)^{-a_k} \sum_{n=0}^{\infty} \prod_{i=1}^{\prime p} \left[\frac{\pi(-1)^n}{\sin[\pi(a_i - a_k)]} \frac{1}{\Gamma(1 - a_i + a_k + n)} \right]$$
$$\times \prod_{j=1}^{q} \left[\frac{(-1)^n}{\pi} \sin[\pi(b_j - a_k)] \Gamma(1 - b_j + a_k - n) \right] \frac{\Gamma(a_k + n)}{n!} (1/z)^n.$$

Further we use the fact that $p = q + 1$ and remove n from the Gamma functions by virtue of $\Gamma(c + n) = \Gamma(c)(c)_n$ to arrive at

$$I = \sum_{k=1}^{p} (-z)^{-a_k} \sum_{n=0}^{\infty} \prod_{i=1}^{\prime p} \frac{\Gamma(a_i - a_k)}{(1 - a_i + a_k)_n} \prod_{j=1}^{q} \frac{(1 - b_j + a_k)_n}{\Gamma(b_j - a_k)} \frac{\Gamma(a_k)(a_k)_n}{n!} (1/z)^n$$
$$= \sum_{k=1}^{p} (-z)^{-a_k} \left[\prod_{i=1}^{\prime p} \Gamma(a_i - a_k) \right] \Gamma(a_k) \left[\prod_{j=1}^{q} \frac{1}{\Gamma(b_j - a_k)} \right]$$
$$\times \sum_{n=0}^{\infty} (a_k)_n \left[\prod_{i=1}^{\prime p} \frac{1}{(1 - a_i + a_k)_n} \right] \left[\prod_{j=1}^{q} (1 - b_j + a_k)_n \right] \frac{1}{n!} (1/z)^n$$
$$= \sum_{k=1}^{p} \frac{\Gamma(a_k) \prod_{i=1}^{\prime p} \Gamma(a_i - a_k)}{\prod_{j=1}^{q} \Gamma(b_j - a_k)} (-z)^{a_k}$$
$$\times \, {}_pF_q \left[a_k, 1 - b_1 + a_k, \ldots, 1 - b_q + a_k; 1 + a_1 + a_k, \ldots, 1 - a_p + a_k; 1/z \right].$$

As required the F-function has $1 + q = p$ parameters of the a and b-type and $p - 1 = q$ parameters of the c-type (because $1 = 1 - a_k + a_k$ parameter is absent). To summarize we obtain the following relation for the analytic continuation

$$
{}_pF_q[(a_i);(b_j);z] = \sum_{k=1}^{p} \prod_{i=1}^{\prime p} \left[\frac{\Gamma(a_i - a_k)}{\Gamma(a_i)} \right] \prod_{j=1}^{q} \left[\frac{\Gamma(b_j)}{\Gamma(b_j - a_k)} \right] (-z)^{-a_k} \tag{2.73}
$$
$$
\times \; {}_pF_q \left[a_k, 1-b_1+a_k, \dots, 1-b_q+a_k; 1+a_1+a_k, \dots, 1-a_p+a_k; 1/z \right],
$$

where recall that $p = q + 1$. In particular, for $q = 1$ and $p = 2$ we immediately recover the result for the conventional hypergeometric series (2.55).

2.8 Examples from Mathematical Physics

2.8.1 Momentum Distribution Function of Interacting Systems

The calculation of particle distribution functions with respect to their quantum numbers is one of the most important tasks in many-body problems. While pure non-interacting bosonic and fermionic systems are subject to Bose and Fermi distribution functions, the situation changes dramatically as soon as the particles start to interact. The momentum distribution function $n(k)$ is usually calculated as a Fourier transform of the equal time Green's function [10]. In one dimension the problem of interacting fermions is under certain conditions analytically solvable and the corresponding Green's function is given by

$$
G(x) = \frac{1}{2\pi} \frac{1}{(-ix+\delta)^{\alpha}}, \tag{2.74}
$$

where δ is a positive infinitesimal and α is a dimensionless interaction parameter. It is $\alpha = 1$ in a non-interacting system and then

$$
n(k) = \int_{-\infty}^{\infty} dx\, e^{ikx}\, G(x) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \frac{e^{ikx}}{-ix+\delta}
$$

can be evaluated by closing in the upper/lower half-plane for positive/negative k. The result, known from the theory of non-interacting Fermi gas, is the Heaviside step-function $n(k) = \Theta(-k)$ defined in (1.26).[18]

For generic α one is confronted with a multivalued integrand. We chose to cut the complex plane along $[-i\delta, -i\infty)$ starting at the branch point $-i\delta$. Next we deform the contour in the way shown in Fig. 2.4. On $C_\pm$ we use $(-z)^{\alpha} = z^{\alpha}\, e^{\mp i\pi\alpha}$. Furthermore the integral along C_ρ vanishes. Then $(ix = z)$

[18] We have chosen the Fermi edge to be located at $k = 0$.

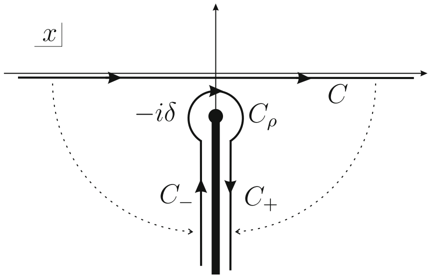

Fig. 2.4 Deformation of the contour for the calculation of $n(k)$

$$
\begin{aligned}
n(k<0) &= \frac{-i}{2\pi}\left[\int_\infty^0 dz\,\frac{e^{-zk}}{(-z)^\alpha} + \int_0^\infty dz\,\frac{e^{-kz}}{(-z)^\alpha}\right] \\
&= -\frac{i}{2\pi}\left[-\int_0^\infty dz\,e^{-kz}\,\frac{e^{-i\pi\alpha}}{z^\alpha} + \int_0^\infty e^{-kz}\,\frac{e^{i\pi\alpha}}{z^\alpha}\right] \\
&= \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty dz\,e^{-kz}\,z^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi}\,|k|^{\alpha-1}\,\Gamma(1-\alpha)\,e^{-|k|\delta}.
\end{aligned}
\tag{2.75}
$$

For positive k the contour can be shifted to infinity along the imaginary positive semiaxis and the integral is identically zero. The limiting case of the non-interacting system is nicely reproduced by the above result as $\sin(\pi\alpha)\Gamma(1-\alpha) = \pi$ for $\alpha \to 1$. The unphysical behaviour of (2.75) for $k \to -\infty$ is due to the fact that the original Green's function (2.74) only captures the long wavelength behaviour of the actual Green's function. Therefore (2.75) is only valid around the Fermi edge.

2.8.2 One–Dimensional Schrödinger Equation

For a genuine one–dimensional quantum mechanical problem, the wave-function must satisfy the equation[19]

$$
-\frac{1}{2}\frac{d^2\psi}{dx^2} + [U(x) - E]\,\psi = 0 \tag{2.76}
$$

for $x \in (-\infty, \infty)$. The boundary conditions satisfied by $\psi(x)$ are that $\psi(x)$ is bounded for $x \to \pm\infty$. According to the Liouville theorem (see Sect. 1.2.5) such solution cannot be an analytic function everywhere. In fact, for most problems the corresponding Schrödinger equation is an equation with several singularities and thus is solved by some version of hypergeometric function.

There is no explicit solution to this equation for an arbitrary potential $U(x)$. So we shall first discuss general properties of the solutions and then provide several solvable examples illustrating general points. We begin by assuming that $U(x)$ is a

[19] We use units in which $\hbar^2/m = 1$.

continuous function of x and that both limits

$$\lim_{x \to \pm\infty} U(x) = U_{1(2)}$$

exist. Without loss of generality we also set $U_1 \leq U_2$. For large enough x we can approximate $U(x)$ in Eq. (2.76) by its limit U_1 or U_2. The asymptotic form of the solution for large values of the variable therefore is

$$\psi(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} \quad \text{for} \quad x \to -\infty \tag{2.77}$$

and

$$\psi(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x} \quad \text{for} \quad x \to +\infty,$$

where $k_1 = \sqrt{2(E - U_1)}$, $k_2 = \sqrt{2(E - U_2)}$ and we set $E > U_2$ for now. We note that as the two expressions above are asymptotic forms of one and the same solution $\psi(x)$ to a linear differential equation, the coefficients must obey a linear relation

$$A_2 = \alpha A_1 + \beta B_1,$$

and similarly $B_2 = \alpha' A_1 + \beta' B_1$, where α and β are, in general complex, coefficients (α' and β' will be related to α and β shortly).

Because Eq. (2.76) is real, if $\psi(x)$ is a solution then its complex conjugate $\psi^*(x)$ is also a solution. This fact has two consequences. First, the asymptotic form of $\psi^*(x)$,

$$\psi^*(x) = A_1^* e^{-ik_1 x} + B_1^* e^{ik_1 x} \quad \text{for} \quad x \to -\infty$$

and

$$\psi^*(x) = A_2^* e^{-ik_2 x} + B_2^* e^{ik_2 x} \quad \text{for} \quad x \to +\infty,$$

only differs from that of $\psi(x)$ by re-labeling the constants. So if $A_2 = \alpha A_1 + \beta B_1$ holds for A_2 then $B_2^* = \alpha B_1^* + \beta A_1^*$, or $B_2 = \beta^* A_1 + \alpha^* B_1$, should hold for B_2. We conclude that $\alpha' = \beta^*$ and $\beta' = \alpha^*$. One arrives at the second consequence of $\psi^*(x)$ being a solution by observing that the functions $\psi(x)$ and $\psi^*(x)$ are linearly independent (unless $A_{1(2)} = \pm B_{1(2)}$). We therefore construct the *Wronskian determinant*

$$W[\psi, \psi^*] = \psi \frac{d\psi^*}{dx} - \psi^* \frac{d\psi}{dx}.$$

For the Schrödinger equation it is a constant:

$$\frac{d}{dx} W[\psi, \psi^*] = \psi \frac{d^2\psi^*}{dx^2} - \psi^* \frac{d^2\psi}{dx^2} = 2[E - U(x)]|\psi|^2 - 2[E - U(x)]|\psi|^2 = 0.$$

From the asymptotic form of $\psi(x)$ and $\psi^*(x)$ then follows the identity

$$k_1(|A_1|^2 - |B_1|^2) = k_2(|A_2|^2 - |B_2|^2).$$

The physical interpretation of $W[\psi, \psi^*]$ is that of a probability flow and the above identity reflects the conservation of it. Expressing here A_2 and B_2 via A_1 and B_1, one obtains a constraint:

$$|\alpha|^2 - |\beta|^2 = \frac{k_1}{k_2}.$$

We see that for $E > U_2$ there are two linearly independent bounded solutions to the Schrödinger equation. In quantum mechanics, the solution of the form e^{ikx} ($k > 0$) is interpreted as a particle wave travelling from left to right and e^{-ikx} — a wave in the opposite direction. Therefore setting $B_2 = 0$ describes a physical situation of an incoming wave (A_1) from the left which is partially reflected (B_1) and partially transmitted (A_2) through the potential $U(x)$. The reflection coefficient R is then defined by the ratio of the reflected to the incoming wave,

$$R = \frac{|B_1|^2}{|A_1|^2} = \frac{|\beta|^2}{|\alpha|^2}, \tag{2.78}$$

while the transmission coefficient T is defined via

$$T = 1 - R = \frac{k_1}{k_2}\frac{1}{|\alpha|^2}.$$

One can show that these coefficients remain the same when the wave is coming from the opposite direction, that accounts for the second linearly independent solution.

As an example we would like to solve Schrödinger equation for a step-like potential of the form

$$U(x) = \frac{U_0}{2}\left[1 + \tanh\left(\frac{x}{2a}\right)\right]. \tag{2.79}$$

With $U_1 = 0$ and $U_2 = U_0$ the above discussion of the asymptotic behaviour is still valid. For the energy $E = \hbar^2 k^2/2m = k^2/2$ we obtain

$$\psi''(x) + \left[k^2 - 2U(x)\right]\psi(x) = 0.$$

In the next step we make the substitution $y = 1/(1+e^{x/a})$ in order to obtain an equation with polynomial coefficients and introduce new parameters $\kappa^2 = (ka)^2 = 2a^2E$, $\lambda^2 = 2a^2U_0$. Then since

$$\frac{d}{dx} = -\frac{y(1-y)}{a}\frac{d}{dy},$$

we obtain

$$y(1-y)\psi'' + (1-2y)\psi' + \left[\frac{\kappa^2}{y(1-y)} - \frac{\lambda^2}{y}\right]\psi = 0.$$

This equation has poles at 0, 1 and ∞ and therefore has hypergeometric solutions. In order to see that explicitly we make the substitution $\psi(y) = y^{\alpha}(1-y)^{\beta} f(y)$, where $\alpha^2 = \lambda^2 - \kappa^2$, and $\beta^2 = -\kappa^2$. This procedure yields

$$y(1-y)\, f'' + [(2\alpha+1) - (2\beta+2\alpha+2)y]\; f' \\ - (\alpha+\beta)(\alpha+\beta+1)\, f = 0.$$

According to (2.51) one particular solution is given by

$$f(y) = C\, F(\alpha+\beta, \alpha+\beta+1; 2\alpha+1; y),$$

where C is a constant. We shall show now, that this solution possesses all required asymptotic properties. For $x \to \infty$ $y \to e^{-x/a} \to 0$ and therefore $f(y) \to C$. For the original wave function we thus obtain $\psi(x) \to C y^{\alpha} \approx C e^{-\alpha x/a}$. Further we are confronted with two different situations:

- $\lambda > \kappa$, then α is a positive real number. In this case the wave function decays exponentially into the step as expected for $E < U_0$.
- $\lambda < \kappa$, then $\alpha = -ik'a$ is purely imaginary. Then $\psi(x) \to Ce^{ik'x}$ with $k'^2 = 2(E - U_0)$. So we have a freely propagating plane wave.

On the other hand, for $x \to -\infty$ $y \to 1$ and therefore $1 - y \approx e^{x/a} \to 0$. Here it is more convenient to switch to the analytic continuation prescription (2.54) and rewrite

$$\begin{aligned} &F(\alpha+\beta, \alpha+\beta+1; 2\alpha+1; y) \\ &= \frac{\Gamma(2\alpha+1)\,\Gamma(-2\beta)}{\Gamma(\alpha-\beta)\,\Gamma(\alpha-\beta+1)}\, F(\alpha+\beta, \alpha+\beta+1; 2\alpha+1; 1-y) \\ &\quad + (1-y)^{-2\beta}\, \frac{\Gamma(2\alpha+1)\,\Gamma(2\beta)}{\Gamma(\alpha+\beta)\,\Gamma(\alpha+\beta+1)} F(\alpha-\beta, \alpha-\beta+1; -2\beta+1; 1-y). \end{aligned}$$

Hence, remembering that $1 - y = e^{x/a}$ we get for $x \to -\infty$

$$\psi(x) \to C \left[\frac{\Gamma(2\alpha+1)\,\Gamma(-2\beta)}{\Gamma(\alpha-\beta)\,\Gamma(\alpha-\beta+1)}\, e^{\beta x/a} + \frac{\Gamma(2\alpha+1)\,\Gamma(2\beta)}{\Gamma(\alpha+\beta)\,\Gamma(\alpha+\beta+1)}\, e^{-\beta x/a} \right]$$

Since $\beta = ika$ we indeed obtain for the asymptotic form of the solution a linear superposition of left/right moving plain waves. By comparing it with (2.77) we can then calculate the reflection coefficient as given in (2.78),

$$R = \left| \frac{\Gamma(2\beta)\,\Gamma(\alpha-\beta)\,\Gamma(\alpha-\beta+1)}{\Gamma(-2\beta)\,\Gamma(\alpha+\beta)\,\Gamma(\alpha+\beta+1)} \right|^2 . \tag{2.80}$$

Let us now again consider two different cases:

- $E < U_0$, then $\beta = i\kappa$ is purely imaginary while α is a positive real number. Then the numerator and denominator of (2.80) are complex conjugate numbers and thus $R = 1$ as expected.
- $E > U_0$, then β remains, of course, the same but $\alpha = -i\sigma$ becomes purely imaginary. Then only $|\Gamma(2\beta)/\Gamma(-2\beta)|^2 = 1$ while for the rest using the recurrence relation (2.2) we obtain

$$R = \left|\frac{(\alpha+\beta)\,\Gamma^2(\alpha-\beta+1)}{(\alpha-\beta)\,\Gamma^2(\alpha+\beta+1)}\right|^2 = \left(\frac{\kappa-\sigma}{\kappa+\sigma}\right)^2 \left[\left|\frac{\Gamma(1-i(\kappa+\sigma))}{\Gamma(1+i(\kappa-\sigma))}\right|^2\right]^2 .$$

Now we can take advantage of the useful relation

$$|\Gamma(1+ix)|^2 = \frac{\pi x}{\sinh(\pi x)}$$

in order to rewrite the result in the following form

$$R = \left[\frac{\sinh\pi(\kappa-\sigma)}{\sinh\pi(\kappa+\sigma)}\right]^2 = \left[\frac{\sinh\pi(k-k')a}{\sinh\pi(k+k')a}\right]^2,$$

where k and k' are the wave numbers on the left/right side of the barrier.

In the limit $a \to 0$ the scattering potential becomes a sharp step. In this limit no explicit solution in terms of hypergeometric function is required. The corresponding results can of course be recovered from the above equations, see for instance [11].

2.8.3 Problems which can be Mapped on the Effective 1D Schrödinger Equation

In some cases the solution of a 1D Schrödinger equation is only possible using a generalisation of the hypergeometric series. For example an equation which arises in the context of a three-body bosonic problem can be written down as [12]

$$\left[\hat{T}(-i\partial_x) - E + e^x\right]\psi(x) = 0, \tag{2.81}$$

where $\hat{T}(-i\partial_x)$ is a differential operator acting as $\hat{T}(-i\partial_x)\psi(x) = \int dx'\, T(x-x')\psi(x')$, where T is the Fourier transform of $\hat{T}$. It can formally be considered as the kinetic energy. Let the Fourier transform of $\hat{L}(x) = \hat{T}(-i\partial_x) - E$ be a rational function,

$$L(k) = P(k)/Q(k),$$

and let $F(k)$ be the image of $\psi(x)$. Then (2.81) is given by

$$P(k)F(k) + Q(k)F(k+i) = 0,$$

and the back transformation yields

$$\left[P(-i\partial_x) + e^x\, Q(-i\partial_x - i)\right]\, \psi(x) = 0.$$

We now want to concentrate on

$$L(k) = \bar{\alpha}_0^2\, \frac{k^2 - s_0^2}{k^2 + \alpha_0^2}\, \frac{k^2 + \beta_1^2}{k^2 + \alpha_1^2}.$$

$L(0) = -E = -\bar{\alpha}_0^2 s_0^2 \beta_1^2/(\alpha_0\alpha_1)^2$ is then the energy eigenvalue. The fact that the above problem is indeed related to the single-particle Schrödingier equation can be demonstrated by the following observation. By expansion of T around $k \to 0$ we see that the leading order term is $\sim k^2$. This is precisely what one obtains for the dispersion relation of a free particle. On the other hand, at large k the dispersion levels off to a constant, which usually happens in the band theory of metals.

With the above assumption in the coordinate representation we obtain

$$\left\{\bar{\alpha}_0^2\, (-\partial_x^2 - s_0^2)(-\partial_x^2 + \beta_1^2) + e^x\, [-(\partial_x + 1)^2 + \alpha_0^2][-(\partial_x + 1)^2 + \alpha_1^2]\right\}\, \psi(x) = 0.$$

In the next step we perform the substitutions $z = e^x/\bar{\alpha}_0^2$ and $\partial_x = z\partial_z \equiv \Delta$ leading to

$$(\Delta + is_0)(\Delta - is_0)(\Delta + \beta_1)(\Delta - \beta_1)\psi(z) = z\prod_{r=0}^{1}(\Delta + 1 + \alpha_r)(\Delta + 1 - \alpha_r)\psi(z).$$

Next we redefine $\psi(z) = z^{i\alpha}\, \phi(z)$, where α is a parameter, then we obtain

$$\begin{aligned}(\Delta + is_0 + i\alpha)(\Delta - is_0 + i\alpha)(\Delta + i\alpha + \beta_1)(\Delta + i\alpha - \beta_1)\phi(z)\\ = z\prod_{r=0}^{1}(\Delta + 1 + \alpha_r + i\alpha)(\Delta + 1 - \alpha_r + i\alpha)\phi(z).\end{aligned}$$

Setting α to four different values we obtain four linearly independent solutions of the original equations.

- Take $\alpha = s_0$, then the equation changes to

$$\begin{aligned}\Delta(\Delta + i2s_0)(\Delta + \beta_1 + is_0)(\Delta - \beta_1 + is_0)\phi(z)\\ = z\prod_{r=0}^{1}(\Delta + 1 + \alpha_r + is_0)(\Delta + 1 - \alpha_r + is_0)\phi(z).\end{aligned}$$

By comparison with (2.72) we then immediately write down the solution

$$\psi(z) = z^{is_0}\,{}_4F_3\,(1+\alpha_0+is_0, 1-\alpha_0+is_0, 1+\alpha_1+is_0, 1-\alpha_1+is_0;$$
$$1+i2s_0, 1+\beta_1+is_0, 1-\beta_1+is_0; -z)$$

- The second independent solution is obtained by the complex conjugation of the above, or by the substitution $s_0 \to -s_0$.
- Next we set $i\alpha = \beta_1$, then the equation is

$$\Delta(\Delta+2\beta_1)(\Delta+\beta_1+is_0)(\Delta+\beta_1-is_0)\phi(z)$$
$$= z\prod_{r=0}^{1}(\Delta+1+\alpha_r+\beta_1)(\Delta+1-\alpha_r+\beta_1)\phi(z).$$

In this case the solution of the original equation is given by

$$\psi(z) = z^{\beta_1}\,{}_4F_3\,(1+\alpha_0+\beta_1, 1-\alpha_0+\beta_1, 1+\alpha_1+\beta_1, 1-\alpha_1+\beta_1;$$
$$1+2\beta_1, 1+\beta_1+is_0, 1+\beta_1-is_0; -z)\,.$$

- The fourth solution is obtained by setting $i\alpha = -\beta_1$. However, the resulting solution is not finite at $z=0$ and we want to discard it.

The full solution is obtained by a linear combination of the above three possibilities. The coefficients are, as usual, set by the boundary conditions. We would like to remind the reader, that this solution is regular for $|z|<1$. The analytic continuation for $|z|>1$ is obtained via the prescription (2.73).

2.9 Problems

Problem 2.1
Upon using identities (2.2) and (2.6) express $|\Gamma(iy)|$ for real y in terms of elementary functions. Compute the asymptotics for $y \to \infty$ and compare it to the Stirling formula. Find the limiting form for $y \to 0$ and explain the result.

Problem 2.2
Euler's formula. Find the value of the product $\prod_{k=1}^{n-1}\Gamma(k/n)$, n being a positive integer. *Hint*: write out the product in the inverse order, multiply and use identity (2.6); to simplify the resulting product of sin's consider the limit $\lim_{z\to 1}(z^n-1)/(z-1)$.

Problem 2.3
Raabe's integral. Calculate the integral

$$I = \int_0^1 dx\ \ln\Gamma(x).$$

Hint: double up the integral, use the transformation $x \leftrightarrow 1-x$ in one of them, reduce the integrand to an elementary function by means of identity (2.6) and integrate by doubling the variable and exploiting the symmetry of the integrand.

Problem 2.4
For $\lambda > 0$, $0 < x < 1$, and $-\pi/2 < \alpha < \pi/2$ evaluate the following integrals:

$$(\mathbf{a}): \quad I_1(x,\alpha) = \int\limits_0^\infty t^{x-1} e^{-\lambda t \cos\alpha} \sin(\lambda t \sin\alpha)\, dt\,,$$

$$(\mathbf{b}): \quad I_2(x,\alpha) = \int\limits_0^\infty t^{x-1} e^{-\lambda t \cos\alpha} \cos(\lambda t \sin\alpha)\, dt\,,$$

$$(\mathbf{c}): \quad I_3(x,\alpha) = \int\limits_0^\infty \frac{\sin(\alpha t)}{t^x}\, dt\,, \qquad (\mathbf{d}): \quad I_4(x,\alpha) = \int\limits_0^\infty \frac{\cos(\alpha t)}{t^x}\, dt$$

in terms of the Gamma function $\Gamma(x)$.

Problem 2.5
Evaluate the integral

$$I_{\mu\nu} = \int\limits_0^\infty (\sinh x)^\mu (\cosh x)^\nu \, dx$$

in terms of the Beta function.

Problem 2.6
Show that the ratio

$$\frac{F(a+1,b;c;z) - F(a,b;c;z)}{F(a+1,b+1;c+1;z)}$$

is an elementary function of z and find this function.

Problem 2.7
Verify the following transformation formulae for the hypergeometric functions:

$$
\begin{aligned}
(\mathbf{a}):\quad F(a,b;c;z) &= (1-z)^{-a}F\left(a, c-b; c; \frac{z}{z-1}\right)\\
&= (1-z)^{-b}F\left(b, c-a; c; \frac{z}{z-1}\right),\\
(\mathbf{b}):\quad F(a,b;c;z) &= (1-z)^{c-a-b}F(c-a, c-b; c; z),
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{c}):\quad F\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right) &= \frac{\sqrt{\pi}}{\Gamma(\frac{1-\nu}{2})\Gamma(\frac{2+\nu}{2})}F\left(-\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{1}{2}; z^2\right)\\
&+ \frac{\sqrt{\pi}\nu z}{\Gamma(\frac{1+\nu}{2})\Gamma(\frac{2-\nu}{2})}F\left(-\frac{\nu-1}{2}, \frac{\nu+2}{2}; \frac{3}{2}; z^2\right).
\end{aligned}
$$

Problem 2.8
Express the following hypergeometric functions

$$
\begin{aligned}
&(\mathbf{a}):\quad F\left(a, b; \frac{a+b+1}{2}; \frac{1}{2}\right),\\
&(\mathbf{b}):\quad F(a, b; a-b+1; -1)
\end{aligned}
$$

in terms of the Gamma functions.

Problem 2.9
The six functions $F(a\pm 1, b; c; z)$, $F(a, b\pm 1; c; z)$, $F(a, b; c\pm 1; z)$ are said to be contiguous to the hypergeometric function $F(a, b; c; z)$. There exists a linear relationship between this function and any two contiguous functions with coefficients which are linear polynomials in z or constants. As an example, verify the following two recurrence relations:

$$
\begin{aligned}
&(\mathbf{a}):\quad cF(a,b;c;z) - (c-a)F(a,b;c+1;z) - aF(a+1,b;c+1;z) = 0,\\
&(\mathbf{b}):\quad cF(a,b-1;c;z) - cF(a-1,b;c;z) + (a-b)zF(a,b;c+1;z) = 0.
\end{aligned}
$$

Problem 2.10
Prove the quadratic transformations:

$$
\begin{aligned}
&(\mathbf{a}):\quad F(2a, 2a+1-c; c; z) = (1+z)^{-2a}F\left(a, a+\tfrac{1}{2}; c; \tfrac{4z}{(1+z)^2}\right),\\
&(\mathbf{b}):\quad F\left(a, a-b+\tfrac{1}{2}; b+\tfrac{1}{2}; z^2\right) = (1+z)^{-2a}F\left(a, b; 2b; \tfrac{4z}{(1+z)^2}\right).
\end{aligned}
$$

Problem 2.11
Determine the Mellin transform and the fundamental strip for the following functions:

$$\begin{aligned}
(\mathbf{a}):\quad & f(t) = (1+t)^{-a},\\
(\mathbf{b}):\quad & f(t) = \ln(1+t),\\
(\mathbf{c}):\quad & f(t) = (1-t)^{-1}.
\end{aligned}$$

Problem 2.12
Ramanujan considered a function defined by the following series:

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{-nx}}{1+e^{-2nx}}.$$

Using the method presented in Sect. 2.2 derive the following identity:

$$f(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{\pi}{x} f(\pi^2/x).$$

Problem 2.13
Use the definitions of the modified Bessel function and of the Macdonald function given by (2.65) on page 96.

(a): Show that for $\mathrm{Re}(\nu + 1/2) > 0$ the function $I_\nu(z)$ has the following integral representation:

$$I_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \cosh(z\cos\varphi) \sin^{2\nu}\varphi \, d\varphi. \tag{2.82}$$

(b): Express

$$I_{\frac{1}{2}}(z), \quad I_{-\frac{1}{2}}(z), \quad K_{\pm\frac{1}{2}}(z)$$

in terms of elementary functions.

Problem 2.14
Express the following integral

$$\mathcal{I}_{i\alpha}(\xi) = \int_1^\infty e^{-\xi t} P_{-\frac{1}{2}+i\alpha}(t)\, dt$$

in terms of the modified Bessel function for $\mathrm{Re}\xi > 0$ and $P_\nu(t) \equiv F(\nu+1, -\nu; 1; \frac{1-t}{2})$ being the special case of the hypergeometric series.[20]

Problem 2.15
The Macdonald function has the following integral representation:

---

[20] This notation is not casual, see Sect. 4.10 later, and especially Eq. (4.55).

$$K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zt}(t^2-1)^{\nu-1/2}dt.$$

Using it calculate the integral

$$g(\nu) = \int_0^\infty e^{-at} K_\nu(\beta t)\, t^{\mu-1} dt$$

under the assumptions $\mathrm{Re}(\mu \pm \nu) > 0$ and $\mathrm{Re}\, a > 0$.

Answers:
Problem 2.1: $|\Gamma(iy)|^2 = \pi/y \sinh \pi y$; $|\Gamma(iy)| = \sqrt{2\pi/y} e^{-\pi y/2}[1 + O(y)]$ as $y \to \pm\infty$; $|\Gamma(iy)| = 1/y + O(y^0)$ as $y \to 0^+$.
Problem 2.2: $(2\pi)^{(n-1)/2}/\sqrt{n}$.
Problem 2.3: $\ln \sqrt{2\pi}$.
Problem 2.4 (a): $\Gamma(x)\sin(\alpha x)/\lambda^x$; **(b):** $\Gamma(x)\cos(\alpha x)/\lambda^x$; **(c):** $\mathrm{sgn}(\alpha)\Gamma(1-x)|\alpha|^{x-1} \cos(\pi x/2)$; **(d):** $\Gamma(1-x)|\alpha|^{x-1}\sin(\pi x/2)$.
Problem 2.5: $\frac{1}{2}B\left(\frac{\mu+1}{2}, \frac{-\nu-\mu}{2}\right)$.
Problem 2.6: bz/c.
Problem 2.8:

$$(\mathbf{a}): \quad \frac{\sqrt{\pi}\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}; \quad (\mathbf{b}): \quad \frac{2^{-a}\sqrt{\pi}\Gamma(1+a-b)}{\Gamma(1+\frac{a}{2}-b)\Gamma(\frac{a+1}{2})}.$$

Problem 2.11 (a): $F_M(s) = \Gamma(s)\Gamma(a-s)/\Gamma(a), \quad 0 < \mathrm{Re}(s) < a$; **(b)**: $F_M(s) = \pi/s\sin(\pi s), \quad -1 < \mathrm{Re}(s) < 0$; **(c)**: $F_M(s) = \pi\cot(\pi s), \quad 0 < \mathrm{Re}(s) < 1$.
Problem 2.13 (b): $\sqrt{2/\pi z}\ \sinh z$, $\sqrt{2/\pi z}\ \cosh z$, $\sqrt{\pi/2z}\ e^{-z}$.
Problem 2.14: $\sqrt{2/\pi\xi}\ K_{i\alpha}(\xi)$.
Problem 2.15:

$$\frac{\sqrt{\pi}(2\beta)^\nu}{\Gamma(\mu+\frac{1}{2})(a+\beta)^{\nu+\mu}}\Gamma(\mu+\nu)\Gamma(\mu-\nu)F\left(\nu+\mu, \nu+\frac{1}{2}; \mu+\frac{1}{2}; \frac{a-\beta}{a+\beta}\right).$$

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