

# Contents

## Part I Markov Semigroups, Basics and Examples

<b>1</b>	<b>Markov Semigroups</b>	<b>3</b>
1.1	Markov Processes and Associated Semigroups	7
1.2	Markov Semigroups, Invariant Measures and Kernels	9
1.3	Chapman-Kolmogorov Equations	16
1.4	Infinitesimal Generators and Carré du Champ Operators	18
1.5	Fokker-Planck Equations	23
1.6	Symmetric Markov Semigroups	24
1.7	Dirichlet Forms and Spectral Decompositions	29
1.8	Ergodicity	32
1.9	Markov Chains	33
1.10	Stochastic Differential Equations and Diffusion Processes	38
1.11	Diffusion Semigroups and Operators	42
1.12	Ellipticity and Hypo-ellipticity	49
1.13	Domains	52
1.14	Summary of Hypotheses (Markov Semigroup)	53
1.15	Working with Markov Semigroups	56
1.16	Curvature-Dimension Condition	70
1.17	Notes and References	74
<b>2</b>	<b>Model Examples</b>	<b>77</b>
2.1	Euclidean Heat Semigroup	78
2.2	Spherical Heat Semigroup	81
2.3	Hyperbolic Heat Semigroup	88
2.4	The Heat Semigroup on a Half-Line and the Bessel Semigroup	92
2.5	The Heat Semigroup on the Circle and on a Bounded Interval	96
2.6	Sturm-Liouville Semigroups on an Interval	97
2.7	Diffusion Semigroups Associated with Orthogonal Polynomials	102
2.8	Notes and References	118

<b>3</b>	<b>Symmetric Markov Diffusion Operators</b>	119
3.1	Markov Triples	120
3.2	Second Order Differential Operators on a Manifold	137
3.3	Heart of Darkness	151
3.4	Summary of Hypotheses (Markov Triple)	168
3.5	Notes and References	173
<b>Part II Three Model Functional Inequalities</b>		
<b>4</b>	<b>Poincaré Inequalities</b>	177
4.1	The Example of the Ornstein-Uhlenbeck Semigroup	178
4.2	Poincaré Inequalities	181
4.3	Tensorization of Poincaré Inequalities	185
4.4	The Example of the Exponential Measure, and Exponential Integrability	187
4.5	Poincaré Inequalities on the Real Line	193
4.6	The Lyapunov Function Method	201
4.7	Local Poincaré Inequalities	206
4.8	Poincaré Inequalities Under a Curvature-Dimension Condition	211
4.9	Brascamp-Lieb Inequalities	215
4.10	Further Spectral Inequalities	220
4.11	Notes and References	230
<b>5</b>	<b>Logarithmic Sobolev Inequalities</b>	235
5.1	Logarithmic Sobolev Inequalities	236
5.2	Entropy Decay and Hypercontractivity	243
5.3	Integrability of Eigenvectors	250
5.4	Logarithmic Sobolev Inequalities and Exponential Integrability	252
5.5	Local Logarithmic Sobolev Inequalities	257
5.6	Infinite-Dimensional Harnack Inequalities	265
5.7	Logarithmic Sobolev Inequalities Under a Curvature-Dimension Condition	268
5.8	Notes and References	273
<b>6</b>	<b>Sobolev Inequalities</b>	277
6.1	Sobolev Inequalities on the Model Spaces	278
6.2	Sobolev and Related Inequalities	279
6.3	Ultracontractivity and Heat Kernel Bounds	286
6.4	Ultracontractivity and Compact Embeddings	290
6.5	Tensorization of Sobolev Inequalities	291
6.6	Sobolev Inequalities and Lipschitz Functions	293
6.7	Local Sobolev Inequalities	296
6.8	Sobolev Inequalities Under a Curvature-Dimension Condition	305
6.9	Conformal Invariance of Sobolev Inequalities	313
6.10	Gagliardo-Nirenberg Inequalities	323
6.11	Fast Diffusion Equations and Sobolev Inequalities	330
6.12	Notes and References	340

## Part III Related Functional, Isoperimetric and Transportation Inequalities

<b>7</b>	<b>Generalized Functional Inequalities</b>	347
7.1	Inequalities Between Entropy and Energy	348
7.2	Off-diagonal Heat Kernel Bounds	355
7.3	Examples	362
7.4	Beyond Nash Inequalities	364
7.5	Weak Poincaré Inequalities	373
7.6	Further Families of Functional Inequalities	382
7.7	Summary for the Model Example $\mu_\alpha$	386
7.8	Notes and References	387
<b>8</b>	<b>Capacity and Isoperimetric-Type Inequalities</b>	391
8.1	Capacity Inequalities and Co-area Formulas	392
8.2	Capacity and Sobolev Inequalities	396
8.3	Capacity and Poincaré and Logarithmic Sobolev Inequalities	399
8.4	Capacity and Further Functional Inequalities	403
8.5	Gaussian Isoperimetric-Type Inequalities Under a Curvature Condition	411
8.6	Harnack Inequalities Revisited	421
8.7	From Concentration to Isoperimetry	425
8.8	Notes and References	429
<b>9</b>	<b>Optimal Transportation and Functional Inequalities</b>	433
9.1	Optimal Transportation	434
9.2	Transportation Cost Inequalities	438
9.3	Transportation Proofs of Functional Inequalities	442
9.4	Hamilton-Jacobi Equations	451
9.5	Hypercontractivity of Solutions of Hamilton-Jacobi Equations	454
9.6	Transportation Cost and Logarithmic Sobolev Inequalities	458
9.7	Heat Flow Contraction in Wasserstein Space	462
9.8	Curvature of Metric Measure Spaces	464
9.9	Notes and References	466

## Appendices

<b>Appendix A</b>	<b>Semigroups of Bounded Operators on a Banach Space</b>	473
A.1	The Hille-Yosida Theory	473
A.2	Symmetric Operators	475
A.3	Friedrichs Extension of Positive Operators	477
A.4	Spectral Decompositions	478
A.5	Essentially Self-adjoint Operators	481
A.6	Compact and Hilbert-Schmidt Operators	483
A.7	Notes and References	485

<b>Appendix B Elements of Stochastic Calculus</b>	487
B.1 Brownian Motion and Stochastic Integrals	487
B.2 The Itô Formula	491
B.3 Stochastic Differential Equations	493
B.4 Diffusion Processes	495
B.5 Notes and References	498
<b>Appendix C Basic Notions in Differential and Riemannian Geometry</b>	499
C.1 Differentiable Manifolds	500
C.2 Some Elementary Euclidean Geometry	502
C.3 Basic Notions in Riemannian Geometry	504
C.4 Riemannian Distance	509
C.5 The Riemannian $\Gamma$ and $\Gamma_2$ Operators	511
C.6 Curvature-Dimension Conditions	513
C.7 Notes and References	518
<b>Afterword</b>	521
Chicken “Gaston Gérard”	521
<b>Notation and List of Symbols</b>	523
<b>Bibliography</b>	527
<b>Index</b>	547

Analysis and Geometry of Markov Diffusion Operators

Bakry, D.; Gentil, I.; Ledoux, M.

2014, XX, 552 p., Hardcover

ISBN: 978-3-319-00226-2