

## Chapter 2

# Boltzmann-Uehling-Uhlenbeck Equation

When an external perturbation is applied to a system, this leaves the equilibrium state. The particles of the system, described by the one-particle distribution function, suffer from many collisions producing a transport of the conserved quantities (energy, momentum and others). The one-particle distribution function is then modified and its time evolution is described by a kinetic equation. By knowing the one-particle distribution function out of equilibrium, and connecting with the hydrodynamic formalism one is able to extract the transport coefficients that govern the relaxation of the fluid to the equilibrium state. We will start by introducing the Boltzmann-Uehling-Uhlenbeck kinetic equation and the method derived by Chapman and Enskog to solve it. For the sake of simplicity, in this chapter we will set the formalism for a one-species gas, ignoring the conserved internal (or flavor-like) charges. Later, in Chaps. 7 and 8 we will attend to these.

### 2.1 $n$ -Particle Distribution Functions and Bogoliubov-Born-Green-Kyrkwood-Yvon Hierarchy

#### 2.1.1 Classical Description

Consider a gas with a large number of particles  $N$  inside a volume  $V$  [1, 2]. Each of these particles is specified at time  $t$  by its position  $\mathbf{r}_i(t)$  (with  $i = 1, \dots, N$ ) and its momentum  $\mathbf{q}_i(t)$ . Therefore the whole system is entirely described by a set of  $6N$  coordinates in the so-called  $\Gamma$ -phase space. A microstate of the system (ensemble of particles with definite positions and momenta at a given time) is specified by a representative point in the  $\Gamma$ -phase space. This microstate must be compatible with the macrostate, defined by some thermodynamic functions as  $N, V, E, T \dots$  The evolution of the system is determined by the Hamilton's equation (the Hamiltonian of the system is a function of  $t, \mathbf{r}_i$  and  $\mathbf{q}_i$ )

$$\dot{\mathbf{q}}_i = -\frac{\partial H(t, \mathbf{r}_i(t), \mathbf{q}_i(t))}{\partial \mathbf{r}_i}, \quad \dot{\mathbf{r}}_i = \frac{\partial H(t, \mathbf{r}_i(t), \mathbf{q}_i(t))}{\partial \mathbf{q}_i} \quad (2.1)$$

and the evolution of the system follows a curve in the  $\Gamma$ -phase space.

For systems with constant total energy, their microstates are restricted to those representative points in the phase space  $(\mathbf{r}_i, \mathbf{q}_i)$  which  $H(t, \mathbf{r}_i, \mathbf{q}_i) = E$ . For open systems there is no such a restriction (apart from the condition of being compatible with the macrostate) and some microstates are easier to reach in the  $\Gamma$ -phase space than others. One can define the phase-space density  $\rho(t, \mathbf{r}_i, \mathbf{q}_i)$  which is the probability for a system to be in a given microstate in the  $\Gamma$ -phase space.

The quantity

$$\prod_{i=1}^N d^3 \mathbf{r}_i d^3 \mathbf{q}_i \rho(t, \mathbf{r}_i, \mathbf{q}_i) \quad (2.2)$$

is the number of accesible microstates that at time  $t$  are contained in the phase-space volume element  $\prod_{i=1}^N d^3 \mathbf{r}_i d^3 \mathbf{q}_i$  centered at  $(\mathbf{r}_i, \mathbf{q}_i)$ .

The evolution of the phase-space density along a phase-space trajectory is given by the Liouville's equation:

$$\frac{d}{dt} \rho(t, \mathbf{r}_i, \mathbf{q}_i) = \frac{\partial}{\partial t} \rho(t, \mathbf{r}_i, \mathbf{q}_i) + \{\rho(t, \mathbf{r}_i, \mathbf{q}_i), H\} = 0, \quad (2.3)$$

where  $H$  is the Hamiltonian of the system. This equation can be obtained from Hamilton's equation (2.1).

Any observable  $O$ , a function of the  $\mathbf{r}_i$  and  $\mathbf{q}_i$  can be averaged over the ensemble as follows:

$$\langle O(t) \rangle = \frac{\int \prod_{i=1}^N d^3 \mathbf{r}_i d^3 \mathbf{q}_i O(\mathbf{r}_i, \mathbf{q}_i) \rho(t, \mathbf{r}_i, \mathbf{q}_i)}{\int \prod_{i=1}^N d^3 \mathbf{r}_i d^3 \mathbf{q}_i \rho(t, \mathbf{r}_i, \mathbf{q}_i)}. \quad (2.4)$$

The building blocks of kinetic theory are the  $n$ -particle distribution functions or joint-probability distributions,  $f(t, \mathbf{x}_1, \mathbf{p}_1, \dots, \mathbf{x}_n, \mathbf{p}_n)$ . A  $n$ -particle distribution function represents the probability of finding at given time  $t$  the particle 1 at  $(\mathbf{x}_1, \mathbf{p}_1)$ , the particle 2 at  $(\mathbf{x}_2, \mathbf{p}_2)$ , and so on up to the particle  $n$ .

These distribution functions can be obtained from the phase-space density function  $\rho(t, \mathbf{r}_i, \mathbf{q}_i)$  by integrating the appropriate remaining coordinates of the phase-space. For example, the one-particle distribution function is defined as:

$$f(t, \mathbf{x}, \mathbf{p}) = \left\langle \sum_i^N \delta(\mathbf{x} - \mathbf{r}_i) \delta(\mathbf{p} - \mathbf{q}_i) \right\rangle. \quad (2.5)$$

To have access to all the distribution functions would provide all the physical information of the system, completely equivalent to knowing the phase-space density function. They can be obtained by solving their equations of evolution (as hard as solving the Liouville equation). The equation for  $f$  is generated by integrating the

Liouville equation over the phase-space coordinates  $\mathbf{r}_n, \mathbf{q}_n, \dots, \mathbf{r}_N, \mathbf{q}_N$  by assuming a particular form of the Hamiltonian (with one- and two- particle interaction potential).

After integrating the Liouville equation, one realizes that the equation of evolution for the  $n$ -particle distribution function is non-linearly coupled with the  $n + 1$ -distribution function (and so on up to  $n = N$ ). This set of coupled integro-differential equations is called the BBGKY (Bogoliubov-Born-Green-Kyrkwood-Yvon) hierarchy of equations.

## 2.2 Kinetic Equation

Taking only the first equation of the BBGKY hierarchy for the one-particle distribution function and performing the approximation of substituting the two-particle distribution function by a product of two one-particle distribution functions one obtains a closed equation for  $f(t, \mathbf{x}, \mathbf{p})$ . The resulting equation is called a kinetic equation.

The assumptions for obtaining the kinetic equation are:

- The fluid is a dilute medium in which only binary collisions occur. Therefore multiple collisions are neglected. However, the binary collisions may be, in principle, inelastic ( $\pi\pi \rightarrow KK$ ) as well as elastic ( $\pi\pi \rightarrow \pi\pi$ ).
- The collision time is much smaller than the mean free time between consecutive collisions. This assumption is valid when the gas is dilute enough. In terms of typical lengths this is expressed as

$$L \gg \lambda_{mfp} \gg R, \quad (2.6)$$

where  $L$  is the size of the system,  $\lambda_{mfp}$  is the mean free path or length between successive collisions and  $R$  is the range of interaction (typically a scattering length or the radius of the particles in the hard-sphere approximation).

- *Stosszahlansatz* or molecular chaos hypothesis. This implies the absence of particle correlations before the collision process takes place. This assumption entails decoupling the first equation in the BBGKY hierarchy by the replacement:

$$f(t, \mathbf{x}_1, \mathbf{p}_1, \mathbf{x}_2, \mathbf{p}_2) \simeq f(t, \mathbf{x}_1, \mathbf{p}_1) f(t, \mathbf{x}_2, \mathbf{p}_2). \quad (2.7)$$

Because from now on we will only work with the one-particle distribution function we will change the notation:

$$f_p(t, \mathbf{x}) \equiv f(t, \mathbf{x}, \mathbf{p}), \quad (2.8)$$

where the subindex  $p$  denote both the momentum dependence of the distribution function and also a label for the particle entering in the elastic scattering process. Considering a classical scattering between two particles, we will denote by  $k_1$  and

$k_2$  the momenta of the two incoming particles; and  $k_3$  and  $p$  the momenta of the outgoing particles.

The time evolution of the one-particle distribution function  $f_1(t, \mathbf{x})$  is given by the kinetic equation, which is of the following type:

$$\frac{df_1}{dt} = C[f_1, f_2]. \quad (2.9)$$

The classical kinetic equation is called the Boltzmann equation and is known since L. Boltzmann derived it in 1872 for a gas of classical particles. The collision operator of the Boltzmann equation in the right-hand side of (2.9) reads explicitly [2–5]

$$C[f_1, f_2] = \frac{1}{(2\pi)^3} \int d\Omega d\mathbf{k}_2 v_{rel} \frac{d\sigma_{12}}{d\Omega} [f_3 f_p - f_1 f_2], \quad (2.10)$$

where  $v_{rel}$  is the relative velocity between the incoming particles, and  $\frac{d\sigma_{12}}{d\Omega}$  the differential cross section of the process. The existence of an equilibrium solution to this equation was proven by Boltzmann in the form of the  $H$ -theorem.

The local  $H$ -theorem follows from that the entropy production at any time-space point is never negative. The entropy production is only vanishing when the solution of the kinetic equation is the local equilibrium distribution function or equilibrium Maxwellian:

$$f_p(t, \mathbf{x}) = n_p(t, \mathbf{x}) \equiv \frac{1}{e^{\frac{p^\alpha u_\alpha(t, \mathbf{x}) - \mu(t, \mathbf{x})}{T(t, \mathbf{x})}}}, \quad (2.11)$$

that satisfies the detailed balance equation

$$n_1(t, \mathbf{x}) n_2(t, \mathbf{x}) = n_3(t, \mathbf{x}) n_p(t, \mathbf{x}). \quad (2.12)$$

### 2.2.1 Wigner Function

So far, the discussion has been purely classical. In quantum theory an analogous derivation can be made, using quantum-mechanical averages instead of (2.4). The analogue to the one-particle distribution function is called Wigner function [6] and it formally coincides with the classical distribution function. Moreover, a global factor due to the quantum mechanical formulation appears in this function:

$$f_p^C(t, x) \rightarrow f_p^Q(t, x) \frac{g}{(2\pi\hbar)^3}, \quad (2.13)$$

where  $g$  accounts for the quantum degeneracy of the particle ( $g = 2$  for electrons due to spin,  $g = 3$  for pions due to isospin or  $g = 2$  for photons due to the

polarization states) and the factor  $1/(2\pi\hbar)^3$  comes from the fact that  $d\mathbf{x}d\mathbf{p}/h^3$  is the number of quantum states in the infinitesimal phase-space volume. Additionally, the collision operator is not written in terms of the cross section but in terms of the scattering matrix elements.

When the Bose-Einstein must be applied (as is the case for pions at moderate temperatures) the kinetic equation is called the Boltzmann-Uehling-Uhlenbeck (BUU) equation. It contains some extra factors that accounts for the Bose-Einstein nature of the particles and that produce an enhancement of the phase space in the available states. To be consistent with our later references we will focus on the BUU equation for  $f_p$ :

$$\frac{df_p}{dt} = C[f_3, f_p], \quad (2.14)$$

where the collision operator of the BUU equation reads explicitly

$$C[f_3, f_p] = \frac{g_3}{1 + \delta_{3,p}} \int d\Gamma_{12,3p} [f_1 f_2 (1 + f_3)(1 + f_p) - f_3 f_p (1 + f_1)(1 + f_2)], \quad (2.15)$$

where  $g_3$  is the degeneracy of the particle 3 and  $1 + \delta_{3,p}$  factor accounts for the possible undistinguishable particles in the final state. The scattering measure is

$$d\Gamma_{12,3p} \equiv \frac{1}{2E_p} |\overline{T}|^2 \prod_{i=1}^3 \frac{d\mathbf{k}_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - k_3 - p). \quad (2.16)$$

The local equilibrium distribution function is the Bose-Einstein function:

$$f_p(t, \mathbf{x}) = n_p(t, \mathbf{x}) \equiv \frac{1}{e^{\frac{p^\alpha u_\alpha(t, \mathbf{x}) - \mu(t, \mathbf{x})}{T(t, \mathbf{x})}} - 1}. \quad (2.17)$$

This function satisfies the detailed balance condition as well:

$$n_1(t, \mathbf{x}) n_2(t, \mathbf{x}) [1 + n_3(t, \mathbf{x})] [1 + n_p(t, \mathbf{x})] = n_3(t, \mathbf{x}) n_p(t, \mathbf{x}) [1 + n_1(t, \mathbf{x})] [1 + n_2(t, \mathbf{x})]. \quad (2.18)$$

In the following, we will denote as  $x$  the space-time four-vector  $(t, \mathbf{x})$  on which the hydrodynamic fields and distribution functions depend.

## 2.3 Chapman-Enskog Expansion

The so-called Chapman-Enskog expansion is one of the several classical methods to obtain an approximate solution of the BUU equation.

In addition to the three length scales defined in Sect. 2.2 one can introduce a characteristic hydrodynamic length  $h$  which is the typical size of the inhomogeneities

of the system [5, 7]. The separation of scales are the following: A particle suffers from a collision with another in a characteristic length being the range of interaction,  $R$ . After that, the particle moves freely a distance of the order of  $\lambda_{mfp}$  until it encounters another particle and collides again. Inside  $h$ , the particle suffers from many collisions. Due to these scatterings the distribution function becomes close to the local equilibrium one. This local equilibrium state is characterized by  $\mu$ ,  $\mathbf{u}$  and  $T$  that vary from one region to another. In a larger time, the particle has travelled distances greater than  $h$  and the differences in the three hydrodynamical fields smooth across the whole system. The gas reaches a state of global equilibrium defined by  $\mu$ ,  $\mathbf{u}$  and  $T$  which do not depend on  $x$ .

We can summarize the hierarchy of scales as

$$L \gg h \gg \lambda_{mfp} \gg R, \quad (2.19)$$

where  $L$  is the typical size of the system. In terms of characteristic times, one can divide the previous inequalities by the thermal velocity  $v \sim \sqrt{T/m}$ .

$$L/v \gg \tau_h \gg \tau_{mft} \gg \tau_R, \quad (2.20)$$

where  $\tau_h = h/v$  is the characteristic time of travel through the inhomogeneities of the system,  $\tau_{mft}$  is the mean free time, and  $\tau_R = R/v$  is the duration of a collision.

In this scenario there are two main time scales governed by a fast and a subsequent slow processes: (1) The fast relaxation from the non-equilibrium initial state to a local equilibrium state, due to many collisions inside  $h$ . The time of local equilibration is of the order of  $\tau_{mft}$ . This stage is called the kinetic regime, sensitive to initial state. (2) The slow relaxation from local to global equilibrium, at distances of several  $h$ . The time needed for this process is of the order of  $\tau_h$ . This process is called the hydrodynamic regime. It does not depend on the initial state but only on the hydrodynamic fields  $T(x)$ ,  $\mu(x)$ ,  $\mathbf{u}(x)$  that depend on the time-space variables.

According to this, one expects that the one-particle distribution function in the second stage depends on space-time through a functional in the hydrodynamical variables:

$$f_p(x) = f_p[T(x), \mu(x), \mathbf{u}(x)]. \quad (2.21)$$

A solution of the BUU equation of this type is called a *normal solution*.

The Chapman-Enskog procedure is a systematic way of constructing a normal solution to the BUU equation in powers of the Knudsen number ( $\text{Kn} = \lambda_{mfp}/h$ ).

To proceed, take the BUU equation (we drop the argument  $(t, \mathbf{x})$  of the distribution function to ease the notation)

$$\frac{df_p}{dt} = C[f_3, f_p], \quad (2.22)$$

separate the convective time derivative and the gradient operator to get

$$\partial_t f_p = -v^i \nabla_i f_p + C[f_3, f_p] \quad (2.23)$$

and divide the right-hand side of the equation by  $f_p$ . The first term of the right-hand side is the inverse characteristic length for the inhomogeneities

$$\left| \frac{p^i}{E_p} \nabla_i \ln f_p \right| \simeq h^{-1} v \quad (2.24)$$

and the second is the inverse mean free path

$$\left| \frac{C[f_3, f_p]}{f_p} \right| \simeq \lambda_{mf}^{-1} v. \quad (2.25)$$

Taking into account the inequalities in Eq. (2.19) we deduce that  $C[f_p, f_p]/f_p$  is much smaller than  $v^i \nabla_i \ln f_p$ . Since  $h^{-1} \simeq |\nabla \ln f_p|$ , the expansion in powers of the Knudsen number is actually equivalent to an expansion in powers of hydrodynamical gradients.

The way of translating the separation of scales into the BUU equation is the following. The normal solution to the kinetic equation is expanded:

$$f_p = f_p^{(0)} + \epsilon f_p^{(1)} + \epsilon^2 f_p^{(2)} + \dots, \quad (2.26)$$

where  $\epsilon$  is the so-called non-uniformity parameter and it measures the relative strength of the gradient. For consistency, it is set to one at the end of the calculation, so it is nothing but a book-keeping parameter that can be interpreted as the Knudsen number, that controls the order of the approximation.

The spatial gradient is formally substituted by,  $\nabla_i \rightarrow \epsilon \nabla_i$ , and the time derivative of the normal solution is expanded

$$\partial_t f_p = \epsilon (\partial_t)^{(1)} f_p + \epsilon^2 (\partial_t)^{(2)} f_p + \dots. \quad (2.27)$$

The action of  $\partial_t^{(i)}$  occurs through the dependence on the hydrodynamic fields

$$\partial_t^{(i)} f_p = \partial_t^{(i)} T \frac{\partial f_p}{\partial T} + \partial_t^{(i)} \mu \frac{\partial f_p}{\partial \mu} + \partial_t^{(i)} \mathbf{u} \cdot \frac{\partial f_p}{\partial \mathbf{u}}. \quad (2.28)$$

The operators  $\partial_t^{(i)} \mu$ ,  $\partial_t^{(i)} T$  and  $\partial_t^{(i)} \mathbf{u}$  are obtained from the macroscopic conservation laws performing the same expansion and equating the terms with equal powers of  $\epsilon$ . Up to order  $\epsilon^1$  they explicitly read

$$\begin{cases} \partial_t^{(0)} T = \partial_t^{(0)} \mu = \partial_t^{(0)} \mathbf{u} = 0, \\ \partial_t^{(1)} T = -T \left( \frac{\partial P}{\partial \epsilon} \right)_n \nabla \cdot \mathbf{V}, \\ \partial_t^{(1)} \mu = - \left[ \mu \left( \frac{\partial P}{\partial \epsilon} \right)_n + \left( \frac{\partial P}{\partial n} \right)_\epsilon \right] \nabla \cdot \mathbf{V}, \\ \partial_t^{(1)} \mathbf{u} = - \frac{\nabla P}{w}. \end{cases} \quad (2.29)$$

One sees that is enough to consider the ideal gas approximation. For higher orders the situation is more complicated and one should take into account the  $\epsilon$ -expansion inside the energy-momentum tensor and four-particle flux.

Then, one substitutes all the previous equations into the BUU equation and identifies terms with equal powers in  $\epsilon$ . One obtains the following hierarchy of equations

$$\begin{cases} 0 &= C[f_3^{(0)}, f_p^{(0)}], \\ \partial_t^{(1)} f_p^{(0)} + \frac{p^i}{E_p} \nabla_i f_p^{(0)} &= C[f_3^{(0)}, f_p^{(1)}] + C[f_3^{(1)}, f_p^{(0)}]. \end{cases} \quad (2.30)$$

The solution of the zeroth-order equation is the local Bose-Einstein function of Eq. (2.17) with arguments depending on time and space (sometimes called Juetner distribution function). This zeroth order approximation reads

$$f_p^{(0)}(x) = n_p(x). \quad (2.31)$$

The next order gives the first nontrivial contribution to the distribution function. In this dissertation we will stop at first order:

$$f_p(x) = n_p(x) + f_p^{(1)}(x). \quad (2.32)$$

The first-order kinetic equation reads

$$\begin{aligned} \partial_t^{(1)} f_p^{(0)} + \frac{p^i}{E_p} \nabla_i f_p^{(0)} &= -\frac{g}{2} \int d\Gamma_{12,3p} (1+n_1)(1+n_2)n_3n_p \\ &\times \left( \frac{f_p^{(1)}}{n_p(1+n_p)} + \frac{f_3^{(1)}}{n_3(1+n_3)} - \frac{f_1^{(1)}}{n_1(1+n_1)} - \frac{f_2^{(1)}}{n_2(1+n_2)} \right), \end{aligned} \quad (2.33)$$

where  $g = 3$  for the pion isospin degeneracy.

Observing the form of the collision operator (with Bose statistics) the *ansatz* for  $f_p^{(1)}$  is conveniently parametrized as

$$f_p^{(1)} = -n_p(1+n_p)\Phi(\mathbf{p}). \quad (2.34)$$

$\Phi(\mathbf{p})$  is an a dimensional function of  $\mathbf{p}$  that will contain an appropriate hydrodynamic gradient depending on the transport coefficient.



One realizes here the main feature in the Chapman-Enskog expansion at first order. The left-hand side of Eq. (2.33) only depends on derivatives of the local equilibrium distribution function (through the hydrodynamic fields) and therefore it does not depend on  $\Phi(\mathbf{p})$ .

### 2.3.1 Left-Hand Side of the BUU Equation

At order  $\epsilon^0$  we have seen that the equation for  $f_p^{(0)}$  reads

$$0 = C[f_3^{(0)}, f_p^{(0)}] \quad (2.35)$$

and the solution is the local Bose-Einstein distribution function. In an arbitrary frame it reads

$$n_p[T(x), \mu(x), u^\alpha(x)] = \frac{1}{e^{\frac{p^\alpha u_\alpha(x) - \mu(x)}{T(x)}} - 1}, \quad (2.36)$$

that reduces to the usual Bose-Einstein distribution function in the comoving frame ( $\mathbf{V} \neq 0, \gamma = 1$ ).

One important remark is that the five independent hydrodynamical fields  $T(x)$ ,  $\mu(x)$  and  $u^i(x)$  are not necessarily the same as in equilibrium and one should fix them appropriately. The prescription to do this is to impose the conditions of fit, that we will describe in the next section.

With the parametrization used in Eq. (2.34),  $f_p = n_p - n_p(1 + n_p)\Phi(\mathbf{p})$  the order  $\epsilon$  reads

$$\begin{aligned} E_p \partial_t^{(1)} n_p(x) + p^i \nabla_i n_p(x) \\ = \frac{g E_p}{2} \int d\Gamma_{12,3p} (1 + n_1)(1 + n_2) n_3 n_p (\Phi_p + \Phi_3 - \Phi_1 - \Phi_2). \end{aligned} \quad (2.37)$$

Note that we have multiplied the equation by  $E_p$  for convenience.

The collision operator has become a linearized operator in  $\Phi_p$ . We focus on the left-hand side of Eq. (2.37) to obtain the final form of this term.

One uses first Eq. (2.28) for the temporal derivative, together with the Euler equations Eq. (2.29), for the derivatives of the hydrodynamical field and the Leibniz rule for the derivative of  $n_p$  with respect to these fields

$$\partial_A n_p(x) = -n_p(x)[1 + n_p(x)] \partial_A [\beta(p^\alpha u_\alpha - \mu)], \quad (2.38)$$

with  $A = T, \mu, u^i$ . The gradient is obtained by using the Leibniz rule with

$$\nabla_i \beta = -\beta^2 \nabla_i T \quad (2.39)$$

and the Gibbs-Duhem relation

$$\nabla_i \mu = -\frac{s}{n} \nabla_i T + \frac{1}{n} \nabla_i P. \quad (2.40)$$

As an intermediate step, we perform the separation into a traceless and “traceful” parts.

$$p^i p^j \partial_i V_j = p^i p^j \left( \frac{1}{2} (\partial_i V_j + \partial_j V_i) - \frac{1}{3} \delta_{ij} \nabla \cdot \mathbf{V} + \frac{1}{3} \delta_{ij} \nabla \cdot \mathbf{V} \right) = p^i p^j (\tilde{V}_{ij} + \frac{1}{3} \delta_{ij} \nabla \cdot \mathbf{V}). \quad (2.41)$$

The first term goes to the shear viscosity and is the “traceless” part because it satisfies  $\sum_i \tilde{V}_{ii} = 0$ . The part that goes with the bulk viscosity is the “traceful” part  $\frac{1}{3} \delta_{ij} \nabla \cdot \mathbf{V}$ .

The final solution reads:

$$E_p \frac{df_p}{dt} = \beta n_p(x) (1 + n_p(x)) \left\{ p^i p^j \tilde{V}_{ij} + \left( \frac{1}{3} \mathbf{p}^2 - E_p^2 v_n^2 - E_p \kappa_\epsilon^{-1} \right) \nabla \cdot \mathbf{V} \right. \\ \left. + \beta \left( E_p - \frac{w}{n} \right) \mathbf{p} \cdot \left( \nabla T - \frac{T}{w} \nabla P \right) \right\}, \quad (2.42)$$

where we have defined the isochorus sound speed and the compressibility at constant energy density

$$v_n^2 = \left( \frac{\partial P}{\partial \epsilon} \right)_n, \quad \kappa_\epsilon^{-1} = \left( \frac{\partial P}{\partial n} \right)_\epsilon. \quad (2.43)$$

These two quantities can be calculated from derivatives of the pressure by the following formulae

$$v_n^2 = \frac{s \chi_{\mu\mu} - n \chi_{\mu T}}{C_V \chi_{\mu\mu}}, \quad (2.44)$$

$$\kappa_\epsilon^{-1} = \frac{n T \chi_{TT} + (n \mu - s T) \chi_{\mu T} - s \mu \chi_{\mu\mu}}{C_V \chi_{\mu\mu}}, \quad (2.45)$$

where  $n = \left( \frac{\partial P}{\partial \mu} \right)_T$ ,  $s = \left( \frac{\partial P}{\partial T} \right)_\mu$ , the susceptibilities

$$\chi_{xy} = \frac{\partial^2 P}{\partial x \partial y} \quad (2.46)$$

and the specific heat

$$C_V = T \left( \frac{\partial s}{\partial T} \right)_V = T \left( \chi_{TT} - \frac{\chi_{\mu T}^2}{\chi_{\mu\mu}} \right). \quad (2.47)$$

One important remark to take into account is that from the previous lines one can see that at first order in the Chapman-Enskog expansion all the thermodynamic functions that appear in the linearized Boltzmann equation should be defined as in equilibrium. That means, that the functions appearing in the left-hand side of the BUU equation (2.42) should be taken as those for an ideal Bose-Einstein gas.

### 2.3.1.1 Shear Viscosity

The shear viscosity appears when the perturbation of the fluid is exclusively by a shear perturbation of the velocity field. For this reason the term associated with the shear viscosity in Eq. (2.42) is

$$p_\mu \partial^\mu n_p(x) \Big|_\eta = \beta n_p(x) (1 + n_p(x)) p^i p^j \tilde{V}_{ij}. \quad (2.48)$$

### 2.3.1.2 Bulk Viscosity

The part of Eq. (2.42) describing uniform compression or expansion of the fluid is related with the bulk viscosity:

$$p_\mu \partial^\mu n_p(x) \Big|_\zeta = \beta n_p(x) (1 + n_p(x)) \left( \frac{1}{3} \mathbf{p}^2 - E_p^2 v_n^2 - E_p \kappa_\epsilon^{-1} \right) \nabla \cdot \mathbf{V}. \quad (2.49)$$

### 2.3.1.3 Heat Conductivity

Finally, the terms of Eq. (2.42) coming with the heat conductivity are those related with gradients in temperature and pressure:

$$p_\mu \partial^\mu n_p(x) \Big|_\kappa = \beta^2 n_p(x) (1 + n_p(x)) \left( E_p - \frac{w}{n} \right) \mathbf{p} \cdot \left( \nabla T - \frac{T}{w} \nabla P \right). \quad (2.50)$$

## 2.4 Conditions of Fit

As pointed out before, the hydrodynamic functions are not necessarily the same as in equilibrium but it is convenient to make this identification. The formal way of doing that is to claim that the energy density, particle density and the particle or energy flux are entirely defined in equilibrium. The three conditions needed for fully specify the hydrodynamical variables are called “conditions of fit” and we detail them here:

- *Condition of fit no. 1:* The energy density is defined as being the same as in equilibrium. In the local reference frame this quantity corresponds to the 00 component of the energy-momentum tensor. Therefore, this amounts in

$$T^{00} = T_{eq}^{00} \rightarrow \tau^{00} = 0. \quad (2.51)$$

Implicitly, this condition defines the temperature field in non-equilibrium to be the same as in equilibrium.

- *Condition of fit no.2:* The particle density is defined as being the same as in equilibrium. This condition makes only sense if the system has a conserved current (particle, charge...). In the local reference frame this corresponds to the 0 component of the four-particle flow

$$n^0 = n_{eq}^0 \rightarrow \nu^0 = 0. \quad (2.52)$$

This condition fixes the chemical potential associated with the conserved current in the non-equilibrium as being the same as in equilibrium.

The hydrodynamical velocity possesses a similar condition to fix. However, one can define it to be parallel to the particle flow (Eckart's choice) or to the energy flow (Landau's choice) because in a relativistic theory both quantities are not necessarily parallel. Even an intermediate choice can be used.

- *Condition of fit no. 3a (Landau or Landau-Lifshitz condition):* The energy flux is defined as being the same as in equilibrium. That makes the velocity field to be parallel as the energy flux. This condition can be applied to a system with or without a conserved current.

$$T^{0i} = T_{eq}^{0i} \rightarrow \tau^{0i} = 0. \quad (2.53)$$

Moreover, in the local rest frame this implies that there is no energy flux at all

$$T^{0i} = 0. \quad (2.54)$$

- *Condition of fit no. 3b (Eckart condition):* The particle flux is defined as in equilibrium. That defines the velocity field to be parallel to the particle flux. This condition can only be applied to a system with a conserved current.

$$n^i = n_{eq}^i \rightarrow \nu^i = 0. \quad (2.55)$$

Moreover, in the local frame this implies that there is no particle flux at all

$$n^i = 0. \quad (2.56)$$

Variable to be defined	Invariant quantity	Expression (arb. frame)	Expression (local rest frame)
$T$	Energy density	$u_\nu u_\nu \tau^{\mu\nu} = 0$	$\tau^{00} = 0$
$\mu$	Particle density	$u_\mu \nu^\mu = 0$	$\nu^0 = 0$
$u^i$	Energy flux	$\Delta^{\mu\nu} T_{\nu\sigma} u^\sigma = 0$	$T^{0i} = 0 \rightarrow \tau^{0i} = 0^a$
$u^i$	Particle flux	$\Delta^{\mu\nu} n_\nu = 0$	$n^i = 0 \rightarrow \nu^i = 0^a$

<sup>a</sup> In equilibrium  $T_{eq}^{0i} = 0$  and  $n_{eq}^i = 0$  because of the symmetry of the integrand in (A.43) and (A.45)

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Theories

Torres-Rincon, J.M.

2014, XIX, 215 p. 68 illus., 20 illus. in color., Hardcover

ISBN: 978-3-319-00424-2