Chapter 2
Background Preliminaries

In this preliminary chapter we will give definitions, descriptions, and formulas, concerning curvature, Euler-Lagrange equations, unconstrained descent optimization, and level sets, all fundamental topics and tools underlying the variational methods of motion analysis described in the subsequent chapters.

2.1 Curvature

Active curve objective functionals of the type we investigate in this book often contain a term which measures the length of a regular closed plane curve, or which integrates a scalar function along such a curve. These terms produce curvature in the objective functional minimization equations. In this section we review some basic facts about curvature of plane curves.

2.1.1 Curvature of a Parametric Curve

A *parametrized differentiable plane curve* is a differentiable map $\mathbf{c} : J \to \mathbb{R}^2$, from an open interval $J \subset \mathbb{R}$ into $\mathbb{R}^2$, i.e., a correspondence which maps each $r \in J$ to a point $(x(r), y(r))$ of the plane in such a way that the coordinate functions $x(r)$ and $y(r)$ are differentiable. The vector $\mathbf{c}'(r) = (x'(r), y'(r))$ of first derivatives of the coordinate functions is the *tangent vector*, or *velocity vector*, of the curve $\mathbf{c}$ at r. A *regular* curve is a differentiable curve for which $\mathbf{c}'(r) \neq 0$ for all r. For $r \in J$, the *arc length* of a regular curve $\mathbf{c}$ from a point r_0 is defined as the function:

$$s(r) = \int_{r_0}^{r} \|\mathbf{c}'(z)\| dz \tag{2.1}$$

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Since $\mathbf{c}'(r) \neq \mathbf{0}$, the arc length function is differentiable and

$$\frac{ds}{dr}(r) = \|\mathbf{c}'(r)\| = \left((x'(r))^2 + (y'(r))^2\right)^{1/2}. \tag{2.2}$$

When r is itself arc length, i.e., $s(r) = r$, then we have $ds/dr(r) = 1$ and, therefore, $\|\mathbf{c}'(r)\| = 1$, prompting the definition: A curve $\mathbf{c} : J \to \mathbb{R}^2$ is said to be parametrized by arc length s if $\|\mathbf{c}'(s)\| = 1 \ \ \forall s \in J$,

Let $\mathbf{c} : r \in J \to (x(r), y(r)) \in \mathbb{R}^2$ be a regular parametrized plane curve, not necessarily by arc length. The parametrization defines two possible orientations of the curve: The orientation along which the parameter grows and the opposite orientation. Let $\mathbf{t}$ be the unit tangent vector of $\mathbf{c}$ associated with the orientation of growing curve parameter (Fig. 2.1). We have:

$$\mathbf{t} = \left(\frac{x'}{\left((x')^2 + (y')^2\right)^{1/2}}, \frac{y'}{\left((x')^2 + (y')^2\right)^{1/2}}\right), \tag{2.3}$$

where the prime symbol designates the derivative with respect to parameter r.

Define now the unit normal vector $\mathbf{n}$ by requiring the basis $(\mathbf{t}, \mathbf{n})$ to have the same orientation as the natural basis (e_1, e_2) of $\mathbb{R}^2$ (Fig. 2.1) and then *define* curvature κ by [1]:

$$\frac{d\mathbf{t}}{ds} = \kappa \mathbf{n}, \tag{2.4}$$

where s is arc length. This definition gives a sign to the curvature, i.e., it can be positive or negative depending on the point of evaluation. This can be quickly verified graphically by drawing a figure such as Fig. 2.1. Changing the orientation of the curve or of $\mathbb{R}^2$ will change the sign of the curvature.

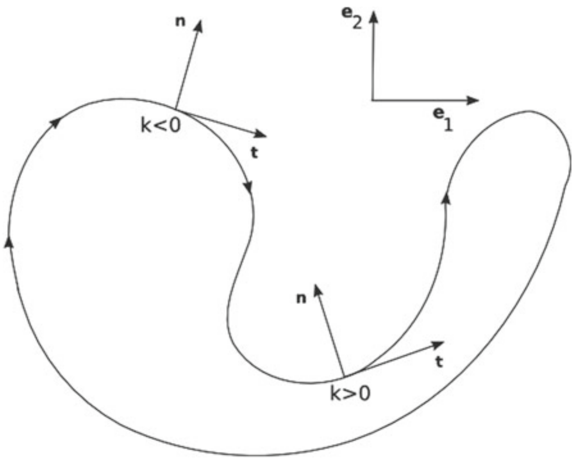

Fig. 2.1 For plane curves, curvature can be given a sign as follows: Let $\mathbf{t}$ be the unit tangent vector and define the unit normal vector $\mathbf{n}$ by requiring the basis $(\mathbf{t}, \mathbf{n})$ to have the same orientation as the natural basis of $\mathbb{R}^2$. Then define curvature κ by $d\mathbf{t}/ds = \kappa\mathbf{n}$, where s is the arc length parameter. The sign of the curvature changes if the orientation of either the curve or of the normal changes

Derivation of the unit tangent vector, with respect to parameter r, gives:

$$\frac{d\mathbf{t}}{dr} = \left(-y' \frac{x'y'' - y'x''}{\left((x')^2 + (y')^2\right)^{3/2}} \ , \ x' \frac{x'y'' - y'x''}{\left((x')^2 + (y')^2\right)^{3/2}}\right)$$
$$= \frac{x'y'' - y'x''}{\left((x')^2 + (y')^2\right)^{3/2}} \left(-y' \ , \ x'\right). \tag{2.5}$$

Using Eq. (2.2), noting that dr/ds is the inverse of $ds/dr \neq 0$, i.e., $dr/ds = 1/\|\mathbf{c}'(r)\| = \left((x'(r))^2 + (y'(r))^2\right)^{-1/2}$, we have:

$$\frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dr}\frac{dr}{ds} = \frac{x'y'' - y'x''}{\left((x')^2 + (y')^2\right)^{3/2}} \left(-\frac{y'}{\left((x')^2 + (y')^2\right)^{1/2}} \ , \ \frac{x'}{\left((x')^2 + (y')^2\right)^{1/2}}\right)$$
$$= \frac{x'y'' - y'x''}{\left((x')^2 + (y')^2\right)^{3/2}}\mathbf{n} \tag{2.6}$$

Comparing Eq. (2.6) to Eq. (2.4), we have the following parametrization independent expression of curvature:

$$\kappa = \frac{x'y'' - y'x''}{\left((x')^2 + (y')^2\right)^{3/2}} \tag{2.7}$$

2.1.2 Curvature of an Implicit Curve

Assume that the level set $\{(x, y)|\phi(x, y) = 0\}$ defines a differentiable parametric curve $\mathbf{c} : r \to \mathbf{c}(r) = (x(r), y(r))$. Then by the chain rule of differentiation:

$$\frac{d}{dr}\phi(\mathbf{c}(r)) = \nabla\phi \cdot \mathbf{v} = 0, \tag{2.8}$$

where $\mathbf{v} = \mathbf{c}'$ is the tangent vector of $\mathbf{c}$. This shows that the gradient of ϕ is perpendicular to the level set curve. Let $\mathbf{n}$ be the unit vector normal to $\mathbf{c}$ defined by:

$$\mathbf{n} = \frac{\nabla\phi}{\|\nabla\phi\|} = \left(\frac{\phi_x}{\left(\phi_x^2 + \phi_y^2\right)^{1/2}}, \frac{\phi_y}{\left(\phi_x^2 + \phi_y^2\right)^{1/2}}\right) \tag{2.9}$$

Let the unit tangent vector $\mathbf{t}$ be defined by requiring the basis $\{\mathbf{t}, \mathbf{n}\}$ to have the same orientation as the natural basis $\{e_1, e_2\}$ of $\mathbb{R}^2$, we have:

$$\mathbf{t} = \left(\frac{\phi_y}{\left(\phi_x^2 + \phi_y^2\right)^{1/2}}, \frac{-\phi_x}{\left(\phi_x^2 + \phi_y^2\right)^{1/2}} \right), \tag{2.10}$$

Curvature can then be defined by Eq. (2.4).

Let s designate arc length. Using the chain rule of differentiation we can write:

$$\frac{d\mathbf{t}}{ds} = \frac{\partial \mathbf{t}}{\partial x}\frac{dx}{ds} + \frac{\partial \mathbf{t}}{\partial y}\frac{dy}{ds}. \tag{2.11}$$

In this expression, $\left(\frac{dx}{ds}, \frac{dy}{ds}\right) = \mathbf{t}$; therefore, substitution in Eq. (2.11) of expression Eq. (2.10) of $\mathbf{t}$ gives:

$$\frac{d\mathbf{t}}{ds} = \frac{1}{\|\nabla\phi\|}\left(\phi_y \frac{\partial \mathbf{t}}{\partial x} - \phi_x \frac{\partial \mathbf{t}}{\partial y}\right). \tag{2.12}$$

The partial derivative with respect to x of the first component of $\mathbf{t}$ evaluates as follows:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\phi_y}{\left(\phi_x^2 + \phi_y^2\right)^{1/2}} \right) &= \frac{\phi_{xy}}{\left(\phi_x^2 + \phi_y^2\right)^{1/2}} - \frac{\phi_y(\phi_x\phi_{xx} + \phi_y\phi_{xy})}{(\phi_x^2 + \phi_y^2)^{3/2}} \\ &= \frac{\phi_{xy}(\phi_x^2 + \phi_y^2) - \phi_y(\phi_x\phi_{xx} + \phi_y\phi_{xy})}{(\phi_x^2 + \phi_y^2)^{3/2}} \\ &= \frac{\phi_{xy}\phi_x^2 - \phi_x\phi_y\phi_{xx}}{(\phi_x^2 + \phi_y^2)^{3/2}} \end{aligned} \tag{2.13}$$

Similarly, we determine that the partial derivative with respect to y of the first component of $\mathbf{t}$ is given by:

$$\frac{\partial}{\partial y}\left(\frac{\phi_y}{\left(\phi_x^2 + \phi_y^2\right)^{1/2}} \right) = \frac{\phi_{yy}\phi_x^2 - \phi_x\phi_y\phi_{xy}}{(\phi_x^2 + \phi_y^2)^{3/2}}. \tag{2.14}$$

Substitution of Eqs. (2.13) and (2.14) back in Eq. (2.12) gives the derivative with respect to s of the first component of $\mathbf{t}$:

$$\frac{dt_1}{ds} = -\frac{\phi_x}{\|\nabla\phi\|}\frac{\phi_{xx}\phi_y^2 - 2\phi_x\phi_y\phi_{xy} + \phi_{yy}\phi_x^2}{(\phi_x^2 + \phi_y^2)^{3/2}} \tag{2.15}$$

We proceed in the same manner to obtain the partial derivatives with respect to x and y of the second component of the tangent vector $\mathbf{t}$:

$$\frac{\partial}{\partial x}\left(\frac{-\phi_x}{\left(\phi_x^2+\phi_y^2\right)^{1/2}}\right)=\frac{\phi_x\phi_y\phi_{xy}-\phi_{xx}\phi_y^2}{(\phi_x^2+\phi_y^2)^{3/2}} \tag{2.16}$$

$$\frac{\partial}{\partial y}\left(\frac{-\phi_x}{\left(\phi_x^2+\phi_y^2\right)^{1/2}}\right)=\frac{\phi_x\phi_y\phi_{yy}-\phi_{xy}\phi_y^2}{(\phi_x^2+\phi_y^2)^{3/2}}, \tag{2.17}$$

which give the derivative with respect to s of the second component of $\mathbf{t}$ by substitution into Eq. (2.12):

$$\frac{dt_2}{ds}=-\frac{\phi_y}{\|\nabla\phi\|}\frac{\phi_{xx}\phi_y^2-2\phi_x\phi_y\phi_{xy}+\phi_{yy}\phi_x^2}{(\phi_x^2+\phi_y^2)^{3/2}} \tag{2.18}$$

Putting together Eq. (2.15) and Eq. (2.18) gives the desired equation:

$$\frac{d\mathbf{t}}{ds}=-\frac{\phi_{xx}\phi_y^2-2\phi_x\phi_y\phi_{xy}+\phi_{yy}\phi_x^2}{(\phi_x^2+\phi_y^2)^{3/2}}\,\mathbf{n} \tag{2.19}$$

Comparing with Eq. (2.4), we have the expression of curvature:

$$\kappa=-\frac{\phi_{xx}\phi_y^2-2\phi_x\phi_y\phi_{xy}+\phi_{yy}\phi_x^2}{(\phi_x^2+\phi_y^2)^{3/2}} \tag{2.20}$$

Curvature can also be expressed as, with our choice of $\mathbf{n}$ and $\mathbf{t}$:

$$\kappa=-\operatorname{div}\left(\frac{\nabla\phi}{\|\nabla\phi\|}\right), \tag{2.21}$$

which can be proved by expanding the righthand side:

$$-\operatorname{div}\left(\frac{\nabla\phi}{\|\nabla\phi\|}\right) \tag{2.22}$$

$$=-\frac{\partial}{\partial x}\left(\frac{\phi_x}{\left(\phi_x^2+\phi_y^2\right)^{1/2}}\right)-\frac{\partial}{\partial y}\left(\frac{\phi_y}{\left(\phi_x^2+\phi_y^2\right)^{1/2}}\right) \tag{2.23}$$

$$=-\frac{\phi_{xx}}{\left(\phi_x^2+\phi_y^2\right)^{1/2}}+\frac{\phi_x(\phi_x\phi_{xx}+\phi_y\phi_{xy})}{(\phi_x^2+\phi_y^2)^{3/2}}-\frac{\phi_{yy}}{\left(\phi_x^2+\phi_y^2\right)^{1/2}}+\frac{\phi_y(\phi_x\phi_{xy}+\phi_y\phi_{yy})}{(\phi_x^2+\phi_y^2)^{3/2}} \tag{2.24}$$

$$= -\frac{\phi_{xx}(\phi_x^2 + \phi_y^2) - \phi_x(\phi_x\phi_{xx} + \phi_y\phi_{xy}) + \phi_{yy}(\phi_x^2 + \phi_y^2) - \phi_y(\phi_x\phi_{xy} + \phi_y\phi_{yy})}{(\phi_x^2 + \phi_y^2)^{3/2}} \tag{2.25}$$

$$= -\frac{\phi_{xx}\phi_y^2 - 2\phi_x\phi_y\phi_{xy} + \phi_{yy}\phi_x^2}{(\phi_x^2 + \phi_y^2)^{3/2}}, \tag{2.26}$$

which is the same expression as given in Eq. (2.20).[1]

2.2 Euler-Lagrange Equations

The active curve objective functionals we will investigate in this book are minimized by solving the corresponding Euler-Lagrange equations.[2] Here following is a review of the basic formulas we will be using, concerning both definite integrals and variable domain integrals. The variable domain integrals we will treat include curve length integrals of a closed regular plane curve, integrals of a scalar function along a closed regular plane curve, or over a closed regular surface in $\mathbb{R}^3$, as well as integrals over bounded regions in $\mathbb{R}^2$ and $\mathbb{R}^3$.

2.2.1 *Definite Integrals*

The purpose in this section is to provide succinct derivations of the basic Euler-Lagrange differential equations for definite integrals. We will follow the presentation of R. Weinstock [2] which requires only basic results in vector calculus [3]. We will first develop the Euler-Lagrange equation corresponding to an integral involving a real function of a real variable:

$$\mathscr{E}(u) = \int_{x_1}^{x_2} g(x, u, u')dx, \tag{2.27}$$

where the endpoints x_1 and x_2 are given real numbers; $u = u(x)$ is a twice differentiable real function; $u' = \frac{du}{dx}$; and g is a function twice differentiable with respect to any of its three arguments, x, u, and u'.

Assume that there exists a twice-differentiable function u satisfying the boundary conditions $u(x_1) = u_1$ and $u(x_2) = u_2$ and which minimizes the integral Eq. (2.27). We want to determine the differential equation which this minimizer u must satisfy. To do this, let $\eta(x)$ be an arbitrary differentiable function which satisfies the endpoint

[1] Note that we could have defined the unit normal of the implicit curve as $\mathbf{n} = -\nabla\phi/\|\nabla\phi\|$ instead of $\mathbf{n} = \nabla\phi/\|\nabla\phi\|$ as in Eq. (2.9), in which case curvature would change sign, i.e., it would have the expression in Eq. (2.20) but without the minus sign. At the same time it would be written $\kappa = \mathrm{div}\,(\nabla\phi/\|\nabla\phi\|)$ rather than with the minus sign as in Eq. (2.21).

[2] The discussions apply as well to the maximization of similar functionals.

conditions $\eta(x_1) = \eta(x_2) = 0$, and define the following one-parameter family of functions U indexed by parameter $\varepsilon \in \mathbb{R}$:

$$U(x, \varepsilon) = u(x) + \varepsilon\eta(x) \tag{2.28}$$

All functions U in this family have the same endpoints as u, i.e., $U(x_1, \varepsilon) = u_1$ and $U(x_2, \varepsilon) = u_2$ for all ε. The minimizer u is the member of the family corresponding to $\varepsilon = 0$, i.e., $U(x, 0) = u(x)$. By definition, there is a neighborhood $\mathscr{U}$ of u where the integral is a minimum at u, i.e., $\mathscr{E}(u) \leq \mathscr{E}(y)\ \forall y \in \mathscr{U}$. We can choose ε in a small enough interval J so that the functions so defined by Eq. (2.28) fall in this neighborhood for all $\varepsilon \in J$. In this case the following integral function of ε:

$$E(\varepsilon) = \int_{x_1}^{x_2} g(x, U(x, \varepsilon), U'(x, \varepsilon))dx, \tag{2.29}$$

with

$$U' = \frac{dU}{dx} = u' + \varepsilon\eta', \tag{2.30}$$

is minimized with respect to ε for $\varepsilon = 0$ and, therefore:

$$\frac{dE}{d\varepsilon}(0) = 0 \tag{2.31}$$

Differentiation under the integral sign (Sect. 2.3) of Eq. (2.29) with respect to parameter ε gives:

$$\begin{aligned}\frac{dE}{d\varepsilon}(\varepsilon) &= \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial U}\frac{\partial U}{\partial \varepsilon} + \frac{\partial g}{\partial U'}\frac{\partial U'}{\partial \varepsilon} \right) dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial U}\eta + \frac{\partial g}{\partial U'}\eta' \right) dx \end{aligned} \tag{2.32}$$

The necessary condition Eq. (2.31) is then written as:

$$\frac{dE}{d\varepsilon}(0) = \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial u}\eta + \frac{\partial g}{\partial u'}\eta' \right) dx = 0 \tag{2.33}$$

Integration by parts of the second term of the integrand gives:

$$\frac{dE}{d\varepsilon}(0) = \int_{x_1}^{x_2} \left[\frac{\partial g}{\partial u} - \frac{d}{dx}\left(\frac{\partial g}{\partial u'} \right) \right] \eta\, dx = 0. \tag{2.34}$$

This equation must hold for all η. Therefore, we have the *Euler-Lagrange equation* which the minimizer u of $\mathscr{E}$ must satisfy:

$$\frac{\partial g}{\partial u} - \frac{d}{dx}\left(\frac{\partial g}{\partial u'}\right) = 0. \tag{2.35}$$

Several Dependent Variables

When the integral involves several dependent real variables $u(x), v(x), \ldots, w(x)$:

$$\mathscr{E}(u) = \int_{x_1}^{x_2} g(x, u, v, \ldots, w, u', v', \ldots, w')dx, \tag{2.36}$$

similar developments yield one Euler-Lagrange equation for each dependent variable:

$$\begin{aligned} \frac{\partial g}{\partial u} - \frac{d}{dx}\left(\frac{\partial g}{\partial u'}\right) &= 0 \\ \frac{\partial g}{\partial v} - \frac{d}{dx}\left(\frac{\partial g}{\partial v'}\right) &= 0 \\ &\cdots \\ \frac{\partial g}{\partial w} - \frac{d}{dx}\left(\frac{\partial g}{\partial w'}\right) &= 0 \end{aligned} \tag{2.37}$$

Several Independent Variables

In subsequent chapters, we will encounter integrals involving scalar functions of two independent variables. Consider an integral of the form:

$$\mathscr{E}(w) = \int_R g(x, y, w, w_x, w_y)dxdy, \tag{2.38}$$

where R is a bounded region of $\mathbb{R}^2$ the boundary ∂R of which is a regular closed plane curve; $w = w(x, y)$ assumes some prescribed values at all points on ∂R; w_x and w_y are the partial derivatives of w; and g is twice continuously differentiable with respect to its arguments. To determine the differential equation which a minimizing function $w(x, y)$ must satisfy, we proceed at first as we did with functions of a single real variable, namely: we consider the following family of functions indexed by real parameter ε:

$$W(x, y, \varepsilon) = w(x, y) + \varepsilon\eta(x, y), \tag{2.39}$$

where η is an arbitrary continuously differentiable real function such that $\eta(x, y) = 0$ on ∂R, so that functions W all have the same boundary values. Then we form the integral function of ε:

$$E(\varepsilon) = \int_R g(x, y, W(x, y, \varepsilon), W_x(x, y, \varepsilon), W_y(x, y, \varepsilon))dxdy, \tag{2.40}$$

and remark that

$$\frac{dE}{d\varepsilon}(0) = 0, \tag{2.41}$$

which would give:

$$\frac{dE}{d\varepsilon}(0) = \int_R \left(\frac{\partial g}{\partial w}\eta + \frac{\partial g}{\partial w_x}\frac{\partial \eta}{\partial x} + \frac{\partial g}{\partial w_y}\frac{\partial \eta}{\partial y} \right) dxdy = 0. \tag{2.42}$$

To continue the derivation we need to apply the Green's theorem to the integrals corresponding to the last two terms of the integrand. We recall the theorem in its most usual form: Let $P(x, y)$ and $Q(x, y)$ be real functions with continuous first partial derivatives in a region R of the plane bounded by a regular closed curve. Then:

$$\int_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy = \int_{\partial R} (Pdy - Qdx). \tag{2.43}$$

We will use the integration by parts expression of this theorem, obtained by setting $P = \eta G$ and $Q = \eta F$ [2]:

$$\int_R \left(G\frac{\partial \eta}{\partial x} + F\frac{\partial \eta}{\partial y} \right) dxdy = -\int_R \eta \left(\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y} \right) dxdy + \int_{\partial R} \eta(Gdy - Fdx). \tag{2.44}$$

In our case, $G = \frac{\partial g}{\partial w_x}$; $F = \frac{\partial g}{\partial w_y}$, and the second integral on the righthand side of Eq. (2.44) is zero because $\eta = 0$ on ∂R. Applying this to Eq. (2.42) gives:

$$\int_R \eta \left[\frac{\partial g}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial w_x} \right) - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial w_y} \right) \right] dxdy = 0. \tag{2.45}$$

This is an equation which must be satisfied for all η, leading to the desired Euler-Lagrange equation:

$$\frac{\partial g}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial w_x} \right) - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial w_y} \right) = 0. \tag{2.46}$$

In the general case of more than two independent variables $x, y, \ldots, z$, Eq. (2.46) generalizes to:

$$\frac{\partial g}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial w_x} \right) - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial w_y} \right) - \cdots - \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial w_z} \right) = 0. \tag{2.47}$$

Example: Let $I : (x, y, t) \in \Omega \times]0, T[\mapsto I(x, y, t) \in \mathbb{R}^+$ be an image sequence and consider the Horn and Schunck optical flow estimation functional:

$$\mathscr{E}(u, v) = \int_{\Omega} (I_x u + I_y v + I_t)^2 dxdy + \lambda \int_{\Omega} (\|\nabla u\|^2 + \|\nabla v\|^2) dxdy$$

where I_x, I_y, I_t are the image spatiotemporal derivatives, $\nabla u, \nabla v$ are the spatial gradients of the optical flow coordinates u, v, and λ is a constant factor to weigh the contribution of the two terms in the objective functional. There are two independent variables, namely the image coordinates x, y and two dependent variables, namely the functions $u(x, y)$, $v(x, y)$. Therefore, we will have two equations, one for u and one for v. We apply Eq. (2.46) to each of u and v with $g(x, y, u, v, u_x, u_y, v_x, v_y) = (I_x u + I_y v + I_t)^2 + \lambda(u_x^2 + u_y^2 + v_x^2 + v_y^2)$ which immediately gives:

$$\begin{aligned} I_x(I_x u + I_y v + I_t) - \lambda \nabla^2 u = 0 \\ I_y(I_x u + I_y v + I_t) - \lambda \nabla^2 v = 0, \end{aligned} \tag{2.48}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator.

Functional derivative: As frequently done in the computer vision literature, we will refer to the lefthand side of the Euler-Lagrange equation of an integral $\mathscr{E}$ corresponding to a dependent real variable u as the *functional derivative* of $\mathscr{E}$ with respect to u, with the notation $\frac{d\mathscr{E}}{du}$ when u is the single argument of $\mathscr{E}$ and $\frac{\partial \mathscr{E}}{\partial u}$ when $\mathscr{E}$ has several arguments.

2.2.2 Variable Domain of Integration

We will derive the functional derivative of functionals which are common in image motion analysis and image segmentation by active contours, and which appear throughout this book, namely integrals over regions enclosed by closed regular plane curves and path integrals of scalar functions over such curves. We will also derive the functional derivative for surface and volume integrals.

Region Integral of a Scalar Function

Let R_γ be the interior of a closed regular plane curve parametrized by arc length, $\gamma : s \in [0, l] \to (x(s), y(s)) \in \mathbb{R}$. The segmentation functionals we will encounter in this book typically contain a term of the form:

$$\mathscr{E}(\gamma) = \int_{R_\gamma} f(x, y) dxdy, \tag{2.49}$$

where f is a scalar function, i.e., independent of γ. The functional depends on γ via its domain of integration which is a function of γ.

To determine the Euler-Lagrange equation corresponding to the minimization of Eq. (2.49) with respect to γ (we assume that the problem is to minimize $\mathscr{E}$ but, of course, the discussion applies to maximization as well), the functional is first transformed into a simple integral as follows using Green's theorem [3]. Let

$$P(x, y) = -\frac{1}{2}\int_0^y f(x, z)dz \tag{2.50}$$

and

$$Q(x, y) = \frac{1}{2}\int_0^x f(z, y)dz \tag{2.51}$$

According to Green's theorem we have:

$$\int_{R_\gamma} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_\gamma Pdx + Qdy \tag{2.52}$$

Since $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = f(x, y)$ we get:

$$\int_{R_\gamma} f(x, y)dxdy = \int_\gamma Pdx + Qdy = \int_0^l \left(Px' + Qy'\right) ds, \tag{2.53}$$

where $x' = \frac{dx}{ds}$ and $y' = \frac{dy}{ds}$. Applying Eq. (2.37) to the last integral in Eq. (2.53), i.e., using:

$$g(s, x, y, x', y') = P(x(s), y(s))x'(s) + Q(x(s), y(s))y'(s), \tag{2.54}$$

we get two equations, one for each component function of γ:

$$\begin{aligned}
\frac{\partial \mathscr{E}}{\partial x} &= \frac{\partial g}{\partial x} - \frac{d}{ds}\left(\frac{\partial g}{\partial x'}\right) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) y' = fy' \\
\frac{\partial \mathscr{E}}{\partial y} &= \frac{\partial g}{\partial y} - \frac{d}{ds}\left(\frac{\partial g}{\partial y'}\right) = \left(-\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) x' = -fx'.
\end{aligned} \tag{2.55}$$

The Green's theorem expression in Eq. (2.52) assumes that curve γ is oriented counter clockwise [3]. With this orientation and since we are using the arc length parametrization, the outward unit normal $\mathbf{n}$ to γ is $\mathbf{n} = (y', -x')$. Therefore, the functional derivatives in Eq. (2.55) can be written in vector form as:

$$\frac{\partial \mathscr{E}}{\partial \gamma} = f\mathbf{n} \tag{2.56}$$

The Length Integral (Two Dimensions)

Another functional which very often appears in the motion analysis formulations in the book is the curve length functional:

$$\mathscr{E}(\gamma) = \int_{\gamma} ds, \tag{2.57}$$

which can be rewritten as:

$$\int_0^l \left(x'^2 + y'^2\right)^{\frac{1}{2}} ds, \tag{2.58}$$

Applying Eq. (2.37) using:

$$g(s, x, y, x', y') = \left(\left(x'(s)\right)^2 + \left(y'(s)\right)^2\right)^{\frac{1}{2}}, \tag{2.59}$$

where s is the arc length parameter, gives:

$$\begin{aligned} \frac{\partial \mathscr{E}}{\partial x} &= \frac{\partial g}{\partial x} - \frac{d}{ds}\left(\frac{\partial g}{\partial x'}\right) = -\frac{d}{ds}\left(\frac{x'}{\left((x')^2 + (y')^2\right)^{1/2}}\right) = -\frac{dx'}{ds} \\ \frac{\partial \mathscr{E}}{\partial y} &= \frac{\partial g}{\partial y} - \frac{d}{ds}\left(\frac{\partial g}{\partial y'}\right) = -\frac{d}{ds}\left(\frac{y'}{\left((x')^2 + (y')^2\right)^{1/2}}\right) = -\frac{dy'}{ds}, \end{aligned} \tag{2.60}$$

where we have used the fact that when a curve $\mathbf{c}$ is parametrized by arc length then $\|\mathbf{c}'\| = 1$. Equation (2.60) are written in vector form as

$$\frac{\partial \mathscr{E}}{\partial \gamma} = -\frac{d\mathbf{t}}{ds}, \tag{2.61}$$

where $\mathbf{t}$ is the unit tangent vector of γ. Using the definition Eq. (2.4) of curvature and assuming the configuration of Fig. 2.1 where the curve is oriented clockwise and the normal outward, we have:

$$\frac{\partial \mathscr{E}}{\partial \gamma} = -\kappa \mathbf{n}, \tag{2.62}$$

If we orient the curve in the opposite direction, i.e., counter clockwise, but leave the normal pointing outward, then:

$$\frac{\partial \mathscr{E}}{\partial \gamma} = \kappa \mathbf{n}, \tag{2.63}$$

where the curvature κ is still given by Eq. (2.7).

Path Integral of a Scalar Function (Two Dimensions)

Consider the following functional:

$$\mathscr{E}(\gamma) = \int_\gamma h ds, \tag{2.64}$$

where h is a scalar function, i.e., independent of γ. For $h = 1$ we have the curve length integral Eq. (2.57) as a special case. We can rewrite this functional as:

$$\mathscr{E}(\gamma) = \int_0^l h \left((x')^2 + (y')^2\right)^{1/2} ds, \tag{2.65}$$

Using

$$g(s, x, y, x', y') = h(x(s), y(s)) \left(\left(x'(s)\right)^2 + \left(y'(s)\right)^2\right)^{\frac{1}{2}}, \tag{2.66}$$

where s is the arc length parameter, the functional derivative of $\mathscr{E}$ with respect to the component x of γ is developed as follows:

$$\begin{aligned} \frac{\partial \mathscr{E}}{\partial x} &= \frac{\partial g}{\partial x} - \frac{d}{ds}\left(\frac{\partial g}{\partial x'}\right) \\ &= h_x - (\nabla h \cdot \mathbf{t})\, x' - h\frac{dx'}{ds}, \end{aligned} \tag{2.67}$$

where $\mathbf{t}$ is the unit tangent vector of γ. Similarly, we have:

$$\frac{\partial \mathscr{E}}{\partial y} = h_y - (\nabla h \cdot \mathbf{t})\, y' - h\frac{dy'}{ds}$$

In vector form, we have:

$$\frac{\partial \mathscr{E}}{\partial \gamma} = \nabla h - (\nabla h \cdot \mathbf{t})\, \mathbf{t} - h\frac{d\mathbf{t}}{ds} \tag{2.68}$$

Since

$$\nabla h - (\nabla h \cdot \mathbf{t})\, \mathbf{t} = (\nabla h \cdot \mathbf{n})\, \mathbf{n} \tag{2.69}$$

and, according to the definition of curvature, $d\mathbf{t}/ds = \kappa \mathbf{n}$, we finally get:

$$\frac{\partial \mathscr{E}}{\partial \gamma} = (\nabla h \cdot \mathbf{n} - h\kappa)\, \mathbf{n}. \tag{2.70}$$

Here also the formula assumes the configuration of Fig. 2.1 where the curve is oriented clockwise and the normal outward. If we orient the curve in the opposite direction, i.e., counter clockwise, but leave the normal pointing outward, then:

$$\frac{\partial \mathscr{E}}{\partial \gamma} = (\nabla h \cdot \mathbf{n} + h\kappa)\, \mathbf{n}. \tag{2.71}$$

Next we give generic derivations of the functional derivatives of surface integrals of a scalar function and volume integrals of a scalar function [4]. Such integrals appear in the objective functional Eq. (4.80) in Chap. 5.

Surface Integral of a Scalar Function

We will now develop the functional derivative of a surface integral of a scalar function [4]:

$$\mathscr{E}_1 (S) = \int_S g\, d\sigma, \tag{2.72}$$

where S is a closed regular surface in $\mathbb{R}^3$ and g is a scalar function independent of S. Let $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ be a cartesian reference system in $\mathbb{R}^3$, and ϕ a parameterization of S:

$$\phi : (r, s) \in [0, l_1] \times [0, l_2] \to \phi(r, s) = (x\,(r, s), y\,(r, s), z\,(r, s)) \in \mathbb{R}^3 \tag{2.73}$$

Let $\mathbf{T}_r$ and $\mathbf{T}_s$ be the following vectors:

$$\begin{aligned} \mathbf{T}_r &= x_r \mathbf{i} + y_r \mathbf{j} + z_r \mathbf{k} \\ \mathbf{T}_s &= x_s \mathbf{i} + y_s \mathbf{j} + z_s \mathbf{k}, \end{aligned} \tag{2.74}$$

where the subscripts on x, y, z indicate partial derivatives. Functional Eq. (2.72) can be rewritten as [1, 3]:

$$\mathscr{E}_1 (S) = \int_0^{l_1} \int_0^{l_2} g\,(x, y, z)\, \|\mathbf{T}_r \times \mathbf{T}_s\|\, drds, \tag{2.75}$$

Let L_1 designate the integrand of $\mathscr{E}_1$:

$$L_1\,(x, y, z, x_r, y_r, z_r, x_s, y_s, z_s, r, s) = g\, \|\mathbf{T}_r \times \mathbf{T}_s\| \tag{2.76}$$

The functional derivatives corresponding to $\mathscr{E}_1$ follow from the formulas:

$$
\begin{aligned}
\frac{\partial \mathscr{E}_1}{\partial x} &= \frac{\partial L_1}{\partial x} - \frac{\partial}{\partial r}\frac{\partial L_1}{\partial x_r} - \frac{\partial}{\partial s}\frac{\partial L_1}{\partial x_s} \\
\frac{\partial \mathscr{E}_1}{\partial y} &= \frac{\partial L_1}{\partial y} - \frac{\partial}{\partial r}\frac{\partial L_1}{\partial y_r} - \frac{\partial}{\partial s}\frac{\partial L_1}{\partial y_s} \\
\frac{\partial \mathscr{E}_1}{\partial z} &= \frac{\partial L_1}{\partial z} - \frac{\partial}{\partial r}\frac{\partial L_1}{\partial z_r} - \frac{\partial}{\partial s}\frac{\partial L_1}{\partial z_s}.
\end{aligned}
\tag{2.77}
$$

Let $\mathbf{n}$ be the unit normal vector that points outwards to the exterior of S, i.e., toward the complement of its interior R_S, and let Φ be an orientation-preserving parametrization so that:

$$
\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = \frac{\mathbf{T}_r \times \mathbf{T}_s}{\|\mathbf{T}_r \times \mathbf{T}_s\|}, \tag{2.78}
$$

in which case:

$$
L_1 = g\,\|\mathbf{T}_r \times \mathbf{T}_s\| = g\,\|\mathbf{N}\|\,. \tag{2.79}
$$

Consider the formula of the first row of Eq. (2.77). We have the following developments:

$$
\begin{aligned}
\frac{\partial L_1}{\partial x} &= g_x \|\mathbf{N}\| \\
\frac{\partial}{\partial r}\frac{\partial L_1}{\partial x_r} &= (\nabla g \cdot \mathbf{T}_r)\,\mathbf{n} \cdot \frac{\partial \mathbf{N}}{\partial x_r} + g\mathbf{n}_r \cdot \frac{\partial \mathbf{N}}{\partial x_r} \\
\frac{\partial}{\partial s}\frac{\partial L_1}{\partial x_s} &= (\nabla g \cdot \mathbf{T}_s)\,\mathbf{n} \cdot \frac{\partial \mathbf{N}}{\partial x_s} + g\mathbf{n}_s \cdot \frac{\partial \mathbf{N}}{\partial x_s}
\end{aligned}
\tag{2.80}
$$

The other two lines of Eq. (2.77) are developed in the same manner and we get:

$$
\begin{aligned}
\frac{\partial \mathscr{E}_1}{\partial x} &= g_x\,\|\mathbf{N}\| - (\nabla g \cdot \mathbf{T}_r)\left(\mathbf{n} \cdot \frac{\partial \mathbf{N}}{\partial x_r}\right) - (\nabla g \cdot \mathbf{T}_s)\,\mathbf{n} \cdot \frac{\partial \mathbf{N}}{\partial x_s} - g\left(\frac{\partial \mathbf{N}}{\partial x_r} \cdot \mathbf{n}_r + \frac{\partial \mathbf{N}}{\partial x_s} \cdot \mathbf{n}_s\right) \\
\frac{\partial \mathscr{E}_1}{\partial y} &= g_y\,\|\mathbf{N}\| - (\nabla g \cdot \mathbf{T}_r)\left(\mathbf{n} \cdot \frac{\partial \mathbf{N}}{\partial y_r}\right) - (\nabla g \cdot \mathbf{T}_s)\,\mathbf{n} \cdot \frac{\partial \mathbf{N}}{\partial y_s} - g\left(\frac{\partial \mathbf{N}}{\partial y_r} \cdot \mathbf{n}_r + \frac{\partial \mathbf{N}}{\partial y_s} \cdot \mathbf{n}_s\right) \\
\frac{\partial \mathscr{E}_1}{\partial z} &= g_z\,\|\mathbf{N}\| - (\nabla g \cdot \mathbf{T}_r)\left(\mathbf{n} \cdot \frac{\partial \mathbf{N}}{\partial z_r}\right) - (\nabla g \cdot \mathbf{T}_s)\,\mathbf{n} \cdot \frac{\partial \mathbf{N}}{\partial z_s} - g\left(\frac{\partial \mathbf{N}}{\partial z_r} \cdot \mathbf{n}_r + \frac{\partial \mathbf{N}}{\partial z_s} \cdot \mathbf{n}_s\right)
\end{aligned}
\tag{2.81}
$$

Since $\mathbf{N} = \mathbf{T_r} \times \mathbf{T_s}$, we further have, looking back at the expression of $\mathbf{T}_r$ and $\mathbf{T}_s$ in Eq. (2.74):

$$
\mathbf{n} \cdot \frac{\partial \mathbf{N}}{\partial x_r} = \mathbf{n} \cdot (\mathbf{i} \times \mathbf{T_s}) = (\mathbf{T_s} \times \mathbf{n}) \cdot \mathbf{i} \tag{2.82}
$$

and, similarly, we have:

$$\mathbf{n} \cdot \frac{\partial \mathbf{N}}{\partial y_r} = (\mathbf{T}_s \times \mathbf{n}) \cdot \mathbf{j} \tag{2.83}$$

$$\mathbf{n} \cdot \frac{\partial \mathbf{N}}{\partial z_r} = (\mathbf{T}_s \times \mathbf{n}) \cdot \mathbf{k}, \tag{2.84}$$

giving the vectorial equation:

$$\begin{bmatrix} \mathbf{n} \cdot \dfrac{\partial \mathbf{N}}{\partial x_r} \\ \mathbf{n} \cdot \dfrac{\partial \mathbf{N}}{\partial y_r} \\ \mathbf{n} \cdot \dfrac{\partial \mathbf{N}}{\partial z_r} \end{bmatrix} = \mathbf{T}_s \times \mathbf{n} \tag{2.85}$$

Similar manipulations give:

$$\begin{bmatrix} \mathbf{n} \cdot \dfrac{\partial \mathbf{N}}{\partial x_s} \\ \mathbf{n} \cdot \dfrac{\partial \mathbf{N}}{\partial y_s} \\ \mathbf{n} \cdot \dfrac{\partial \mathbf{N}}{\partial z_s} \end{bmatrix} = \mathbf{n} \times \mathbf{T}_r, \tag{2.86}$$

and

$$\begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial x_r} \cdot \mathbf{n}_r \\ \dfrac{\partial \mathbf{N}}{\partial y_r} \cdot \mathbf{n}_r \\ \dfrac{\partial \mathbf{N}}{\partial z_r} \cdot \mathbf{n}_r \end{bmatrix} = \mathbf{T}_s \times \mathbf{n}_r; \begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial x_s} \cdot \mathbf{n}_s \\ \dfrac{\partial \mathbf{N}}{\partial y_s} \cdot \mathbf{n}_s \\ \dfrac{\partial \mathbf{N}}{\partial z_s} \cdot \mathbf{n}_s \end{bmatrix} = \mathbf{n}_s \times \mathbf{T}_r \tag{2.87}$$

We substitute Eqs. (2.85)–(2.87) in Eq. (2.81) to get the following vectorial expression of the functional derivative of $\mathscr{E}_1$:

$$\frac{\partial \mathscr{E}_1}{\partial \mathbf{x}} = \|\mathbf{N}\| \, \nabla g - (\nabla g \cdot \mathbf{T}_r)\,(\mathbf{T}_s \times \mathbf{n}) - (\nabla g \cdot \mathbf{T}_s)\,(\mathbf{n} \times \mathbf{T}_r) - g\,(\mathbf{T}_s \times \mathbf{n}_r + \mathbf{n}_s \times \mathbf{T}_r)\,, \tag{2.88}$$

where $\mathbf{x} = (x, y, z)$. This is not yet the expression we want and proceed to further developments. We decompose ∇g in the first term of the right-hand side of equation Eq. (2.88) in the basis $\left(\frac{\mathbf{T}_r}{\|\mathbf{T}_r\|}, \frac{\mathbf{T}_s}{\|\mathbf{T}_s\|}, \mathbf{n}\right)$, and we express $\mathbf{n}_r$ and $\mathbf{n}_s$ as a linear combination of $\mathbf{T}_r$ and $\mathbf{T}_s$ [1]:

$$\begin{aligned} \mathbf{n}_r &= a_{11}\mathbf{T}_r + a_{12}\mathbf{T}_s \\ \mathbf{n}_s &= a_{12}\mathbf{T}_r + a_{22}\mathbf{T}_s, \end{aligned} \tag{2.89}$$

which, by substitution in Eq. (2.88) gives:

$$\begin{aligned}\frac{\partial \mathscr{E}_1}{\partial \mathbf{x}} = & \ \|\mathbf{N}\| \left((\nabla g \cdot \mathbf{n})\, \mathbf{n} + (\nabla g \cdot \mathbf{t}_r)\, \mathbf{t}_r + (\nabla g \cdot \mathbf{t}_s)\, \mathbf{t}_s \right) \\ & - (\nabla g \cdot \mathbf{t}_r)\, (\mathbf{t}_s \times \mathbf{n})\, \|\mathbf{t}_s\|\, \|\mathbf{t}_r\| - (\nabla g \cdot \mathbf{t}_s)\, (\mathbf{n} \times \mathbf{t}_r)\, \|\mathbf{T}_s\|\, \|\mathbf{T}_r\| \\ & - g\, (a_{11} \mathbf{T}_s \times \mathbf{T}_r + a_{22} \mathbf{T}_s \times \mathbf{T}_r) \end{aligned} \tag{2.90}$$

Using circular permutations of the identity: $\mathbf{t}_r \times \mathbf{t}_s = \mathbf{n}$, and the definition of the mean curvature:

$$\kappa = \frac{1}{2}(a_{11} + a_{22}), \tag{2.91}$$

we finally get the desired expression of the functional derivative of the integral of a scalar function Eq. (2.72):

$$\frac{\partial \mathscr{E}_1}{\partial \mathbf{x}} = (\nabla g \cdot \mathbf{n} + 2 g \kappa)\, \mathbf{N} \tag{2.92}$$

Volume Integral of a Scalar Function

We will now develop the functional derivative of a volume integral of a scalar function [4]:

$$\mathscr{E}_2\, (S) = \int_{V_S} f\, d\rho, \tag{2.93}$$

where V_S is the volume bounded by S. We will first transform $\mathscr{E}_2$ into a surface using the Gauss' divergence theorem. To do this, let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be the vector field defined by:

$$\begin{aligned} P\,(x, y, z) &= \tfrac{1}{3} \int_0^x f\,(\lambda, y, z)\, d\lambda \\ Q\,(x, y, z) &= \tfrac{1}{3} \int_0^y f\,(x, \lambda, z)\, d\lambda \\ R\,(x, y, z) &= \tfrac{1}{3} \int_0^z f\,(x, y, \lambda)\, d\lambda \end{aligned} \tag{2.94}$$

Then:

$$\operatorname{div}\mathbf{F} = P_x + Q_y + R_z = f, \tag{2.95}$$

where subscripts indicate partial derivation. Using the Gauss divergence theorem we have [3]:

$$
\begin{aligned}
\int_{R_S} f\,d\rho = \int_{V_S} \operatorname{div}\mathbf{F}\,d\rho &= \int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma \\
&= \int_0^{l_1}\int_0^{l_2} \mathbf{F}\cdot\mathbf{n}\,\|\mathbf{T}_r\times\mathbf{T}_s\|\,dr\,ds,
\end{aligned}
\tag{2.96}
$$

where $\mathbf{n}$ is the outward unit normal to S. Designate by $L_2(r, s, x, y, z, x_r, y_r, z_r, x_s, y_s, z_s)$ the integrand of the last integral in Eq. (2.96). Just as with $\mathscr{E}_1$, the functional derivative of $\mathscr{E}_2$ with respect to $\mathbf{x} = (x, y, z)$ follows the formulas:

$$
\begin{aligned}
\frac{\partial \mathscr{E}_2}{\partial x} &= \frac{\partial L_2}{\partial x} - \frac{\partial}{\partial r}\frac{\partial L_2}{\partial x_r} - \frac{\partial}{\partial s}\frac{\partial L_2}{\partial x_s} \\
\frac{\partial \mathscr{E}_2}{\partial y} &= \frac{\partial L_2}{\partial y} - \frac{\partial}{\partial r}\frac{\partial L_2}{\partial y_r} - \frac{\partial}{\partial s}\frac{\partial L_2}{\partial y_s} \\
\frac{\partial \mathscr{E}_2}{\partial z} &= \frac{\partial L_2}{\partial z} - \frac{\partial}{\partial r}\frac{\partial L_2}{\partial z_r} - \frac{\partial}{\partial s}\frac{\partial L_2}{\partial z_s}
\end{aligned}
\tag{2.97}
$$

Developing $\mathbf{N}$ as:

$$
\mathbf{N} = \mathbf{T}_r \times \mathbf{T}_s = \begin{pmatrix} x_r \\ y_r \\ z_r \end{pmatrix} \times \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = \begin{pmatrix} y_r z_s - y_s z_r \\ -x_r z_s + x_s z_r \\ x_r y_s - x_s y_r \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix},
\tag{2.98}
$$

we find that:

$$
\begin{aligned}
\frac{\partial L_2}{\partial x} &= P_x N_1 + Q_x N_2 + R_x N_2 \\
\frac{\partial}{\partial r}\frac{\partial L_2}{\partial x_r} &= -Q_x x_r z_s - Q_y y_r z_s - Q_z z_r z_s + R_x x_r y_s + R_y y_r y_s + R_z z_r y_s \\
\frac{\partial}{\partial s}\frac{\partial L_2}{\partial x_s} &= Q_x x_s z_r + Q_y y_s z_r + Q_z z_s z_r - R_x x_s y_r - R_y y_s y_r - R_z z_s y_r
\end{aligned}
\tag{2.99}
$$

Using Eq. (2.98), substitution of Eq. (2.99) back in Eq. (2.97) gives:

$$
\frac{\partial \mathscr{E}_2}{\partial x} = P_x N_1 + Q_y N_1 + R_z N_1 = f N_1
\tag{2.100}
$$

Similar developments yield:

$$
\begin{aligned}
\tfrac{\partial \mathscr{E}_2}{\partial y} &= f N_2 \\
\tfrac{\partial \mathscr{E}_2}{\partial z} &= f N_3,
\end{aligned}
\tag{2.101}
$$

and, finally, this gives the desired result:

$$\frac{\partial \mathcal{E}_2}{\partial \mathbf{x}} = f\mathbf{N} \tag{2.102}$$

An objective functional we will study in Chap. 5 for motion tracking in the spatiotemporal domain [4, 5] has the form:

$$\mathcal{E}(S) = \int_{V_S} f\, d\rho + \int_S g\, d\sigma \tag{2.103}$$

According to the formulas Eqs. (2.92) and (2.102), the Euler-Lagrange equation corresponding to this functional is:

$$(f + \nabla g \cdot \mathbf{n} + 2g\kappa)\, \mathbf{n} = 0 \tag{2.104}$$

2.3 Differentiation Under the Integral Sign

In addition to having functions as arguments, the integrands of the objective functionals we will study in this book can also depend on a parameter. Their minimization with respect to the parameter uses the *differentiation under the integral sign*, in which the differentiation and integration operators are interchanged, i.e., the derivative of the integral with respect to the parameter is the integral of the integrand derivative with respect to the parameter. In its elementary calculus version, sufficient to us, the theorem of differentiation under the integral sign is as follows:

Let $J = [a, b]$ be a compact interval of $\mathbb{R}$ and A a compact subset of $\mathbb{R}^N$. Let $(\alpha, \mathbf{x}) \to f(\alpha, \mathbf{x})$ be a continuous real function on $J \times A$. If f has a continuous partial derivative $\frac{\partial f}{\partial \alpha}(\alpha, \mathbf{x})$ on $J \times A$, then the real function:

$$\mathcal{E}(\alpha) = \int_A f(\mathbf{x}, \alpha) d\mathbf{x} \tag{2.105}$$

is C^1 on J and:

$$\frac{d\mathcal{E}}{d\alpha}(\alpha) = \frac{d}{d\alpha}\left(\int_A f(\mathbf{x}, \alpha) d\mathbf{x}\right) = \int_A \frac{\partial f}{\partial \alpha}(\mathbf{x}, \alpha) d\mathbf{x} \tag{2.106}$$

2.4 Descent Methods for Unconstrained Optimization

Descent methods, sometimes also called greedy methods, for unconstrained optimization of an objective function with respect to an argument, are iterative methods which decrease the objective function at each iteration by incremental modification of the argument.

2.4.1 Real Functions

Let $f : \mathbf{x} \in \mathbb{R}^N \rightarrow f(\mathbf{x}) \in \mathbb{R}$ be a C^1 real function of which we want to determine an unconstrained local minimum, assuming such a minimum exists. To do so, consider $\mathbf{x}$ to be a C^1 function of (algorithmic) time, $\mathbf{x} : \tau \geq 0 \rightarrow \mathbf{x}(\tau) \in \mathbb{R}^N$, and let g be the composition of f and $\mathbf{x}$: $g(\tau) = f(\mathbf{x}(\tau))$. We have

$$\frac{dg}{d\tau} = \nabla f \cdot \frac{d\mathbf{x}}{d\tau} \tag{2.107}$$

Therefore, if we vary $\mathbf{x}$ from an initial position $\mathbf{x}_0$ according to the evolution equation:

$$\frac{d\mathbf{x}}{d\tau}(\tau) = -\alpha \nabla f(\mathbf{x}(\tau)), \quad \alpha \in \mathbb{R}^+, \tag{2.108}$$

then f will always decrease because:

$$\frac{df}{d\tau}(\mathbf{x}(\tau)) = \frac{dg}{d\tau}(\tau) = -\alpha \|\nabla f(\mathbf{x}(\tau))\|^2 \leq 0, \tag{2.109}$$

and will eventually reach a local minimum. More generally, if we vary $\mathbf{x}$ in direction $\mathbf{d}$ according to:

$$\begin{aligned} \frac{d\mathbf{x}}{d\tau}(\tau) &= -\alpha(\tau)\mathbf{d}(\mathbf{x}(\tau)) \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned} \tag{2.110}$$

where $\alpha(\tau) \in \mathbb{R}^+$ and $\nabla f \cdot \mathbf{d} > 0$, then f will vary according to

$$\frac{df}{d\tau}(\mathbf{x}(\tau)) = \frac{dg}{d\tau}(\tau) = -\alpha(\tau)\nabla f(\mathbf{x}(\tau)) \cdot \mathbf{d}(\mathbf{x}(\tau)) \ \leq 0 \tag{2.111}$$

Methods of unconstrained minimization based on Eq. (2.110) are called *descent* methods. When $\mathbf{d} = \nabla f$ is used it is the *gradient*, or *fastest*, descent. The scaling function α is often predetermined. For instance, $\alpha(\tau) = $ constant, or $\alpha(\tau) = 1/\tau$. In general, numerical descent methods are implemented as [6]:

1. $k = 0; \quad \mathbf{x}^0 = \mathbf{x}_0$
2. Repeat until convergence

$$\begin{aligned} \mathbf{d}^k &= \mathbf{d}(\mathbf{x}^k) \\ \alpha_k &= \arg\min_{\alpha \geq 0} f(\mathbf{x}^k - \alpha \mathbf{d}^k) \\ \mathbf{x}^{k+1} &= \mathbf{x}^k - \alpha_k \mathbf{d}^k \\ k &\leftarrow k + 1 \end{aligned}$$

Vectorial functions $F = (f_1, ..., f_n)^t$ are processed similarly by treating each component real function f_i as described above.

2.4.2 Integral Functionals

Consider the problem of minimizing functional Eq. (2.27):

$$\mathscr{E}(u) = \int_{x_1}^{x_2} g(x, u, u')dx,$$

where $u = u(x)$ and u' is the derivative of u with respect to x. To do so, let u vary in time, i.e., u is embedded in a one-parameter family of functions indexed by (algorithmic) time τ, and consider the time-dependent functional:

$$\mathscr{E}(u, \tau) = \int_{x_1}^{x_2} g(x, u(x, \tau), u'(x, \tau))dx. \tag{2.112}$$

The derivative of $\mathscr{E}$ with respect to the time parameter τ develops as:

$$\begin{aligned}
\frac{\partial \mathscr{E}}{\partial \tau} &= \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial g}{\partial u'}\frac{\partial u'}{\partial \tau} \right) dx \\
&= \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial g}{\partial u'}\frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial x}\right) \right) dx \\
&= \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial g}{\partial u'}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \tau}\right) \right) dx
\end{aligned}$$

Integration by parts of the second term of the integrand yields:

$$\frac{\partial \mathscr{E}}{\partial \tau} = \frac{\partial g}{\partial u'}\frac{\partial u}{\partial \tau}\Big]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial u'}\right) \right) \frac{\partial u}{\partial \tau}dx \tag{2.113}$$

Assuming the endpoint conditions

$$\frac{\partial u}{\partial \tau}(x_1, \tau) = \frac{\partial u}{\partial \tau}(x_2, \tau) \;\; \forall \tau, \tag{2.114}$$

we finally obtain:

$$\frac{\partial \mathscr{E}}{\partial \tau} = \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial u'}\right) \right) \frac{\partial u}{\partial \tau}dx \tag{2.115}$$

Therefore, when u varies according to the evolution equation:

$$\frac{\partial u}{\partial \tau} = -\left(\frac{\partial g}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial u'}\right)\right), \tag{2.116}$$

i.e.,

$$\frac{\partial u}{\partial \tau} = -\frac{\partial \mathscr{E}}{\partial u}, \tag{2.117}$$

it implies that:

$$\frac{\partial \mathscr{E}}{\partial \tau} = -\int_{x_1}^{x_2} \left(\frac{\partial g}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial u'}\right)\right)^2 \leq 0 \tag{2.118}$$

Therefore, $\mathscr{E}$ continually decreases and, starting from an initial approximation $u(0) = u_0$, u will converge to a local minimum of $\mathscr{E}$, assuming such a minimum exists. Evolution Eq. (2.116) is the fastest descent equation to minimize functional Eq. (2.112). Functionals of several dependent variables are processed similarly.

Example: Let $I : \Omega \subset \mathbb{R}^2 \to L \subset \mathbb{R}^+$ be an image and $\gamma : s \in [0, 1] \to (x(s), y(s)) \in \mathbb{R}$ a closed regular plane curve. Let $R_1 = R_\gamma$ be the interior of γ and $R_2 = R_\gamma^c$ its complement. Consider minimizing the following functional [7]:

$$\mathscr{E}(\gamma, \mu_1, \mu_2) = \int_{R_1} (I - \mu_1)^2 \, dxdy + \int_{R_2} (I - \mu_2)^2 \, dxdy + \lambda \int_\gamma ds, \tag{2.119}$$

where λ is a real constant and μ_1, μ_2 are real parameters. This is an image segmentation functional the minimization of which will realize a piecewise constant two-region partition of the image. The first two terms are data terms which evaluate the deviation of the image from a constant representation by μ_1 in R_1 and μ_2 in its complement R_2, and the length integral is a regularization term to promote shorter, smoother curves γ. The minimization of $\mathscr{E}$ can be done by an iterative two-step greedy algorithm which repeats two consecutive steps until convergence, one step to minimize with respect to the parameters μ_1 and μ_2 with γ fixed, and the other to minimize with respect to γ with μ_1, μ_2 fixed, i.e., assumed independent of γ. Minimization with respect to the parameters, with γ given, is done by setting the derivative of $\mathscr{E}$ with respect to each parameter to zero. The derivatives are obtained by differentiation under the integral sign and we have:

$$\frac{\partial \mathscr{E}}{\partial \mu_i} = \frac{\partial}{\partial \mu_i} \int_{R_i} (I - \mu_i)^2 \, dxdy = \int_{R_i} \frac{\partial}{\partial \mu_i} \left((I - \mu_i)^2\right) dxdy = 0, \quad i = 1, 2. \tag{2.120}$$

This immediately gives μ_i to be the mean value of I in R_i:

$$\mu_i = \frac{\int_{R_i} I(x, y) \, dxdy}{\int_{R_i} dxdy}, \quad i = 1, 2. \tag{2.121}$$

The minimization of $\mathscr{E}$ with respect to γ assuming μ_1, μ_2 fixed, independent of γ thereof, can be done by embedding γ in a one-parameter family of curves $\gamma : s, \tau \in [0, 1] \times \mathbb{R}^+ \rightarrow \gamma(s, \tau) = (x(s, \tau), y(s, \tau), \tau) \in \Omega \times \mathbb{R}^+$ indexed by algorithmic time τ and using the corresponding (Euler-Lagrange) descent equation:

$$\frac{\partial \gamma}{\partial \tau} = -\frac{\partial \mathscr{E}}{\partial \gamma}. \tag{2.122}$$

Note that each of the component functions of γ satisfies the endpoint conditions Eq. (2.114). Orienting γ counterclockwise and its unit normal $\mathbf{n}$ to point away from its interior, writing the data term of R_2 as $\int_{\Omega}(I-\mu_2)^2 dxdy - \int_{R_1}(I-\mu_2)^2 dxdy$, and using the basic formulas of Eq. (2.56) and Eq. (2.63), we obtain the partial differential equation governing the evolution of γ in (algorithmic) time:

$$\frac{\partial \gamma}{\partial \tau} = -\left((I-\mu_1)^2 - (I-\mu_2)^2 + \lambda\kappa\right)\mathbf{n}, \tag{2.123}$$

Instead of taking μ_1, μ_2 fixed when deriving the minimization equation with respect to γ, one can substitute their expression Eq. (2.121) in the functional and then derive the equation, thereby accounting for the dependence of the parameters on γ, and then opting for gradient descent. In this particular case of functional, however, the terms in the calculations due to the dependence on γ cancel out and one ends up with the same equation as when simply assuming the parameters fixed, i.e., independent of curve variable γ. This is a general result for a dependence of parameters on γ of the type in Eq. (2.121) [8].

In the computer vision literature a curve such as γ is called an *active curve* or *active contour*, and a partial differential equation such as Eq. (2.122) is referred to as its *evolution equation*. It moves in the direction of its normal at every one of its points and at the speed specified by the factor multiplying $\mathbf{n}$ in the evolution equation.

A direct implementation of the evolution equation which would discretize the curve and move each of its points *explicitly* would, in general, run into insurmountable numerical difficulties. The *level set* implementation, which we take up next, is an efficient way of realizing curve evolution without the numerical ills of the explicit implementation.

2.5 Level Sets

The *level set method* [9, 10] is for problems of moving interfaces, for curves, or surfaces in higher dimensions, that are moved by a differential equation which affects their shape. From a general point of view, it is, therefore, about optimizing the shape of curves and surfaces. In the problems we address, curves and surfaces are made to move so as to adhere to the boundary of desired regions in an image. For example, to detect moving objects in an image sequence, an active curve can be made to move so as to coincide with image boundaries of high image motion contrast, which is a characteristic of moving object contours, and in motion-based image segmentation a number of such curves can be made to evolve so as to adhere to the boundary of distinctly moving objects in space.

For a "nice" smooth curve which keeps its shape approximately during motion, it is natural to think of following a number of maker points on the curve, simply moving them from their current position for a time step and then interpolating the resulting particle positions. However, such a simple means of processing a moving curve will generally come to unresolvable numerical ills. There are several reasons for this. The most obvious is perhaps the fact that the evolution equation can cause a change in the topology of the curve. For instance, the curve can *split* into two pieces or more. By following the points explicitly, i.e., individually, there is no general way to detect when this splitting occurs, in which case the curve in its constituent parts cannot be recovered and further processing of its motion will be unstable and arbitrarily erroneous. Similar numerical instability problems will assuredly occur when distinct curves *merge* to form a single one.

Two less obvious but nevertheless serious, and common, difficulties with explicit following of marker points on a moving curve are *fans* and *shocks*. Consider, as illustrated in Fig. 2.2a, a curve forming a corner initially and moving "outward" in the direction of its normal at constant speed, say at unit speed. The particles on the horizontal side of the corner move straight up and those on the vertical side straight to the left, all transported a unit distance away from their initial position. Between these two sets of makers, a gap, or fan, has developed where there is no information about the shape of the curve because there are no particles to move in that place. The gap will widen with every move outward and the process of following the curve can break down quickly.

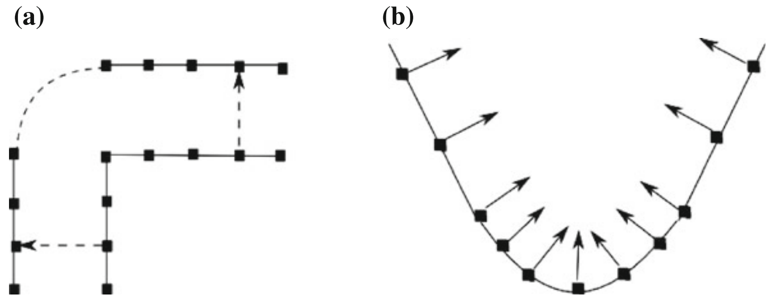

Fig. 2.2 Fanning and shocks cause instability when trying to follow the movement of curves via tracking explicitly a set of markers points on them. **a** Fanning: A corner moving outward in the direction of its normal at constant speed. The particles on the horizontal side of the corner move straight up and those on the vertical side straight to the left. Between these two sets of makers, a gap, or fan, has developed where we have no information about the actual shape of the curve. **b** shock: A curve with two straight segments, one on each side of a curved portion. When the marker points on the curved portion are moved inward, they are brought closer to each other and, with continued inward motion, will eventually meet and create a shock

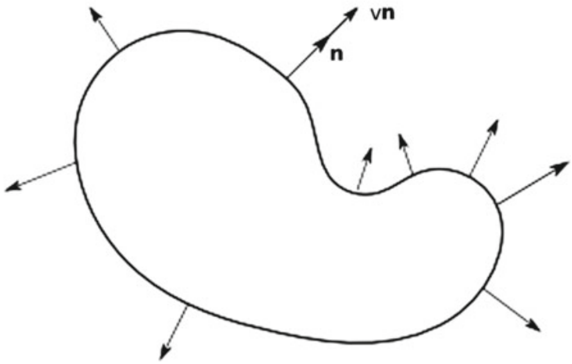

Fig. 2.3 A closed regular curve moving outward at speed V in the direction of its normal **n**. Both **n** and V are functions of position on the curve

Shocks can appear when a contour initially curved as a fan is moved to "retract". This is illustrated in Fig. 2.2b, which basically shows the fanning example of Fig. 2.2a with time running in the opposite direction. The curve has two straight segments, one on each side of a curved portion. The particles in this curved portion will be brought closer to each other by inward motion and will eventually be so close as to meet and create a shock which will occult any previous ordering of the markers and, therefore, cause numerical griefs of various sorts.

The level set method, instead, moves active curves in a numerically stable manner. It deals efficiently with changes in the moving curve topology and with conditions such as fans and shocks when these occur. The basic idea is to describe the moving curve not explicitly by markers points on it but *implicitly* by a level set of a surface $\phi(x, y)$, the zero level set for instance, in which case the curve is represented for all practical purposes by $\phi(x, y) = 0$.

Let Γ be the set of closed regular plane curves $\gamma : s \in [0, 1] \rightarrow \gamma(s) \in \Omega$. For the purpose of describing its motion, an active contour is represented by a one-parameter family of curves in Γ indexed by algorithmic time t, i.e, a function $\gamma : s, \tau \in [0.1] \times \mathbb{R}^+ \rightarrow \gamma(s, \tau) = (x(s, \tau), y(s, \tau), \tau)) \in \Omega \times \mathbb{R}^+$ such that $\forall \tau$ curve $\gamma_\tau : s \rightarrow (x(s, \tau), y(s, \tau)$ is in Γ).

Consider a curve $\gamma \in \Gamma$ moving according to a velocity vector which is in the direction of its normal at each point (Fig. 2.3):

$$\frac{\partial \gamma}{\partial \tau} = V\mathbf{n}, \tag{2.124}$$

where V is the speed of motion. In the level set method, γ is described implicitly by the zero level set of a function $\phi : \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}$ (Fig. 2.4):

$$\forall s, \tau \quad \phi(\gamma(s, \tau)) = \phi(x(s, \tau), y(s, \tau), \tau) = 0 \tag{2.125}$$

With ϕ sufficiently smooth, taking the total derivative of Eq. (2.125) with respect to time gives:

$$\frac{\partial \phi}{\partial \tau} = \nabla \phi \cdot \frac{\partial \gamma}{\partial \tau} + \frac{\partial \phi}{\partial \tau} = 0 \tag{2.126}$$

Using Eq. (2.124), we get:

$$\frac{\partial \phi}{\partial \tau} = -V \nabla \phi \cdot \mathbf{n} \tag{2.127}$$

With the convention that $\mathbf{n}$ is oriented outward and ϕ is positive inside its zero level set, negative outside, the normal $\mathbf{n}$ is given by:

$$\mathbf{n} = -\frac{\nabla \phi}{\|\nabla \phi\|}, \tag{2.128}$$

and substitution of Eq. (2.128) in Eq. (2.127) yields the evolution equation of ϕ:

$$\frac{\partial \phi}{\partial \tau} = V \|\nabla \phi\| \tag{2.129}$$

The analysis above applies to points on the level set zero of ϕ. Therefore, one must define *extension velocities* [10] to evolve the level set function elsewhere. Several possibilities have been envisaged. For instance, the extension velocity at a point has been taken to be the velocity of the evolving curve point closest to it. Extension velocities have also been defined so that the level set function is at all times the distance function from the evolving curve. In image segmentation and motion analysis problems, of the sort we have in this book, such extension velocities may not be easily implemented. When an expression of velocity is valid for all level sets, which is the case in just about all the active curve motion analysis methods in this book, then one can simply use this expression in all of the image domain, i.e., to evolve ϕ everywhere on its definition domain.

Regardless of what the extension velocities are chosen to be, the computational burden can be significantly lightened by restricting processing to a band around the active contour [10], a scheme called *narrow banding*.

By definition, an active curve γ can be recovered any time as the zero level set of its level set function ϕ and this is regardless of variations in its topology. The level set function always remains a function (Fig. 2.4), thereby assuring continued stable processing; stable processing is preserved also in the presence of fans and socks. In motion analysis problems which include motion-based image segmentation, and we will address some in this book, another advantage of the level set implementation is that region membership of points, i.e., the information as to which motion region they belong, is readily available from the sign of the level set functions.

There are several articles and books on the subject of level sets. The book of Sethian [10] is about efficient and numerically stable implementation of the level method, with examples from different domains of applications, including image analysis.

The velocities we will encounter in the forthcoming chapters of this book have components of one of three distinct types:

Type 1: V is a function of the curvature of the evolving curve. The component $\lambda\kappa$ in Eq. (2.123) is of this type.

Type 2: V is of the form $\mathbf{F} \cdot \mathbf{n}$ where $\mathbf{F}$ is a vector field dependent on position and possibly time but not on the curve. The term $\nabla h \cdot \mathbf{n}$ in Eq. (2.70), would it appear in a curve evolution equation, would be of this type. Such terms are called advection speeds in [10].

Type 3: V is a scalar function which depends on position and time but is not of the other two types. The velocity component $\left((I-\mu_1)^2 - (I-\mu_2)^2\right)$ in Eq. (2.123), corresponding to the objective functional data terms, is of this type.

Velocities of the types 1, 2, and 3 are discretized differently as summarized below [10]. The velocity of a curve evolution in motion analysis is, in general, a linear combination of velocities of the three types. The discretization of Eq. (2.129) can be written in the following manner:

$$\phi_{ij}^{k+1} = \phi_{ij}^{k} + \triangle t \begin{cases} +V_{ij}^{k}\left((D_{ij}^{0x})^2 + (D_{ij}^{0y})^2\right)^{\frac{1}{2}} & \text{for type 1} \\ -\left(\max(F_{1ij}^{k},0)D_{ij}^{-x} + \min(F_{1ij}^{k},0)D_{ij}^{+x}\right. \\ \left. +\max(F_{2ij}^{k},0)D_{ij}^{-y} + \min(F_{2ij}^{k},0)D_{ij}^{+y}\right) & \text{for type 2} \\ -\left(\max(V_{ij}^{k},0)\nabla^{+} + \min(V_{ij}^{k},0)\nabla^{-}\right) & \text{for type 3} \end{cases} \tag{2.130}$$

where i, j are indices on the discretization grid of Ω, k is the iteration index, F_1, F_2 are the coordinates of $\mathbf{F}$ appearing in the general expression of terms of type 2. Finite difference $x-$derivative operators D^{+x} (forward scheme), D^{-x} (backward scheme), and D^{0x} (central scheme), are applied to ϕ at i, j and iteration k, i.e., D_{ij}^{+x}, D_{ij}^{-x}, D_{ij}^{0x} in Eq. (2.130) stand for $D^{+x}(\phi^k)_{ij}$, $D^{-x}(\phi^k)_{ij}$, $D^{0x}(\phi^k)_{ij}$ and are given by:

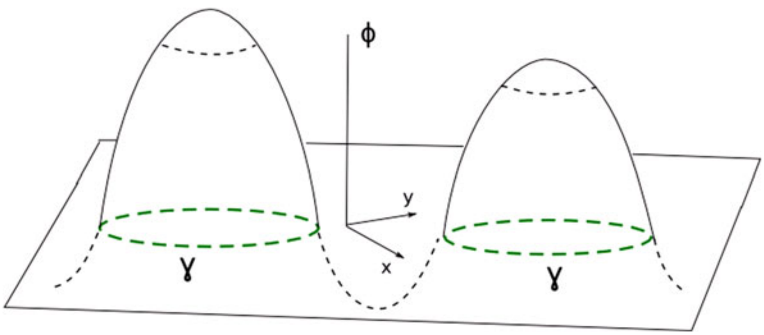

Fig. 2.4 The active curve γ is represented implicitly by the zero level of the level set function ϕ. Regardless of variations in the topology of γ, ϕ remains a function, thereby allowing stable curve evolution, unaffected by changes in the curve topology, fanning, and shocks. In this figure, γ has split into two component curves but ϕ remains a function

$$\begin{aligned} D_{ij}^{+x} &= \phi_{i+1,j}^k - \phi_{ij}^k \\ D_{ij}^{-x} &= \phi_{ij}^k - \phi_{i-1,j}^k \\ D_{ij}^{0x} &= \frac{1}{2}(\phi_{i+1,j}^k - \phi_{i-1,j}^k) \end{aligned} \tag{2.131}$$

Similar formulas and comments apply to the $y-$derivative operators D^{+y}, D^{-y}, and D^{0y}. Finally, operators ∇^+ and ∇^- are defined by:

$$\begin{aligned} \nabla^+ &= \Big(\max(D_{ij}^{-x},0)^2 + \min(D_{ij}^{+x},0)^2 \\ &\quad + \max(D_{ij}^{-y},0)^2 + \min(D_{ij}^{+y},0)^2\Big)^{\frac{1}{2}} \\ \nabla^- &= \Big(\max(D_{ij}^{+x},0)^2 + \min(D_{ij}^{-x},0)^2 \\ &\quad + \max(D_{ij}^{+y},0)^2 + \min(D_{ij}^{-y},0)^2\Big)^{\frac{1}{2}} \end{aligned} \tag{2.132}$$

The time step size Δt is adjusted for the experimentation at hand; it may vary for different applications. As a rule of thumb, one can chose its value so that the movement of the curve is approximately one pixel or less everywhere on the image positional array at each iteration.

References

1. M.P. Do Carmo, *Differential Geometry of Curves and Surfaces* (Prentice Hall, Englewood Cliffs, 1976)
2. R. Weinstock, *Calculus of Variations* (Dover, New York, 1974)
3. J.E. Marsden, A.J. Tromba, *Vector Calculus* (W. H. Freeman and Company, New York, 1976)
4. A. Mitiche, R. Feghali, A. Mansouri, Motion tracking as spatio-temporal motion boundary detection. J. Robot. Auton. Syst. **43**, 39–50 (2003)
5. R. El-Feghali, A. Mitiche, Spatiotemporal motion boundary detection and motion boundary velocity estimation for tracking moving objects with a moving camera: a level sets pdes approach with concurrent camera motion compensation. IEEE Trans. Image Process. **13**(11), 1473–1490 (2004)
6. M. Minoux, *Programmation mathématique*, vol. 1 (Dunod, Paris, 1983)
7. T. Chan, L. Vese, Active contours without edges. IEEE Trans. Image Process. **10**(2), 266–277 (2001)
8. G. Aubert, M. Barlaud, O. Faugeras, S. Jehan-Besson, Image segmentation using active contours: calculus of variations or shape gradients? SIAM J. Appl. Math. **63**(6), 2128–2154 (2003)
9. S. Osher, J. Sethian, Front propagation with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations. J. Comput. Phys. **79**, 12–49 (1988)
10. J.A. Sethian, *Level set Methods and Fast Marching Methods* (Cambridge University Press, Cambridge, 1999)

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