

Ricci Bounds for Euclidean and Spherical Cones

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Abstract We prove generalized lower Ricci bounds for Euclidean and spherical cones over complete Riemannian manifolds. These cones are regarded as complete metric measure spaces. In general, they will be neither manifolds nor Alexandrov spaces. We show that the Euclidean cone over an n -dimensional Riemannian manifold whose Ricci curvature is bounded from below by $n - 1$ satisfies the curvature-dimension condition $\text{CD}(0, n + 1)$ and that the spherical cone over the same manifold fulfills the curvature-dimension condition $\text{CD}(n, n + 1)$. More generally, for each $N > 1$ we prove that the condition $\text{CD}(N - 1, N)$ for a weighted Riemannian space is equivalent to the condition $\text{CD}(0, N + 1)$ for its N -Euclidean cone as well as to the condition $\text{CD}(N, N + 1)$ for its N -spherical cone.

1 Introduction

In two similar but independent approaches, the second author [14, 15] and Lott and Villani [8, 9] presented a concept of generalized lower Ricci curvature bounds for metric measure spaces (M, d, m) . The full strength of this concept appears if the condition $\text{Ric}(M, d, m) \geq K$ is combined with a kind of upper bound N on the dimension. This leads to the so-called curvature-dimension condition $\text{CD}(K, N)$ which can be formulated in terms of optimal transportation for each pair of numbers $K \in \mathbb{R}$ and $N \in [1, \infty)$.

A complete Riemannian manifold satisfies $\text{CD}(K, N)$ if and only if its Ricci curvature is bounded from below by K and its dimension from above by N .

A broad variety of geometric and functional analytic results can be deduced from the curvature-dimension condition $\text{CD}(K, N)$. Among them are the

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Brunn-Minkowski inequality and the theorems by Bishop-Gromov, Bonnet-Myers and Lichnerowicz. Moreover, the condition $\text{CD}(K, N)$ is stable under convergence with respect to the L^2 -transportation distance d .

1.1 Statement of the Main Results

Let M be a complete n -dimensional Riemannian manifold (with Riemannian distance d and Riemannian volume $d\mathfrak{m} = d\text{vol}$). The *Euclidean cone* $\text{Con}(M) = M_r \times [0, \infty)$ over M is defined as the quotient of the product $M \times [0, \infty)$ obtained by identifying all points in the fiber $M \times \{0\}$. This point is called the origin O of the cone. It is equipped with a metric d_{Con} defined by the cosine formula

$$d_{\text{Con}}((x, s), (y, t)) = \sqrt{s^2 + t^2 - 2st \cos(d(x, y) \wedge \pi)},$$

and with a measure m_n defined as the product $d\mathfrak{m}_n(x, s) := d\mathfrak{m}(x) \otimes s^n ds$.

Theorem 1. *The Ricci curvature of M is bounded from below by $n - 1$ and there holds $\text{diam}(M) \leq \pi$ if and only if the metric measure space $(\text{Con}(M), d_{\text{Con}}, m_n)$ satisfies the curvature-dimension condition $\text{CD}(0, n + 1)$.*

Note that in dimensions $n \neq 1$ the diameter bound $\text{diam}(M) \leq \pi$ is redundant: it follows from the Ricci bound.

The heuristic interpretation of the assertion in the theorem is that the Euclidean cone – regarded as a metric measure space – has non-negative Ricci curvature in a generalized sense. Note that already in 1982, Cheeger and Taylor [3, 6] observed that the punctured Euclidean cone $\text{Con}(M) \setminus \{O\}$ constructed over a compact n -dimensional Riemannian manifold M with $\text{Ric} \geq n - 1$ is a $(n + 1)$ -dimensional Riemannian manifold with $\text{Ric} \geq 0$. Note, however, that the sectional curvature might be unbounded from below (and above). Thus in general $\text{Con}(M)$ will *not* be an Alexandrov space. Moreover, $\text{Con}(M)$ in general is not a manifold and, of course, $\text{Con}(M) \setminus \{O\}$ is not complete. In particular, the Ricci curvature in the classical sense is not defined in its singularity O .

Actually, we will prove a significantly more general result:

Theorem 2. *For any real number $N > 1$, the $\text{CD}(N - 1, N)$ condition for a weighted Riemannian manifold is equivalent to the $\text{CD}(0, N + 1)$ condition for the associated N -Euclidean cone.*

It is an open question whether analogous assertions hold true with an arbitrary metric measure space (M, d, \mathfrak{m}) in the place of the weighted Riemannian manifold M . A partial result towards this conjecture was derived by Ohta [11] for metric measure spaces satisfying the so-called *measure contraction property* $\text{MCP}(K, N)$, a property being slightly weaker than the curvature-dimension condition $\text{CD}(K, N)$.

Remark 1. If a complete separable metric measure space (M, d, \mathfrak{m}) satisfies the measure contraction property $\text{MCP}(N - 1, N)$ for some $N \geq 1$ and if $\text{diam}(M) \leq \pi$

(which follows from the previous condition if $N \neq 1$) then its N -Euclidean cone $(\text{Con}(\mathbf{M}), d_{\text{Con}}, m_N)$ satisfies the measure contraction property $\text{MCP}(0, N + 1)$.

As a second main result we deduce a generalized lower Ricci bound for the *spherical cone* $\Sigma(\mathbf{M}) = \mathbf{M}_{\sin(r)} \times [0, \pi]$ over the compact Riemannian manifold \mathbf{M} . It can be defined as the quotient of the product space $\mathbf{M} \times [0, \pi]$ obtained by contracting all points in the fiber $\mathbf{M} \times \{0\}$ to the south pole \mathcal{S} and all points in the fiber $\mathbf{M} \times \{\pi\}$ to the north pole \mathcal{N} . It is endowed with a metric d_Σ defined via

$$\cos(d_\Sigma(p, q)) = \cos s \cos t + \sin s \sin t \cos(d(x, y) \wedge \pi)$$

for $p = (x, s), q = (y, t) \in \Sigma(\mathbf{M})$ and with a measure $d\hat{m}_n(x, s) := d\text{vol}(x) \otimes (\sin^n s ds)$.

- Theorem 3.** (i) *The Ricci curvature of \mathbf{M} is bounded from below by $n - 1$ and $\text{diam}(\mathbf{M}) \leq \pi$ if and only if the metric measure space $(\Sigma(\mathbf{M}), d_\Sigma, \hat{m}_n)$ satisfies the curvature-dimension condition $\text{CD}(n, n + 1)$.*
- (ii) *A weighted Riemannian manifold satisfies the curvature-dimension condition $\text{CD}(N - 1, N)$ for a given real number $N > 1$ if and only if the associated N -spherical cone satisfies the curvature-dimension condition $\text{CD}(N, N + 1)$.*

Note that the analogous results holds true for generalized lower bounds for the sectional curvature.

Remark 2 (see e.g. [2], Theorem 4.7.1, 10.2.3). Let (\mathbf{M}, d) be a complete length metric space with $\text{diam}(\mathbf{M}) \leq \pi$.

- (i) Then (\mathbf{M}, d) has curvature bounded from below by 1 in the sense of Alexandrov if and only if the Euclidean cone $(\text{Con}(\mathbf{M}), d_{\text{Con}})$ has nonnegative curvature in the sense of Alexandrov.
- (ii) Moreover, (\mathbf{M}, d) has curvature bounded from below by 1 in the sense of Alexandrov if and only if the spherical cone $(\text{Con}(\mathbf{M}), d_{\text{Con}})$ has curvature bounded from below by 1 in the sense of Alexandrov.

Note that the diameter bound is redundant if \mathbf{M} is not one-dimensional.

Metric cones play an important role in the study of limits of Riemannian manifolds. Assume for instance that (\mathbf{M}, d) is the Gromov-Hausdorff limit of a sequence of complete n -dimensional Riemannian manifolds whose Ricci curvature is uniformly bounded from below. Then in the non-collapsed case, every tangent cone $T_x \mathbf{M}$ is a metric cone $\text{Con}(S_x \mathbf{M})$ with $\text{diam}(S_x \mathbf{M}) \leq \pi$ [4, 5]. The latter we would expect from the diameter estimate by Bonnet-Myers if $\text{Ric} \geq n - 2$ on $S_x \mathbf{M}$ which in turn is consistent with the formal assertion ‘ $\text{Ric} \geq 0$ on $T_x \mathbf{M}$ ’.

1.2 Basic Definitions and Notations

Throughout this paper, (\mathbf{M}, d) always will denote a complete separable metric space (\mathbf{M}, d) and m a locally finite measure on $(\mathbf{M}, \mathcal{B}(\mathbf{M}))$ with full support. That is, for

all $x \in \mathbf{M}$ and all sufficiently small $r > 0$ the volume $\mathbf{m}(B_r(x))$ of balls centered at x is positive and finite. To avoid pathologies, we assume that \mathbf{M} has more than one point. Such a triple $(\mathbf{M}, \mathbf{d}, \mathbf{m})$ will henceforth be called *metric measure space*.

The metric space (\mathbf{M}, \mathbf{d}) is called a *length space* iff $\mathbf{d}(x, y) = \inf \text{Length}(\gamma)$ for all $x, y \in \mathbf{M}$, where the infimum runs over all curves γ in \mathbf{M} connecting x and y . (\mathbf{M}, \mathbf{d}) is called a *geodesic space* if and only if every two points $x, y \in \mathbf{M}$ are connected by a curve γ with $\mathbf{d}(x, y) = \text{Length}(\gamma)$. Distance minimizing curves of constant speed are called *geodesics*. The space of all geodesics $\gamma : [0, 1] \rightarrow \mathbf{M}$ will be denoted by $\Gamma(\mathbf{M})$.

(\mathbf{M}, \mathbf{d}) is called *non-branching* if for every tuple (z, x_0, x_1, x_2) of points in \mathbf{M} for which z is a midpoint of x_0 and x_1 as well as of x_0 and x_2 , it follows that $x_1 = x_2$. $\mathcal{P}_2(\mathbf{M}, \mathbf{d})$ denotes the L^2 -Wasserstein space of probability measures μ on $(\mathbf{M}, \mathcal{B}(\mathbf{M}))$ with finite second moments which means that $\int_{\mathbf{M}} \mathbf{d}^2(x_0, x) d\mu(x) < \infty$ for some (hence all) $x_0 \in \mathbf{M}$. The L^2 -Wasserstein distance $\mathbf{d}_W(\mu_0, \mu_1)$ between two probability measures $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbf{M}, \mathbf{d})$ is defined as

$$\mathbf{d}_W(\mu_0, \mu_1) = \inf \left\{ \left(\int_{\mathbf{M} \times \mathbf{M}} \mathbf{d}^2(x, y) d\mathbf{q}(x, y) \right)^{1/2} : \mathbf{q} \text{ coupling of } \mu_0 \text{ and } \mu_1 \right\}.$$

Here the infimum ranges over all *couplings* of μ_0 and μ_1 , i.e. over all probability measures on $\mathbf{M} \times \mathbf{M}$ with marginals μ_0 and μ_1 . Equipped with this metric, $\mathcal{P}_2(\mathbf{M}, \mathbf{d})$ is a complete separable metric space. The subspace of \mathbf{m} -absolutely continuous measures is denoted by $\mathcal{P}_2(\mathbf{M}, \mathbf{d}, \mathbf{m})$.

Definition 1. (i) A subset $\mathcal{E} \subset \mathbf{M} \times \mathbf{M}$ is called *\mathbf{d}^2 -cyclically monotone* if and only if for any $k \in \mathbb{N}$ and for any family $(x_1, y_1), \dots, (x_k, y_k)$ of points in \mathcal{E} the inequality

$$\sum_{i=1}^k \mathbf{d}^2(x_i, y_i) \leq \sum_{i=1}^k \mathbf{d}^2(x_i, y_{i+1})$$

holds with the convention $y_{k+1} = y_1$.

(ii) Given probability measures μ_0, μ_1 on \mathbf{M} , a probability measure \mathbf{q} on $\mathbf{M} \times \mathbf{M}$ is called *optimal coupling* of them iff \mathbf{q} has marginals μ_0 and μ_1 and

$$\mathbf{d}_W^2(\mu_0, \mu_1) = \int_{\mathbf{M} \times \mathbf{M}} \mathbf{d}^2(x, y) d\mathbf{q}(x, y).$$

(iii) A probability measure ν on $\Gamma(\mathbf{M})$ is called *optimal path measure* (or dynamical optimal transference plan) iff the probability measure $(e_0, e_1)_* \nu$ on $\mathbf{M} \times \mathbf{M}$ is an optimal coupling of the probability measures $(e_0)_* \nu$ and $(e_1)_* \nu$ on \mathbf{M} .

Here and in the sequel $e_t : \Gamma(\mathbf{M}) \rightarrow \mathbf{M}$ for $t \in [0, 1]$ denotes the evaluation map $\gamma \mapsto \gamma_t$. Moreover, for each measurable map $f : \mathbf{M} \rightarrow \mathbf{M}'$ and each measure μ on \mathbf{M} the push forward (or image measure) of μ under f will be denoted by $f_* \mu$.

From [14, Lemma 2.11], [16, Theorem 5.10] we quote:

- Lemma 1.** (i) For each pair $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbf{M}, \mathbf{d})$ there exists an optimal coupling \mathbf{q} .
(ii) The support of any optimal coupling \mathbf{q} is a \mathbf{d}^2 -cyclically monotone set.
(iii) If \mathbf{M} is geodesic then for each pair $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbf{M}, \mathbf{d})$ there exists an optimal path measure with given initial and terminal distribution: $(e_0)_* \nu = \mu_0$ and $(e_1)_* \nu = \mu_1$.
(iv) Given any optimal path measure ν as above, a geodesic $(\mu_t)_{t \in [0,1]}$ in $\mathcal{P}_2(\mathbf{M}, \mathbf{d})$ connecting μ_0 and μ_1 is given by

$$\mu_t := (e_t)_* \nu.$$

- (v) If (\mathbf{M}, \mathbf{d}) is a non-branching space, then for each pair of geodesics γ, γ' in the support of an optimal path measure we have:

$$\gamma_{1/2} = \gamma'_{1/2} \implies \gamma = \gamma'.$$

1.3 The Curvature-Dimension Condition

Definition 2. Given $K \in \mathbb{R}$ and $N \in [1, \infty)$, the condition $\text{CD}(K, N)$ states that for each pair $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbf{M}, \mathbf{d}, \mathbf{m})$ there exist an optimal coupling \mathbf{q} of $\mu_0 = \rho_0 \mathbf{m}$ and $\mu_1 = \rho_1 \mathbf{m}$ and a geodesic $\mu_t = \rho_t \mathbf{m}$ in $\mathcal{P}_2(\mathbf{M}, \mathbf{d}, \mathbf{m})$ connecting them such that

$$\begin{aligned} \int_{\mathbf{M}} \rho_t^{1-1/N'} d\mathbf{m} &\geq \\ \int_{\mathbf{M} \times \mathbf{M}} \left[\tau_{K, N'}^{(1-t)}(\mathbf{d}(x_0, x_1)) \rho_0^{-1/N'}(x_0) + \tau_{K, N'}^{(t)}(\mathbf{d}(x_0, x_1)) \rho_1^{-1/N'}(x_1) \right] d\mathbf{q}(x_0, x_1) \end{aligned} \quad (1)$$

for all $t \in (0, 1)$ and all $N' \geq N$.

In the case $K > 0$, the volume distortion coefficients $\tau_{K, N}^{(t)}(\cdot)$ for $t \in (0, 1)$ are defined by

$$\tau_{K, N}^{(t)}(\theta) = t^{1/N} \cdot \left[\frac{\sin\left(\sqrt{\frac{K}{N-1}} t \theta\right)}{\sin\left(\sqrt{\frac{K}{N-1}} \theta\right)} \right]^{1-1/N}$$

if $0 \leq \theta < \sqrt{\frac{N-1}{K}}\pi$ and by $\tau_{K,N}^{(t)}(\theta) = \infty$ if $\theta \geq \sqrt{\frac{N-1}{K}}\pi$. In the case $K < 0$ an analogous definition applies with $\sin\left(\sqrt{\frac{K}{N-1}} \dots\right)$ replaced by $\sinh\left(\sqrt{\frac{-K}{N-1}} \dots\right)$. In the case $K = 0$ simply

$$\tau_{0,N}^{(t)}(\theta) = t.$$

Therefore, the condition $\text{CD}(0, N)$ just asserts that for each $N' \geq N$ the Rényi entropy

$$\mathbf{S}_{N'}(v_t | \mathbf{m}) := - \int_{\mathbf{M}} \rho_t^{1-1/N'} d\mathbf{m}$$

is convex in $t \in [0, 1]$.

Replacing the volume distortion coefficients $\tau_{K,N}^{(t)}(\cdot)$ by slightly smaller coefficients $\sigma_{K,N}^{(t)}(\cdot)$ in the definition of $\text{CD}(K, N)$ leads to the reduced curvature-dimension condition $\text{CD}^*(K, N)$, a condition introduced and studied in [1, 7].

The definitions of the condition $\text{CD}(K, N)$ in [15] and [8] slightly differ. We follow the notation of [15]. For non-branching spaces, both concepts coincide. In this case, it suffices to verify (1) for $N' = N$ since this already implies (1) for all $N' \geq N$. Even more, the condition (1) can be formulated as a pointwise inequality.

Lemma 2 ([8, 15, 16]). *A nonbranching metric measure space $(\mathbf{M}, \mathbf{d}, \mathbf{m})$ satisfies the curvature dimension condition $\text{CD}(K, N)$ for given numbers K and N if and only if for each pair $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbf{M}, \mathbf{d}, \mathbf{m})$ there exist an optimal path measure ν with initial and terminal distributions $(e_0)_* = \mu_0$, $(e_1)_* = \mu_1$ such that for ν -a.e. $\gamma \in \Gamma(\mathbf{M})$ and all $t \in (0, 1)$*

$$\rho_t^{-1/N}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(\dot{\gamma}) \cdot \rho_0^{-1/N}(\gamma_0) + \tau_{K,N}^{(t)}(\dot{\gamma}) \cdot \rho_1^{-1/N}(\gamma_1) \quad (2)$$

where $\dot{\gamma} := \mathbf{d}(\gamma_0, \gamma_1)$ and ρ_t denotes the Radon-Nikodym density of $(e_t)_*\nu$ with respect to \mathbf{m} .

Lemma 3. *Assume that a metric measure space $(\mathbf{M}, \mathbf{d}, \mathbf{m})$ satisfies the curvature dimension condition $\text{CD}(N-1, N)$ for some number $N > 1$.*

- (i) *Then the diameter of \mathbf{M} is bounded by π .*
- (ii) *Moreover, for every $x \in \mathbf{M}$ the set $\mathbf{M}_x := \{x' \in \mathbf{M} : \mathbf{d}(x, x') = \pi\}$ of antipodes of x consists of at most one point.*

Assertion (i), the ‘generalized Bonnet-Myers theorem’ was proven in [15]. Assertion (ii) is due to Ohta [10, Theorem 4.5].

Now let us have closer look on the curvature-dimension condition in the case of weighted Riemannian spaces. Let be given a complete n -dimensional manifold \mathbf{M} equipped with its Riemannian distance \mathbf{d} and with a weighted measure

$d\mathbf{m}(x) = e^{-V(x)} d\text{vol}_{\mathbf{M}}(x)$ for some function $V : \mathbf{M} \rightarrow \mathbb{R}$. Then for each real number $N > n$ the N -Ricci tensor is defined as

$$\text{Ric}_x^{N,V}(v, v) := \text{Ric}_x(v, v) + \left[\text{Hess } V - \frac{1}{N-n} \nabla V \otimes \nabla V \right]_x(v, v).$$

For $N = n$ we define

$$\text{Ric}_x^{N,V}(v, v) := \begin{cases} \text{Ric}_x(v, v) + \text{Hess } V_x(v, v), & \text{if } \nabla V(v) = 0 \\ -\infty & \text{else.} \end{cases}$$

For $1 \leq N < n$ we define $\text{Ric}_x^{N,V}(v, v) := -\infty$ for all $v \neq 0$.

Lemma 4 ([8, 15]). *The weighted Riemannian space $(\mathbf{M}, d, \mathbf{m})$ satisfies the condition $\text{CD}(K, N)$ if and only if $\text{Ric}^{N,V} \geq K$ on \mathbf{M} in the sense that*

$$\text{Ric}_x^{N,V}(v, v) \geq K \cdot \|v\|_{T_x}^2$$

for all $x \in \mathbf{M}$ and all $v \in T_x \mathbf{M}$.

2 Euclidean Cones over Metric Measure Spaces

Definition 3 (N -Euclidean cone). For a metric measure space $(\mathbf{M}, d, \mathbf{m})$ and any $N \in [1, \infty)$, the N -Euclidean cone $(\text{Con}(\mathbf{M}), d_{\text{Con}}, \mathbf{m}_N)$ is a metric measure space defined as follows:

- (i) $\text{Con}(\mathbf{M}) := \mathbf{M} \times [0, \infty) / \mathbf{M} \times \{0\}$
- (ii) For $(x, s), (x', t) \in \mathbf{M} \times [0, \infty)$

$$d_{\text{Con}}((x, s), (x', t)) := \sqrt{s^2 + t^2 - 2st \cos(d(x, x') \wedge \pi)}$$

- (iii) $d\mathbf{m}_N(x, s) := d\mathbf{m}(x) \otimes s^N ds$.

The point $\mathbf{O} := \mathbf{M} \times \{0\} \in \text{Con}(\mathbf{M})$ is called origin of the cone.

The most prominent example in this setting is the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, endowed with its intrinsic Riemannian distance and with the Riemannian volume measure on it. In other words, $d(x, y)$ is the Euclidean angle between the rays from the origin $0 \in \mathbb{R}^{n+1}$ to the points x and y on the unit sphere of \mathbb{R}^{n+1} . Each $\xi \in \mathbb{R}^{n+1} \setminus \{0\}$ can be uniquely written as $\xi = (x, r)$ with $r \in (0, \infty)$ and $x \in \mathbb{S}^n$, namely, $r = |\xi|$ and $x = \frac{\xi}{|\xi|}$.

The definition of the metric d_{Con} and the measure \mathbf{m}_n ensures that the n -Euclidean cone over \mathbb{S}^n is the Euclidean space \mathbb{R}^{n+1} equipped with the Euclidean metric and the Lebesgue measure expressed in spherical coordinates.

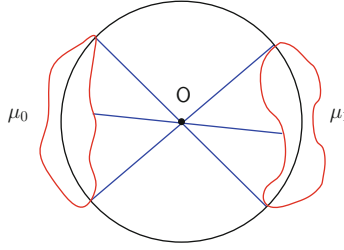


Fig. 1 Mass transport through O

Conjecture 1. A metric measure space (M, d, m) satisfies the curvature-dimension condition $CD(N - 1, N)$ for some real number $N \geq 1$ and $\text{diam}(M) \leq \pi$ (which follows from the previous condition if $N \neq 1$) if and only if the N -Euclidean cone $(\text{Con}(M), d_{\text{Con}}, m_N)$ satisfies the curvature-dimension condition $CD(0, N + 1)$.

Conjecture 1 is true for every weighted Riemannian space. The proof is based on two ingredients:

- (a) Optimal transports on the cone never transport mass through the origin – provided the base space M satisfies an appropriate CD condition.
- (b) Optimal transports on the punctured cone $\text{Con}_0(M)$ satisfy the CD condition implied by the Ricci bound for the incomplete, weighted Riemannian manifold $\text{Con}_0(M)$. The latter in turn is equivalent to a Ricci bound for the complete weighted Riemannian manifold M .

Property (a) will be proven as a result of independent interest for general metric measure spaces (Fig. 1).

Theorem 4. Assume that the metric measure space (M, d, m) satisfies the curvature-dimension condition $CD(N - 1, N)$ for some $N \geq 1$ and that $\text{diam}(M) \leq \pi$ (which follows from the previous condition if $N \neq 1$). Let v be any optimal path measure on the Euclidean cone $(\text{Con}(M), d_{\text{Con}})$.

- (i) For every $t \in (0, 1)$ there exists at most one geodesic $\gamma \in \text{supp}[v]$ with $\gamma_t = O$.
- (ii) For every $r > 0$ there exists at most one $x \in M$ such that $\gamma_0 = (x, r)$ is the initial point of some geodesic $\gamma \in \text{supp}[v] \cap \Gamma_O$ where

$$\Gamma_O := \{\gamma \in \Gamma(\text{Con}(M)) : \gamma_t = O \text{ for some } t \in (0, 1)\}.$$

- (iii) If $(e_0)_* v \ll m_N$ then v gives no mass to geodesics through O :

$$v(\Gamma_O) = 0.$$

Proof. (i) Fix $t \in (0, 1)$ and assume that two geodesics $\gamma, \gamma' \in \text{supp}[v]$ have the origin as common t -intermediate point, i.e. $\gamma_t = \gamma'_t = O$. Then $\gamma_0 = (x_0, tr)$, $\gamma_1 = (x_1, (1 - t)r)$ for some $x_0, x_1 \in M$ and with $r = \dot{\gamma} = d_{\text{Con}}(\gamma_0, \gamma_1)$.

Similarly, $\gamma'_0 = (x'_0, tr')$, $\gamma'_1 = (x'_1, (1-t)r')$ for some $x'_0, x'_1 \in \mathbb{M}$ and with $r' = \dot{\gamma}'$. If $r > 0$ then x_0 and x_1 are antipodes of each other (i.e. $d(x_0, x_1) = \pi$). Similarly, for x'_0 and x'_1 . (See e.g. Lemma 6 for a detailed proof in the more sophisticated case of spherical cones.)

Cyclic monotonicity implies

$$0 \leq d_{Con}^2(\gamma_0, \gamma'_1) + d_{Con}^2(\gamma'_0, \gamma_1) - d_{Con}^2(\gamma_0, \gamma_1) - d_{Con}^2(\gamma'_0, \gamma'_1).$$

On the other hand, a simple application of the triangle inequality yields

$$\begin{aligned} d_{Con}^2(\gamma_0, \gamma'_1) + d_{Con}^2(\gamma'_0, \gamma_1) - d_{Con}^2(\gamma_0, \gamma_1) - d_{Con}^2(\gamma'_0, \gamma'_1) \\ \leq [tr + (1-t)r']^2 + [tr' + (1-t)r]^2 - r^2 - r'^2 \\ = -2t(1-t)(r - r')^2. \end{aligned}$$

Hence, $r = r'$.

With this at hand, a more precise calculation yields

$$\begin{aligned} 0 &\leq d_{Con}^2(\gamma_0, \gamma'_1) + d_{Con}^2(\gamma'_0, \gamma_1) - d_{Con}^2(\gamma_0, \gamma_1) - d_{Con}^2(\gamma'_0, \gamma'_1) \\ &= 2r^2 [t^2 + (1-t)^2 - t(1-t) \cos d(x_0, x'_1) - t(1-t) \cos d(x'_0, x_1)] - 2r^2 \\ &= -2r^2 t(1-t) [2 + \cos d(x_0, x'_1) + \cos d(x'_0, x_1)]. \end{aligned}$$

Thus $d(x_0, x'_1) = d(x'_0, x_1) = \pi$. That is, x_0 and x'_1 are antipodes (as well as x'_0 and x_1). Since antipodes in \mathbb{M} are unique (Lemma 3(ii)) we conclude that $x_0 = x'_0$ and $x_1 = x'_1$. Thus $\gamma_0 = \gamma'_0$ and $\gamma_1 = \gamma'_1$.

In most cases of interest, geodesics are uniquely determined by their initial and terminal points. In these case, we are done. The general case, requires an additional argument. An optimal path measure ν not only induces an optimal coupling $(e_0, e_1)_* \nu$ between its initial and terminal distribution $(e_0)_* \nu$ and $(e_1)_* \nu$. More generally, the measure $(e_\sigma, e_\tau)_* \nu$ will be an optimal coupling of $(e_\sigma)_* \nu$ and $(e_\tau)_* \nu$ for each $0 \leq \sigma \leq \tau \leq 1$. For each $\sigma \in (0, t)$ one can choose $\tau \in (t, 1)$ (and vice versa) such that $(e_t)_* \nu$ is a t -intermediate point of $(e_\sigma)_* \nu$ and $(e_\tau)_* \nu$. Hence, the previous argument will imply that $\gamma_\sigma = \gamma'_\sigma$ and $\gamma_\tau = \gamma'_\tau$. This finishes the proof.

- (ii) Assume $\gamma, \gamma' \in \text{supp}[\nu] \cap \Gamma_0$ with $\gamma_0 = (x_0, r)$ and $\gamma'_0 = (x'_0, r)$. The fact that γ passes through the origin implies that $\gamma_1 = (x_1, r_1)$ with $x_1 \in \mathbb{M}$ being an antipode of x_0 , i.e. $d(x_0, x_1) = \pi$. Similarly, $\gamma'_1 = (x'_1, r'_1)$ with $d(x'_0, x'_1) = \pi$. The radii r_1, r'_1 are arbitrary positive numbers. Cyclic monotonicity implies

$$\begin{aligned}
0 &\leq d_{Con}^2(\gamma_0, \gamma'_1) + d_{Con}^2(\gamma'_0, \gamma_1) - d_{Con}^2(\gamma_0, \gamma_1) - d_{Con}^2(\gamma'_0, \gamma'_1) \\
&= r^2 + r_1'^2 - 2rr_1' \cos d(x_0, x'_1) + r^2 + r_1^2 - 2rr_1 \cos d(x'_0, x_1) \\
&\quad - (r + r_1)^2 - (r + r'_1)^2 \\
&= -2rr_1' [1 + \cos d(x_0, x'_1)] - 2rr_1 [1 + \cos d(x'_0, x_1)].
\end{aligned}$$

Hence, $d(x_0, x'_1) = \pi$. That is, x_0 and x'_1 are antipodes (as well as x'_0 and x_1 , which has been observed before). Uniqueness of antipodes in \mathbf{M} implies $x_0 = x'_0$.

- (iii) Let us assume that $\nu(\Gamma_O) > 0$. Then without restriction we even may assume that ν is supported by Γ_O . (Otherwise, replace ν by its restriction onto the set Γ_O .) Since $m_N(O) = 0$ we may also assume that $\gamma_0 \neq O$ and $\gamma_1 \neq O$ for ν -a.e. γ .

The previous part (ii) asserts that for each $r > 0$ there exists at most one point $x_0 = f(r) \in \mathbf{M}$ such that $(f(r), r)$ is the initial point γ_0 of some geodesic $\gamma \in \text{supp}[\nu] \cap \Gamma_O$. Thus the measure $\mu_0 := (e_0)_* \nu$ is concentrated on the set $C_f := \{(f(r), r) \in \text{Con}(\mathbf{M}) : r > 0\}$.

The curvature-dimension condition for the base space \mathbf{M} implies that \mathbf{m} has no atoms. Hence,

$$m_N(C_f) = 0$$

and therefore $\mu_0 \not\ll m_N$. □

According to the previous result, we know that – under the given curvature-dimension assumptions – optimal path measures on an Euclidean cone never will transport mass through the origin. It therefore suffices to study optimal transports on the *punctured cone*

$$C_0 := \text{Con}(\mathbf{M}) \setminus \{O\}.$$

To analyze such transports, we restrict ourselves to base spaces \mathbf{M} which are (weighted) Riemannian manifolds. Our results crucially will rely on the fact that in this case the punctured cone C_0 is a incomplete(!) Riemannian manifold and that the Ricci curvature of it can be calculated explicitly. More precisely, the punctured n -Euclidean cone is a Riemannian manifold whereas the punctured N -Euclidean cone is a weighted Riemannian manifold.

Lemma 5. (i) *The punctured Euclidean cone C_0 is an $(n + 1)$ -dimensional Riemannian manifold. For $(x, r) \in C_0$ with $x \in \mathbf{M}$ and $r > 0$ the tangent space $T_{(x,r)}C_0$ can be parametrized as $T_x\mathbf{M} \oplus \mathbb{R}$ with $\|(v, t)\|_{T_{(x,r)}}^2 = r^2 \|v\|_{T_x}^2 + t^2$. Moreover, for $(v, t) \in T_{(x,r)}C_0$ with $v \in T_x\mathbf{M}$ and $t \in \mathbb{R}$ we have the identity*

$$\text{Ric}_{(x,r)}((v, t), (v, t)) = \text{Ric}_x(v, v) - (n - 1)\|v\|_{T_x}^2.$$

In particular, $\text{Ric} \geq 0$ on C_0 if and only if $\text{Ric} \geq n - 1$ on \mathbf{M} .

- (ii) The punctured N -Euclidean cone \mathbf{C}_0 is a weighted $(n + 1)$ -dimensional Riemannian manifold with measure $d\mathbf{m}_N(x, r) = r^N dr d\mathbf{vol}_M(x) = e^{-W(r)} d\mathbf{m}_n(x, r)$ where $W(r) = -(N - n) \log r$ and $d\mathbf{m}_n(x, r) = d\mathbf{vol}_{\mathbf{C}_0}(x, r) = r^n dr d\mathbf{vol}_M(x)$ denotes the Riemannian volume measure on \mathbf{C}_0 . For each $N \geq n$, the $(N + 1)$ -Ricci tensor satisfies

$$\mathbf{Ric}_{(x,r)}^{N+1,W}((v, t), (v, t)) = \mathbf{Ric}_x(v, v) - (N - 1)\|v\|_{T_x}^2.$$

In particular, $\mathbf{Ric}^{N+1,W} \geq 0$ on \mathbf{C}_0 if and only if $\mathbf{Ric} \geq N - 1$ on \mathbf{M} .

- (iii) More generally, let $N \geq 1$ and let the n -dimensional Riemannian manifold \mathbf{M} be equipped with the weighted measure $d\mathbf{m}(x) = e^{-V(x)} d\mathbf{vol}_M(x)$ for some $V : \mathbf{M} \rightarrow \mathbb{R}$ and let the punctured cone \mathbf{C}_0 be equipped with the measure

$$d\mathbf{m}_N(x, r) = r^N dr d\mathbf{m}(x) = e^{-V(x)-W(r)} d\mathbf{vol}_{\mathbf{C}_0}(x, r)$$

with (as before) $W(r) = -(N - n) \log r$ and $d\mathbf{vol}_{\mathbf{C}_0}(x, r) = r^n dr d\mathbf{vol}_M(x)$. Then

$$\mathbf{Ric}_{(x,r)}^{N+1,V+W}((v, t), (v, t)) = \mathbf{Ric}_x^{N,V}(v, v) - (N - 1)\|v\|_{T_x}^2. \quad (3)$$

In particular, $\mathbf{Ric}^{N+1,V+W} \geq 0$ on \mathbf{C}_0 if and only if $\mathbf{Ric}^{N,V} \geq N - 1$ on \mathbf{M} .

Proof. Assertion (i) is a classical result due to Cheeger and Taylor [3, 6]. Assertion (ii) is the particular case of (iii) with $V = 0$ and $N \geq n$.

(iii): For arbitrary $V(x, r) = V(x)$ and $W(x, r) = W(r)$ depending only on the radial coordinate $r \in \mathbb{R}$ or on the basic coordinate $x \in M$, respectively, we have

$$\nabla W_{(x,r)}(v, t) = h'(0), \quad [\mathbf{Hess} W]_{(x,r)}((v, t), (v, t)) = h''(0)$$

for all $(x, r) \in \mathbf{C}_0$ and all $(v, t) \in T_{(x,r)}\mathbf{C}_0$ where $h(s) := W\left(\sqrt{(r + st)^2 + s^2 r^2}\|v\|_{T_x}^2\right)$. Moreover,

$$[\nabla V \otimes \nabla W]_{(x,r)}((v, t), (v, t)) = \nabla V_x(v) \cdot W'(r) \cdot t$$

for all $(x, r) \in \mathbf{C}_0$ and all $(v, t) \in T_{(x,r)}\mathbf{C}_0$ as well as

$$[\mathbf{Hess} V]_{(x,r)}((v, t), (v, t)) = f''(0) = [\mathbf{Hess} V]_x(v, v) - 2\nabla V_x(v) \cdot \frac{t}{r}.$$

Here the expressions on the LHS always have to be interpreted as quantities on the $(n + 1)$ -dimensional manifold \mathbf{C}_0 whereas the expressions on the RHS are the original data on the basic n -dimensional manifold \mathbf{M} and

$$f(s) = V \left(\exp_x \left(\frac{v}{\|v\|_{T_x}} \cdot \arctan \frac{rs\|v\|_{T_x}}{r + st} \right) \right).$$

For the particular choice of $W(x, r) = -(N - n) \log r$, explicit calculations yield

$$\left[\text{Hess } W - \frac{1}{N - n} \nabla W \otimes \nabla W \right]_{(x,r)} ((v, t), (v, t)) = -(N - n) \|v\|_{T_x}^2.$$

Hence, in the case when $N > n$, together with the identity from (i)

$$\begin{aligned} & \text{Ric}_{(x,r)}^{N+1,V+W}((v, t), (v, t)) \\ &= \text{Ric}_{(x,r)}((v, t), (v, t)) \\ & \quad + \left[\text{Hess}(V + W) - \frac{1}{N - n} \nabla(V + W) \otimes \nabla(V + W) \right]_{(x,r)}((v, t), (v, t)) \\ &= \text{Ric}_x(v, v) - (n - 1) \|v\|_{T_x}^2 + \left[\text{Hess } V - \frac{1}{N - n} \nabla V \otimes \nabla V \right]_x(v, v) - (N - n) \|v\|_{T_x}^2 \\ &= \text{Ric}_x^{N,V}(v, v) - (N - 1) \|v\|_{T_x}^2. \end{aligned}$$

The case $N = n$ follows from an analogous computation and our definition of N -Ricci tensor. In the case $N < n$, by definition $\text{Ric}^{N,V}$ as well as $\text{Ric}^{N+1,V+W}$ are $-\infty$. \square

Theorem 5. *Let be given a complete n -dimensional manifold \mathbf{M} equipped with its Riemannian distance \mathbf{d} and with a weighted measure $d\mathbf{m}(x) = e^{-V(x)} d\text{vol}_{\mathbf{M}}(x)$ for some function $V : \mathbf{M} \rightarrow \mathbb{R}$. Then for each real number $N > 1$ the following statements are equivalent:*

- (i) *The weighted Riemannian space $(\mathbf{M}, \mathbf{d}, \mathbf{m})$ satisfies the condition $\text{CD}(N - 1, N)$.*
- (ii) *The N -Euclidean cone $(\text{Con}(\mathbf{M}), \mathbf{d}_{\text{Con}}, \mathbf{m}_N)$ satisfies the condition $\text{CD}(0, N + 1)$.*

Proof. Each of the CD -conditions under consideration will imply that $\dim_{\mathbf{M}} \leq N$. Hence, without restriction $N \geq n$.

(ii) \Rightarrow (i): The condition $\text{CD}(0, N + 1)$ for the N -Euclidean cone $(\text{Con}(\mathbf{M}), \mathbf{d}_{\text{Con}}, \mathbf{m}_N)$ implies that this condition holds *locally* on the punctured cone. For this (incomplete) weighted Riemannian manifold, however, the local curvature-dimension condition $\text{CD}_{\text{loc}}(0, N + 1)$ is equivalent to nonnegativity of the $(N + 1)$ -Ricci tensor $\text{Ric}^{N+1,V+W}$ on \mathbf{C}_0 , see Lemma 4. Due to the previous Lemma 5(iii), this implies $\text{Ric}_{\mathbf{M}}^{N,V} \geq N - 1$. For the (complete) weighted Riemannian space $(\mathbf{M}, \mathbf{d}, \mathbf{m})$, the latter in turn is equivalent to $\text{CD}(N - 1, N)$.

(i) \Rightarrow (ii): Let probability measures μ_0 and μ_1 on $\text{Con}(\mathbf{M})$ be given, absolutely continuous with respect to \mathbf{m}_N . According to Theorem 4, any optimal path

measure ν with marginal distributions $(e_0)_*\nu = \mu_0$ and $(e_1)_*\nu = \mu_1$ will give no mass to geodesics through the origin. In other words, ν -almost every geodesic will stay within the punctured cone \mathbf{C}_0 .

According to Lemma 5(iii), assertion (i) implies that the $(N + 1)$ -Ricci tensor $\text{Ric}^{N+1, V+W}$ on the weighted Riemannian space \mathbf{C}_0 is nonnegative. Hence, classical arguments based on Jacobi field calculus – exactly the same as used to deduce Lemma 4 – will imply that (2) holds true with $K = 0$ for ν -a.e. geodesic γ which remains within \mathbf{C}_0 . That is, $\text{CD}(0, N + 1)$ holds true on $\text{Con}(\mathbf{M})$. \square

Corollary 1. *Given a complete n -dimensional manifold \mathbf{M} (equipped with its Riemannian distance \mathbf{d} and its Riemannian volume $d\mathbf{m} = d\text{vol}_{\mathbf{M}}$) and a real number $N \geq 1$. Then the following statements are equivalent:*

- (i) $\text{Ric} \geq N - 1$ on \mathbf{M} , $\dim_{\mathbf{M}} \leq N$ and $\text{diam}(\mathbf{M}) \leq \pi$ (the latter follows from the Ricci and dimension bounds if $N \neq 1$);
- (ii) The space $(\mathbf{M}, \mathbf{d}, \mathbf{m})$ satisfies the curvature-dimension condition $\text{CD}(N - 1, N)$ and $\text{diam}(\mathbf{M}) \leq \pi$ (which follows from the CD condition if $N \neq 1$);
- (iii) The N -Euclidean cone $(\text{Con}(\mathbf{M}), \mathbf{d}_{\text{Con}}, \mathbf{m}_N)$ satisfies $\text{CD}(0, N + 1)$.

Proof. The equivalence (i) \Leftrightarrow (ii) is well-known. Moreover, it is well-known that for each $N > 1$ the condition $\text{CD}(N - 1, N)$ implies $\text{diam}(\mathbf{M}) \leq \pi$. See Lemmas 3 and 4.

In the case $N \neq 1$, the equivalence (ii) \Leftrightarrow (iii) for Riemannian spaces follows from the more general assertion of Theorem 5 for *weighted* Riemannian spaces. Indeed, the arguments there also apply to the case $N = 1$. It only remains to prove that (iii) in the case $N = 1$ implies $\text{diam}(\mathbf{M}) \leq \pi$.

Assume the contrary: i.e. \mathbf{M} is a circle or an interval of $\text{diam}(\mathbf{M}) > \pi$. Then there exist non-empty intervals $I, J \subset \mathbf{M}$ of length $R > 0$ such that $d(x, y) > \pi$ for all $x \in I, y \in J$. Thus for all $x \in I, y \in J$ and $r \in (0, \infty)$ the origin \mathbf{O} will be the unique midpoint of (x, r) and (y, r) in $\text{Con}(\mathbf{M})$. Moreover, for each pair $(x, s) \in I \times [1 - \epsilon, 1]$ and $(y, t) \in J \times [1 - \epsilon, 1]$ in $\text{Con}(\mathbf{M})$ the midpoint will lie in the domain $B_\epsilon := ((I \cup J) \times (0, \epsilon]) \cup \{\mathbf{O}\}$.

Let μ_0 and μ_1 be the ‘uniform distributions’ on $I_\epsilon := I \times [1 - \epsilon, 1]$ and let $J_\epsilon := J \times [1 - \epsilon, 1]$, resp., i.e. $d\mu_0 = C_\epsilon 1_{I_\epsilon} d\mathbf{m}_N$, $d\mu_1 = C_\epsilon 1_{J_\epsilon} d\mathbf{m}_N$ with suitable $C_\epsilon \geq \frac{1}{R\epsilon}$. Then respectively their Renyi entropy satisfies

$$-S_{N+1}(\mu_0|\mathbf{m}_N) = -S_{N+1}(\mu_1|\mathbf{m}_N) = C_\epsilon^{-\frac{1}{N+1}} \leq (R\epsilon)^{-\frac{1}{N+1}} = c\epsilon^{-\frac{1}{N+1}}.$$

On the other hand, the midpoint $\mu_{1/2}$ of μ_0 and μ_1 is supported on B_ϵ . Hence, its Renyi entropy is bounded from below by the Renyi entropy of the uniform distribution on B_ϵ :

$$S_{N+1}(\mu_{1/2}|\mathbf{m}_N) \geq S_{N+1}(C'_\epsilon 1_{B_\epsilon} \mathbf{m}_N|\mathbf{m}_N) = -C'^{-\frac{1}{N+1}}_\epsilon = -c'\epsilon.$$

Note that $C'^{-1}_\epsilon = m_N(B_\epsilon) = 2R \cdot \int_0^\epsilon r^N dr = \frac{2R}{N+1} \epsilon^{N+1}$. Thus, choosing ϵ sufficiently small we obtain

$$S_{N+1}(\mu_{1/2}|\mathbf{m}_N) \gg \frac{1}{2}(S_{N+1}(\mu_0|\mathbf{m}_N) + S_{N+1}(\mu_1|\mathbf{m}_N))$$

which contradicts the $\text{CD}(0, N + 1)$ condition. \square

Example 1. Let $\mathbf{M} = \left(\frac{1}{\sqrt{3}}\mathbb{S}^2\right) \times \left(\frac{1}{\sqrt{3}}\mathbb{S}^2\right)$.

- (i) Then the Euclidean cone over \mathbf{M} – more precisely, the metric measure space $(\text{Con}(\mathbf{M}), d_{\text{Con}}, m_4)$ – satisfies the curvature-dimension condition $\text{CD}(0, 5)$.
- (ii) On the other hand, the Euclidean cone over \mathbf{M} – more precisely, the metric space $(\text{Con}(\mathbf{M}), d_{\text{Con}})$ – is not an Alexandrov space: the sectional curvature on the punctured cone \mathbf{C}_0 is unbounded from below (and above) in any punctured neighborhood of the origin 0 .

Proof. (i) Given $x, y \in \frac{1}{\sqrt{3}}\mathbb{S}^2$, let u_1, u_2 be an orthonormal basis of $T_x(\frac{1}{\sqrt{3}}\mathbb{S}^2)$ and v_1, v_2 be an orthonormal basis of $T_y(\frac{1}{\sqrt{3}}\mathbb{S}^2)$. Then an orthonormal basis of $T_{(x,y)}\mathbf{M} = T_x(\frac{1}{\sqrt{3}}\mathbb{S}^2) \oplus T_y(\frac{1}{\sqrt{3}}\mathbb{S}^2)$ is given by $\{\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2\}$ with $\tilde{u}_i = (u_i, 0)$ and $\tilde{v}_i = (0, v_i)$. In this basis

$$\text{Sec}_{(x,y)}(\tilde{u}_1, \tilde{u}_2) = 3, \quad \text{Sec}_{(x,y)}(\tilde{u}_1, \tilde{v}_1) = 0, \quad \text{Sec}_{(x,y)}(\tilde{u}_1, \tilde{v}_2) = 0$$

and analogously for any other basis vector in the place of \tilde{u}_1 . Hence, in particular,

$$\text{Ric}_{(x,y)}(\xi, \xi) = 3$$

for each $\xi \in \{\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2\}$ and thus for each $\xi \in T_{(x,y)}\mathbf{M}$.

- (ii) Thus according to Theorem 5 the Euclidean cone satisfies the $\text{CD}(0, 5)$ condition.
- (iii) Given $r > 0$ an orthonormal basis of $T_{(x,y,r)}\mathbf{C}_0 = T_x(\frac{1}{\sqrt{3}}\mathbb{S}^2) \oplus T_y(\frac{1}{\sqrt{3}}\mathbb{S}^2) \oplus \mathbb{R}$ is given by $\{\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2, \hat{w}\}$ with $\hat{u}_i = \frac{1}{r}(u_i, 0, 0)$, $\hat{v}_i = \frac{1}{r}(0, v_i, 0)$ and $\hat{w} = (0, 0, 1)$. In this basis

$$\begin{aligned} \text{Sec}_{(x,y,r)}(\hat{u}_1, \hat{u}_2) &= \frac{2}{r^2}, & \text{Sec}_{(x,y,r)}(\hat{u}_1, \hat{v}_1) &= -\frac{1}{r^2}, \\ \text{Sec}_{(x,y,r)}(\hat{u}_1, \hat{v}_2) &= -\frac{1}{r^2}, & \text{Sec}_{(x,y,r)}(\hat{u}_1, \hat{w}) &= 0 \end{aligned} \tag{4}$$

and analogously for \tilde{u}_2, \tilde{v}_1 or \tilde{v}_2 in the place of \tilde{u}_1 . Of course, this in particular implies $\text{Ric}_{(x,y,r)}(\xi, \xi) = 0$ for each $\xi \in T_{(x,y,r)}\mathbf{C}_0$, see Lemma 5. \square

3 Spherical Cones over Metric Measure Spaces

There are further objects with famous Euclidean ancestors – among them is the *spherical cone* or *suspension* over a topological space M . We begin with a familiar example: In order to construct the Euclidean sphere \mathbb{S}^{n+1} out of its equator \mathbb{S}^n we add two poles \mathcal{S} and \mathcal{N} and connect them via semicircles, the *meridians*, through every point in \mathbb{S}^n .

In the general case of abstract spaces M , we consider the product $M \times [0, \pi]$ and contract each of the fibers $\mathcal{S} := M \times \{0\}$ and $\mathcal{N} := M \times \{\pi\}$ to a point, the *south* and the *north pole*, respectively. The resulting space is denoted by $\Sigma(M)$ and is called the spherical cone over M .

Definition 4 (*N-spherical cone*). The *N-spherical cone* $(\Sigma(M), d_\Sigma, \hat{m}_N)$ over a metric measure space (M, d, m) is the metric measure space defined as follows:

- (i) $\Sigma(M) := M \times [0, \pi] \setminus M \times \{0\}, M \times \{\pi\}$
- (ii) For $(x, s), (x', t) \in M \times [0, \pi]$

$$\cos(d_\Sigma((x, s), (x', t))) := \cos s \cos t + \sin s \sin t \cos(d(x, x') \wedge \pi)$$
- (iii) $d\hat{m}_N(x, s) := dm(x) \otimes (\sin^N s ds)$.

For a nice introduction and detailed information about Euclidean and spherical cones over metric spaces we refer to [2].

Lemma 6. Assume that $\text{diam}(M) \leq \pi$. Let $\gamma : [0, 1] \rightarrow \Sigma(M)$ be a non-constant geodesic with endpoints $\gamma_0 = (x_0, r_0)$ and $\gamma_1 = (x_1, r_1)$ in $\Sigma(M)$. If $\gamma_t = \mathcal{S}$ for some $t \in (0, 1)$, then x_0 and x_1 are antipodes in M .

Proof. Due to the definition of d_Σ , it holds that $r_0 = d_\Sigma(\gamma_0, \gamma_t) = t d_\Sigma(\gamma_0, \gamma_1)$ as well as $r_1 = d_\Sigma(\gamma_t, \gamma_1) = (1 - t) d_\Sigma(\gamma_0, \gamma_1)$ and consequently, $r_1 = \frac{1-t}{t} r_0$. Inserting this equality into the expression for $\cos\left(\frac{r_0}{t}\right)$ we obtain

$$\cos\left(\frac{r_0}{t}\right) = \cos(d_\Sigma(\gamma_0, \gamma_1)) = \cos r_0 \cos\left(\frac{1-t}{t} r_0\right) + \sin r_0 \sin\left(\frac{1-t}{t} r_0\right) \cos(d(x_0, x_1)).$$

Since $\text{diam}(M) \leq \pi$ by assumption, this leads to

$$\begin{aligned} \cos(d(x_0, x_1)) &= \frac{\cos\left(\frac{r_0}{t}\right) - \cos r_0 \cos\left(\frac{1-t}{t} r_0\right)}{\sin r_0 \sin\left(\frac{1-t}{t} r_0\right)} \\ &= \frac{\cos\left(\frac{r_0}{t}\right) - \frac{1}{2} [\cos\left(\frac{2t-1}{t} r_0\right) + \cos\left(\frac{r_0}{t}\right)]}{\frac{1}{2} [\cos\left(\frac{2t-1}{t} r_0\right) - \cos\left(\frac{r_0}{t}\right)]} \\ &= \frac{\frac{1}{2} [\cos\left(\frac{r_0}{t}\right) - \cos\left(\frac{2t-1}{t} r_0\right)]}{\frac{1}{2} [\cos\left(\frac{2t-1}{t} r_0\right) - \cos\left(\frac{r_0}{t}\right)]} = -1. \end{aligned}$$

That is, $d(x_0, x_1) = \pi$. □

Theorem 6. Assume that the metric measure space $(\mathbf{M}, \mathbf{d}, \mathbf{m})$ satisfies the curvature-dimension condition $\mathbf{CD}(N - 1, N)$ for some $N \geq 1$ and that $\text{diam}(\mathbf{M}) \leq \pi$ (which follows from the previous condition if $N \neq 1$). Let ν be any optimal path measure on the spherical cone $(\Sigma(\mathbf{M}), \mathbf{d}_\Sigma)$ satisfying $(e_0)_* \nu \ll m_N$. Then ν gives no mass to geodesics through the poles:

$$\nu(\Gamma_{\mathcal{S}}) = \nu(\Gamma_{\mathcal{N}}) = 0$$

where $\Gamma_{\mathcal{S}} := \{\gamma \in \Gamma(\Sigma(\mathbf{M})) : \gamma_t \in \mathcal{S} \text{ for some } t \in (0, 1)\}$ and analogously $\Gamma_{\mathcal{N}}$ with \mathcal{N} in the place of \mathcal{S} .

Proof. We follow the argumentation in the proof of assertion (iii) of Theorem 4. Assume that $\nu(\Gamma_{\mathcal{S}}) > 0$. Then without restriction we even may assume that $\nu(\Gamma_{\mathcal{S}}) = 1$. According to Lemma 7 below, for each $r \in (0, \pi)$ there exists at most one point $f(r) \in \mathbf{M}$ such that $(f(r), r) \in \Sigma(\mathbf{M})$ is the initial point γ_0 of some geodesic $\gamma \in \text{supp}[\nu]$. Hence, $\mu_0 := (e_0)_* \nu$ is concentrated on the set $C_f := \{(f(r), r) \in \Sigma(\mathbf{M}) : r \in (0, \pi)\}$.

The curvature-dimension condition for $(\mathbf{M}, \mathbf{d}, \mathbf{m})$ implies that \mathbf{m} has no atoms and thus

$$\hat{m}_N(C_f) = 0$$

which contradicts the assumption $\mu_0 \ll \hat{m}_N$. Hence, $\nu(\Gamma_{\mathcal{S}}) = 0$. Analogously, we deduce $\nu(\Gamma_{\mathcal{N}}) = 0$. \square

Lemma 7. Under the assumptions of the previous theorem, for every $r \in (0, \pi)$ there exists at most one $x \in \mathbf{M}$ such that $\gamma_0 = (x, r) \in \Sigma(\mathbf{M})$ is the initial point of some geodesic $\gamma \in \text{supp}[\nu] \cap \Gamma_{\mathcal{S}}$.

Proof. Assume $\gamma, \gamma' \in \text{supp}[\nu] \cap \Gamma_{\mathcal{S}}$ with $\gamma_0 = (x_0, r)$ and $\gamma'_0 = (x'_0, r)$. According to Lemma 6, the fact that γ passes through the south pole implies that $\gamma_1 = (x_1, r_1)$ with $x_1 \in \mathbf{M}$ being an antipode of x_0 , i.e. $\mathbf{d}(x_0, x_1) = \pi$. Similarly, $\gamma'_1 = (x'_1, r'_1)$ with $\mathbf{d}(x'_0, x'_1) = \pi$. The radii r_1, r'_1 are arbitrary numbers in $(0, \pi)$.

By the very definition of \mathbf{d}_Σ , taking into account that the diameter of \mathbf{M} is bounded by π ,

$$\begin{aligned} & \mathbf{d}_\Sigma^2(\gamma_0, \gamma'_1) + \mathbf{d}_\Sigma^2(\gamma'_0, \gamma_1) \\ &= \arccos^2[\cos r \cdot \cos r'_1 + \sin r \cdot \sin r'_1 \cdot \cos \mathbf{d}(x_0, x'_1)] \\ & \quad + \arccos^2[\cos r \cdot \cos r_1 + \sin r \cdot \sin r_1 \cdot \cos \mathbf{d}(x'_0, x_1)] \\ & \stackrel{(*)}{\leq} \arccos^2[\cos r \cdot \cos r'_1 - \sin r \cdot \sin r'_1] + \arccos^2[\cos r \cdot \cos r_1 - \sin r \cdot \sin r_1] \\ &= (r + r'_1)^2 + (r + r_1)^2 = \mathbf{d}_\Sigma^2(\gamma'_0, \gamma'_1) + \mathbf{d}_\Sigma^2(\gamma_0, \gamma_1) \end{aligned}$$

with equality in (*) if and only if $d(x_0, x'_1) = d(x'_0, x_1) = \pi$, that is, if and only if x_0 and x'_1 are antipodes and x'_0 and x_1 are antipodes. Therefore, d_Σ^2 -cyclical monotonicity implies $x_0 = x'_0$. \square

From now on, let us again focus on weighted Riemannian spaces, that is, \mathbf{M} is a complete, n -dimensional manifold equipped with its Riemannian distance d and with a measure $d\mathbf{m}(x) = e^{-V(x)}d\text{vol}_M(x)$. A crucial fact for our argumentation is that the punctured cone $\Sigma_0 := \Sigma(\mathbf{M}) \setminus \{\mathcal{S}, \mathcal{N}\}$ is given as a warped product $\mathbf{M}_{\sin(r)} \times (0, \pi)$ for which the Ricci curvature can be calculated explicitly.

Lemma 8. (i) *The punctured spherical cone Σ_0 is an incomplete $(n+1)$ -dimensional Riemannian manifold whose tangent space $T_{(x,r)}\Sigma_0$ at $(x, r) \in \Sigma_0$ with $x \in \mathbf{M}$ and $0 < r < \pi$ can be parametrized as $T_{(x,r)}\Sigma_0 = T_x\mathbf{M} \oplus \mathbb{R}$ and whose metric tensor is given by $\|(v, t)\|_{T_{(x,r)}}^2 = \sin^2 r \cdot \|v\|_{T_x}^2 + t^2$ for $(v, t) \in T_{(x,r)}\Sigma_0$. Furthermore, we have the equality*

$$\text{Ric}_{(x,r)}((v, t), (v, t)) = \text{Ric}_x(v, v) + (1 - n \cos^2 r) \cdot \|v\|_{T_x}^2 + n t^2.$$

In particular, $\text{Ric} \geq n$ on Σ_0 if and only if $\text{Ric} \geq n - 1$ on \mathbf{M} .

(ii) *Now let us consider the punctured N -spherical cone over the weighted Riemannian manifold \mathbf{M} . That is, given any real $N > 1$ put $W(x, r) = -(N - n) \log \sin r$ and $V(x, r) = V(x)$. Then*

$$\text{Ric}_{(x,r)}^{N+1, V+W}((v, t), (v, t)) - N \|(v, t)\|_{T_{x,r}}^2 = \text{Ric}_x^{N, V}(v, v) - (N - 1) \|v\|_{T_x}^2. \quad (5)$$

In particular, $\text{Ric}^{N+1, V+W} \geq N$ on Σ_0 if and only if $\text{Ric}^{N, V} \geq N - 1$ on \mathbf{M} .

Proof. The formula for the Ricci tensor in (i) is well-known, see [12], Corollary 7.43, or e.g. [13]. Note that

$$\begin{aligned} \text{Ric}_{(x,t)}((v, t), (v, t)) - \text{Ric}_x(v, v) &= (1 - n \cos^2 r) \cdot \|v\|_{T_x}^2 + n t^2 \\ &= n \|(v, t)\|_{T_{(x,r)}}^2 - (n - 1) \|v\|_{T_x}^2. \end{aligned}$$

The proof of assertion (ii) follows the lines of argumentation in the previous case of Euclidean cones – with appropriate modifications. For arbitrary $V(x, r) = V(x)$ and $W(x, r) = W(r)$ as above (depending only on the radial coordinate $r \in \mathbb{R}$ or on the basic coordinate $x \in M$, respectively) we have as before

$$[\nabla V \otimes \nabla W]_{(x,r)}((v, t), (v, t)) = \nabla V_x(v) \cdot W'(r) \cdot t$$

for all $(x, r) \in \Sigma_0$ and all $(v, t) \in T_{(x,r)}\Sigma_0$ and

$$\nabla W_{(x,r)}(v, t) = h'(0), \quad [\text{Hess } W]_{(x,r)}((v, t), (v, t)) = h''(0)$$

where now

$$h(s) := W(\arccos[\cos(r + st) \cdot \cos(s \cdot \sin r \cdot \|v\|_{T_x})]).$$

Moreover,

$$[\text{Hess } V]_{(x,r)}((v, t), (v, t)) = f''(0) = [\text{Hess } V]_x(v, v) - 2\nabla V_x(v) \cdot \cot(r) \cdot t$$

where

$$f(s) = V\left(\exp_x\left(\frac{v}{\|v\|_{T_x}} \cdot \arctan \frac{\tan(\sin(r) \cdot s\|v\|_{T_x})}{\sin(r + st)}\right)\right).$$

For the particular choice of $W(x, r) = -(N - n) \log \sin(r)$, some lengthy calculation yields

$$\begin{aligned} \left[\text{Hess } W - \frac{1}{N - n} \nabla W \otimes \nabla W \right]_{(x,r)}((v, t), (v, t)) &= (N - n) [t^2 - \cos^2(r) \|v\|_{T_x}^2] \\ &= (N - n) [\|(v, t)\|_{T(x,r)}^2 - \|v\|_{T_x}^2]. \end{aligned}$$

Hence, together with the identity from (i)

$$\begin{aligned} \text{Ric}_{(x,r)}^{N+1,V+W}((v, t), (v, t)) &= \text{Ric}_{(x,r)}((v, t), (v, t)) \\ &\quad + \left[\text{Hess}(V + W) - \frac{1}{N - n} \nabla(V + W) \otimes \nabla(V + W) \right]_{(x,r)}((v, t), (v, t)) \\ &= \text{Ric}_x(v, v) + n \|(v, t)\|_{T(x,r)}^2 - (n - 1) \|v\|_{T_x}^2 \\ &\quad + \left[\text{Hess } V - \frac{1}{N - n} \nabla V \otimes \nabla V \right]_x(v, v) + (N - n) [\|(v, t)\|_{T(x,r)}^2 - \|v\|_{T_x}^2] \\ &= \text{Ric}_x^{N,V}(v, v) - (N - 1) \|v\|_{T_x}^2 + N \|(v, t)\|_{T(x,r)}^2. \end{aligned} \quad \square$$

Theorem 7. *Let be given a complete n -dimensional manifold \mathbf{M} equipped with its Riemannian distance \mathbf{d} and with a weighted measure $d\mathbf{m}(x) = e^{-V(x)} d\text{vol}_{\mathbf{M}}(x)$ for some function $V : \mathbf{M} \rightarrow \mathbb{R}$. Then for each real number $N \geq 1$ the following statements are equivalent:*

- (i) *The weighted Riemannian space $(\mathbf{M}, \mathbf{d}, \mathbf{m})$ has $\text{diam}(\mathbf{M}) \leq \pi$ and satisfies the condition $\text{CD}(N - 1, N)$.*
- (ii) *The N -spherical cone $(\Sigma(\mathbf{M}), \mathbf{d}_\Sigma, \mathbf{m}_N)$ satisfies the condition $\text{CD}(N, N + 1)$.*

Proof. This is essentially the same argumentation as in the proof of Theorem 5, now with Lemma 8 instead of Lemma 5. Again we may assume without restriction that $N \geq n$.

(i) \Rightarrow (ii): Let probability measures μ_0 and μ_1 on $\Sigma(\mathbf{M})$ be given, absolutely continuous with respect to \hat{m}_N . According to Theorem 6, any optimal path measure ν with marginal distributions $(e_0)_*\nu = \mu_0$ and $(e_1)_*\nu = \mu_1$ will give no mass to geodesics through the poles. In other words, ν -almost every geodesic will stay within the punctured cone Σ_0 .

According to Lemma 8, assertion (i) implies that the $(N + 1)$ -Ricci tensor $\text{Ric}^{N+1, V+W}$ on the weighted Riemannian space Σ_0 is bounded from below by N . Hence, classical arguments based on Jacobi field calculus – exactly the same as used to deduce Lemma 4 – will imply that (2) holds true with $K = N$ for ν -a.e. geodesic γ which remains within Σ_0 . That is, $\text{CD}(N, N + 1)$ holds true on $\Sigma(\mathbf{M})$.

(ii) \Rightarrow (i): First the curvature-dimension condition $\text{CD}(N, N + 1)$ for the N -spherical cone $(\Sigma(\mathbf{M}), d_\Sigma, \hat{m}_N)$ implies that this condition holds *locally* on the punctured cone Σ_0 . For this (incomplete) weighted Riemannian manifold, however, the local curvature-dimension condition $\text{CD}_{loc}(N, N + 1)$ is equivalent to the bound $\text{Ric}^{N+1, V+W} \geq N$ for the $(N + 1)$ -Ricci tensor on Σ_0 , see Lemma 4. Due to Lemma 8, this implies $\text{Ric}_M^{N, V} \geq N - 1$. For the (complete) weighted Riemannian space (\mathbf{M}, d, m) , the latter in turn is equivalent to $\text{CD}(N - 1, N)$.

Finally, in the case $N = 1$ it remains to prove that (ii) implies the diameter bound $\text{diam}(\mathbf{M}) \leq \pi$. This can be achieved by means of a straightforward adaptation of the argument from the proof of Corollary 1. \square

Corollary 2. *The n -spherical cone $(\Sigma(\mathbf{M}), d_\Sigma, \nu)$ over a complete n -dimensional Riemannian manifold $(\mathbf{M}, d, \text{vol})$ satisfies $\text{CD}(n, n + 1)$ if and only if $\text{Ric} \geq n - 1$ on \mathbf{M} and $\text{diam}(\mathbf{M}) \leq \pi$.*

Theorem 7 allows to apply the Lichnerowicz theorem [8] in order to obtain a lower bound on the spectral gap of the Laplacian on the spherical cone:

Corollary 3 (Lichnerowicz estimate, Poincaré inequality). *Let $(\Sigma(\mathbf{M}), d_\Sigma, \hat{m}_n)$ be the n -spherical cone of a compact n -dimensional Riemannian manifold $(\mathbf{M}, d, \text{vol})$ with $\text{Ric} \geq n - 1$ and $\text{diam}(\mathbf{M}) \leq \pi$. Then for every $f \in \text{Lip}(\Sigma(\mathbf{M}))$ fulfilling $\int_{\Sigma(\mathbf{M})} f \, d\hat{m}_n = 0$ the following inequality holds true:*

$$\int_{\Sigma(\mathbf{M})} f^2 d\hat{m}_n \leq \frac{1}{n+1} \int_{\Sigma(\mathbf{M})} |\nabla f|^2 d\hat{m}_n.$$

The Lichnerowicz estimate implies that the Laplacian Δ on the spherical cone $(\Sigma(\mathbf{M}), d_\Sigma, \hat{m}_n)$ defined by the identity

$$\int_{\Sigma(\mathbf{M})} f \cdot \Delta g \, d\hat{m}_n = - \int_{\Sigma(\mathbf{M})} \nabla f \cdot \nabla g \, d\hat{m}_n$$

admits a spectral gap λ_1 of size at least $n + 1$,

$$\lambda_1 \geq n + 1.$$

An analogous statement – with N in the place of n – holds true for the Laplacian on the N -spherical cone over a weighted n -dimensional Riemannian manifold satisfying $\text{Ric}^{N,V} \geq N - 1$.

Extension to (κ, N) -Cones

Let us finally mention that there is a canonical extension of the concept of cones which covers both, the Euclidean cones and the spherical cones.

Definition 5. Given a metric measure space (M, d, m) and numbers $\kappa \in \mathbb{R}$, $N \in (0, \infty)$ we define the (κ, N) -cone over (M, d, m) to be the metric measure space $(\bar{M}, \bar{d}, \bar{m})$ with

- (i) $\bar{M} := M \times [0, \infty)$ if $\kappa \leq 0$ and $\bar{M} := M \times [0, \pi/\sqrt{\kappa}]$ if $\kappa > 0$ where all the points $(x, 0)$, $x \in M$, have to be identified as well as – in the case $\kappa > 0$ – all the points $(x, \pi/\sqrt{\kappa})$.
- (ii) For $(x, s), (y, t) \in \bar{M}$

$$\bar{d}((x, s), (y, t)) := \mathcal{C}_\kappa^{-1}(\mathcal{C}_\kappa(s) \cdot \mathcal{C}_\kappa(t) + \kappa \cdot \mathcal{S}_\kappa(s) \cdot \mathcal{S}_\kappa(t) \cdot \cos(d(x, y) \wedge \pi)) \quad (6)$$

where

$$\begin{aligned} \mathcal{C}_\kappa(r) &= \cos(\sqrt{\kappa} r), \quad \mathcal{S}_\kappa(r) = \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} r) \text{ if } \kappa > 0 \quad \text{and} \\ \mathcal{C}_\kappa(r) &= \cosh(\sqrt{-\kappa} r), \quad \mathcal{S}_\kappa(r) = \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} r) \text{ if } \kappa < 0. \end{aligned}$$

In the case $\kappa = 0$, the metric \bar{d} will be defined as in Definition 3. Indeed, the formula (6) leads in the limit $\kappa \rightarrow 0$ to the definition of d_{Con} .

- (iii) $d\bar{m}(x, s) := dm(x) \otimes (\mathcal{S}_\kappa(s))^N ds$.

The metric space (\bar{M}, \bar{d}) obtained as such a cone over a metric space (M, d) is discussed in detail in [2]. In the case $\kappa = 0$ it is simply the Euclidean cone and in the case $\kappa = 1$ it is the spherical cone. In the case $\kappa = -1$, the cone is also called *hyperbolic cone* based on (M, d) . Without too much effort, our previous results extend to the general case of (κ, N) -cones over weighted Riemannian spaces (M, d, m) . Indeed, the case $\kappa > 0$ is just a rescaling of the case $\kappa = 1$. Replacing all \sin and \cos by \sinh and \cosh (e.g. in Lemma 8) allows to switch from the case $\kappa > 0$ to the case $\kappa < 0$.

Theorem 8. *Given a complete n -dimensional manifold M equipped with its Riemannian distance d and with a weighted measure $d\mathbf{m}(x) = e^{-V(x)}d\text{vol}_M(x)$ for some function $V : M \rightarrow \mathbb{R}$. Then for all $\kappa \in \mathbb{R}$ and $N \geq 1$ the following statements are equivalent:*

- (i) *The weighted Riemannian space (M, d, \mathbf{m}) has $\text{diam}(M) \leq \pi$ and satisfies the condition $\text{CD}(N - 1, N)$.*
- (ii) *The (κ, N) -cone $(\bar{M}, \bar{d}, \bar{\mathbf{m}})$ satisfies the condition $\text{CD}(\kappa \cdot N, N + 1)$.*

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