

Chapter 2

The Standard Format for Adaptive Logics

The purpose of this section is to introduce the reader to ALs with a special eye on the modeling of defeasible reasoning. The standard format of ALs has been introduced by Diderik Batens (see e.g. [1, 2] for a systematic study). As will be shown in the following, for the standard format a rich meta-theory is available which equips ALs with many desirable properties and at the same time provides a unifying framework to ALs.

2.1 The Standard Format

The basic idea behind ALs is to interpret a given set of premises “as normally as possible”. Depending on the application this may have different meanings. Let me give some examples:

- (i) In applications in which we are confronted with inconsistent information we may want to interpret the premises as consistently as possible.
- (ii) In applications in which we are confronted with conflicting norms and obligations we may want to interpret the premises as non-conflicting as possible.

There are three elements that constitute ALs in the standard format:

1. the lower limit logic **LLL**,
2. the set of abnormalities Ω , and
3. the adaptive strategy: reliability or minimal abnormality.

AL^r denotes the AL defined by $\langle \mathbf{LLL}, \Omega, \text{reliability} \rangle$ and **AL^m** denotes the AL defined by $\langle \mathbf{LLL}, \Omega, \text{minimal abnormality} \rangle$. By **AL** I will refer to either of the two.

In the following sections I will introduce each element of the standard format, beginning with the lower limit logic.

2.2 The Lower Limit Logic

ALs employ and strengthen a monotonic logic **LLL**, their so-called *lower limit logic*. This logic is a reflexive, transitive, monotonic and compact logic that has a characteristic semantics. Hence we have:

- *Reflexivity*: $\Gamma \subseteq Cn_{\mathbf{LLL}}(\Gamma)$.
- *Transitivity*: If $\Gamma' \subseteq Cn_{\mathbf{LLL}}(\Gamma)$ then $Cn_{\mathbf{LLL}}(\Gamma') \subseteq Cn_{\mathbf{LLL}}(\Gamma)$.
- *Monotonicity*: $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{LLL}}(\Gamma \cup \Gamma')$.
- *Compactness*: If $A \in Cn_{\mathbf{LLL}}(\Gamma)$ then there is a finite $\Gamma' \subseteq \Gamma$ such that $A \in Cn_{\mathbf{LLL}}(\Gamma')$.

For instance in application (i) lower limit logics are of interest that are inconsistency-tolerant. That is to say, logics that do not validate the ‘ex contradictione quodlibet’ principle:

$$(A \wedge \neg A) \supset B \quad (\text{ECQ})$$

Were we to employ a logic as the lower limit logic that validates (ECQ) then **AL** would trivialize premise sets that contain $A \wedge \neg A$.

Candidates serving as lower limit logic are for instance **CLuN** (see [3]), **CLuNs** (see [4]) or da Costa’s **C_i** systems (see [5, 6]). Note though that not all ALs that model reasoning on the basis of conflicting information are based on subclassical lower limit logics. Indeed, by translating the input Γ for instance to $\Gamma^\diamond = \{\diamond A \mid A \in \Gamma\}$ one can use classical modal logics as lower limits (see e.g., [7, 8]). We offer a more simple non-modal approach with a “dummy operator” that precedes premises in Sect. 2.4 and in [9].¹

For application (ii) systems of interest are deontic logics that are conflict-tolerant, i.e. logics that do not cause deontic explosion given deontic conflicts. Where **OA** indicates the obligation to bring about A , the deontic explosion principle (**D-EX**) is given by

$$(\text{OA} \wedge \text{O}\neg A) \supset \text{OB} \quad (\text{D-EX})$$

Examples of logics that do not validate (**D-EX**) are Lou Goble’s **P** (see e.g. [10–12]) or his **DPM** systems (see e.g. [13–15]).

The lower limit logic constitutes the core of an AL in two senses. Semantically, an AL selects from the **LLL**-models of a given premise set models that are “sufficiently normal” according to a given standard of normality. The latter is characterized by the other two elements of ALs, the abnormalities and the adaptive strategy as will be demonstrated below.

Syntactically, all the rules of the proof theory of **LLL** are applicable. As a consequence, everything that is provable in **LLL** is also provable in the adaptive system. As will be explicated later, ALs enhance the static proof theory of **LLL** by a dynamic element, that in many cases allows for additional consequences.

¹ Note also that all lower limit logics used in applications in parts II–IV of this book are supraclassical.

Where **LLL** is defined over a language \mathcal{L} , we write \mathcal{W} for the set of well-formed formulas in \mathcal{L} . The consequence relation of **LLL** is hence a mapping $\wp(\mathcal{W}) \rightarrow \wp(\mathcal{W})$.

For the adaptive meta-theory it is very useful to extend the language of **LLL** by classical connectives, written in a “checked way”, e.g. $\check{\neg}$ and $\check{\vee}$. We denote the enriched language by \mathcal{L}^+ and the corresponding set of well-formed formulas by \mathcal{W}^+ , where \mathcal{W}^+ is the $\langle \check{\neg}, \check{\vee}, \check{\wedge}, \check{\supset}, \check{\equiv} \rangle$ -closure of \mathcal{W} . Note that this means that none of the “checked connectives” occurs within the scope of the connectives of \mathcal{L} . For instance, where \rightarrow is a connective of \mathcal{L} , $\check{\neg}(A \rightarrow B)$ is a formula in \mathcal{W}^+ , but $(\check{\neg} A) \rightarrow B$ is not.

Let **LLL**⁺ be the logic that is the result of superimposing the classical symbols on **LLL**. Namely, **LLL**⁺ takes over the axiomatization of **LLL** and restricts the rules and axioms of **LLL** to formulas in \mathcal{W} . Moreover, the classical axioms for the checked connectives are defined for all formulas in \mathcal{W}^+ .² Semantically the internal structure of the **LLL**-models may be kept. Similarly as for the axiomatization, the semantic clauses of **LLL** are restricted to formulas of \mathcal{L} , while for the checked symbols we have $M \models \check{\neg} A$ iff $M \not\models A$, $M \models A \check{\vee} B$ iff $M \models A$ or $M \models B$, etc. Thus, it will not be necessary to formally distinguish between **LLL**-models and **LLL**⁺-models.

In the adaptive meta-theory the derivability relation \vdash_{LLL^+} plays an essential role. However, it is customarily denoted by “ \vdash_{LLL} ”. Hence, the reader should not be surprised to find formulas in $\mathcal{W}^+ \setminus \mathcal{W}$ on the left- or right-hand-side of \vdash_{LLL} . In order not to depart too much from the literature on ALs, I will adopt this convention while providing the reader unfamiliar with ALs with this warning.³ Similarly there are two consequence relations corresponding to **LLL** and **LLL**⁺. We define, where $\Gamma \subseteq \mathcal{W}$, $Cn_{\text{LLL}}^{\mathcal{L}}(\Gamma) =_{\text{df}} \{A \in \mathcal{W} \mid \Gamma \vdash_{\text{LLL}} A\}$ and, where $\Gamma \subseteq \mathcal{W}^+$, $Cn_{\text{LLL}}^+(\Gamma) =_{\text{df}} \{A \in \mathcal{W}^+ \mid \Gamma \vdash_{\text{LLL}^+} A\}$. Where I skip the superscript either of the two readings may be applied.

2.3 The Abnormalities

In Sect. 1.1, I have characterized a defeasible inference as an inference that is supported by its premises ‘ceteris normalibus’ (cf. Fig. 1.1). The inference is warranted if and as long as there is no reason to suppose that certain abnormalities that violate the ceteris normalibus condition are the case (cf. Fig. 1.2). ALs formalize this principle.

² Often bridge principles need to be added. E.g., where \vee is a classical disjunction in \mathcal{L} , the axiom $(A \check{\vee} B) \equiv (A \vee B)$ is added to ensure the equivalence between the two classical disjunctions.

³ Note that the “checked” classical connectives are added even in the case that **LLL** already contains classical corresponding symbols. The reason is of a rather technical nature: it is to ensure that a formula is derivable already at a finite stage of an adaptive proof (cf. Section 2.7 and the discussion in Section 4.9.3 of [2]).

Abnormalities are characterized by a logical form F in the enriched language \mathcal{L}^+ . Formulas of this form are supposed to be **LLL**-contingent, i.e. $\not\vdash_{\text{LLL}} F$ and $\not\vdash_{\text{LLL}} \neg F$. By Ω we denote the set of all formulas of the form F .

For our application (i) abnormalities may have the form of inconsistencies, $A \wedge \neg A$. For application (ii) abnormalities may have the form of deontic conflicts, $\bigcirc A \wedge \bigcirc \neg A$.

To interpret the premises “as normally as possible” means to interpret the premises in such a way that as few abnormalities as possible are validated. We will see that semantically ALs select **LLL**-models of a given premise set that are “sufficiently normal” in terms of the abnormalities they validate. Proof-theoretically the idea is to apply certain rules conditionally, namely on the condition that certain abnormalities are false. These points are realized and disambiguated by the last element, the adaptive strategy.

2.4 The Adaptive Strategy

Adaptive strategies are the technically most involving aspect of ALs. Currently two strategies are part of the standard format: the minimal abnormality strategy and the reliability strategy. Together with the abnormalities they substantiate what it means to interpret premises as “normally as possible”.

I will introduce a “toy” application in order to explicate the different intuitions behind the two strategies.

Let us model the defeasible reasoning of a detective. Suppose a murder happened. There are two witnesses. One states that the major suspect Mr. X entered the scene of crime right before the lethal shot was heard throughout the whole neighborhood. Another one states that he saw the major suspect leaving the scene of crime directly after the shot was heard. Moreover, our detective has the information that nobody else was at the scene of crime shortly before and shortly after the time of the killing.

We model the fact that there is evidence available for A by $\bigcirc A$ (e.g., some witness may state A , A may be the result of forensic investigations, etc.). A \bigcirc -less formula A expresses that A is a fact, or that there is definite proof for A , or that our detective accepts A as fact. Since we want to keep things simple we treat \bigcirc as a dummy operator and hence don’t attach any logical properties to \bigcirc . As a lower limit logic we employ classical propositional logic **CL** equipped with \bigcirc . Let this logic be named **CL_o**.⁴ The semantics of **CL_o** is like the semantics for **CL**, just besides the usual assignment function v that assigns to each propositional letter a truth value, we also use an enhanced assignment function v_o that (independently from v) associates each well-formed formula with a truth-value. Truth in a model M is defined as usual for the classical operators:

⁴ In [9] we show that **CL_o** gives rise to very simple ALs that represent the Rescher-Manor consequence relations [16].

- $M \models A$ where A is a propositional letter iff $v(A) = 1$
- $M \models \neg A$ iff $M \not\models A$
- $M \models A \vee B$ iff $M \models A$ or $M \models B$
- and similar for the other classical connectives.

The \circ operator is characterized by

- $M \models \circ A$ iff $v_\circ(A) = 1$.

The idea is that

- (a) if our detective has evidence for A , — $\circ A$;
- (b) and as long as there is no reason to assume that A is not the case, — $\neg \circ \neg A$,

then the detective is warranted to defeasibly infer that A is the case. Of course, \mathbf{CL}_\circ is a monotonic system. We will in a moment strengthen it in a nonmonotonic adaptive way.

But let us return to our detective. Assume he has the following evidence:

- shortly before and shortly after the time of death nobody but the victim was at the scene of crime, — $\circ n$;
- that Mr. X entered the scene of crime alone right before the shot, — $\circ a$;
- that Mr. X left the scene of crime alone right after the shot, — $\circ b$.

Moreover, we presuppose that for some reason our detective accepts that if nobody else was at the scene of crime shortly before and shortly after the crime, but Mr. X entered the scene of crime alone right before the shot was heard, then he must be the murderer: $(a \wedge n) \supset c$. Similarly, $(b \wedge n) \supset c$.

What makes the situation more complicated is that our detective has definite proof that at least one of the witnesses has been bribed by one of Mr. X's enemies in order to fake a witness statement. Hence, since one of the witnesses lies, we have $\neg a \vee \neg b$. What should our detective conclude?⁵

2.4.1 The Reliability Strategy

If she takes a cautious stance, she will not conclude that Mr. X is the murderer since after all, both of the witnesses may be bribed. Let us elaborate a bit on this stance.

I have already mentioned that semantically ALs select from the lower limit logic models of the given premises the ones that are “sufficiently normal” with respect to a certain standard of normality. The latter is characterized by the abnormalities and the adaptive strategy.

The abnormalities for our application are cases where our detective has evidence for A but A is not the case. Hence $\Omega_\circ = \{\circ A \wedge \neg A\}$. Let henceforth \mathbf{CL}_\circ^r be the AL defined by the triple:

⁵ I do of course not claim that the modeling of the defeasible reasoning of our detective by \mathbf{CL}_\circ is by any means optimal. It is however sufficiently intuitive and simple in order to serve as a toy application for introducing the basic concepts and mechanisms of ALs.

1. lower limit logic: \mathbf{CL}_o
2. abnormalities: Ω_o
3. strategy: reliability

2.4.1.1 The Semantics

Let us first take a look at the semantics. What \mathbf{CL}_o -models of the premise set

$$\Gamma_1 = \{on, (a \wedge n) \supset c, (b \wedge n) \supset c, \circ a, \circ b, \neg a \vee \neg b\}$$

should be selected according to the cautious rationale of our detective?

An important notion is the so-called *abnormal part* of a model. It consists of all the abnormalities validated by a given model M , in symbols

$$\text{Ab}(M) = \{A \in \Omega \mid M \models A\}$$

For our applications the abnormal part of an \mathbf{CL}_o -model M is thus, $\text{Ab}(M) = \{A \in \Omega_o \mid M \models A\}$. I will in the remainder of this section abbreviate abnormalities $\circ A \wedge \neg A$ by $!A$. Note that in \mathbf{CL}_o we have the following:

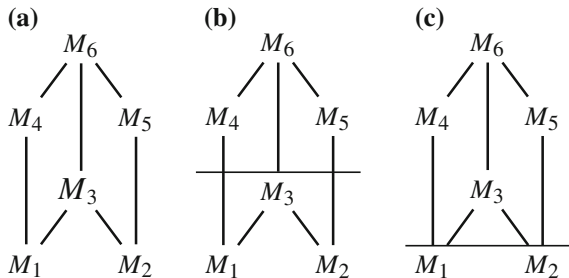
$$\circ A, \circ B, \neg A \vee \neg B \vdash_{\mathbf{CL}_o} !A \checkmark !B$$

Hence, in every \mathbf{CL}_o -model of Γ_1 at least one of the abnormalities $!a$ and $!b$ is valid. Let us focus for our discussion on the following models of Γ_1 :⁶

The abnormal part imposes a strict partial order $\sqsubset_{\text{Ab}}^\Gamma$ on the lower limit logic models of a given premise set Γ where $M \sqsubset_{\text{Ab}}^\Gamma M'$ iff $\text{Ab}(M) \subset \text{Ab}(M')$. Similarly, we define the partial order $\sqsubseteq_{\text{Ab}}^\Gamma$ on the lower limit logic models of Γ by: $M \sqsubseteq_{\text{Ab}}^\Gamma M'$ iff $\text{Ab}(M) \subseteq \text{Ab}(M')$. For our six models this is illustrated in Fig. 2.1a.

Interpreting the premises “as normally as possible” first of all means that in cases in which we have no reason to suppose that an abnormality $!A$ occurs, we should

Fig. 2.1 **a** An excerpt of the partial order $\sqsubset_{\text{Ab}}^\Gamma$ on the \mathbf{CL}_o -models of Γ_1 ; **b** under the line are reliable models; **c** under the line are minimal abnormal models



⁶ I do not exhaustively characterize these models by means of what formulas they validate. However, it is obvious that models such as M_1 to M_6 exist.

presume that $!A$ is not the case. Take for instance our premise on . Since the premises give no reason for supposing $\neg n$ (we will make this formally precise in a moment) the semantic selection corresponding to the reasoning of our detective neglects models M_4 , M_5 and M_6 since these models validate the abnormality $on \wedge \neg n$.

This can be made more precise by introducing another central notion for ALs: *minimal Dab-consequences*. Where $\Delta \subseteq \Omega$ is a finite and non-empty set of abnormalities, adaptive logicians use $\text{Dab}(\Delta)$ as a notation for the classical disjunction of members in Δ : $\bigvee \Delta$. Where $\Delta = \emptyset$ the string ' $\check{\vee} \text{Dab}(\Delta)$ ' denotes the empty string. The minimal Dab-consequences derivable from a given premise set Γ are all $\text{Dab}(\Delta)$ for which (i) $\Gamma \vdash_{\text{LLL}} \text{Dab}(\Delta)$ and (ii) there is no $\Delta' \subset \Delta$ such that $\Gamma \vdash_{\text{LLL}} \text{Dab}(\Delta')$. For a minimal Dab-consequence $\text{Dab}(\Delta)$ we know that in each LLL-model of Γ at least one of the abnormalities in Δ is validated. Due to the minimality of Δ there is no $\Delta' \subset \Delta$ with the same property. Where $\text{Dab}(\Delta_1), \text{Dab}(\Delta_2), \dots$ are the minimal Dab-consequences, the set of *unreliable abnormalities* is $U(\Gamma) = \Delta_1 \cup \Delta_2 \dots$.

Indeed, there is no reason to assume that an abnormality is true in case it is not unreliable. After all, in this case it is not a disjunct of any minimal Dab-consequence. Of course, it may still be a disjunct of a *non-minimal* Dab-consequence. However, just as there is no reason to believe that it rains just because we can derive "It rains or it is windy" from "It is windy", there is no reason to believe that an abnormality is true just because by means of addition we can add it as a disjunct to a Dab-formula.

In our example the only minimal Dab-consequence is $!a \check{\vee} !b$. Hence, $U(\Gamma_1) = \{!a, !b\}$. Note that $!n \notin U(\Gamma_1)$. The idea is to select only lower limit models that validate only abnormalities in $U(\Gamma)$, i.e. models M that satisfy $\text{Ab}(M) \subseteq U(\Gamma)$. We call these models the *reliable models of Γ* . Models M_1 , M_2 and M_3 satisfy this requirement with respect to our premise set Γ_1 (see Fig. 2.1b).

As discussed above, the cautious rationale underlying the reliability strategy also takes into account the possibility that both of our witnesses have been bribed. Hence both abnormalities, $!a$ and $!b$ may be valid. Models M_1 , M_2 and M_3 validate at least one of the two abnormalities. M_3 validates both of them. Note that in the model M_3 , c is not validated. After all, the interpretation offered by M_3 treats both a and b as unreliable and thus in this interpretation neither $(a \wedge n) \supset c$ nor $(b \wedge n) \supset c$ can be used for deriving c . Hence, our cautious detective does not (tentatively) conclude that Mr. X is the murderer.

Generically the semantic consequence relation for the reliability strategy is defined as follows.

Definition 2.4.1. Where $\mathcal{M}_{\text{ALr}}(\Gamma)$ is the set of all reliable LLL-models of Γ ,

$$\Gamma \Vdash_{\text{ALr}} A \text{ iff for all } M \in \mathcal{M}_{\text{ALr}}(\Gamma), M \models A.$$

Note that we have $\Gamma_1 \not\models_{\text{CL}_o} c$ since the reliable model M_3 does not validate c .

Given the definition of reliable models we immediately get the following representational theorem (where $\Gamma \check{\sim}_{\text{df}} \{ \check{\sim} A \mid A \in \Gamma \}$):

Theorem 2.4.1. *Where $\Gamma \subseteq \mathcal{W}^+$: $\Gamma \Vdash_{\mathbf{AL}^r} A$ iff $\Gamma \cup (\Omega \setminus U(\Gamma))^{\checkmark} \Vdash_{\mathbf{LLL}} A$.*

By the compactness of **LLL** this implies:

Corollary 2.4.1. *Where $\Gamma \subseteq \mathcal{W}^+$: $\Gamma \Vdash_{\mathbf{AL}^r} A$ iff there is a $\Delta \subseteq \Omega \setminus U(\Gamma)$ such that $\Gamma \Vdash_{\mathbf{LLL}} A \checkmark \text{Dab}(\Delta)$.*

2.4.1.2 The Proof Theory

Let me now show how the reliability strategy is realized by adaptive proofs. The adaptive proof format enhances the static proofs of the lower limit logic by an additional column in which conditions are attached to proof lines. Conditions are finite and possibly empty sets of abnormalities. A line in a proof consists of a line number, a formula, a justification, and a condition. The central feature of adaptive proofs is that they apply certain rules conditionally. Let me explicate this again by our example.

Note first that in **CL_o** the following rules are *not* valid:

$$\text{If } \circ A, \text{ then } A. \quad (2.1)$$

$$\text{If } \circ A \text{ and } A \supset B, \text{ then } B. \quad (2.2)$$

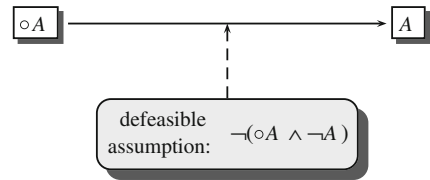
However, the following is valid⁷:

$$\circ A \vdash_{\mathbf{CL}_o} A \checkmark !A \quad (2.3)$$

$$\circ A, A \supset B \vdash_{\mathbf{CL}_o} B \checkmark !A \quad (2.4)$$

Hence, by (2.3), given $\circ A$ either A or the abnormality $!A$ is the case. Our AL enables conditional applications of rules (2.1) and (2.2). That is to say, from $\circ A$, A is derived “on the condition $\{!A\}$ ”, or from $\circ A$ and $A \supset B$, B is derived “on the condition $\{!A\}$ ”. Roughly the idea is to apply rules (2.1) and (2.2) on the condition that $!A$ can be considered not to be the case (see Fig. 2.2). This is still an ambiguous phrase and has different readings according to the two strategies.

Fig. 2.2 Conditional inference



⁷ In order to reduce notational clutter I will often omit set brackets on the left hand side of \vdash .

For the reliability strategy this is spelled out as follows: deriving A “on the condition Δ ” means that A is derived on the condition that no member of Δ is unreliable. Let us have a look at a proof fragment:

1 $\circ n$	PREM	\emptyset
2 $(a \wedge n) \supset c$	PREM	\emptyset
3 $(b \wedge n) \supset c$	PREM	\emptyset
4 $\circ a$	PREM	\emptyset
5 $\circ b$	PREM	\emptyset
6 $\neg a \vee \neg b$	PREM	\emptyset
¹⁰ 7 a	4; RC	$\{!a\}$
8 n	1; RC	$\{!n\}$
¹⁰ 9 c	2, 7, 8; RU	$\{!a, !n\}$
10 $!a \checkmark !b$	4, 5, 6; RU	\emptyset

The first thing to notice is that, although for our applications we are interested in the adaptive consequence relation over the language \mathcal{L} that characterizes our lower limit logic, the adaptive proofs are formulated in the enriched language \mathcal{L}^+ . As the reader will see, this plays an important role in the modeling of defeasible reasoning in adaptive proofs. The proofs are governed by three generic rules: PREM, RU, and RC. Let us have a look at them separately.

At lines 1–6 premises are introduced. This is enabled by a generic premise introduction rule:

$$\text{If } A \in \Gamma : \quad \frac{\vdots \quad \vdots}{A \quad \emptyset} \quad (\text{PREM})$$

Beside the premise introduction rule there are two other generic rules characterizing adaptive proofs: the unconditional rule RU and the conditional rule RC. Via RU the adaptive proofs come with all of the deductive power of the lower limit logic:

$$\text{If } A_1, \dots, A_n \vdash_{\text{LLL}} B : \quad \frac{\begin{array}{cc} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \end{array}}{B \quad \Delta_1 \cup \dots \cup \Delta_n} \quad (\text{RU})$$

Note that the conditions of the used lines are carried forward.

The core and finesse of adaptive proofs comes with the conditional rule. It has been illustrated by means of the rules (2.1) and (2.2) above. In general the rule reads as follows⁸:

⁸ Note that, as already mentioned earlier, I stick with the customary usage of \vdash_{LLL} in RU and RC as denoting the derivability relation \vdash_{LLL^+} characterizing the strengthened lower limit logic that operates on \mathcal{L}^+ .

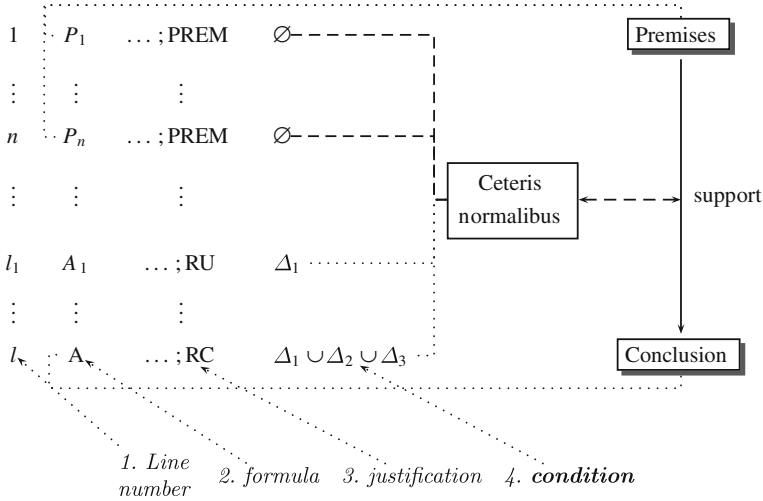


Fig. 2.3 Schematic illustration of an adaptive proof

$$\begin{array}{c}
 \text{If } A_1, \dots, A_n \vdash_{\text{LLL}} B \checkmark \text{Dab}(\Theta) : \\
 \begin{array}{cc}
 A_1 & \Delta_1 \\
 \vdots & \vdots \\
 A_n & \Delta_n \\
 \hline
 B & \Delta_1 \cup \dots \cup \Delta_n \cup \Theta
 \end{array}
 \end{array}
 \quad (\text{RC})$$

At lines 7 and 8 we have conditional applications of rule (2.1). Take for instance line 7: the idea here is to derive defeasibly a from $\circ a$ on the condition $\{\circ a \wedge \neg a\}$. That is to say, from the fact that our detective has a good reason to assume a she derives a on the condition that not- a is not the case. The ceteris normalibus condition of this type of defeasible inference is that whenever there is a good reason to assume some a then, normally, $\neg a$ should not hold. In Fig. 2.3 our generic scheme for defeasible inferencing from Fig. 1.1 is related to the proof format of ALs.

At line 10 in our proof from Γ_1 the only minimal Dab-consequence is derived on the empty condition. At this point something important happens: the conditions of lines 7 and 9 are violated. After all, $\neg a$ turned out to be unreliable at line 10. In adaptive proofs, lines the conditions of which have been violated, are marked. The marking indicates that the second elements of these lines are not considered to be derived. Indeed, as long as the marking persists, the ceteris normalibus condition that guarantees the support from the premises is violated.

Before I give a formal definition of the marking, it is important to note that markings are dynamic. They may come and go. In order to see this, suppose for the moment that our detective has definite proof that the second witness has been bribed and thus has been lying. Where $\Gamma_2 = \Gamma_1 \cup \{\neg b\}$, we add the following lines to the proof from Γ_2 :

$$\begin{array}{ll}
 11 \neg b & \text{PREM } \emptyset \\
 12 \neg b & 5, 11; \text{RU } \emptyset
 \end{array}$$

What is remarkable here is that adding $\neg b$ to our premises leads to an alteration of the unreliable abnormalities. Now $!b$ is the only minimal Dab-consequence and $U(\Gamma_2) = \{!b\}$. Hence, the conditions of lines 7 and 9 can now be considered to be reliable. Consequently, these lines are unmarked at line 12.

At different stages of the proof the ‘minimal Dab-formulas’⁹ that are derivable are different. By analyzing a premise set in a proof, our insight in the premises grows and hence what is considered as an unreliable formula at a certain stage of the proof may change. Hence, in order to define the marking in such a way that it mirrors the dynamics of the defeasible reasoning that is modeled, we need to define the set of unreliable formulas such that it is relative to the stage of the current proof.

We say $\text{Dab}(\Delta)$ is a *minimal Dab-formula at stage s* of a proof iff

- (i) $\text{Dab}(\Delta)$ has been derived on the empty condition at stage s , and
- (ii) for all $\Delta' \subset \Delta$, $\text{Dab}(\Delta')$ has not been derived on the empty condition at stage s .

Moreover, where $\text{Dab}(\Delta_1), \text{Dab}(\Delta_2), \dots$ are the minimal Dab-formulas at stage s , the set of unreliable formulas at stage s is $U_s(\Gamma) = \Delta_1 \cup \Delta_2 \cup \dots$. The marking for the reliability strategy is defined as follows:

Definition 2.4.2 (Marking for the Reliability Strategy). Line i is marked at stage s iff, where Δ is its condition, $\Delta \cap U_s(\Gamma) \neq \emptyset$.

Note that, on the one hand, marked lines may be unmarked at a later stage of a proof. On the other hand, unmarked lines may be marked at a later stage. Suppose our detective has definite proof that also the first witness has been bribed. In this case the conditions of line 7 and 9 are violated again.

13 $\neg a$	PREM \emptyset
14 $!a$	4, 13; RU \emptyset

At this stage of the proof, $U_{14}(\Gamma_3) = \{!a, !b\}$, where $\Gamma_3 = \Gamma_2 \cup \{\neg a\}$. Hence, according to Definition 2.4.2, lines 7 and 9 are marked again at line 14.

Given a marking definition (the one for reliability introduced above or the one for minimal abnormality that is going to be introduced in the next section), the following definitions characterize the notion of derivation in adaptive dynamic proofs. The first definition concerns a dynamic notion of derivation:

Definition 2.4.3. A formula A has been *derived at stage s* of an adaptive proof, iff, at that stage, A is the second element of some unmarked line i .

In order to define a syntactic consequence relation we need a static, non-relative notion of derivability. This is provided by the following definition.

Definition 2.4.4 (Final derivability). A is *finally derived* from Γ on a finite line i of a proof at stage s iff

- (i) A has been derived at stage s at line i ;

⁹ A precise meaning will be given to this notion in a moment.

- (ii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.

This definition can be interpreted in terms of an argumentation game where the proponent has a winning strategy in case her argument is able to withstand criticism (see [17]). Condition (i) says that the proponent is supposed to produce an argument for A by means of deriving it with an assumption that is not violated at some line l (otherwise the corresponding line would be marked). Now the opponent may respond and offer criticism. That is, he may derive Dab-formulas such that the proponent's argument is retracted (i.e., marked). However, our proponent is given the chance to reply: she repels the criticism in case she can further extend the proof such that her assumption is safe again and hence line l is unmarked. In case she is able to repel any possible criticism, she has a winning strategy and A is said to be finally derived.

This account fits in nicely with dialectical accounts of defeasible reasoning. For instance, Blair argued in his [18] that the view that “a valid inference is one whose justifying warrant can withstand criticism” (p. 116) and that “[t]he concepts of defeasibility and presumption are dialectical concepts” (p. 115) is common among many prominent theorists that deal with defeasible arguments such as Toulmin (see [19]), Wellman (see [20]), Rescher (see [21, chapter 3]), Pollock (see [22]), and Walton (see [23]).

Definition 2.4.5. $\Gamma \vdash_{\mathbf{AL}^r} A$ iff A is finally \mathbf{AL}^r -derivable from Γ .

Take for instance line 8 of our proof from Γ_1 . There is no possible extension of the proof from Γ_1 that leads to the marking of this line. Hence, n is finally derivable from Γ_1 . However, there is no way to finally derive a or b from Γ_1 .

Note that for the reliability strategy the extensions referred to in point (ii) of Definition 2.4.4 can be restricted to the finite ones (see e.g. [2]).

The following theorem shows that A is derivable from Γ iff it is derivable on a condition Δ consisting of reliable formulas.

Theorem 2.4.2. *Where $\Gamma \subseteq \mathcal{W}$: $\Gamma \vdash_{\mathbf{AL}^r} A$ iff there is a $\Delta \subseteq \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \text{Dab}(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$.*

I will not provide any meta-proofs for the theorems and lemmas in this chapter for the following two reasons. On the one hand, the meta-theory for the standard format that is presented in this chapter has been proven by Diderik Batens (e.g., in his seminal [1]). On the other hand, most of these results will follow as corollaries of the results presented in Chap. 5: there we introduce a generalization of the standard format and provide all the proofs for the meta-theory.

By making use of some basic properties of **LLL** we can alternatively characterize \mathbf{AL}^r as follows (where $\Gamma \checkmark =_{\text{df}} \{\checkmark A \mid A \in \Gamma\}$)¹⁰:

Corollary 2.4.2. *Where $\Gamma \subseteq \mathcal{W}$: $\Gamma \vdash_{\mathbf{AL}^r} A$ iff $\Gamma \cup (\Omega \setminus U(\Gamma)) \checkmark \vdash_{\mathbf{LLL}} A$.*

Finally, we have the following completeness and soundness result:

Theorem 2.4.3. *Where $\Gamma \subseteq \mathcal{W}$: $\Gamma \vdash_{\mathbf{AL}^r} A$ iff $\Gamma \Vdash_{\mathbf{AL}^r} A$.*

Although the derivability relation $\vdash_{\mathbf{AL}}$ is defined over \mathcal{L}^+ , for applications we are mainly interested in the consequence set restricted to premises and consequences over the language \mathcal{L} that characterizes our lower limit logic. However, for meta-theoretical insights also the enhanced consequence relation is of interest. Hence, we define, where $\Gamma \subseteq \mathcal{W}$, $Cn_{\mathbf{AL}}^{\mathcal{L}}(\Gamma) =_{\text{df}} \{A \in \mathcal{W} \mid \Gamma \vdash_{\mathbf{AL}} A\}$ and, where $\Gamma \subseteq \mathcal{W}^+$, $Cn_{\mathbf{AL}}^{\mathcal{L}^+}(\Gamma) =_{\text{df}} \{A \in \mathcal{W}^+ \mid \Gamma \vdash_{\mathbf{AL}} A\}$. I will also often omit the superscript, namely in cases in which both readings apply.

2.4.2 The Minimal Abnormality Strategy

We proceed analogous to the discussion of the reliability strategy: we first have a look at the semantics and then at the proof theory for the minimal abnormality strategy.

2.4.2.1 The Semantics

The minimal abnormality strategy is ‘bolder’ in comparison to the reliability strategy. Semantically the name is nearly self-explanatory. The *minimally abnormal models* are selected, i.e. the minimal elements of the partial order $\sqsubseteq_{\mathbf{Ab}}^{\Gamma}$. In yet other words, all the **LLL**-models of a given premise set Γ that validate a minimal set of abnormalities. An **LLL**-model of Γ is a minimally abnormal model of Γ iff for all **LLL**-models M' of Γ , $\text{Ab}(M') \not\subseteq \text{Ab}(M)$. Note that $\text{Ab}(M_1), \text{Ab}(M_2) \subset \text{Ab}(M_3)$ (see Fig. 2.1c). Hence, for the minimal abnormality strategy the reliable model M_3 is not selected.

For the minimal abnormality strategy “interpreting the premises as normally as possible” is read in a more rigorous way compared to the reliability strategy. The idea is to select **CL**_o-models that validate as few abnormalities as possible. Given our (only) minimal Dab-consequence of Γ_1 , $!a \checkmark !b$, models are selected that validate only one of the two unreliable abnormalities.

The semantic consequence relation for minimal abnormality is defined as follows.

¹⁰ There is a $\Delta \subseteq \Omega \setminus U(\Gamma)$ for which $\Gamma \vdash_{\mathbf{LLL}} A \checkmark \text{Dab}(\Delta)$ iff [by the deduction theorem] there is a $\Delta \subseteq \Omega \setminus U(\Gamma)$ for which $\Gamma \cup \Delta \checkmark \vdash_{\mathbf{LLL}} A$ iff [by the compactness and monotonicity of **LLL**] $\Gamma \cup (\Omega \setminus U(\Gamma)) \checkmark \vdash_{\mathbf{LLL}} A$.

Definition 2.4.6. Where $\mathcal{M}_{\mathbf{AL}^m}(\Gamma)$ is the set of all minimally abnormal **LLL**-models of Γ ,

$$\Gamma \Vdash_{\mathbf{AL}^m} A \text{ iff for all } M \in \mathcal{M}_{\mathbf{AL}^m}(\Gamma), M \models A.$$

It is important to notice that the existence of minimally abnormal models is guaranteed.

Theorem 2.4.4. $\sqsubset_{\mathbf{Ab}}^\Gamma$ is smooth (alias stoppered).¹¹

Immediate consequences of this are:

Corollary 2.4.3.

- (i) If Γ has **LLL**-models then there are minimally abnormal models of Γ . (Reassurance)
- (ii) For every **LLL**-model M of Γ either M is minimally abnormal or there is an **LLL**-model M' of Γ that is minimally abnormal and for which $\text{Ab}(M') \subset \text{Ab}(M)$. (Strong Reassurance)

Moreover, it can be shown that every minimally abnormal model of Γ is also reliable. That is to say,

Theorem 2.4.5. $\mathcal{M}_{\mathbf{AL}^m}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL}^r}(\Gamma)$.

Hence, points (i) and (ii) in Corollary 2.4.3 also apply to reliable models.

Note that in our example all the minimally abnormal models of Γ_1 either validate $!a$ or $!b$ as the only abnormality. Hence, in all minimally abnormal models c is validated. This demonstrates that the minimal abnormality strategy is ‘bolder’ than the reliability strategy since $\Gamma_1 \Vdash_{\mathbf{CL}_o^m} c$ while $\Gamma_1 \not\Vdash_{\mathbf{CL}_o^r} c$.

Before I introduce the proof theory for minimal abnormality let me draw the reader’s attention to a remarkable fact. Where $\text{Dab}(\Delta_1), \text{Dab}(\Delta_2), \dots$ are the minimal **Dab**-consequences of Γ , let $\Sigma(\Gamma) = \{\Delta_1, \Delta_2, \dots\}$. A *choice set* of $\Sigma(\Gamma)$ is a set that contains a member from each Δ_i . Let $\Phi(\Gamma)$ be set of the minimal choice sets of $\Sigma(\Gamma)$, i.e. all choice sets $\varphi \subseteq \Omega$ of $\Sigma(\Gamma)$ such that there is no choice set $\varphi' \subseteq \Omega$ of $\Sigma(\Gamma)$ for which $\varphi' \subset \varphi$.¹²

The next theorem shows that each minimally abnormal model validates a minimal choice set as its abnormal part and vice versa, for each minimal choice set φ there is a minimally abnormal model that validates φ as its abnormal part.

Theorem 2.4.6. Where $\Gamma \subseteq \mathcal{W}^+$ and $\mathcal{M}_{\mathbf{LLL}}(\Gamma)$ is non-empty.

¹¹ A binary relation $< \subseteq X \times X$ is smooth (resp. stoppered) iff for every $a \in X$, either a is minimal or there is a $<$ -minimal $b \in X$ for which $b < a$. The smoothness property will also play an important role when the standard format is generalized in Chap. 5 where we will—inter alia—prove this statement.

¹² Properties of choice sets that are useful in the context of ALs are inquired in the technical Appendix A.

- (i) $\mathcal{M}_{\mathbf{AL}^m}(\Gamma) = \bigcup_{\varphi \in \Phi(\Gamma)} \{M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) \mid Ab(M) = \varphi\}.$
- (ii) $\varphi \in \Phi(\Gamma)$ iff there is an $M \in \mathcal{M}_{\mathbf{AL}^m}(\Gamma)$ for which $Ab(M) = \varphi.$

This immediately implies a representational theorem:

Theorem 2.4.7. *Where $\Gamma \subseteq \mathcal{W}^+$: $\Gamma \Vdash_{\mathbf{AL}^m} A$ iff for each $\varphi \in \Phi(\Gamma)$, $\Gamma \cup (\Omega \setminus \varphi) \check{\Vdash}_{\mathbf{LLL}} A.$*

By the compactness of **LLL** this implies:

Corollary 2.4.4. *Where $\Gamma \subseteq \mathcal{W}^+$: $\Gamma \Vdash_{\mathbf{AL}^m} A$ iff for each $\varphi \in \Phi(\Gamma)$ there is a $\Delta \subseteq \Omega \setminus \varphi$ for which $\Gamma \Vdash_{\mathbf{LLL}} A \check{\vee} \text{Dab}(\Delta).$*

With the help of the minimally abnormal models we are able to give an alternative definition for the semantic selection for the reliability strategy.

Lemma 2.4.1. *Where \mathcal{M} is a set of **LLL**-models, define*

$$\Psi(\mathcal{M}) = \bigcup \{Ab(M) \mid M \text{ is minimally abnormal in } \mathcal{M}\}.$$

Where $\Gamma \subseteq \mathcal{W}^+$: M is a reliable **LLL**-model of Γ iff $Ab(M) \subseteq \Psi(\mathcal{M}_{\mathbf{LLL}}(\Gamma)).$

This characterization is attractive from a model-theoretic perspective since it is formulated independent of the consequence relation of the **LLL** which was used in the original definition in order to characterize the set $U(\Gamma)$. It is formulated only in terms of properties of the **LLL**-models of Γ , just like the definition of the semantic selection for the minimal abnormality strategy.

2.4.2.2 The Proof Theory

The proof theory for minimal abnormality differs from the one for reliability only with respect to the marking definition. We again employ the generic rules PREM, RU and RC.

As we have seen above, there is a direct link between the minimal choice sets (of $\Sigma(\Gamma)$) and the minimally abnormal interpretations of Γ provided by the minimally abnormal models. Also in the proof theory we will make use of this link. At any stage of the proof we are interested in the question which assumptions can be considered justified and which not. The information that we use in order to judge this is given by the minimal **Dab**-formulas that have been derived so far. While the reliability strategy considered each disjunct of a minimal **Dab**-formula as “unreliable”, the minimal abnormality strategy is less skeptical. Let us first illustrate this by means of a simple example, and then make things more precise by making use of the notion of choice sets.

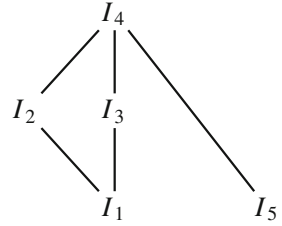
Suppose we have the following excerpt from a proof at some stage s (where we denote abnormalities by preceding them with “!”):

$$\begin{array}{ll} l \ C & \dots \{!A\} \\ l' \ C & \dots \{!B\} \end{array}$$

Table 2.1 Possible interpretations of $\{!A \checkmark !B, !A \checkmark !C\}$

	$!A$	$!B$	$!C$
I_1	1	0	0
I_2	1	1	0
I_3	1	0	1
I_4	1	1	1
I_5	0	1	1

Fig. 2.4 Ordering of the interpretations in Table 2.1 in terms of abnormal parts



$$\begin{array}{ll}
 I'' \quad !A \checkmark !B & \dots \emptyset \\
 I''' \quad !A \checkmark !C & \dots \emptyset
 \end{array}$$

Suppose further that $!A \checkmark !B$ and $!A \checkmark !C$ are the only minimal **Dab**-formulas derived at stage s . The possible interpretations of these formulas are listed in Table 2.1. The corresponding ordering in terms of abnormal parts is illustrated in Fig. 2.4.

We have two minimally abnormal interpretations of these formulas: one I_1 according to which $!A$ is true, another one I_5 according to which $!B$ is true. Let us have a look at the formula C . Since both conditions on which it is derived contain unreliable abnormalities these lines are marked according to the reliability strategy. The situation is different for the minimal abnormality strategy. The reason is that the assumption expressed by the condition $\{!A\}$ is true in I_5 and the assumption expressed by the condition $\{!B\}$ is true in I_1 . In other words, in each minimal abnormal interpretation of our minimal **Dab**-formulas derived so far C is justified.

Now, how does that relate to choice sets? Where $\text{Dab}(\Delta_1), \text{Dab}(\Delta_2), \dots$ are the minimal **Dab**-formulas at stage s of a proof from Γ , the choice sets of $\Sigma_s(\Gamma) = \{\Delta_1, \Delta_2, \dots\}$ give us exactly the possible interpretations of the minimal **Dab**-formulas derived so far. Hence, the minimal of these choice sets exactly correspond to the minimally abnormal interpretations of these minimal **Dab**-formulas.

In view of this, the marking of the minimal abnormality strategy will exactly mirror the idea of the semantics: we only take into account the minimally abnormal interpretations of the given premises—now contextualized to a given stage of the proof—and only claims that are justified in each of these interpretations are taken to be consequences at a given stage of the proof. This is realized by the following marking definition: where $\Phi_s(\Gamma)$ is the set of all minimal choice sets of $\Sigma_s(\Gamma)$ we define

Definition 2.4.7 (Marking for the Minimal Abnormality Strategy). Line i is marked at stage s iff, where A is derived on the condition Δ at line i ,

- (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or
- (ii) for some $\varphi \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.

Another way to interpret the marking definition is in terms of an argumentation game. Suppose the proponent derives a formula A on a line with condition Δ at stage s . Each minimal choice set $\varphi \in \Phi_s(\Gamma)$ represents a minimally abnormal interpretation of the Dab-formulas derived at stage s : each $B \in \varphi$ is true in this interpretation while each $B \in \Omega \setminus \varphi$ is false. Each minimal choice set φ thus represents a potential counter-argument against the defeasible assumption used by our proponent in order to derive A (namely that all members of Δ are false). φ is a counter-argument in case the defeasible assumption, i.e. the condition of line l , contains elements of φ . In this case the assumption of line l is not valid in the interpretation offered by φ .

In case there is no minimally abnormal interpretation φ in which the assumption holds (see point (i)), the proponent cannot defend herself and her inference to A is retracted in terms of being marked. But suppose there is a φ such that $\Delta \cap \varphi = \emptyset$. In this case there is at least one minimally abnormal interpretation in which the assumption of our proponent holds. But what about minimally abnormal interpretations in which the assumption does not hold, i.e. some $\varphi \in \Phi_s(\Gamma)$ for which $\varphi \cap \Delta \neq \emptyset$? In this case the proponent has to offer for each such φ another argument whose assumption is valid in φ (see point (ii)). If she is able to do so, i.e. if she is able to defend herself against all counter-arguments, then her argument is justified and hence line l is not marked at stage s .

In sum: suppose our proponent derived A on the assumption Δ at line l .

- *Is the argument at line l defensible?*

Our proponent should be able to at least pinpoint one minimal abnormal interpretation of the Dab-formulas derived so far in which the assumption Δ holds.

- *Is the claim A justifiable?*

For each counter-argument of our opponent, i.e. each minimally abnormal interpretation I of the Dab-formulas derived so far, she has to have an argument for A with an assumption that is valid in I .

If both questions are answered to the positive, our proponent wins the argumentation game at this stage. Otherwise, the opponent wins and line l is marked.¹³

¹³ The terminological distinction between defensible and justified arguments is borrowed from abstract argumentation. Given a set of abstract entities (arguments) and an attack relation between them, there are various rationales according to which we can select arguments. (These rationales are called *extension types* in Part III.) If an argument is in all selections (that satisfy the criteria imposed by the rationale) it is called *justified*, if it is in some selection it is called *defensible*, if it is in no selection it is called *overruled*. See also the detailed discussion in [24]. The situation is analogous in our case: an argument for the claim A offered at a line l with an assumption expressed by the condition Δ is called justified if the assumption is valid in all minimally abnormal interpretations of the Dab-formulas (at the present stage), it is defensible if the assumption is valid in some minimally

Let us close this discussion by having another look at a proof from Γ_1 , this time applying the marking definition for minimal abnormality.

1	$\circ n$	PREM	\emptyset
2	$(a \wedge n) \supset c$	PREM	\emptyset
3	$(b \wedge n) \supset c$	PREM	\emptyset
4	$\circ a$	PREM	\emptyset
5	$\circ b$	PREM	\emptyset
6	$\neg a \vee \neg b$	PREM	\emptyset
¹⁰ 7	a	4; RC	$\{!a\}$
¹⁰ 8	b	5; RC	$\{!b\}$
9	n	1; RC	$\{!n\}$
10	$!a \check{\vee} !b$	4, 5, 6; RU	\emptyset
11	c	2, 4, 9; RC	$\{!a, !n\}$
12	c	3, 5, 9; RC	$\{!b, !n\}$

Note that lines 11 and 12 are not marked as they would be according to the reliability strategy. For instance the condition of line 11 does (i) not intersect with all minimal choice sets in $\Phi_{12}(\Gamma_1) = \{\{!a\}, \{!b\}\}$ and (ii) it is not the case that there is a minimal choice set $\varphi \in \Phi_{12}(\Gamma_1)$ such that all conditions on which c has been derived intersect with φ . The reason for (ii) is that c is also derived on the condition $\{!b, !n\}$ at line 12. Indeed, c is valid in all minimally abnormal models of Γ_1 .

A different situation occurs with respect to line 7. Its condition, and in fact all conditions on which a can be derived, intersect with the minimal choice set $\{!a\}$. Indeed, in the minimally abnormal model M_1 with abnormal part $\{!a\}$, a is not validated. An analogous argument applies to line 8.

Note that for our example the minimal choice sets $\Phi(\Gamma_1)$ are $\{!a\}$ and $\{!b\}$. Hence c is finally derivable.

The following theorem makes the link between the minimal choice sets and the adaptive consequences.

Theorem 2.4.8. *Where $\Gamma \subseteq \mathcal{W}$: $\Gamma \vdash_{\mathbf{AL}^m} A$ iff for every $\varphi \in \Phi(\Gamma)$ there is a $\Delta \subseteq \Omega$ for which $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \text{Dab}(\Delta)$.*

Finally, we have the following completeness and soundness result:

Theorem 2.4.9. *Where $\Gamma \subseteq \mathcal{W}$: $\Gamma \vdash_{\mathbf{AL}^m} A$ iff $\Gamma \Vdash_{\mathbf{AL}^m} A$.*

(Footnote 13 continued)

abnormal interpretation of the Dab -formulas. The line is marked in case its argument is not justified. In Sect. 2.8 we present an alternative approach where the marking takes place in case an argument is not defensible and relate the two approaches to what is often called the skeptical and the credulous approach to defeasible reasoning.

2.4.3 A Special Case: The Simple Strategy

Sometimes we deal with cases in which both standard strategies, reliability and minimal abnormality, coincide. These are cases in which all minimal Dab-consequences of the lower limit logic **LLL** are abnormalities. That is to say, every minimal Dab-consequence $\text{Dab}(\Delta)$ is such that Δ is a singleton. Let us call a premise set Γ for which all Dab-consequences are abnormalities, a *simple premise set*.

Where Γ is a simple premise set, it is straightforward to check that in this case $\Phi(\Gamma) = \{U(\Gamma)\}$ and, moreover, that in this case both strategies lead to the same consequence set.

Simple premise sets allow for a simplification of the adaptive strategy: the so-called *simple strategy*.

2.4.3.1 The Semantics

Let us first take a look at the semantics. Given a simple premise set Γ it is easy to see that all the minimally abnormal **LLL**-models M of Γ are such that $A \in \text{Ab}(M)$ iff A is verified by every **LLL**-model of Γ . This is equivalent to: $\text{Ab}(M) = \{A \in \Omega \mid \Gamma \vdash_{\text{LLL}} A\}$ resp. $\text{Ab}(M) = \{A \in \Omega \mid \Gamma \Vdash_{\text{LLL}} A\}$ resp. $\text{Ab}(M) = \bigcap_{M' \in \mathcal{M}_{\text{LLL}}(\Gamma)} \text{Ab}(M')$. The same holds for all the reliable **LLL**-models of Γ . This motivates the following definition:

Definition 2.4.8. An **LLL**-model M of Γ is *simple* iff $\text{Ab}(M) = \{A \in \Omega \mid \Gamma \Vdash_{\text{LLL}} A\}$.

Theorem 2.4.10. Where Γ is a simple premise set, the following points are equivalent:

- (i) A is verified by all simple models of Γ
- (ii) A is verified by all reliable models of Γ
- (iii) A is verified by all minimally abnormal models of Γ

Definition 2.4.9. $\Gamma \Vdash_{\text{AL}}^s A$ iff A is verified by all simple models of Γ .

2.4.3.2 The Proof Theory

Derivations are again governed by the generic rules PREM, RU, and RC. What changes and is simplified is the marking definition.

Definition 2.4.10 (Marking for the Simple Strategy). Line i is marked at stage s iff, where Δ is its condition, stage s contains a line on which an $A \in \Delta$ has been derived on the empty condition.

Final derivability is defined as for reliability and minimal abnormality. Hence, $\Gamma \vdash_{\text{AL}^s} A$ iff A is finally derivable from Γ (with respect to the marking for the simple strategy).

Theorem 2.4.11. *Where $\Gamma \subseteq \mathcal{W}$ is a simple premise set, $\Gamma \vdash_{\text{AL}^r} A$ iff $\Gamma \vdash_{\text{AL}^m} A$ iff $\Gamma \vdash_{\text{AL}^s} A$.*

Theorem 2.4.12. *Where $\Gamma \subseteq \mathcal{W}$ is a simple premise set, $\Gamma \Vdash_{\text{AL}^s} A$ iff $\Gamma \vdash_{\text{AL}^s} A$.*

2.5 Modeling Defeasibility in Adaptive Proofs

In this section we enhance our understanding of how ALs model defeasible reasoning. We start off with taking another look at dynamics in Sect. 2.5.1. Then, in Sect. 2.5.2, we compare the derivative power of the two strategies in view of so-called floating conclusions. Finally, in Sect. 2.5.3 we relate ALs to so-called plausible reasoning and a related problem concerning contraposition.

2.5.1 Internal and External Dynamics

As has been demonstrated above, formulas are derived conditionally in adaptive proofs. An unmarked line may be marked at a later stage of the proof and a marked line may be unmarked.¹⁴ This is analogous to the tentative way of arriving at conclusions in defeasible reasoning, where we infer some A from some premises presuming that the circumstances satisfy some *ceteris normalibus* condition in order for the inference to be warranted. In ALs this is made explicit, on the one hand, by specifying what counts as an abnormality and, on the other hand, by specifying the exact nature of the normality condition by the adaptive strategy. In the adaptive proofs formulas are derived on conditions that are sets of abnormalities and the adaptive strategy specifies when the condition is met or violated. The marking definition that is characterized by the adaptive strategy determines when a formula counts as derived and when not.

We have distinguished between two types of dynamics. On the one hand, there is the internal dynamics according to which we may have to retract inferences in view of new insights gained by means of analyzing the given premises. On the other hand, there is the external dynamics according to which we may have to retract inferences in view of new information given by means of new premises.

The internal dynamics is modeled by the marking dynamics of AL proofs. We start off with a specific set of premises and analyze and reason on the basis of them with the help of the three generic rules PREM, RU, and RC. As we have seen, informed

¹⁴ Note that when I speak of lines “being/getting marked” this should in no way be misunderstood as being an activity that is up to a decision by a user of the logic. The marking is characterized by the marking definition in a perfectly deterministic way.

by the minimal **Dab**-formulas derived at a specific stage, some inferences may be retracted by means of marking the corresponding line, while some inferences which were previously marked may be reinstated since the marking is removed. Since the retraction mechanism is fully determined by the analysis of the given premise set this is clearly an instance of the internal dynamics of defeasible reasoning.

As pointed out already, the external dynamics is mirrored by the nonmonotonicity of the consequence relation: sometimes new information may lead to the situation in which some formula that was previously a consequence is not anymore a consequence as soon as the new information is considered. I already discussed that the primary focus in the research on defeasible reasoning is on the external rather than the internal dynamics. ALs are nonmonotonic, so they obviously reflect the external dynamics as well. However, the question arises whether ALs add anything interesting when explicating the external dynamics which distinguishes them from other formal models. Here it is useful to distinguish between two ways in which a formal model **L** exhibits an external dynamics:

1. **L** is nonmonotonic: some previous output may not anymore be output given additional input. Hence, **L** can be said to *be externally dynamic*.
2. **L** *models the rationale* underlying the external dynamics by means of a procedural explication of the reasoning process that causes some previous consequences to cease to be consequences given new input.

My claim is that it is point 2 where ALs offer an essential contribution. Suppose our detective starts reasoning with the premise set $\Gamma_1 = \{\text{on}, (a \wedge n) \supset c, (b \wedge n) \supset c, \text{oa}, \text{ob}\}$. The following proof \mathcal{P}_1 explicates her reasoning on the basis of the reliability strategy and Γ_1 :

1 on	PREM \emptyset
2 $(a \wedge n) \supset c$	PREM \emptyset
3 $(b \wedge n) \supset c$	PREM \emptyset
4 oa	PREM \emptyset
5 ob	PREM \emptyset
6 c	1,2,4; RC $\{!a, !n\}$
7 c	1,3,5; RC $\{!b, !n\}$

Suppose at some point she gets new information which contains the definite proof that one of the witnesses was bribed and thus lied, she just doesn't know which one: $\neg a$ or $\neg b$. Instead of starting her reasoning process again from scratch from the enriched premise set $\Gamma_2 = \Gamma_1 \cup \{\neg a \vee \neg b\}$, she can continue her reasoning process \mathcal{P}_1 as follows in a proof \mathcal{P}_2 from Γ_2 :

\vdots	\vdots
⁹ 6 c	1,2,4; RC $\{!a, !n\}$
⁹ 7 c	1,3,5; RC $\{!b, !n\}$
8 $\neg a \vee \neg b$	PREM \emptyset
9 $!a \vee !b$	4,5,8; RU \emptyset

The new information causes the marking of lines 6 and 7: while c was a consequence from Γ_1 it ceases to be a consequence given the new information $\neg a \vee \neg b$. Reusing and extending the proof \mathcal{P}_1 resulting in \mathcal{P}_2 explicates the reasoning process that leads to the retraction of the previous inferences resulting in c : hence it provides an understanding as to why our detective previously inferred c (given only Γ_1) and then she gave up on it (given Γ_2).

Moreover, ALs are also able to explicate cases of *reinstatements*: i.e., cases in which c is a consequence of Γ_1 , ceases to be a consequence of Γ_2 , and then is a consequence of Γ_3 again (where $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3$). Let us demonstrate this by extending our example further.

Suppose that some informant provides our detective with the information that indeed the second witness has been bribed: $\neg b$. Hence, our premise set is now $\Gamma_3 = \Gamma_2 \cup \{\neg b\}$. Again, our detective can base her reasoning on the previous reasoning process and thus reuse \mathcal{P}_2 and extend it in the following way leading to a proof \mathcal{P}_3 from Γ_3 :

⋮	⋮
6 c	1,2,4; RC $\{!a, !n\}$
¹¹ 7 c	1,3,5; RC $\{!b, !n\}$
8 $\neg a \vee \neg b$	PREM \emptyset
9 $!a \vee !b$	4,5,8; RU \emptyset
10 $\neg b$	PREM \emptyset
11 $!b$	5,10; RU \emptyset

Note that c at line 6 is reinstated in view of the new evidence. The reason is that $!a \vee !b$ is not anymore a minimal **Dab**-formula in view of $!b$ at line 11. Again, looking at the sequence $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ we see a detailed explication of the dynamics of her reasoning process: in \mathcal{P}_1 we see the rationale behind accepting the inference at line 6 as finally derived since the condition was reliable (meaning it only contained reliable abnormalities), in \mathcal{P}_2 the inference was retracted since the condition contained an unreliable abnormality, finally in \mathcal{P}_3 the inference is safe again since the condition is reliable again.

Note that where $\Gamma \subset \Gamma'$: an **AL**-proof from Γ is also an **AL**-proof from Γ' . This is the technical reason why our detective may reuse a previous proof (fragment) from Γ when reasoning on the basis of an enriched premise set Γ' , as happened in the transition from \mathcal{P}_1 to \mathcal{P}_2 and from \mathcal{P}_2 to \mathcal{P}_3 in our example.

Note finally that, according to the given presentation, the way ALs explicate the external dynamics of defeasible reasoning is analogous to the way they explicate the internal dynamics: namely by a retraction mechanism that is implemented by means of (un-)marking lines. The difference is that in the case of the external dynamics we make a transition from a proof \mathcal{P} from Γ to a proof \mathcal{P}' from Γ' by reusing \mathcal{P} , while the internal dynamics occurs in one and the same proof. The analogous treatment is in no way surprising: after all, both dynamics are based on the fact that new insights may cause previous defeasible inferences to be retracted and the only difference concerns the source of the new insights. In the case of internal dynamics it

is based on a better understanding of the given premises, while in the case of external dynamics it is based on new input. In practice both dynamics occur often as part of the same reasoning activities: think for instance of learning processes. Hence, the fact that there is a clear link between the nonmonotonicity of the consequence relation of ALs and the internal dynamics is an argument in favor of the unifying power of ALs as a formal model for defeasible reasoning.

2.5.2 Comparing the Strategies

We have seen that the standard format offers two strategies: the reliability and the minimal abnormality strategy. The latter offers for many examples a ‘bolder type’ of reasoning. That is to say, it offers a consequence relation that, in many examples, gives rise to more consequences compared to the one for reliability. This was illustrated by our example: while the reliability strategy corresponds to a rationale that refrains from drawing the conclusion that Mr. X is the murderer, according to the minimal abnormality strategy our detective concludes that Mr. X is the murderer.

We have distinguished the two strategies by means of their different handling of minimal Dab-consequences. For the reliability strategy it was sufficient that (a part of) the condition of a conditional application of a rule was unreliable, i.e. part of a minimal Dab-consequence, in order to invalidate the application. In contrast, for the bolder minimal abnormality strategy there are cases in which some A is derived on a condition Δ that involves unreliable abnormalities but is nevertheless not marked. Recall that by the minimal abnormality strategy our detective derives that Mr. X is the murderer. We have seen that in each minimally abnormal model she can rely on one of the two witnesses which is due to the fact that $a \vee b$ is valid in all minimally abnormal models. In contrast, the fact that $!a \vee !b$ is a minimal Dab-consequence of Γ_1 makes all the conditions on which c is derived unreliable and hence it is not derivable that Mr. X is the murderer according to the reliability strategy.

Scholars in defeasible reasoning sometimes distinguish between two basic types of conflicts:

1. a conflict between a defeasible inference and a “hard fact” (i.e., a premise) or any formula that can be inferred from the premises by means of non-defeasible rules;
2. a conflict between two defeasible inferences.

The first type of conflict is to be resolved by retracting the defeasible inference. Recall that in our proof from Γ_1 we derived n at line 8 by a defeasible inference on the basis of rule (2.1) on the condition $\{!n\}$:

$$8 \quad n \qquad 1; \text{RC} \quad \{!n\}$$

Now suppose we introduce $\neg n$ as a hard fact by a new premise and let $\Gamma_4 = \Gamma_1 \cup \{\neg n\}$:

$$\begin{array}{ll}
11 \neg n & \text{PREM } \emptyset \\
12 !n & 1, 11; \text{RU } \emptyset
\end{array}$$

In this case line 8 gets marked. It is easy to see that this generalizes for all ALs in standard format. Say A has been derived conditionally at line i and some B has been derived on the empty condition. Suppose moreover that $B \vdash_{\text{LLL}} \checkmark A$. Then line i is marked. This follows directly with the following derivable rule:

$$\frac{\begin{array}{c} A \\ \checkmark A \end{array} \quad \begin{array}{c} \Delta \\ \Delta' \end{array}}{\text{Dab}(\Delta \cup \Delta') \emptyset} \quad (\text{RD})$$

It is easy to see that, where Δ is the condition of a line l , and $\text{Dab}(\Delta)$ is derived on the condition \emptyset then l is marked according to both adaptive strategies.

RD is a consequence of the following lemma:

Lemma 2.5.1 (Conditions Lemma). *An AL-proof from Γ contains a line at which A is derived on the condition Δ iff $\Gamma \vdash_{\text{LLL}} A \checkmark \text{Dab}(\Delta)$.*¹⁵

The lemma gives immediately rise to the following rule:

$$\frac{A \quad \Delta}{A \checkmark \text{Dab}(\Delta) \emptyset} \quad (\text{RA})$$

We now discuss the second conflict type: conflicts between defeasible inferences. Again, a look at the derived rule RD helps us to understand how ALs handle such a conflict. It expresses that whenever we have a conflict between two claims, one derived on the condition Δ on line l and another one derived on the condition Δ' on line l' , then we can derive (unconditionally) that one of the abnormalities in $\Delta \cup \Delta'$ is true. If there are no other minimal disjunctions of abnormalities in the proof and if there are no alternative arguments for our two claims, this means that according to both strategies both lines l and l' are retracted. However, the handling of such conflicts is not fully analogous with respect to the two strategies. This will be demonstrated in the following example.

Suppose a reliable although not infallible witness reports that

- Mr. X wore a long black coat in the bar in which he was seen half an hour before the murder. — $\circ l$

Another reliable although not infallible source however witnesses that

- Mr. X wore a short dark blue jacket and black trousers at the same time. — $\circ j$

Obviously $\neg(l \wedge j)$, since both cannot be the case. Moreover, we have

- If Mr. X was dressed in a long black coat, then he wore dark clothes. — $l \supset m$

¹⁵ This is proven under the same name in [2, Chap. 4].

- If Mr. X was dressed in a short dark blue jacket and black trousers, then he wore dark clothes. — $j \supset m$

Let us have a look at a proof segment with the minimal abnormality strategy from $\Gamma_{fc} = \{\circ l, \circ j, \neg(l \wedge j), l \supset m, j \supset m\}$:

1 $\circ l$	PREM	\emptyset
2 $\circ j$	PREM	\emptyset
3 $\neg(l \wedge j)$	PREM	\emptyset
4 $l \supset m$	PREM	\emptyset
5 $j \supset m$	PREM	\emptyset
¹² 6 l	1; RC	$\{!l\}$
7 m	4, 6; RU	$\{!l\}$
¹² 8 j	2; RC	$\{!j\}$
9 m	5, 8; RU	$\{!j\}$
¹² 10 $l \wedge j$	6, 8; RU	$\{!l, !j\}$
11 $\neg(l \wedge j)$	3; RU	\emptyset
12 $!l \checkmark !j$	10, 11; RD	\emptyset
13 $l \vee j$	6; RU	$\{!l\}$
14 $l \vee j$	8; RU	$\{!j\}$

Note that lines 7, 9, 13 and 14 are marked according to the reliability strategy, however they are unmarked according to the minimal abnormality strategy. Indeed, $l \vee j$ as well as m are finally derivable according to the minimal abnormality strategy. Note that for each choice set $\varphi \in \Phi_{14}(\Gamma_{fc}) = \{\{!l\}, \{!j\}\}$, $l \vee j$ is derived on a condition that has an empty intersection with φ . It is easy to see that there is no extension of the proof in which lines 13 and 14 are marked.

Conclusions such as m are often referred to as *floating conclusions*. Although no sequence of defeasible inferences leading to the conclusion m is valid in every selected model, in each of them at least one of these sequences is such that all the conditions of the rules constituting the sequence are valid. Note that there are two types of minimally abnormal models, one with abnormal part $\{!l\}$ and one with abnormal part $\{!j\}$. In the latter models none of the conditions of the sequence of inferences leading to the derivation of m explicated at lines 1, 4, 6 and 7 are violated. Similarly, in the former type of models none of the conditions of the sequence of inferences leading to the derivation of m explicated at lines 2, 5, 8 and 9 are violated.

In sum, according to the minimal abnormality strategy we get floating conclusions, while reliability blocks them.

2.5.3 Adaptive Logics and Plausible Reasoning

In this section we will demonstrate in which sense ALs model plausible reasoning and discuss a related problem that has to do with contraposition.

2.5.3.1 ALs Model Plausible Reasoning

As has become clear, ALs formally model defeasible reasoning by means of inferences based on assumptions. In the literature we can see two approaches to assumption-based reasoning:

- (a) In the first approach the concrete assumption made in a defeasible inference is left unspecified or implicit. What is used is a defeasible inference rule. One way to realize this is for instance with a connective $A \rightarrow B$ to which a defeasible Modus Ponens rule is applicable so that B is defeasibly derived given A .
- (b) In the other approach the assumptions that are associated with a defeasible inference are made explicit. Often this is expressed in the object language, e.g., $A \wedge \neg ab_1 \supset B$. Given A and $\neg ab_1$ we can apply Modus Ponens to derive B .

Moreover, various scholars (see [25–27]) make a difference between two types of reasoning:

- (1) *Defeasible Reasoning* “as unsound (but still rational) reasoning on a solid basis” [27, p. 262]; and
- (2) *Plausible Reasoning* “as sound (i.e., deductive) reasoning on an uncertain basis” [27, p. 262].

Hereby, (a) is often associated with (1), while (b) is associated with (2). The reason for the latter is that once we have explicit abnormality assumptions we can use the material implication as a conditional and Modus Ponens as an inference rule, whereas the (uncertain) abnormality assumptions are added as additional premises to the premise set. In the former case defeasible rules are applied to the premise set which is taken for granted (i.e., certain).

By now it is obvious that ALs belong to category (b): after all, normality assumptions are made explicit in the fourth column of adaptive proofs. The assumptions are generated by applications of the RC rule and stated in the fourth column of the proof. We have seen that the minimal Dab-consequences together with a rationale provided by the adaptive strategy determine which assumptions are considered safe and which not.

Let us now take a closer look at where ALs fall according to the second distinction.

Recall that the consequence relation of ALs is reflexive and yet (most frequently) nonmonotonic. This seems to indicate that we have a case of (1) where the reflexivity mirrors the “solid basis” and the nonmonotonicity mirrors the “unsound (but still rational)” reasoning.

But we should be more careful with our analysis. After all, the conditional inferences by means of the RC rule can be thought of as having the form of a classical deduction, i.e., of disjunctive syllogism: from $A \checkmark \text{Dab}(\Delta)$ and the assumption $\neg \text{Dab}(\Delta)$ derive by means of disjunctive syllogism A . Under this perspective ALs implement plausible reasoning in the following way. We have two premise sets, Γ and Ω^{\checkmark} .¹⁶ Γ provides a solid basis, while Ω^{\checkmark} is an uncertain basis consisting of normality

¹⁶ Recall that $\Omega^{\checkmark} = \{\neg A \mid A \in \Omega\}$.

assumptions.¹⁷ Whereas PREM only allows for the introduction of premises from the solid base, RC is a way of introducing premises from the uncertain base in such a way that (i) a record is held of the used uncertain premises in the fourth column of the proof, and (ii) the introduced normality assumptions are immediately applied in an instance of disjunctive syllogism (as described above). Viewed in this way, we only have a ‘deductive’ logic in which we formally distinguish between two types of premises. The adaptive marking then handles which parts of the uncertain basis may be considered safe in specific inferences and retracts inferences that are based on unsafe assumptions. Let us demonstrate this with a familiar example. On the left side we have a usual AL proof, on the right side a reconstruction that is more explicitly in the style of plausible reasoning and in which RC is replaced by an argument that makes use of disjunctive syllogism (DS) (where $!A =_{\text{df}} \circ A \wedge \neg A$):

1	$\circ n$	PREM	\emptyset	$\circ n$		PREM1	\emptyset
2	$\circ a$	PREM	\emptyset	$\circ a$		PREM1	\emptyset
3	$\circ b$	PREM	\emptyset	$\circ b$		PREM1	\emptyset
4	$(a \wedge n) \supset c$	PREM	\emptyset	$(a \wedge n) \supset c$		PREM1	\emptyset
5	$(b \wedge n) \supset c$	PREM	\emptyset	$(b \wedge n) \supset c$		PREM1	\emptyset
6*				$n \checkmark !n$		1;RU	\emptyset
6''					$\checkmark !n$	PREM2	$\{\checkmark !n\}$
6	n	1; RC	$\{!n\}$	n		6'',6'',DS	$\{\checkmark !n\}$
7*				$c \checkmark !a \checkmark !n$		1,2,4;RU	\emptyset
$\checkmark 7''$					$\checkmark !a$	PREM2	$\{!a\}$
7	c	2,4,6; RC	$\{!a, !n\}$	c		6'',7'',7'',DS	$\{\checkmark !a, \checkmark !n\}$
8*				$a \checkmark !a$		2; RU	\emptyset
$\checkmark 8''$				a		7'',8'',DS	$\{\checkmark !a\}$
9*	a	2; RC	$\{!a\}$	a		1,3,5; RU	\emptyset
$\checkmark 9''$				$c \checkmark !b \checkmark !n$		PREM2	$\{\checkmark !b\}$
9	c	1,3,5; RC	$\{!b, !n\}$	c		6'',9'',9'',DS	$\{\checkmark !b, \checkmark !n\}$
10	$\neg a \vee \neg b$	PREM	\emptyset	$\neg a \vee \neg b$		PREM1	\emptyset
11	$!a \checkmark !b$	2,3,10;RU	\emptyset	$!a \checkmark !b$		2,3,10;RU	\emptyset

On the right side we use two premise introduction rules: PREM1 for the premises in the solid base Γ and PREM2 for the premises in the uncertain premise set Ω^{\checkmark} . We use an additional “boxed” column to introduce these premises for the sake of transparency. In the last column we keep a record of the used “uncertain” premises. RU is a generic rule for all the non-defeasible (i.e., deductive) inferences that stem from the lower limit logic. DS is disjunctive syllogism (we could have also just written RU since DS is valid in the lower limit logic enriched by the “checked connectives”). The question which parts of the uncertain premise set can be considered safe for a given inference is analogous to the determination of the marking of lines. For instance, according to the minimal abnormality strategy a line l with formula A and a record $\Delta \subset \Omega^{\checkmark}$ is marked at stage s iff, (i) there is no $\varphi \in \Phi_s(\Gamma)$ for which

¹⁷ A similar distinction can be found for instance in the **ASPIC**⁺-framework [27, 28] where we find an ‘ordinary’ knowledge base \mathcal{K}_p that is uncertain and an ‘axiomatic’ solid knowledge base \mathcal{K}_n .

$\varphi \checkmark \cap \Delta = \emptyset$, or (ii) for some $\varphi \in \Phi_s(\Gamma)$ there is no line l' with formula A and a record Θ such that $\varphi \checkmark \cap \Theta = \emptyset$.

Altogether we now have a proof with only deductive inference steps and premise introduction where the marking retracts inferences based on unsafe premises in the uncertain premise set $\Omega \checkmark$. Given this perspective ALs explicate plausible reasoning.

2.5.3.2 A Problem with Contraposition?

Formal models that explicate plausible reasoning have come under some criticism due to the fact that for the deductive rules which are used also their contraposition is available (most recently in Prakken [27] and Caminada in [29]). For instance, Prakken gives the following example (illustrated in Fig. 2.5a):

1. Birds normally fly: $b \wedge \neg ab_1 \supset f$
2. Penguins normally don't fly: $p \wedge \neg ab_2 \supset \neg f$
3. All penguins are birds: $p \supset b$
4. Penguins are abnormal birds with respect to flying: $p \supset ab_1$
5. Tweety is observed as a penguin: o
6. Animals that are observed as penguins are normally penguins: $o \wedge \neg ab_3 \supset p$

Now Prakken observes that we can construct an argument against applying 5 and 6 to infer p by means of applying contraposition to 4 and 6:

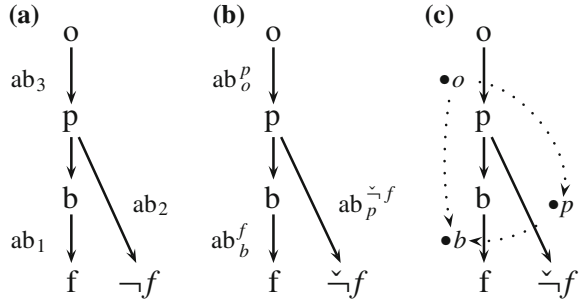
- 4'. $\neg ab_1 \supset \neg p$
- 6'. $o \wedge \neg p \supset ab_3$

Were contraposition not available this move would be blocked. Also Caminada states that given contraposition is available for the defeasible inference rules the principle “to keep the effects of possible conflicts as local as possible” [29, p. 113] (see also Hage [30, p. 109]) is violated. Note that besides the obvious conflict between f and $\neg f$, 4' also introduces a conflict between p and $\neg p$. While Caminada argues that contraposition should only be blocked in what he calls constitutive reasoning while it is “perfectly reasonable” in epistemic reasoning,¹⁸ The example seems to indicate that Prakken would go further. He argues in [27] that contraposition is “a property which is too strong for default statements”.¹⁹

Given the above analysis of ALs as a formal model for plausible reasoning we should expect a similar scenario. And indeed, contraposition is available for conditional inferences in ALs in the following sense. Suppose we can derive B from A defeasibly on the condition Δ by RC. That means: $A \vdash_{\text{LLL}} B \checkmark \text{Dab}(\Delta)$. But

¹⁸ Caminada calls upon the distinction between epistemic and constitutive reasons in Hage [30, p. 60]: “Epistemic reasons are reasons for believing in facts that obtain independent of the reasons that plead for or against believing them. Constitutive reasons, on the contrary, influence the very existence of their conclusions.”

¹⁹ In [27] Prakken also argues against ad hoc solutions such as to strictly prioritize perceptual evidence since “the strength of perceptive inferences is highly context-dependent.” (see his footnote 10) or to model perceptual inferences in a non-defeasible way.

Fig. 2.5 Illustrations of the examples

then we also have $\checkmark B \vdash_{\text{LLL}} \checkmark A \checkmark \text{Dab}(\Delta)$ and we can thus derive $\checkmark A$ from $\checkmark B$ on the same condition Δ .

An obstacle in reconstructing examples as the one above by Prakken in an AL is that there is not one unique way to express it in ALs. Both, defaults and a defeasible Modus Ponens mechanism that models default inferencing, may be represented in various ways: for instance, in Part II we use a conditional that satisfies the so-called KLM properties. Alternatively we could use material implication $A \supset B$ preceded by a dummy operator $\circ(A \supset B)$ and adaptively activate them as much as possible by making use of the abnormality $\circ(A \supset B) \wedge \neg(A \supset B)$.²⁰ Let us thus stay on a more schematic level: suppose we have the following proof fragment from some premise set Γ (illustrated in Fig. 2.5b)

\vdots	\vdots
$l_0 \ p \supset b$	PREM \emptyset
$l_1 \ o$	PREM \emptyset
$l_2 \ p$	$\dots, l_1; \text{RC } \{\text{ab}_o^p\}$
$l_3 \ b$	$l_0, l_2; \text{RU } \{\text{ab}_o^p\}$
$l_4 \ \neg f$	$\dots, l_2; \text{RC } \{\text{ab}_o^p, \text{ab}_p^{\neg f}\}$
$l_5 \ f$	$\dots, l_3; \text{RC } \{\text{ab}_o^p, \text{ab}_b^f\}$

It is easy to see that in view of the lines l_4 and l_5 we can derive

$$l_6 \ \text{ab}_o^p \checkmark \text{ab}_p^{\neg f} \checkmark \text{ab}_b^f \quad l_4, l_5; \text{RD } \emptyset$$

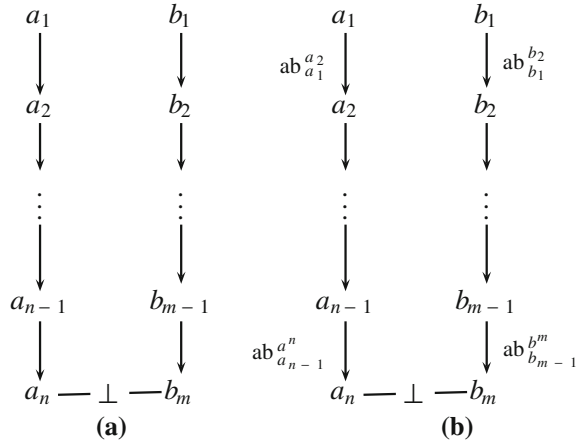
In view of line l_6 all our conditional derivations on lines l_2, \dots, l_5 are marked. Moreover, in view of the proof fragment: $\Gamma \vdash_{\text{LLL}} p \checkmark (\neg f \checkmark \text{ab}_p^{\neg f})$ and $\Gamma \vdash_{\text{LLL}} b \checkmark (f \checkmark \text{ab}_b^f)$. Since $p \supset b \in \Gamma$, $\Gamma \vdash_{\text{LLL}} p \checkmark (f \checkmark \text{ab}_b^f)$. By simple manipulations, $\Gamma \vdash_{\text{LLL}} \checkmark p \checkmark \text{ab}_p^{\neg f} \checkmark \text{ab}_b^f$. Hence, we can produce the line

$$l_7 \ \checkmark p \quad \dots \ \{\text{ab}_p^{\neg f}, \text{ab}_b^f\}$$

In sum, the conflict between f and $\neg f$ is not isolated. Note that in view of line l_6 all the conditional inferences, including line l_2 with p , are marked. This shows that

²⁰ Both, Joke Meheus and Erik Weber independently suggested this in a conversation.

Fig. 2.6 Prakken's problem: abstract representation



the conflict between f and $\neg f$ spreads by effecting the defeasible inferences at lines l_2 and l_3 as well, since the corresponding abnormalities are involved in the minimal Dab-formula. Moreover, other conflicts are derivable such as the one between p and $\sim p$ (however, the corresponding lines are marked).

Let us conclude this discussion with various remarks.

1. In a more abstract phrasing the problem Prakken points out for plausible reasoning (that makes use of rules for which contraposition is available) is as follows. Suppose we have two sequences of rules $a_1 \rightarrow a_2 \rightarrow \dots \rightarrow a_n$ and $b_1 \rightarrow b_2 \rightarrow \dots \rightarrow b_m$ such that (i) $\vdash \neg(a_n \wedge b_m)$ and (ii) we have both a_1 and b_1 (see Fig. 2.6a). Due to the availability of contraposition we can construct an argument against any of the a_i (where $1 < i \leq n$): $b_1 \rightarrow b_2 \rightarrow \dots \rightarrow b_m \rightarrow \neg a_n \rightarrow \neg a_{n-1} \rightarrow \dots \rightarrow \neg a_i$. Hence, instead of the obvious conflict in a_n resp. b_m we suddenly end up with a conflict in each a_i . Since the same holds for all b_i (where $1 < i \leq m$) and since we are interested in a consistent consequence set we cannot—in view of symmetry—conclude any a_i nor any b_i (given a_1 and b_1).

We have shown by means of an example that whenever we have analogous sequences of conditional inferences in ALs (see Fig. 2.6b) we can (i) construct conditional arguments for each $\sim a_i$ and (ii) derive a Dab-formula which contains all the abnormalities in the conditions in the sequences:

$$\text{Dab} \left(\{ \text{ab}_{a_1}^{a_2}, \dots, \text{ab}_{a_{n-1}}^{a_n}, \text{ab}_{b_1}^{b_2}, \dots, \text{ab}_{b_{m-1}}^{b_m} \} \right) \quad (2.5)$$

In case this Dab-formula is minimal and there are no alternative ways to obtain an inference for a_i , the conditional inference for a_i is marked and hence a_i is not a consequence. In this sense, Prakken's scenario is reproduced in ALs.

2. However, in many concrete ALs the problem is nevertheless avoided. Take for instance the ALs for default inferencing in Chap. 6. A default rule is represented by $a \rightsquigarrow b$ where \rightsquigarrow is axiomatized by means of the KLM-properties. Moreover, the logic models a defeasible Modus Ponens as follows: from A and $A \rightsquigarrow B$ infer B unless

we have $\bullet A$. The latter, $\bullet A$, expresses that the given circumstances are unusual for the proposition A which may be witnessed by the truth of some C that is less normal than A (this can be expressed by $(A \vee C) \rightsquigarrow \neg C$ in view of the KLM-properties).²¹

A proof for Prakken's example may look as follows:

1 $o \rightsquigarrow p$	PREM \emptyset
2 $p \supset b$	PREM \emptyset
3 $p \rightsquigarrow \neg f$	PREM \emptyset
4 $b \rightsquigarrow f$	PREM \emptyset
5 o	PREM \emptyset
¹² 6 p	1,5; RC $\{\bullet o\}$
¹² 7 b	2,6; RU $\{\bullet o\}$
¹² 8 $\neg f$	3,6; RC $\{\bullet o, \bullet p\}$
¹² 9 f	4,7; RC $\{\bullet o, \bullet b\}$
10 $\bullet o \checkmark \bullet p \checkmark \bullet b$	8,9; RD \emptyset

So far it seems as if the problem is reproduced since in view of the minimal Dab-formula at line 12 our conditional inferences at lines 6–9 are marked.

The way the problem is avoided in this system is that \bullet is ‘inherited’ along \rightsquigarrow -sequences: if $A \rightsquigarrow B$ and $\bullet A$ then $\bullet B$. Indeed, according to the KLM-properties, if $A \rightsquigarrow B$ then B is at least as normal as A . Hence, if A is excepted (i.e., we have an abnormal situation relative to A) then B is excepted as well (see Fig. 2.5c for an illustration: the dotted line indicates the ‘inheritance’). Thus, the Dab-formula at line 12 is not minimal, but can be shortened to $\bullet p \checkmark \bullet b$ (and if we accept that $p \rightsquigarrow b$ we can further shorten it to $\bullet b$). In any case this will lead to the unmarking of lines 6 and 7 (resp. also to the unmarking of line 8).

We can conclude from this that although—in principle—the fact that contraposition is available for the conditional inferences in adaptive logics can cause the problem pointed out by Prakken, in concrete ALs it may nevertheless be avoided due to specific properties of the lower limit logic that may lead to the shortening of the Dab-formula (2.5).

3. Although we do get the “right” consequences in the example above (such as p), some may still argue that some of the inferences for which the logic allows (irrespective whether the corresponding lines are marked) are based on contraposing default inferencing. E.g., the logic allows for the following inference:

13 $\neg p$	2–4; RU $\{\bullet p, \bullet b\}$
-------------	------------------------------------

Of course, given our discussion above, this line gets marked. However, the mere fact that the logic allows for the inference may for some already be counter-intuitive. In view of this it is a research challenge to see whether the standard format for ALs can be adjusted in a way that allows for defeasible inferences that cannot be contraposed.

²¹ For a detailed technical definition of the system see Chap. 6. An intuitive demonstration is enough for the purpose of the current section.

4. More research needs to be done on the question in which application contexts contraposition is a desired property of defeasible inferences. Caminada did an important step in clarifying this issue in [29]: it seems very plausible that in many contexts of epistemic reasoning contraposition is applicable while in many contexts of constitutive reasoning it is not. Nevertheless, examples such as the one by Prakken discussed above may indicate that the demarcation is not that smooth (see also [31]).

2.6 Some Properties of ALs in Standard Format

One of the merits of the standard format for ALs is that it comes along with many nice properties. For any concrete logic formulated in this format, these properties do not have to be proven since they have been shown generically to hold for any AL in standard format. Let me introduce some of these properties in this section. Later, in Chap. 4, I will point out some more specific properties related to premise sets. Most of the properties that are presented in this section will be shown to hold for a more general setting in Chap. 5. There and in the respective Appendix the reader can find meta-proofs.

2.6.1 Properties of the Adaptive Consequence Relations

The first result shows that the semantic and the syntactic consequence relations define identical relations. Indeed, by Theorem 2.4.3 and Theorem 2.4.9 we have the following soundness and completeness result for both adaptive strategies:

Theorem 2.6.1. (Soundness and Completeness of AL). *Where $\Gamma \subseteq \mathcal{W}$: $\Gamma \vdash_{\text{AL}} A$ iff $\Gamma \Vdash_{\text{AL}} A$.*

Soundness even holds for premise sets with “checked connectives”²²:

Theorem 2.6.2. *Where $\Gamma \subseteq \mathcal{W}^+$: $\Gamma \vdash_{\text{AL}} A$ implies $\Gamma \Vdash_{\text{AL}} A$.*

The completeness doesn’t hold for premise sets with “checked connectives”, as is shown in Sect. 2.7.

By Definition 2.4.4, final derivability concerns finite stages of an adaptive proof. However, it is important to notice that it is essential for the minimal abnormality strategy that the extensions of the proof referred to in Definition 2.4.4ii may be infinite. Indeed, as demonstrated in [1, p. 229], there are premise sets for which it is true that for every way to finally derive some A at some line i there is an extension of the proof that leads to the marking of line i such that only an infinite further extension leads to the unmarking of line i .

²² We prove the corresponding theorem for the generalized standard format in Chap. 5 (see Corollary 5.4.3).

Although final derivability does not ensure the stability of a line i at which some A is finally derived with respect to its marking, for infinite proofs it is guaranteed that there is a stage from which on line i is unmarked and remains so.

Theorem 2.6.3. *Where $\Gamma \subseteq \mathcal{W}$: $\Gamma \vdash_{\mathbf{AL}} A$ iff A is derivable at an unmarked line of an **AL**-proof from Γ that is stable with respect to that line.*²³

The next theorem states certain properties concerning the strength of the adaptive consequence relation. It shows that **AL** is a reflexive and supraclassical (with respect to the enriched language) strengthening of **LLL**. Moreover, the ‘bolder’ minimal abnormality strategy leads indeed always to at least as many consequences (w.r.t. \subset) as the reliability strategy.

Theorem 2.6.4. *Where $\Gamma \subseteq \mathcal{W}^+$:*

- (i) $\Gamma \subseteq \text{Cn}_{\mathbf{AL}}(\Gamma)$ (Reflexivity)
- (ii) $\text{Cn}_{\mathbf{CL}}^{\mathcal{L}^+}(\Gamma) \subseteq \text{Cn}_{\mathbf{AL}}^{\mathcal{L}^+}(\Gamma)$ (Supraclassicality)
- (iii) $\text{Cn}_{\mathbf{LLL}}(\Gamma) \subseteq \text{Cn}_{\mathbf{AL}^r}(\Gamma) \subseteq \text{Cn}_{\mathbf{AL}^m}(\Gamma)$.

The next theorem states some closure properties of the adaptive consequence set. The central result is that the adaptive consequences are a fixed point. If the **AL** is again applied to its own consequence set of some premise set Γ , nothing new will be derived. This is a desirable property. Suppose the idealized case that our detective at the end of the day reached all the final conclusions Γ' based on some premises Γ . It would be rather strange if next morning the same reasoning applied to $\Gamma \cup \Gamma'$ would lead her to new conclusions since she did not gather any new evidence. If the fixed point property would not hold she might never reach a final set of conclusions for her case.

Theorem 2.6.5. *Where $\Gamma \subseteq \mathcal{W}$:*

- (i) $\text{Cn}_{\mathbf{LLL}}(\text{Cn}_{\mathbf{AL}}(\Gamma)) = \text{Cn}_{\mathbf{AL}}(\Gamma)$ (Redundancy of **LLL** with respect to **AL**)
- (ii) $\text{Cn}_{\mathbf{AL}}(\text{Cn}_{\mathbf{LLL}}(\Gamma)) = \text{Cn}_{\mathbf{AL}}(\Gamma)$.
- (iii) $\mathcal{M}_{\mathbf{AL}}(\Gamma) = \mathcal{M}_{\mathbf{AL}}(\text{Cn}_{\mathbf{AL}}(\Gamma))$ and hence $\text{Cn}_{\mathbf{AL}}(\Gamma) = \text{Cn}_{\mathbf{AL}}(\text{Cn}_{\mathbf{AL}}(\Gamma))$.
(Fixed Point/Idempotence)

Beside **LLL** that defines the monotonic core and the lower limit of the adaptive strengthening, there is also an upper limit logic **ULL**. The upper limit logic explicates the standard of normality of an **AL**. An **AL** can be seen as interpreting a premise set in terms of its upper limit logic “as much as possible”. For premise sets that do not give rise to abnormalities, i.e. premise sets Γ for which no Dab-formulas are in the **LLL**-consequence set of Γ , the **AL**-consequences are identical to the **ULL**-consequences. Such premise sets are called *normal*. This can be defined for instance in the following way: Γ is *normal* iff $U(\Gamma) = \emptyset$. Evidently, given a normal premise

²³ A proof from Γ is stable with respect to a line l iff the status of the marking (marked vs. unmarked) of line l remains the same for every possible extension of the proof. This is shown in Appendix B for the more generic setting in which n **AL**s are sequentially combined (see Corollary B.2.3): **AL**s in standard format are a border case in which $n = 1$.

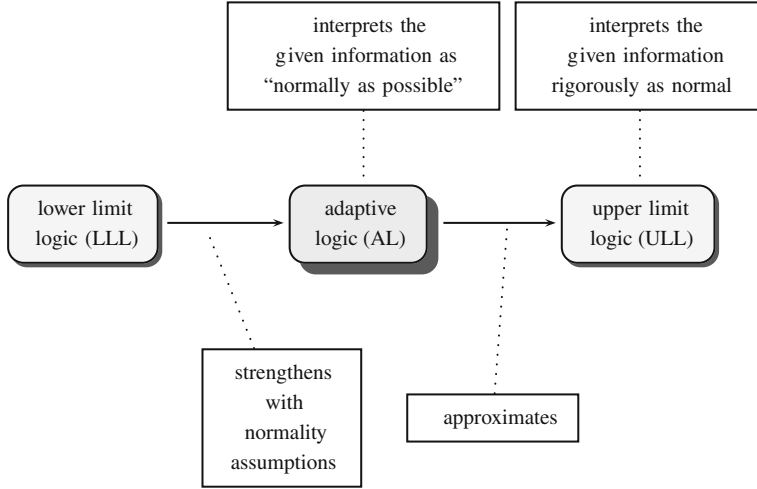


Fig. 2.7 The relationship between **LLL**, **AL**, and **ULL**

set, we expect an AL to realize its standard of normality. The upper limit logic is defined as follows:

Definition 2.6.1. Where $\Omega^{\check{}} =_{\text{df}} \{ \check{A} \mid A \in \Omega \}$, **ULL** is characterized by the following consequence relation:

$$\begin{aligned} Cn_{\text{ULL}}^{\mathcal{L}^+}(\Gamma) &=_{\text{df}} Cn_{\text{LLL}}^{\mathcal{L}^+}(\Gamma \cup \Omega^{\check{}}) \\ Cn_{\text{ULL}}^{\mathcal{L}} &=_{\text{df}} \mathcal{W} \cap Cn_{\text{LLL}}^{\mathcal{L}^+}(\Gamma \cup \Omega^{\check{}}) \end{aligned}$$

Moreover, $\mathcal{M}_{\text{ULL}}(\Gamma) =_{\text{df}} \mathcal{M}_{\text{LLL}}(\Gamma \cup \Omega^{\check{}})$.

The following results show that **ULL** is indeed an upper limit to **AL**.

Theorem 2.6.6. Where $\Gamma \subseteq \mathcal{W}^+$:

- (i) $Cn_{\text{AL}}(\Gamma) \subseteq Cn_{\text{ULL}}(\Gamma)$.
- (ii) $\mathcal{M}_{\text{ULL}}(\Gamma) \subseteq \mathcal{M}_{\text{AL}}(\Gamma)$.
- (iii) If Γ is normal, then $Cn_{\text{AL}}(\Gamma) = Cn_{\text{ULL}}(\Gamma)$ and $\mathcal{M}_{\text{AL}}(\Gamma) = \mathcal{M}_{\text{ULL}}(\Gamma)$.

Altogether we have (see also the illustration in Fig. 2.7),

Corollary 2.6.1. Where $\Gamma \subseteq \mathcal{W}^+$:

- (i) $Cn_{\text{LLL}}(\Gamma) \subseteq Cn_{\text{AL}^r}(\Gamma) \subseteq Cn_{\text{AL}^m}(\Gamma) \subseteq Cn_{\text{ULL}}(\Gamma)$
- (ii) $\mathcal{M}_{\text{ULL}}(\Gamma) \subseteq \mathcal{M}_{\text{AL}^m}(\Gamma) \subseteq \mathcal{M}_{\text{AL}^r}(\Gamma) \subseteq \mathcal{M}_{\text{LLL}}(\Gamma)$

The properties featured in the next theorem are “cautious” weakenings of properties that often characterize monotonic logics.

Theorem 2.6.7. *Where $\Gamma, \Gamma' \subseteq \mathcal{W}$:*

- (i) *If $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ then $Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{AL}}(\Gamma)$. (Cautious Cut / Cumulative Transitivity)*
- (ii) *If $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ then $Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$. (Cautious Monotonicity / Cumulative Monotonicity)*
- (iii) *If $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$, then $Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') = Cn_{\mathbf{AL}}(\Gamma)$. (Cumulative Indifference / Cumulativity)*

Cumulative indifference is a strengthening of the Fixed-Point property since it entails the latter (given the reflexivity of **AL**).²⁴ Moreover, it is a very desirable property itself. Suppose the case on which our detective is working is of a very complicated nature. Let the given evidence be Γ . Suppose further that in the evening she arrives at some—but due to the complicated nature of the case not all—final conclusions Γ' . That is to say, for every $A \in \Gamma'$ she is able to guarantee that no further analysis of Γ will lead to a withdrawal of A . For the adaptive proof this means A is finally derivable. Cumulative indifference guarantees that in the next morning she can reason on, on the basis of $\Gamma \cup \Gamma'$, i.e. she can rely on the insights she won the day before. This has the advantage that, technically speaking, once she established that some A is finally derivable, she doesn't have to keep track of the maybe very complicated conditions that enabled her to arrive at A , but rather she may introduce A as a premise in an adaptive proof from $\Gamma \cup \Gamma'$.

The nonmonotonicity of ALs can easily be demonstrated by the case of our detective that was introduced in Sect. 2.4. Let in the following $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$. Note that

$$\Gamma_1 \cup \{\neg b\} \vdash_{\mathbf{CL}_o^{\mathbf{x}}} c.$$

However, enhancing our premise set by $\{\neg a\}$, we have

$$\Gamma_1 \cup \{\neg b\} \cup \{\neg a\} \not\vdash_{\mathbf{CL}_o^{\mathbf{x}}} c.$$

Also Cut/Transitivity is a property that does not hold for ALs in general. For instance we have $\neg a \vdash_{\mathbf{CL}_o^{\mathbf{x}}} c \vee \neg a$ and $c \vee \neg a \vdash_{\mathbf{CL}_o^{\mathbf{x}}} c$. However, $\neg a \not\vdash_{\mathbf{CL}_o^{\mathbf{x}}} c$.

More generally the following can be proven.

Theorem 2.6.8.²⁵ *Where $\Gamma \subseteq \mathcal{W}$: If $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}}(\Gamma)$, then*

- (i) **AL** *is nonmonotonic, and*
- (ii) **AL** *is non-transitive.*

There are other properties that are often discussed in the context of defeasible reasoning and nonmonotonic logics: Rational Monotonicity and Rational Distributivity. The next section will demonstrate that these properties are not generically validated in ALs. At the same time it will demonstrate that this is rather an advantage of ALs since these properties are not without critical counter-examples.

²⁴ The proof is trivial and left to the reader.

²⁵ This is shown in [2].

2.6.2 Some Remarks on Computational Complexity

In this section I offer only some brief remarks on the computational complexity of ALs rather than providing a detailed survey of the given results. It would take significant space to spell out the technical preliminaries of complexity studies in the realm of the arithmetic hierarchy and hence lead us too far off the main course of the present study. I will instead provide pointers to the relevant literature for the interested reader.

While for most well-known formal accounts for nonmonotonic and defeasible reasoning there are thorough studies investigating complexity-related issues, such studies are sparse for ALs. Only rather recently some key results have been published. There is the critical study by Horsten and Welch [32] which caused two replies by adaptive logicians: [33] and [34]. Horsten and Welsh demonstrate that for some premise sets the consequence relation of the inconsistency-adaptive logic \mathbf{CLuN}^r is Σ_3^0 -hard in the arithmetic hierarchy. They argue that in view of this result the usefulness of ALs as a tool that explicates defeasible reasoning is put into question. In the technical study [33] Verdée proves that the minimal abnormality variant of the same inconsistency-adaptive logic \mathbf{CLuN}^m falls into an even higher complexity class within the analytic hierarchy (he proves Π_1^1 -completeness). Nevertheless, in a reply to the philosophical worries of Horsten and Welsh in [34] Batens et. al argue that such a high complexity class is to be expected from any serious formal attempt to capture the complexity of actual defeasible reasoning. It is not surprising then that many formal systems for defeasible reasoning fall in similar complexity classes (see e.g., [35, 36] for circumscription, [37, 38] for (generalized) closed world assumption).

Recently, Odintsov and Speranski contributed one paper [39] studying the complexity of some inconsistency-adaptive logics where they reaffirm and generalize some of the previous results. Finally, there is a forthcoming study [40] by them where these complexity results for the \mathbf{CLuN} -based ALs are shown to hold generally for ALs in the standard format. For instance, the complexity upper bound Σ_3^0 -for the reliability strategy and Π_1^1 -for the minimal abnormality strategy are generalized for ALs in the standard format. The authors also investigate several interesting special cases (such as the case where $\Phi(\Gamma)$ is finite which is relevant for instance for our study of sequential combinations of ALs in Sect. 3.2.2).

2.6.3 Excursus on the Rational Properties

2.6.3.1 Rational Monotonicity

Besides cautious monotonicity there is another, in comparison stronger, weakening of monotonicity: rational monotonicity.

$$\text{If } A \in Cn_L(\Gamma) \text{ and } A \notin Cn_L(\Gamma \cup \{B\}), \text{ then } \neg B \in Cn_L(\Gamma) \quad (\text{RM})$$

The idea behind Rational Monotonicity is that, if adding B to the premise set Γ leads to nonmonotonicity, then $\neg B$ should be a consequence of Γ .

Rational Monotonicity is not a generic property of the consequence relation of ALs in standard format. Counter-examples are easily found. Rational Monotonicity, although an intuitive property in many cases, has also been criticized. In order to demonstrate the criticism and the fact that ALs do not in general validate Rational Monotonicity we “translate” an example by Stalnaker (see [41]) into the language of \mathbf{CL}_o . Suppose some reliable though not infallible source S_1 tells us that

- Bizet is a French composer, — $\circ f_B$;
- Satie is a French composer, — $\circ f_S$
- Verdi is an Italian composer, — $\circ i_V$.

Another reliable though not infallible source S_2 tells us that

- Verdi and Bizet are compatriots, — $\circ c_{V,B}$.

According to yet another reliable though not infallible source S_3 ,

- Verdi and Satie are compatriots, — $\circ c_{V,S}$.

Obviously the following is valid: $c_{V,B} \supset (\neg i_V \vee \neg f_B)$ and $c_{V,S} \supset (\neg i_V \vee \neg f_S)$. Let our premise set Γ_{RM} comprise sources S_1 and S_2 and thus be

$$\Gamma_{\text{RM}} = \{\circ f_B, \circ f_S, \circ i_V, \circ c_{V,B}, c_{V,B} \supset (\neg i_V \vee \neg f_B), c_{V,S} \supset (\neg i_V \vee \neg f_S)\}.$$

The following proof fragment demonstrates how f_S can be derived from Γ_{RM} .

1 $\circ f_B$	PREM	\emptyset
2 $\circ f_S$	PREM	\emptyset
3 $\circ i_V$	PREM	\emptyset
4 $\circ c_{V,B}$	PREM	\emptyset
5 $c_{V,B} \supset (\neg i_V \vee \neg f_B)$	PREM	\emptyset
6 $c_{V,S} \supset (\neg i_V \vee \neg f_S)$	PREM	\emptyset
7 $c_{V,B} \supset (!i_V \vee !f_B)$	1, 3, 5; RU	\emptyset
8 $!c_{V,B} \checkmark !i_V \checkmark !f_B$	4, 7; RU	\emptyset
9 f_S	2; RC	$\{!f_S\}$

The minimal choice sets for Γ_{RM} are $\Phi(\Gamma_{\text{RM}}) = \{\{!c_{V,B}\}, \{!i_V\}, \{!f_B\}\}$ and the set of unreliable abnormalities is $U(\Gamma_{\text{RM}}) = \{!c_{V,B}, !i_V, !f_B\}$. Hence,

$$\Gamma_{\text{RM}} \vdash_{\mathbf{CL}_o^{\text{m}}} f_S, \text{ and} \quad (2.6)$$

$$\Gamma_{\text{RM}} \vdash_{\mathbf{CL}_o^{\text{r}}} f_S. \quad (2.7)$$

Note further that there is a minimal abnormal \mathbf{CL}_o -model M of Γ_{RM} such that $\text{Ab}(M) = \{!i_V\}$ and $M \models \circ c_{V,S}$. Hence

$$\Gamma_{\text{RM}} \not\vdash_{\text{CL}_o^{\text{m}}} \checkmark \circ c_{V,S} \quad (2.8)$$

$$\Gamma_{\text{RM}} \not\vdash_{\text{CL}_o^{\text{r}}} \checkmark \circ c_{V,S} \quad (2.9)$$

Suppose now we take into account our source S_3 and add $\circ c_{V,S}$ to our premise set Γ_{RM} . In this case we add the following lines to a proof from $\Gamma_{\text{RM}} \cup \{\circ c_{V,S}\}$:

10 $\circ c_{V,S}$	PREM \emptyset
11 $c_{V,S} \supset (!i_V \vee !f_S)$	2, 3, 6; RU \emptyset
12 $!c_{V,S} \checkmark !i_V \checkmark !f_S$	10, 11; RU \emptyset

Note that at line 12 we have the following minimal choice sets of $\Gamma_{\text{RM}} \cup \{\circ c_{V,S}\}$,

$$\Phi_{12}(\Gamma_{\text{RM}} \cup \{\circ c_{V,S}\}) = \{\{!c_{V,B}, !c_{V,S}\}, \{!c_{V,B}, !f_S\}, \{!i_V\}, \{!c_{V,S}, !f_B\}\}.$$

Moreover, the set of unreliable abnormalities is

$$U_{12}(\Gamma_{\text{RM}} \cup \{\circ c_{V,S}\}) = \{!c_{V,B}, !c_{V,S}, !f_S, !i_V, !f_B\}.$$

Hence, at this stage of the proof line 9 is marked according to both strategies. It is easy to see that there is no extension of the proof that leads to the unmarking of line 9. Since there is no other way to derive f_S we have

$$\Gamma_{\text{RM}} \cup \{\circ c_{V,S}\} \not\vdash_{\text{CL}_o^{\text{m}}} f_S, \text{ and} \quad (2.10)$$

$$\Gamma_{\text{RM}} \cup \{\circ c_{V,S}\} \not\vdash_{\text{CL}_o^{\text{r}}} f_S. \quad (2.11)$$

Altogether this shows that Rational Monotonicity is not valid in ALs. For reliability this is demonstrated by (2.7), (2.9) and (2.11), for minimal abnormality strategy it is demonstrated by (2.6), (2.8) and (2.10). As has been argued by Stalnaker, this is also the intuitive behavior.

2.6.3.2 Rational Distributivity

Similarly, ALs do not in general validate Rational Distributivity:

$$\text{If } A \notin \text{Cn}_{\text{L}}(\Gamma \cup \{B\}) \text{ and } A \notin \text{Cn}_{\text{L}}(\Gamma \cup \{C\}), \text{ then } A \notin \text{Cn}_{\text{L}}(\Gamma \cup \{B \checkmark C\}) \quad (\text{RD})$$

Consider the following example. A usually reliable, though not infallible source S_1 tells us that

- Peter had 6 points at the exam,— $\circ p_6$;
- Sue had 5 points at the exam,— $\circ s_5$;
- Anne had 4 points at the exam,— $\circ a_4$.

Another also reliable but not infallible source S_2 informs us that

- Peter was the worst in the exam,— $\circ p_w$.

Now suppose that yet another reliable but not infallible source S_3 states that

- Anne was the best in the exam,— $\circ a_b$.

Obviously we have $p_w \supset (\neg p_6 \vee (\neg s_5 \wedge \neg a_4))$ and $a_b \supset (\neg a_4 \vee (\neg p_6 \wedge \neg s_5))$. Let Γ_{RD} comprise only source S_1 . Hence,

$$\Gamma_{RD} = \{\circ p_6, \circ s_5, \circ a_4, p_w \supset (\neg p_6 \vee (\neg s_5 \wedge \neg a_4)), a_b \supset (\neg a_4 \vee (\neg p_6 \wedge \neg s_5))\}$$

The following proof fragment demonstrates that s_5 is not derivable from $\Gamma_{RD} \cup \{\circ p_w\}$:

1 $\circ p_6$	PREM	\emptyset
2 $\circ s_5$	PREM	\emptyset
3 $\circ a_4$	PREM	\emptyset
4 $\circ p_w$	PREM	\emptyset
5 $p_w \supset (\neg p_6 \vee (\neg s_5 \wedge \neg a_4))$	PREM	\emptyset
6 $!p_w \vee !p_6 \vee (!s_5 \wedge !a_4)$	1, 2, 3, 4, 5; RU	\emptyset
7 $!p_w \checkmark !p_6 \checkmark !s_5$	6; RU	\emptyset
8 $!p_w \checkmark !p_6 \checkmark !a_4$	6; RU	\emptyset
7 s_5	2; RC	$\{!s_5\}$

Note that the minimal choice sets at this stage of the proof are $\Phi_9(\Gamma_{RD} \cup \{\circ p_w\}) = \{\{!p_w\}, \{!p_6\}, \{!s_5, !a_4\}\}$. It is easy to see that $\Phi(\Gamma_{RD} \cup \{\circ p_w\}) = \Phi_9(\Gamma_{RD} \cup \{\circ p_w\})$. Since the only way to derive s_5 is on the condition $\{!s_5\}$, s_5 is not derivable. Thus,

$$\Gamma_{RD} \cup \{\circ p_w\} \not\vdash_{\mathbf{CL}_o^m} s_5 \quad (2.12)$$

Similarly as above, $U(\Gamma_{RD} \cup \{\circ p_w\}) = \{!p_w, !p_6, !a_4, !s_5\}$ and hence

$$\Gamma_{RD} \cup \{\circ p_w\} \not\vdash_{\mathbf{CL}_o^r} s_5 \quad (2.13)$$

Analogously it can be shown that

$$\Gamma_{RD} \cup \{\circ a_b\} \not\vdash_{\mathbf{CL}_o^m} s_5 \quad (2.14)$$

$$\Gamma_{RD} \cup \{\circ a_b\} \not\vdash_{\mathbf{CL}_o^r} s_5 \quad (2.15)$$

The following proof fragment demonstrates that s_5 is derivable from $\Gamma_{RD} \cup \{\circ p_w \checkmark \circ a_b\}$ for both adaptive strategies and hence that Rational Distributivity does not in general hold for ALs.

1 $\circ p_6$	PREM	\emptyset
2 $\circ s_5$	PREM	\emptyset
3 $\circ a_4$	PREM	\emptyset
4 $\circ a_b \checkmark \circ p_w$	PREM	\emptyset
5 $a_b \supset (\neg a_4 \vee (\neg p_6 \wedge \neg s_5))$	PREM	\emptyset
6 $p_w \supset (\neg p_6 \vee (\neg s_5 \wedge \neg a_4))$	PREM	\emptyset

$$\begin{array}{lll}
7 \ !a_b \vee !a_4 \vee (!p_6 \wedge !s_5) \vee !p_w \vee 1, 2, 3, 4, 5, 6; \text{RU } \emptyset & & \\
\ !p_6 \vee (!s_5 \wedge !a_4) & & \\
8 \ !a_b \check{\vee} !a_4 \check{\vee} !p_w \check{\vee} !p_6 & 7; \text{RU} & \emptyset \\
9 \ s_5 & 2; \text{RC} & \{!s_5\}
\end{array}$$

Note that $\Phi_9(\Gamma_{\text{RD}} \cup \{\circ p_w \check{\vee} \circ a_b\}) = \{\{!a_b\}, \{!a_4\}, \{!p_w\}, \{!p_6\}\}$. Again, it is easy to show that $\Phi(\Gamma_{\text{RD}} \cup \{\circ p_w \check{\vee} \circ a_b\}) = \Phi_9(\Gamma_{\text{RD}} \cup \{\circ p_w \check{\vee} \circ a_b\})$. Moreover $U(\Gamma_{\text{RD}} \cup \{\circ p_w \check{\vee} \circ a_b\}) = \{!a_b, !p_w, !a_4, !p_6\}$. Hence, line 9 is finally derived. Thus,

$$\Gamma_{\text{RD}} \cup \{\circ p_w \check{\vee} \circ a_b\} \vdash_{\text{CL}_o^{\text{m}}} s_5 \quad (2.16)$$

$$\Gamma_{\text{RD}} \cup \{\circ p_w \check{\vee} \circ a_b\} \vdash_{\text{CL}_o^{\text{r}}} s_5 \quad (2.17)$$

Note that (2.12), (2.14) and (2.16) show that Rational Distributivity does not hold for CL_o^{m} and hence that it does not in general hold for ALs with minimal abnormality strategy. Moreover, (2.13), (2.15) and (2.17) show that it also does not hold for CL_o^{m} . Hence Rational Distributivity does not in general hold for ALs that employ the reliability strategy. The example shows that in some cases this is as desired. Although Rational Distributivity holds for a great variety of examples, there are some where it fails. In order for ALs to be a generic framework for defeasible reasoning it is desirable that ALs provide means to handle the latter cases in an intuitive way.

The fact that properties such as Rational Monotonicity and Rational Distributivity do not in general hold for ALs does not mean that ALs may not be used in order to characterize reasoning forms that explicate such properties. It only means that the characterization has to be realized under a translation (see Sect. 4.4).

2.7 The Necessity of Superimposing Classical Connectives

The reader may have the impression that, given a supraclassical lower limit logic **LLL**, the superimposing of the classical “checked” connectives is redundant.²⁶ Since all the ALs introduced in the following parts of this book are based on supraclassical lower limit logics it is important to avoid this confusion. For instance, some may think that Dab-formulas $\text{Dab}(\Delta) = A_1 \check{\vee} \dots \check{\vee} A_n$ can be simply expressed by $A_1 \vee \dots \vee A_n$ where \vee is the classical disjunction that is expressible in **LLL** (due to it being supraclassical). This impression may be further strengthened by the fact that in many papers that feature supraclassical lower limit logics checked symbols do not occur.

I was convinced of the redundancy of the checked connectives in cases in which **LLL** is supraclassical until I encountered the following example. It can be presented in a schematic and abstract form. We only need to presuppose that **LLL** is supraclassical and that abnormalities are denoted by $!A$ and formulated in \mathcal{L} . Let $\{!A_1, !A_2, \dots\}$

²⁶ See Sect. 2.2.

be the set of all abnormalities in Ω . Let our premise set be $\Gamma = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = \{!A_i \vee !A_j \mid 1 \leq i < j\}$$

$$\Gamma_2 = \left\{ \bigwedge_{1 \leq i < j \leq n} (!A_i \vee !A_j) \supset (A \vee !A_{n-1}) \mid 1 < n \right\}$$

Note that $\Phi(\Gamma) = \{\varphi_i \mid i \in \mathbb{N}\}$ where $\varphi_i = \Omega \setminus \{!A_i\}$. Moreover $\Gamma \vdash_{\text{LLL}} A \vee !A_i$ for every $i \in \mathbb{N}$. Let M be a minimal abnormal model of Γ . By Theorem 2.4.6, there is a φ_i such that $\text{Ab}(M) = \varphi_i$. Hence $M \models \neg !A_i$. Since $M \models A \vee !A_i$, $M \models A$. Hence $\Gamma \Vdash_{\text{AL}^m} A$.

In the following I will show that if formulas such as $!A_1 \vee !A_2$ are treated as Dab-formulas then the consequence set is not complete with respect to the semantics.

The reader is warned: In the following discussion I will incorrectly(!) treat formulas of the type $!A_1 \vee \dots \vee !A_n$ as Dab-formulas.

The problem is the following: (†) A cannot be produced as the second element of a finite line i such that at some finite stage s line i is unmarked. In other words, at every finite stage s all conditional derivations of A are marked. Definition 2.4.4 requires (a) that A is the second element of a line i and (b) that there is a finite stage s at which line i is unmarked. Hence, A is not finally derived in any AL^m -proof from Γ and thus $\Gamma \not\vdash_{\text{AL}^m} A$. Thus, AL^m is not complete for premise sets that contain checked connectives.

To illustrate (†) let me go a bit through a sample proof from Γ ²⁷:

1	$!A_1 \vee !A_2$	PREM	\emptyset
2	$(!A_1 \vee !A_2) \supset (A \vee !A_1)$	PREM	\emptyset
3	$A \vee !A_1$	1, 2; RU	\emptyset
¹ 4	A	3; RC	$\{!A_1\}$
5	$!A_1 \vee !A_3$	PREM	\emptyset
6	$!A_2 \vee !A_3$	PREM	\emptyset
7	$\bigwedge_{1 \leq i < j \leq 3} (!A_i \vee !A_j) \supset (A \vee !A_2)$	PREM	\emptyset
8	$A \vee !A_2$	1, 5, 6, 7; RU	\emptyset
⁶ 9	A	8; RC	$\{!A_2\}$

Note that at line 4 the minimal choice sets are $\{\{!A_1\}, \{!A_2\}\}$. Since there is a minimal choice set that intersects with all conditions on which A has been derived so far, namely $\{!A_1\}$ line 4 is marked.

²⁷ This example is formulated for the minimal abnormality strategy. A similar example was presented by Frederik Van De Putte in [42].

An analogous argument applies to line 9. The minimal choice sets are now $\{\{!A_1, !A_2\}, \{!A_1, !A_3\}, \{!A_2, !A_3\}\}$. Again, the choice set $\{!A_1, !A_2\}$ intersects with all the conditions of lines at which A has been derived, namely $\{!A_1\}$ and $\{!A_2\}$.

The problem is: it is only possible to derive A on the condition $\{!A_n\}$ at some stage s if all $!A_i \vee !A_j \in \Gamma$ where $1 \leq i, j \leq n+1$ have been introduced in the proof. But then some $\varphi_n \supseteq \{!A_1, \dots, !A_n\}$ is a minimal choice set in $\Phi_s(\Gamma)$ and hence all the conditions of lines at which A has been introduced are marked since they intersect with φ_n .

This shows that at every finite stage of the proof every line that features A as second element is marked. By Definition 2.4.4, A is not finally derived and hence $\Gamma \not\vdash_{\mathbf{AL}^m} A$.

When I confronted Diderik Batens with this “problem” and the proof fragment above, he reminded me of the role of the superimposed “checked” classical connectives.²⁸ Recall that (a) premises are supposed to be formulated in \mathcal{L} , and (b) Dab-formulas $\text{Dab}(\Delta)$ are defined by $\check{\vee} \Delta$. As a consequence, lines 4 and 9 are unmarked in the proof above since for any stage s in the proof fragment above, $\Phi_s(\Gamma) = \{\emptyset\}$. Indeed, A is finally derived at line 4. In order to see this suppose line 4 is marked in an extension of the proof above. We extend the proof further in such a way that all formulas in Γ_1 are derived by PREM and that A is derived on any condition $!A_i$ where $i \in \mathbb{N}$. It is easy to see that (i) there is such an extension, (ii) that line 4 is unmarked at this stage, and (iii) that the marking remains stable from this stage on.

Of course, given a supraclassical **LLL**, whenever $\bigvee \Delta$ is produced at line l on the empty condition in an **AL**-proof from some premise set Γ' then also $\text{Dab}(\Delta)$ is derivable on the empty condition, say on the next line l' . Hence, adaptive logicians often *conventionally* formulate object-level proofs in such a way that the marking is “shortcut”: the marking is as if at line l the Dab-formula $\text{Dab}(\Delta)$ has been derived and the derivation of the actual Dab-formula $\text{Dab}(\Delta)$ is omitted in the presented proof. By treating formulas of the type $\bigvee \Delta$ as Dab-formulas, no $\check{\vee}$ connectives occur in the proofs which simplifies the presentation. In most cases, unlike the example above, this procedure is harmless in the sense that it produces the correct consequences. Obviously such proofs can be translated in a straightforward way into formally correct object-level proofs (by just adding a line l' featuring $\text{Dab}(\Delta)$ on the empty condition, whenever at a line l , $\bigvee \Delta$ has been derived on the empty condition).

I will follow this convention throughout most of the following parts of this book.

Finally, it should be mentioned that **AL** is always sound and complete for any premise set $\Gamma \subseteq \mathcal{W}^+$ if we first close Γ under **LLL**⁺:

Theorem 2.7.1. *Where $\Gamma = \text{Cn}_{\mathbf{LLL}}^+(\Gamma)$: $\Gamma \Vdash_{\mathbf{AL}} A$ iff $\Gamma \vdash_{\mathbf{AL}} A$.*

Proof. “ \Leftarrow ”: this follows by Theorem 2.6.2.

²⁸ Indeed, he had already written a draft for a section for his forthcoming book (see [2, Part 4]) that discusses this problem with a similar example.

“ \Rightarrow ”: Let $\Gamma \Vdash_{\mathbf{AL}^r} A$. Hence, for all $M \in \mathcal{M}_{\mathbf{AL}^r}(\Gamma)$, $M \models A$. Hence, for all $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ for which $\text{Ab}(M) \subseteq U(\Gamma)$, $M \models A$. Thus, $\Gamma \cup (\Omega \setminus U(\Gamma))^{\checkmark} \Vdash_{\mathbf{LLL}} A$. By the compactness of \mathbf{LLL} there is a finite $\Delta \subseteq \Omega \setminus U(\Gamma)$ such that $\Gamma \cup \Delta^{\checkmark} \Vdash_{\mathbf{LLL}} A$. Thus, $\Gamma \Vdash_{\mathbf{LLL}} A \checkmark \text{Dab}(\Delta)$. By the completeness of \mathbf{LLL} , $A \checkmark \text{Dab}(\Delta) \in \text{Cn}_{\mathbf{LLL}}^{\mathcal{L}^+}(\Gamma)$ and thus also $\checkmark A \checkmark \text{Dab}(\Delta) \in \text{Cn}_{\mathbf{LLL}}^{\mathcal{L}^+}(\Gamma)$. We now prove A in an \mathbf{AL}^r -proof from $\Gamma = \text{Cn}_{\mathbf{LLL}}^{\mathcal{L}^+}(\Gamma)$ as follows: We introduce $\checkmark A \checkmark \text{Dab}(\Delta)$ on line 1 by PREM. Then we derive A on the condition Δ by RC on line 2. Since no Dab -formulas are derived at this stage, line 2 is unmarked. Suppose line 2 is marked in an extension of the proof at some stage s . For each minimal Dab -formula Θ at stage s for which $\Delta \cap \Theta \neq \emptyset$ there is a $\Theta' \subset \Theta$ such that $\text{Dab}(\Theta')$ is a minimal Dab -consequence of $\text{Cn}_{\mathbf{LLL}}^{\mathcal{L}^+}(\Gamma)$ and $\Theta' \cap \Delta = \emptyset$. This holds since $\Delta \subseteq \Omega \setminus U(\Gamma) = \Omega \setminus U(\text{Cn}_{\mathbf{LLL}}^{\mathcal{L}^+}(\Gamma))$. We extend the proof by introducing $\text{Dab}(\Theta')$ for all these Θ 's. Let the resulting stage be s' . Obviously, by the construction, $U_{s'}(\Gamma) \cap \Delta = \emptyset$.

The proof for minimal abnormality is similar and left to the reader. \square

By Theorem 2.6.1 and Theorem 2.7.1 we immediately get:

Corollary 2.7.1. *Where $\Gamma \subseteq \mathcal{W}$ or $\Gamma = \text{Cn}_{\mathbf{LLL}}^{\mathcal{L}^+}(\Gamma)$: $\Gamma \Vdash_{\mathbf{AL}} A$ iff $\Gamma \vdash_{\mathbf{AL}} A$.*

2.8 Normal Selections: A ‘Credulous’ Strategy that is not in the Standard Format

The difference between the two standard strategies manifests itself in the fact that one, reliability, models a more ‘cautious’ and the other one, minimal abnormality, a ‘bolder’ style of defeasible reasoning. That is to say, the consequence relation for minimal abnormality is in many cases stronger than the one for reliability. However, there is also a more rigorous way of distinguishing between credulous and skeptical reasoning in the context of logics that model defeasible reasoning which can be found (under different names) in various well-known systems such as default logic, inheritance networks, abstract argumentation, Input/Output logic, the maximal consistent subset approach, etc.

Join Approach A is a *skeptical* consequence of Γ iff A is valid in/implied by/etc. all models/extensions/maximal consistent subsets/etc. of Γ

Meet Approach A is a *credulous* consequence of Γ iff A is valid in/implied by/etc. some interpretation/extension/maximal consistent subset/etc. of Γ .

Obviously, what is modeled by ALs in standard format (such as it is currently defined) is the former, *skeptical* notion. However, there is also an adaptive strategy that is in the spirit of the second, *credulous* notion.

According to the *normal selections strategy* A is a semantic consequence of Γ iff it is valid in a specific set of selected models of Γ . The latter sets are equivalence classes of \mathbf{LLL} -models that have the same abnormal part. Where $M \sim M'$

Fig. 2.8 The quotient structure $\mathcal{M}_{\mathbf{AL}^m}(\Gamma)/\sim$

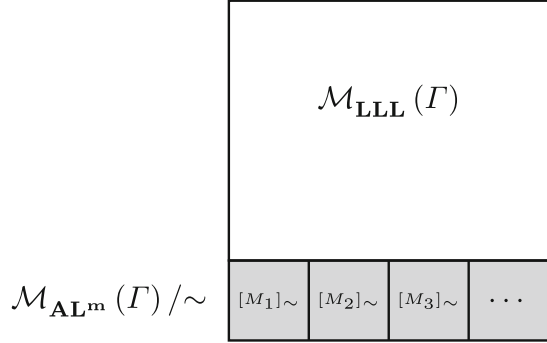


Table 2.2 Equivalence class $[M]_{\sim} \in \mathcal{M}_{\mathbf{AL}^m}(\Gamma)/\sim$ represents a set of models

	M_1	M_2	M_3	M_4	M_5	M_6
$M \models$	$\neg a, b,$ c, n	$a, \neg b,$ c, n	$\neg a, \neg b,$ $\neg c, n$	$\neg a, b,$ $c, \neg n$	$a, \neg b,$ $c, \neg n$	$\neg a, \neg b,$ $\neg c, \neg n$
$\text{Ab}(M) =$	$\{!a\}$	$\{!b\}$	$\{!a, !b\}$	$\{!a, !n\}$	$\{!b, !n\}$	$\{!a, !b, !n\}$

iff $\text{Ab}(M) = \text{Ab}(M')$, $\mathcal{M}_{\mathbf{AL}^m}(\Gamma)/\sim$ is the quotient structure defined by the equivalence relation \sim on the set of all \mathbf{AL}^m -models of Γ ,²⁹ (see Fig. 2.8) and $[M]_{\sim} = \{M' \in \mathcal{M}_{\mathbf{AL}^m}(\Gamma) \mid M \sim M'\}$ we can define

Definition 2.8.1. $\Gamma \Vdash_{\mathbf{AL}^n} A$ iff there is a $[M]_{\sim} \in \mathcal{M}_{\mathbf{AL}^m}(\Gamma)/\sim$ such that for all $M' \in [M]_{\sim}$, $M' \models A$.

Alternatively this can be expressed by: $\Gamma \Vdash_{\mathbf{AL}^n} A$ iff there is a $M \in \mathcal{M}_{\mathbf{AL}^m}(\Gamma)$ such that for all $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ for which $\text{Ab}(M') = \text{Ab}(M)$, $M' \models A$ (Fig. 2.8).

Each equivalence class $[M]_{\sim} \in \mathcal{M}_{\mathbf{AL}^m}(\Gamma)/\sim$ represents a set of models that interpret the premise set Γ “as normally as possible”. In each equivalence class this is realized in a different way. In our example M_1 and M_2 (see Table 2.2) belong to different equivalence classes. For instance a model M_7 for which $\text{Ab}(M_7) = \{!a\}$ belongs to the same equivalence class as M_1 . We have for instance $\Gamma_1 \Vdash_{\mathbf{CL}_o^n} a$ since for all $M \in [M_2]_{\sim}$, $M \models a$.

Each equivalence class offers a specific minimally abnormal interpretation of the given Dab-consequences. If we find one interpretation such that A is validated by all models that share this interpretation, A is considered a consequence. This distinguishes the normal selections strategy from both the reliability and the minimal abnormality strategy where A had to be valid in *all* models that offer sufficiently normal interpretations (so, the reliable resp. the minimally abnormal models). Thus, this makes the normal selections strategy more similar to the ‘meet’-approach that is characteristic for credulous consequence relations, while the strategies of the stan-

²⁹ The fact that \sim is a equivalence relation on $\mathcal{M}_{\mathbf{AL}^m}(\Gamma)$ can be easily shown and is left to the reader.

dard format are more similar to the ‘join’-approach behind skeptical consequence relations. For instance a is valid in all models that have the (minimally) abnormal part $\{!b\}$ and hence it is a consequence according to the normal selections strategy. Note also that $\Gamma \not\models_{\text{CL}_o^m} A$, i.e. a is not a consequence according to the minimal abnormality strategy. For instance the minimally abnormal model M_1 does not verify a .

With Theorem 2.4.6 we immediately get:

Theorem 2.8.1. $\Gamma \Vdash_{\text{AL}^n} A$ iff there is a $\varphi \in \Phi(\Gamma)$ such that for all $M \in \{M' \in \mathcal{M}_{\text{LLL}}(\Gamma) \mid \text{Ab}(M') = \varphi\}, M \models A$.

Similarly, in the proof theory the idea is, that if A is derivable on an assumption that is not violated in some minimal abnormal interpretation of the Dab-consequences then A can be considered a consequence. This is realized by means of the following marking definition:

Definition 2.8.2 (Marking for normal selections, variant 1). A line l with condition Δ is marked at stage s , iff for all $\varphi \in \Phi_s(\Gamma)$, $\Delta \cap \varphi \neq \emptyset$.

In other words, a line with the condition Δ is *unmarked* in case there is a $\varphi \in \Phi(\Gamma)$ such that $\Delta \cap \varphi = \emptyset$. In the terminology of Sect. 2.4.2.2 a line l is unmarked in case the argument at line l is defensible.³⁰

Otherwise the proof theory is the same as in the standard format: we again have the three generic rules PREM, RU, and RC.

The good news is that this marking condition can be simplified in a way that no reference need to be made to minimal choice sets:

Definition 2.8.3 (Marking for Normal Selections, variant 2). Line l is marked at stage s iff, where Δ is the condition of line l , $\text{Dab}(\Delta')$ has been derived on the empty condition at stage s for some $\Delta' \subseteq \Delta$.³¹

In Appendix A we show (merely on the basis of set-theoretic insights into choice sets) that

Corollary 2.8.1. Where $\Delta \subseteq \Omega$ is finite and $\Gamma \subseteq \mathcal{W}^+$:

- (i) there is a $\varphi \in \Phi_s(\Gamma)$ such that $\Delta \cap \varphi = \emptyset$ iff there is no minimal Dab-formula $\text{Dab}(\Theta)$ at stage s such that $\Theta \subseteq \Delta$;
- (ii) there is a $\varphi \in \Phi(\Gamma)$ such that $\Delta \cap \varphi = \emptyset$ iff there is no minimal Dab-consequence $\text{Dab}(\Theta)$ such that $\Theta \subseteq \Delta$.

Note that (i) immediately implies the equivalence of the marking definitions.

³⁰ The distinction between the skeptical and the credulous approach has been discussed in relation to the distinction between justified and defensible arguments in [24, Sect. 4.3].

³¹ Yet another way of phrasing the marking definition in such a way that it leads to the same adaptive consequences is by: Line l with condition Δ is marked at stage s iff $\text{Dab}(\Delta)$ is derived on the empty condition at stage s . Obviously, if we can derive $\text{Dab}(\Delta')$ for some $\Delta' \subseteq \Delta$ at stage s on the empty condition we can also derive $\text{Dab}(\Delta)$ on the empty condition and so eventually mark line l according to the marking definition, variant 2.

Considering the second marking definition it is evident that once a line is marked, it will never be unmarked in a proof. Recall that this is unlike the marking in the standard format where a line may be marked at some point of the proof but get unmarked again at a later stage.

Let us have a simple demonstration by means of our detective case:

1	$\circ n$	PREM	\emptyset
2	$(a \wedge n) \supset c$	PREM	\emptyset
3	$(b \wedge n) \supset c$	PREM	\emptyset
4	$\circ a$	PREM	\emptyset
5	$\circ b$	PREM	\emptyset
6	$\neg a \vee \neg b$	PREM	\emptyset
7	a	4; RC	$\{!a\}$
8	b	5; RC	$\{!b\}$
9	n	1; RC	$\{!n\}$
10	$!a \checkmark !b$	4, 5, 6; RU	\emptyset
11	c	2, 4, 9; RC	$\{!a, !n\}$
¹⁰ 12	$a \wedge b$	7, 8; RU	$\{!a, !b\}$
13	$\neg b$	6, 7; RU	$\{!a\}$
¹⁰ 14	$b \wedge \neg b$	8, 13; RU	$\{!a, !b\}$

The first difference to the strategies of the standard format concerns lines 7 and 8: both are marked according to reliability and minimal abnormality but not according to normal selections. Similar as in the standard format lines 12 and 14 get marked: after all, the disjunction of the members of the condition of these lines has been derived at line 10 (cf. marking variant 2). What is most remarkable is that by means of normal selections we can derive both b (line 8) and $\neg b$ (line 13): for each respective condition Δ there is a minimal choice set (note that $\Phi_{14}(\Gamma) = \{\{!a\}, \{!b\}\}$) that has an empty intersection with Δ . However, line 14 with the formula $b \wedge \neg b$ gets marked. Obviously, there is no minimally abnormal interpretation which validates both abnormalities in the conditions: $!a$ and $!b$.

Final derivability is defined as usual (see Definition 2.4.4). Hence, we define $\Gamma \vdash_{\mathbf{AL}^n} A$ iff A is finally derivable in a \mathbf{AL}^n proof from Γ .

Given the equivalence of our two marking definitions, it is not surprising that we get two corresponding representational theorems for the syntactic consequence relation.

Theorem 2.8.2. *Where $\Gamma \subseteq \mathcal{W}$ or $\Gamma = \text{Cn}_{\mathbf{LLL}}^{\mathcal{L}^+}(\Gamma)$: $\Gamma \vdash_{\mathbf{AL}^n} A$ iff there is a $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \checkmark \text{Dab}(\Delta)$ and $\Gamma \not\vdash_{\mathbf{LLL}} \text{Dab}(\Delta)$.*

Theorem 2.8.3. *Where $\Gamma \subseteq \mathcal{W}$ or $\Gamma = \text{Cn}_{\mathbf{LLL}}^{\mathcal{L}^+}(\Gamma)$: $\Gamma \vdash_{\mathbf{AL}^n} A$ iff there is a $\Delta \subseteq \Omega$ such that (a) $\Gamma \vdash_{\mathbf{LLL}} A \checkmark \text{Dab}(\Delta)$, and (b) for some $\varphi \in \Phi(\Gamma)$, $\varphi \cap \Delta = \emptyset$.*

Finally, this gives us soundness and completeness³²:

³² The proof is straight-forward in view of Theorem 2.8.1 and Theorem 2.8.3. In Chap. 5 we prove a generalized version of Theorem 2.8.4.

Theorem 2.8.4. *Where $\Gamma \subseteq \mathcal{W}$ or $\Gamma = \text{Cn}_{\text{LLL}}^{\mathcal{L}^+}(\Gamma)$: $\Gamma \vdash_{\text{AL}^n} A$ iff $\Gamma \Vdash_{\text{AL}^n} A$.*

As a concluding remark it should be mentioned that the normal selections strategy can be represented by means of the simple strategy under a translation. This will be demonstrated in a future paper together with Joke Meheus.

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Adaptive Logics for Defeasible Reasoning
Applications in Argumentation, Normative Reasoning
and Default Reasoning

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2014, XVIII, 438 p. 34 illus., Hardcover

ISBN: 978-3-319-00791-5