Chapter 2
Plane Poiseuille Flow of Incompressible Bipolar Viscous Fluids

2.1 Introduction

In Sect. 1.4 we introduced the model of an incompressible, nonlinear, bipolar viscous fluid; this model, which is consistent with the basic principles of continuum mechanics and thermodynamics, as delineated in Sect. 1.4, is based on the following constitutive hypotheses for the Cauchy stress tensor τ_{ij} and the first multipolar stress tensor τ_{ijk}:

$$\tau_{ij} = -p\delta_{ij} + 2\mu_0(\epsilon + e_{ij}e_{ij})^{-\alpha/2} - 2\mu_1\Delta e_{ij} \tag{2.1a}$$

and

$$\tau_{ijk} = 2\mu_1\frac{\partial e_{ij}}{\partial x_k} \tag{2.1b}$$

where p is the pressure, $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ is the rate of deformation tensor, $\boldsymbol{v}$ is the fluid velocity field, and μ_0, μ_1, ϵ, and α are the constitutive constants, the first three of which are positive while α, in this chapter, will be assumed to satisfy $0 < \alpha < 1$. In a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with smooth boundary $\partial\Omega$, the constitutive hypotheses (2.1a,b) yield (see Sect. 1.4) the following initial-boundary value problem (take the density $\rho \equiv 1$):

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + 2\nabla\cdot(\mu(|\boldsymbol{e}|)\boldsymbol{e})\\ - 2\mu_1\nabla\cdot(\Delta\boldsymbol{e}) + \boldsymbol{f},\end{aligned} \tag{2.2a}$$

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$$\nabla \cdot \boldsymbol{v} = 0, \tag{2.2b}$$

$$\boldsymbol{v} = \boldsymbol{0}, \quad \tau_{ijk}\nu_j\nu_k - \tau_{jkl}\nu_j\nu_k\nu_l\nu_i = 0, \tag{2.2c}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}). \tag{2.2d}$$

The system of partial differential equations (2.2a), (2.2b) holds in $\Omega \times [0, T)$, $T > 0$, while the initial condition (2.2d) is assumed to hold in Ω, at $t = 0$, and the boundary data (2.2c) is specified for $(\boldsymbol{x}, t) \in \partial\Omega \times [0, T)$, with $\boldsymbol{\nu}$ the exterior unit normal to $\partial\Omega$ at time t. In (2.2a), $\boldsymbol{f}$ specifies the external body force/volume in Ω while

$$\mu(|\boldsymbol{e}|) = 2\mu_0(\epsilon + |\boldsymbol{e}|^2)^{-\alpha/2} \tag{2.3}$$

is the nonlinear viscosity function. For $\alpha = 0$, $\mu_1 = 0$, and in the absence of the second set of boundary conditions in (2.2c), the system (2.2a)–(2.2d) reduces to the specification of the standard initial-boundary value problem for the (incompressible) Navier–Stokes equations.

In Sect. 1.5.2 we considered, for the bipolar model (2.2a)–(2.2d), the most standard of all classical problems in fluid dynamics, namely, the problem of Poiseuille flow between parallel plates whose location, in the Cartesian coordinate system (x_1, x_2, x_3), is at $x_2 = \pm a$, with $a > 0$. As is customary in considering plane Poiseuille flow between parallel plates we assumed, in Sect. 1.6, a velocity field (for steady flow) of the form

$$v_1 = v_1(x_2), \quad v_2 = 0, \quad v_3 = 0, \tag{2.4}$$

In this case, with $\boldsymbol{f} = \boldsymbol{0}$, the steady plane Poiseuille flow of an incompressible, bipolar, viscous fluid satisfies (see Sect. 1.5.2) the following boundary-value problem, where $p_1 = \dfrac{\partial p}{\partial x_1}$ is a constant:

$$\mu_0\left[\left(\epsilon + \frac{1}{2}v_1'^2(x_2)\right)^{-\alpha/2} v_1'(x_2)\right]' - \mu_1 v_1''''(x_2) = p_1 \tag{2.5a}$$

$$v_1(\pm a) = 0, \quad v_1''(\pm a) = 0. \tag{2.5b}$$

Explicit solutions for the non-Newtonian problem derived from (2.5a,b) by setting $\epsilon = 0$, $\mu_1 = 0$, and deleting the second set of boundary conditions in (2.5b), were obtained in (1.5.2) and compared with the standard solution obtained for the Navier–Stokes model (i.e., with the case $\alpha = 0$).

In this chapter we will treat, in depth, the behavior of both steady and time-dependent plane Poiseuille flow solutions for the incompressible, bipolar, fluid flow model. We begin in Sect. 2.2 by considering the problems of existence, uniqueness, and continuous dependence for the generalization of the nonlinear boundary-value problem (2.5a,b) in which the constant p_1 is replaced by a function $f \in L^2(-a, a)$.

In (2.5a), $\mu_1 > 0$ and we are interested in the behavior of solutions of (2.5a,b), not only as $\epsilon \to 0^+$, but also as $\mu_1 \to 0^+$; any continuous dependence result, in this case, can not hold in the C^2 sense (as boundary-layer theory comes into effect at that level of smoothness) but will be shown to hold in the norm of $C^{1+\delta}$ for $0 < \delta < \frac{1}{2}$. More explicit results for the boundary-value problem are delineated in Sect. 2.3. Suppose we display the dependence of the solution of (2.5a,b) on ϵ and μ_1, for a fixed $\alpha \in (0, 1)$, by writing $v_1 = u(x_2; \epsilon, \mu_1)$ and, for the same fixed α, write $u(x_2; 0, 0) = u_0(x_2)$; then, it will be shown in Sect. 2.3 that with $y = x_2$,

$$u(y; \epsilon, \mu_1) > 0, \quad -a \leq y < 0; \qquad u'(y; \epsilon, \mu_1) < 0, \quad 0 < y \leq a \tag{2.6a}$$

with $u''(y; \epsilon, \mu_1) \leq 0, -a < y < a$, for all $\epsilon, \mu_1 \geq 0$, Also

$$u'''(-a; \epsilon, \mu_1) < 0, \tag{2.6b}$$

$$u'(-a; \epsilon, \mu_1) = -u'(a; \epsilon, \mu_1) \tag{2.6c}$$

for all $\epsilon, \mu_1 \geq 0$ and

$$\left|u'(y; \epsilon, 0) - u_0'(y)\right| < \left(1 + \frac{1}{\sqrt{1-\alpha}}\right)\sqrt{\epsilon} \tag{2.6d}$$

for $y \in [-a, a]$, $\epsilon > 0$. It is also proven in 2.3 that $\exists C_+$, C_1, C_2, positive and independent of both ϵ and μ_1, such that

$$\begin{aligned} -\left(1 + \frac{1}{\sqrt{1-\alpha}}\right) a\sqrt{\epsilon} - \frac{\sqrt{aC_2}}{1-\alpha}\left(\sqrt{\epsilon} + C_1\right)^{\alpha} \mu_1^{*1/2} &\leq u(y; \epsilon, \mu_1) - u_0(y) \\ &\leq \left(1 + \frac{1}{\sqrt{1-\alpha}}\right) a\sqrt{\epsilon} + \frac{aC_+}{1-\alpha}\left(\sqrt{\epsilon} + C_1\right)^{\alpha} \mu_1^* \end{aligned} \tag{2.7}$$

with $\mu_1^* = \mu_1/\mu_0$. In Sect. 2.4 we reconsider the problem of uniqueness in relation to the nonlinear boundary-value problem (2.5a,b). If we denote the region between the parallel plates at $y = \pm a$ by

$$\Omega_a = \{(x, y, z) \mid y \in [-a, a],\ a > 0,\ -\infty < x, z < \infty\} \tag{2.8}$$

and the uniquely determined vector field corresponding to the solution of (2.5a,b) by

$$\boldsymbol{v}^P = (u(y; \epsilon, \mu_1), 0, 0) \tag{2.9}$$

where we again set $v_1 = u$, $x_2 = y$, then it will be proven in Sect. 2.4 that for μ_1 sufficiently large $\boldsymbol{v}^P$ is, in fact, the unique equilibrium solution $\boldsymbol{v}$ of (2.2a)–(2.2d) in Ω_a which satisfies $\boldsymbol{v} - \boldsymbol{v}^P \in \boldsymbol{H}^4(\Omega_a)$. The existence of other equilibrium solutions of (2.2a)–(2.2d) in Ω_a, beyond the Poiseuille vector field (2.9), when μ_1 is not sufficiently large, is an open problem. In Sect. 2.5 we take up the problem of existence and asymptotic stability of time-dependent Poiseuille flow in the domain Ω_a; specifically we ask whether or not there exists, globally in time, smooth Poiseuille solutions of (2.2a)–(2.2d) in $\Omega_a \times [0, T)$, $T > 0$, of the form

$$\boldsymbol{v}^P(\boldsymbol{x}, t) = (v(y, t; \epsilon, \mu_1), 0, 0). \tag{2.10}$$

We show, in Sect. 2.5 that there exists a unique weak solution to the corresponding initial-boundary value problem which is of class $C^{4,1}(y, t)$ on $(-a, a) \times [0, T)$, for any $T > 0$, in which case the weak solution is actually a classical solution of the problem. Finally, it is also demonstrated in Sect. 2.5 that the unique steady Poiseuille flow solution (2.9) is linearly asymptotically stable, as well as asymptotically stable, within the class of all flows in $\Omega_a \times [0, T)$, $T > 0$, of the Poiseuille type (2.10).

2.2 Existence, Uniqueness, and Continuous Dependence for Steady Poiseuille Flow

We consider, in this section, a slight generalization of the boundary-value problem (2.5a,b) associated with the steady flow of an incompressible, bipolar, viscous fluid in a parallel-wall channel, namely,

$$-\left[\frac{u'(y)}{(\epsilon + u'^2(y))^{\alpha/2}}\right]' + \mu_1 u''''(y) = f(y), \quad -a < y < a, \tag{2.11a}$$

$$u(\pm a) = u''(\pm a) = 0 \tag{2.11b}$$

where we have written $y = x_2$, $u = v_1$, and where $f \in L^2(-a, a)$. Our first basic result is the following existence and uniqueness theorem:

Theorem 2.1. *Let* $V = H_0^{\frac{3}{2}+\delta}(-a, a)$, *with* $0 < \delta < \dfrac{1}{2}$, *let* $B_M(0)$ *be the ball of radius* $M > 0$ *in* V, *and set*

$$W_M = B_M(0) \cap H^2(-a, a). \tag{2.12}$$

Then, for M *sufficiently large, there exists a unique solution* $u \in W_M$ *of the boundary-value problem* (2.11a,b).

Proof. By virtue of standard embedding results (Appendix A) W_M is compact in V for any $\delta < 1/2$. For $v \in V$ we define

$$L_v u = -\left[\frac{u'}{(\epsilon + v'^2)^{\alpha/2}}\right]' + \mu_1 u''''. \tag{2.13}$$

Then, for fixed $v \in V$, the linear boundary-value problem

$$\begin{cases} L_v u = f, \quad -a < y < a, & \text{(2.14a)} \\ u(\pm a) = u''(\pm a) = 0 & \text{(2.14b)} \end{cases}$$

has, as a consequence of the Lax-Milgram Lemma (see Appendix A), a unique solution $u \in H^2(-a, a)$ for which

$$||u||_{H^2(-a,a)} \le c||f||_{L^2(-a,a)} \tag{2.15}$$

with $c > 0$ independent of u. Let $T : v \to u$ where u is the unique solution of (2.14a,b). For any given $f \in L^2(-a, a)$ it is a direct consequence of (2.15) that $\exists M > 0$ sufficiently large such that $T : W_M \to W_M$. We want to show that T is a continuous map. For $v, w \in W_M$, let $u_1 = Tv$, $u_2 = Tw$; then

$$-\left[\frac{u_1'}{(\epsilon + v'^2)^{\alpha/2}}\right]' + \left[\frac{u_2'}{(\epsilon + w'^2)^{\alpha/2}}\right]' + \mu_1[u_1 - u_2]'''' = 0. \tag{2.16}$$

Multiplying (2.16) by $u_1 - u_2$, integrating over $(-a, a)$, and then integrating by parts we obtain, in view of (2.14b),

$$\mu_1 \int_{-a}^{a} [(u_1 - u_2)''(y)]^2\, dy + \int_{-a}^{a} \frac{u_1'(y)(u_1 - u_2)'(y)}{(\epsilon + v'^2(y))^{\alpha/2}}\, dy - \int_{-a}^{a} \frac{u_2'(y)(u_1 - u_2)'(y)}{(\epsilon + w'^2(y))^{\alpha/2}}\, dy = 0 \tag{2.17}$$

or

$$\mu_1||u_1 - u_2||^2_{H^2(-a,a)} + \int_{-a}^{a} \frac{(u_1 - u_2)'(y)u_1'(y)[(\epsilon + w'^2(y))^{\alpha/2} - (\epsilon + v'^2(y))^{\alpha/2}]}{(\epsilon + v'^2(y))^{\alpha/2}(\epsilon + w'^2(y))^{\alpha/2}}\, dy + \int_{-a}^{a} \frac{[(u_1 - u_2)'(y)]^2}{(\epsilon + w'^2(y))^{\alpha/2}}\, dy = 0. \tag{2.18}$$

As $u_1, u_2 \in H^2(-a, a)$, and $u_1', u_2' \in L^\infty(-a, a)$, we may estimate the first integral in (2.18) from above, and drop the (nonnegative) second integral, so as to obtain an estimate of the form

$$||u_1 - u_2||_{H^2(-a,a)} \le c_1 \left[\int_{-a}^{a} \left(\left|w'(y)\right|^\alpha - \left|v'(y)\right|^\alpha\right)^2 dy\right]^{1/2} \tag{2.19}$$

for some $c_1 > 0$; in obtaining (2.19) we have also employed the mean value theorem in the integrand of the first integral in (2.18). The continuity of T follows directly from the estimate (2.19). By the Schauder fixed-point theorem it now follows that there exists, for $M > 0$ sufficiently large, a unique $u \in W_M$ such that $u = Tu$ and, thus, we have established the existence of a unique solution of (2.11a,b) for arbitrary $\mu_1 > 0$. □

For the second of the fundamental results relative to the boundary-value problem (2.11a,b) we assume that $f(y)$ is a constant, say, $f(y) = K$, $-a \le y \le a$; this clearly covers the case of the constant pressure gradient p_1 in (2.5a). Also, we define $\bar{u} = \bar{u}(y)$ to be the unique solution of (2.11a) for $\mu_1 = 0$, and $f(y) = K$, which is subject to the boundary conditions $\bar{u}(\pm a) = 0$. We state the following result, which highlights the continuous dependence of the solution of (2.1a,b), established in Theorem 2.1, on the positive constitutive parameter μ_1:

Theorem 2.2. *For fixed $\epsilon > 0$, and $\alpha \in (0,1)$, denote by $u_{\mu_1}(x)$ the unique solution of (2.11a,b) with $f(x) = K$. Then, as $\mu_1 \to 0^+$,*

$$u_{\mu_1} \to \bar{u}, \text{ in } C^{1+\delta} \tag{2.20}$$

for $0 < \delta < \dfrac{1}{2}$.

Proof. From Theorem 2.1, we infer, for $M > 0$ sufficiently large, the existence of a unique solution $u_{\mu_1} \in W_M$ of the boundary-value problem

$$-\left[\frac{u'_{\mu_1}(y)}{(\epsilon + u'^2_{\mu_1}(y))^{\alpha/2}}\right]' + \mu_1 u''''_{\mu}(y) = K, \quad -a < y < a, \tag{2.21a}$$

$$u_{\mu_1}(\pm a) = 0, \quad u''_{\mu_1}(\pm a) = 0. \tag{2.21b}$$

We now set $v_{\mu_1} = u'_{\mu_1}$ in (2.21a) and then integrate the resulting equation over $(-a, x)$, $x < a$, so as to obtain

$$-\frac{v_{\mu_1}(y)}{(\epsilon + v^2_{\mu_1}(y))^{\alpha/2}} + \mu v''_{\mu_1}(y) = K(y + a) - A_{\mu_1} \tag{2.22}$$

with

$$A_{\mu_1} = \frac{v_{\mu_1}(-a)}{(\epsilon + v^2_{\mu_1}(-a))^{\alpha/2}} - \mu_1 v''_{\mu_1}(-a). \tag{2.23}$$

In view of the boundary conditions (2.21b), $v'_{\mu_1}(-a) = v'_{\mu_1}(a) = 0$ and

$$\int_{-a}^{a} v_{\mu_1}(y)\, dy = u_{\mu_1}(0) - u_{\mu_1}(-a) = 0. \tag{2.24}$$

We now multiply (2.22) by $v_{\mu_1}(y)$, integrate over $(-a, a)$, and then integrate by parts to obtain

$$\int_{-a}^{a} \frac{v_{\mu_1}^2(y)}{(\epsilon + v_{\mu_1}^2(y))^{\alpha/2}}\, dx + \mu_1 \int_{-a}^{a} v_{\mu_1}'^2(y)\, dx = -K \int_{-a}^{a} y v_{\mu_1}(y)\, dy \tag{2.25}$$

where we have used (2.24). Now, let

$$E_\epsilon = \left\{ y \mid v_{\mu_1}^2(y) > \epsilon \right\}. \tag{2.26}$$

Then $\forall y \in E_\epsilon$, $v_{\mu_1}^2/(\epsilon + v_{\mu_1}^2)^{\alpha/2} > \beta_1 v_{\mu_1}^{2-\alpha}$, $\beta_1 = 2^{-\alpha/2} > 0$; similarly, as $u_{\mu_1}' \in L^\infty(-a, a)$, on $E_\epsilon^c = [-a, a]/E_\epsilon$, $\exists \beta_2, \rho > 0$ such that $v_{\mu_1}^{2-\alpha} \le \beta_2 \epsilon^\rho$. Therefore

$$\begin{aligned} \int_{-a}^{a} v_{\mu_1}^{2-\alpha}(y)\, dy &= \int_{E_\epsilon} v_{\mu_1}^{2-\alpha} dy + \int_{E_\epsilon^c} v_{\mu_1}^{2-\alpha}\, dy \\ &\le \frac{1}{\beta_1} \int_{E_\epsilon} \frac{v_{\mu_1}^2}{(\epsilon + v_{\mu_1}^2)^{\alpha/2}} dy + \beta_2 \epsilon^\rho \ \mathrm{meas}(E_\epsilon^c) \\ &\le \frac{1}{\beta_1} \int_{-a}^{a} \frac{v_{\mu_1}^2}{(\epsilon + v_{\mu_1}^2)^{\alpha/2}} dy + \beta_3 \end{aligned} \tag{2.27}$$

Using the last estimate in (2.27) in (2.25) we have

$$\int_{-a}^{a} v_{\mu_1}^{2-\alpha}(y)\, dy + \frac{\mu}{\beta_1} \int_{-a}^{a} v_{\mu_1}'^2(y)\, dy \le \frac{K}{\beta_1} \int_{-a}^{a} |y| \big|v_{\mu_1}(y)\big|\, dy + \beta_3. \tag{2.28}$$

By virtue of the Hölder Inequality, (2.28) yields

$$\int_{-a}^{a} \left|v_{\mu_1}\right|^{2-a} dy \le \frac{K}{\beta_1} \left[\int_{-a}^{a} |y|^{(2-\alpha)/(1-\alpha)} dy \right]^{(1-\alpha)/(2-\alpha)} \left[\int_{-a}^{a} \left|v_{\mu_1}\right|^{2-\alpha} dy \right]^{1/(2-\alpha)} + \beta_3. \tag{2.29}$$

For arbitrary $\delta > 0$, we now use Young's inequality (see appendix A)

$$|a| \cdot |b| \le \delta |a|^p + \delta^{-1/(p-1)} |b|^{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

on the right-hand side of (2.29) with $p = 2 - \alpha$; for δ chosen sufficiently small we obtain from (2.29) an estimate of the form

$$\int_{-a}^{a} v_{\mu_1}^{2-\alpha}(y)\, dy \le a_1 \int_{-a}^{a} |y|^{(2-\alpha)/(1-\alpha)} dy + a_2 \tag{2.30}$$

with $a_1, a_2 > 0$. To obtain our next set of estimates we multipy (2.22) by $v''_{\mu_1}(y)$, integrate over $(-a, a)$, and integrate by parts; inasmuch as $v'_{\mu_1}(-a) = v'_{\mu_1}(a) = 0$ we easily find that

$$\int_{-a}^{a} \left[\frac{v_{\mu_1}(y)}{(\epsilon + v_{\mu_1}^2(y))^{\alpha/2}} \right]' v'_{\mu_1}(y)\, dy + \mu_1 \int_{-a}^{a} (v''_{\mu_1}(y))^2\, dy = -K \int_{-a}^{a} v'_{\mu_1}(y)\, dy$$
$$= K[v_{\mu_1}(-a) - v_{\mu_1}(a)]$$

or

$$\int_{-a}^{a} \frac{v'^2_{\mu_1}(y)}{(\epsilon + v_{\mu_1}^2(y))^{\alpha/2}} \cdot \left[\frac{\epsilon + (1-\alpha) v_{\mu_1}^2(y)}{\epsilon + v_{\mu_1}^2(y)} \right] dy + \mu_1 \int_{-a}^{a} (v''_{\mu_1}(y))^2 dy = K[v_{\mu_1}(-a) - v_{\mu_1}(a)]. \tag{2.31}$$

Now, $\forall \alpha$ with $0 < \alpha < 1$, $\exists \bar{c}_1, \bar{c}_2 > 0$ such that

$$\bar{c}_1 \le \frac{\epsilon + (1-\alpha)\eta}{\epsilon + \eta} \le \bar{c}_2, \quad \forall \eta \ge 0 \tag{2.32}$$

where the $\bar{c}_i$, $i = 1, 2$, depend on α but not on η. Applying (2.32) to (2.31), with $\eta = v_{\mu_1}^2$, we get the estimate

$$\int_{-a}^{a} \frac{v'^2_{\mu_1}(y)}{(\epsilon + v_{\mu_1}^2(y))^{\alpha/2}} dy + \frac{\mu_1}{\bar{c}_1} \int_{-a}^{a} (v''_{\mu_1}(y))^2 dy \le \frac{K}{\bar{c}_1} [v_{\mu_1}(-a) - v_{\mu_1}(a)]$$
$$\le \bar{c}_3 \max_{[-a,a]} \left| v_{\mu_1}(y) \right|. \tag{2.33}$$

We now set

$$\Psi(v_{\mu_1}) = \int_0^{v_{\mu_1}} \frac{ds}{(\epsilon + s^2)^{\alpha/4}}. \tag{2.34}$$

Then it follows directly from (2.34) that

$$\int_{-a}^{a} \left(\frac{d}{dx} \Psi(v_{\mu_1}(y)) \right)^2 dx \le \bar{c}_3 \max_{[-a,a]} \left| v_{\mu_1}(y) \right|. \tag{2.35}$$

As $\Psi(v_{\mu_1})$ is an even function, and $1/(\epsilon + s^2)^{\alpha/4} \le 1/(s^2)^{\alpha/4}$, we have that

$$\left| \Psi(v_{\mu_1}) \right| = \Psi(\left| v_{\mu_1} \right|) \le \left(1 - \frac{\alpha}{2} \right)^{-1} \left| v_{\mu_1} \right|^{1-\alpha/2} \le 4 \left| v_{\mu_1} \right|^{1-\alpha/2}. \tag{2.36}$$

Therefore, by virtue of our previous estimate (2.30), $\exists \Psi_0 > 0$ (const.) such that

$$\int_{-a}^{a} \Psi^2(v_{\mu_1}(y))dx \leq \Psi_0. \tag{2.37}$$

Now, $\forall w \in H^1[-a,a]$, and $\forall \delta > 0$, $\exists c_\delta > 0$ such that

$$\max_{[-a,a]} |w| \leq \delta \left(\int_{-a}^{a} w'^2(y)\, dy \right)^{1/2} + c_\delta \left(\int_{-a}^{a} w^2(y)\, dy \right)^{1/2} \tag{2.38}$$

(see, e.g., [Lio1] Lemma 5.1); applying (2.38) with $w = \Psi(v_{\mu_1})$, and making use of both (2.35) and (2.37), we find that for some $d_\delta > 0$,

$$\max_{[-a,a]} \left|\Psi(v_{\mu_1})\right| \leq \delta \left[\max_{[-a,a]} \left|v_{\mu_1}(y)\right| \right]^{1/2} + d_\delta. \tag{2.39}$$

Our goal now is to show that for some $c > 0$,

$$\max_{[-a,a]} \left|v_{\mu_1}(y)\right| \leq c \left[\max \left\{ 1, \max_{[-a,a]} \left|\Psi(v_{\mu_1})\right|^{\beta} \right\} \right] \tag{2.40}$$

with c independent of v_{μ_1}, and $\beta = 1/(1-\alpha/2)$. To this end we define, for $s \in R^1$,

$$F(s) = \Psi(s) - ks^{1-\alpha/2} \tag{2.41}$$

where k is chosen so that $\Psi(1) > k$. Thus, $F(1) \geq 0$, while

$$F'(s) = \frac{1}{(\epsilon + s)^{\alpha/4}} - k\left(1 - \frac{\alpha}{2}\right) s^{-\alpha/2}$$

so that, for k chosen sufficiently small, $F'(s) \geq 0$, $\forall s \in R^1$. Consequently, $F(s) \geq 0$, $\forall s \geq 1$ so that

$$ks^{1/\beta} \leq \Psi(s), \quad \forall s \geq 1 \tag{2.42}$$

with $\beta = 1/(1-\alpha/2)$. Employing (2.42) in (2.39) we obtain

$$\max_{[-a,a]} \left|\Psi(v_{\mu_1})\right| \leq \delta' \left[\max \left\{ 1, \max_{[-a,a]} \left|\Psi(v_{\mu_1})\right|^{\beta/2} \right\} \right] + d_\delta \tag{2.43}$$

with $\delta' = \delta k^{-\beta/2}$. If $\max_{[-a,a]} \left|\Psi(v_{\mu_1})\right|^{\beta/2} > 1$, then

$$\max_{[-a,a]} \left|\Psi(v_{\mu_1})\right| - \delta' \max_{[-a,a]} \left|\Psi(v_{\mu_1})\right|^{\beta/2} \leq d_\delta$$

or, as $\beta/2 = 1/(2-\alpha) < 1$, for $0 < \alpha < 1$,

$$(1-\delta') \max_{[-a,a]} \left|\Psi(v_{\mu_1})\right| \le d_\delta. \tag{2.44}$$

Therefore, for δ chosen sufficiently small, it follows that $\exists C > 0$ such that (recall that $\left|\Psi(v_{\mu_1})\right| = \Psi(\left|v_{\mu_1}\right|)$):

$$\max_{[-a,a]} \left[\Psi(\left|v_{\mu_1}(y)\right|)\right] \le C. \tag{2.45}$$

Clearly, an estimate of the form (2.45) also follows from (2.43) if $\max\limits_{[-a,a]} \left|\Psi(v_{\mu_1})\right|^{\beta/2} \ge 1$. Now, the estimate (2.40) is a direct consequence of (2.42), and the use of (2.45) in (2.42) then produces a bound of the form

$$\max_{[-a,a]} \left|v_{\mu_1}(y)\right| \le C'$$

for some $C' > 0$. Thus, by virtue of (2.33), we have, for some $C > 0$,

$$\int_{-a}^{a} \left[\frac{d}{dx}\Psi(v_{\mu_1}(y))\right]^2 dx + \mu \int_{-a}^{a} (v''_{\mu_1}(y))^2 dy \le C. \tag{2.46}$$

Combining (2.46) with (2.37), it follows that $\exists \tilde{C}$, independent of μ_1, such that

$$\left|\left|\Psi(v_{\mu_1})\right|\right|_{H^1(-a,a)} \le \tilde{C}. \tag{2.47}$$

Therefore, $\exists \Psi^0 \in H^1(-a,a)$ such that

$$\Psi(v_{\mu_1}) \to \Psi^0, \text{ in } H^1(-a,a), \text{ as } \mu_1 \to 0^+ \tag{2.48}$$

and, by virtue of (2.46), we also note that

$$\mu v''_{\mu_1} \to 0, \text{ in } L^2(-a,a), \text{ as } \mu_1 \to 0^+. \tag{2.49}$$

In view of (2.48), and Theorem 2.1, for some $\bar{\Psi}$ we have

$$\Psi(v_{\mu_1}) \to \bar{\Psi}, \text{ in } C^{0,\delta}, \text{ for } 0 < \delta < 1/2, \text{ as } \mu_1 \to 0^+. \tag{2.50}$$

But Ψ, being monotone, is invertible, and as $\Psi^{-1} \in C^1(R^1)$ we find that

$$v_{\mu_1} \to \bar{u}, \text{ in } C^{0,\delta}, \text{ for } 0 < \delta < 1/2, \text{ as } \mu_1 \to 0^+ \tag{2.51}$$

Finally, as $u'_{\mu_1} = v_{\mu_1}$ we have, for the unique solution $u_{\mu_1}(y)$ of (2.21a,b), that

$$v_{\mu_1} \to \bar{u}, \text{ in } C^{1+\delta}, \text{ for } 0 < \delta < 1/2, \text{ as } \mu_1 \to 0^+ \tag{2.52}$$

with $\bar{u}$ the unique solution of (2.21a), with $\mu = 0$, subject to the boundary conditions $\bar{u}(\pm a) = 0$; this concludes the demonstration of the continuous dependence of u_{μ_1} on μ_1, as $\mu_1 \to 0^+$, in the $C^{1+\delta}$ norm, for $0 < \delta < 1/2$. □

Remarks. If $\mu_1 = 0$, then (2.5a) reduces to

$$\left[\left(\epsilon + \frac{1}{2}v_1'^2(x_2)\right)^{-\alpha/2} v_1'(x_2)\right]' = p_1/\mu_0$$

so that for some real constant γ,

$$\left(\epsilon + \frac{1}{2}v_1'^2(x_2)\right)^{-\alpha/2} v_1'(x_2) = g_\gamma(x_2) \tag{2.53}$$

where

$$g_\gamma(x_2) = \left(\frac{p_1}{\mu_0}\right) x_2 + \gamma.$$

If we now set $W_\epsilon(x_2) = \epsilon + \dfrac{1}{2}v_1'^2(x_2)$, then it follows from (2.53) that W_ϵ satisfies the transcendental algebraic equation

$$W_\epsilon^{1-\alpha} - \epsilon W_\epsilon^\alpha = \frac{1}{2}g_\gamma^2; \quad \epsilon > 0,\ 0 < \alpha < 1 \tag{2.54}$$

whose solutions are easily seen to be dependent continuously on ϵ as $\epsilon \to 0^+$. Thus, solutions of (2.11a), with $f(y) = p_1$ and $\mu_1 = 0$, subject to $u(\pm a) = 0$, depend continuously on ϵ; that the same continuous dependence with respect to ϵ holds for the full boundary-value problem (2.11a,b), with $\mu_1 > 0$ and $f(y) = p_1$, follows from the detailed estimates in Sect. 2.3.

2.3 Estimates and Generalized Reynolds Numbers for Steady Plane Poiseuille Flow

We will continue here the practice of using the notation $u = v_1$, and $y = x_2$, in which case (2.5a,b) assumes the equivalent form

$$\mu_0\left[\left(\epsilon + \frac{1}{2}u'^2(y)\right)^{-\alpha/2} u'(y)\right]' - \mu_1 u''''(y) = p_1, \quad -a < y < a \tag{2.55a}$$

$$u(\pm a) = 0, \quad u''(\pm a) = 0 \tag{2.55b}$$

where we have written $u(y)$ in lieu of $u(y; \epsilon, \mu_1)$, while for $\epsilon = \mu_1 = 0$ the above fourth-order boundary-value problem reduces to

$$\mu_0 \left[\left(\frac{1}{2} u_0'^2(y) \right)^{-\alpha/2} u_0'(y) \right]' = p_1, \tag{2.56a}$$

$$u_0(\pm a) = 0 \tag{2.56b}$$

where $u_0(y) = u(y; 0, 0)$. Whenever it is deemed important to avoid any possible confusion, we will make explicit the dependence of the solution u of (2.55a,b) on ϵ and μ_1.

In this section we continue the study of plane equilibrium Poiseuille flows of incompressible, isothermal, bipolar fluids initiated in Sect. 2.2. Through the use of dimensional analysis applied to (2.55a) we isolate the natural counterparts of the Reynolds number associated with the Navier–Stokes theory. We then investigate, in greater detail than was done in Sect. 2.2, properties of the solutions $u_0(\cdot)$ of equations (2.56a,b) and use the solutions to compute the associated mean velocity, maximum velocity, volume flow, and pressure drop. Finally, although a continuous dependence result for solutions of (2.55a,b), (in $C^{1+\delta}$, $0 < \delta < 1/2$) as $\epsilon, \mu_1 \to 0^+$, was established in Sect. 2.2 precise estimates of the errors incurred by setting $\epsilon = \mu_1 = 0$ and using, in place of $u(\cdot; \epsilon, \mu_1)$ the solutions $u_0(\cdot)$ of equations (2.56a,b) were not presented there; such estimates are derived here and are subsequently employed to establish the related estimates for the volume flow, etc. It is hoped (and expected) that such estimates will eventually serve as a guide for the formulation of experiments directed at the determination of the constitutive constants in the model.

2.3.1 Generalized Reynolds Numbers for Plane Poiseuille Flow of a Bipolar Fluid

In this subsection we indicate the appropriate form which a dimensionless version of the evolution equation associated with (2.55a) assumes and, in the process, are led to the definition of generalized Reynolds numbers that are connected with plane Poiseuille flows of an incompressible bipolar fluid. Employing a standard approach (and not taking, a priori, the density $\rho = 1$) we set

$$\bar{y} = \frac{y}{a}, \quad \bar{t} = \frac{V}{a} t, \quad \bar{u} = \frac{u}{V}, \quad \bar{p} = \frac{p}{\rho V^2} \tag{2.57}$$

in the evolution equation for plane Poiseuille flow of an incompressible bipolar fluid,

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial y}+\mu_0\frac{\partial}{\partial y}\left\{\left[\epsilon+\left(\frac{\partial u}{\partial y}\right)^2\right]^{-\alpha/2}\frac{\partial u}{\partial y}\right\}-\mu_1\frac{\partial^4 u}{\partial y^4}. \tag{2.58}$$

Here V is a measure of the mean- or far-field velocity associated with the flow and $u=u(y,t)$, where we have supressed the dependence of u on ϵ and μ_1. With $\bar{u}(\bar{y},\bar{t})=u(a\bar{y},a\bar{t}/V)$, $\bar{p}(\bar{y})=p(a\bar{y})$, elementary calculations yield

$$\frac{\partial u}{\partial t}=\frac{V^2}{a}\frac{\partial\bar{u}}{\partial\bar{t}}, \tag{2.59a}$$

$$\frac{\partial p}{\partial y}=\frac{\rho V^2}{a}\frac{\partial\bar{p}}{\partial\bar{y}}, \tag{2.59b}$$

$$\frac{\partial u}{\partial y}=\frac{V}{a}\frac{\partial\bar{u}}{\partial\bar{y}}, \tag{2.59c}$$

$$\frac{\partial^4 u}{\partial y^4}=\frac{V}{a^4}\frac{\partial^4\bar{u}}{\partial\bar{y}^4}, \tag{2.59d}$$

the substitution of which into (2.58) produces

$$\left(\frac{\rho V^2}{a}\right)\frac{\partial\bar{u}}{\partial\bar{t}}=-\left(\frac{\rho V^2}{a}\right)\frac{\partial\bar{p}}{\partial\bar{y}}+\frac{\mu_0}{a}\frac{\partial}{\partial\bar{y}}\left\{\left[\epsilon+\frac{V^2}{a^2}\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2\right]^{-\frac{\alpha}{2}}\left(\frac{V}{a}\right)\frac{\partial\bar{u}}{\partial\bar{y}}\right\}-\frac{\mu_1 V}{a^4}\frac{\partial^4\bar{u}}{\partial\bar{y}^4}. \tag{2.60}$$

After multiplying (2.60) by $\dfrac{a}{\rho V^2}$, and setting

$$\nu_0=\frac{\mu_0}{\rho},\quad \nu_1=\frac{\mu_1}{\rho} \tag{2.61}$$

(2.60) becomes

$$\frac{\partial\bar{u}}{\partial\bar{t}}=-\frac{\partial\bar{p}}{\partial\bar{y}}+\frac{\nu_0}{aV}\frac{\partial}{\partial\bar{y}}\left\{\left[\epsilon+\frac{V^2}{a^2}\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2\right]^{-\frac{\alpha}{2}}\frac{\partial\bar{u}}{\partial\bar{y}}\right\}-\frac{\nu_1}{a^3V}\frac{\partial^4\bar{u}}{\partial\bar{y}^4}. \tag{2.62}$$

In as much as ϵ must have the dimension of a velocity gradient squared,

$$\bar{\epsilon}=\epsilon a^2/V^2 \tag{2.63}$$

is dimensionless; using the definition (2.63) of $\bar{\epsilon}$ we may now rewrite (2.62) as

$$\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial y}+\frac{\nu_0}{aV}\left(\frac{a}{V}\right)^{\alpha}\frac{\partial}{\partial y}\left\{\left[\epsilon+\left(\frac{\partial u}{\partial y}\right)^2\right]^{-\frac{\alpha}{2}}\frac{\partial u}{\partial y}\right\}-\frac{\nu_1}{a^3V}\frac{\partial^4 u}{\partial y^4} \tag{2.64}$$

where we have dropped the superposed bars from y, t, p, u, and ϵ. The dimensionless version (2.64) of (2.58) leads naturally to the definition of two generalized Reynolds numbers that are associated with plane Poiseuille flow of a bipolar viscous fluid, namely,

$$R_0^{(\alpha)} = \frac{V^{\alpha+1}}{\nu_0 a^{\alpha-1}}, \quad R_1 = \frac{a^3 V}{\nu_1}. \tag{2.65}$$

Using the definitions (2.65), the evolution equation for $u(y,t)$ assumes the form

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial y} + \frac{1}{R_0^{(\alpha)}}\frac{\partial}{\partial y}\left\{\left[\epsilon + \left(\frac{\partial u}{\partial y}\right)^2\right]^{-\frac{\alpha}{2}}\frac{\partial u}{\partial y}\right\} - \frac{1}{R_1}\frac{\partial^4 u}{\partial y^4}. \tag{2.66}$$

For $\alpha = \mu_1 = 0$, clearly $R_0^{(0)} = Va/\nu_0$ and $R_1^{-1} = 0$, so that (2.66) reduces to the standard dimensionless form for plane Poiseuille flow within the context of the Navier–Stokes formulation, with $R_0^{(0)}$ being the usual Reynolds number.

2.3.2 *The Poiseuille Flow for $\epsilon = \mu_1 = 0$*

In this subsection we will look, in greater detail than that which was done in Sect. 1.7.1, at the behavior of the solution of (2.56a,b). In the next subsection our interest will be in obtaining estimates which relate the behavior of the solution of the boundary-value problem (2.56a,b) to that of the solution $u(y;\epsilon,\mu_1)$ of (2.55a,b); the quantities of particular interest to us will be the volume flow

$$Q_{\epsilon,\mu_1} = \int_{-a}^{a} u(y;\epsilon,\mu_1)\,dy \tag{2.67a}$$

which for $\epsilon = \mu_1 = 0$ has the form

$$Q_{0,0} \equiv Q_0 = \int_{-a}^{a} u_0(y)\,dy \tag{2.67b}$$

the mean velocity

$$\bar{u}_{\epsilon,\mu_1} = \frac{1}{2a} Q_{\epsilon,\mu_1} \tag{2.68a}$$

and its counterpart for $\epsilon = \mu_1 = 0$, i.e.,

$$\bar{u}_0 = \frac{1}{2a} Q_0 \tag{2.68b}$$

and the friction factors

$$f_{\epsilon,\mu_1} = \frac{4\tau_{12}(\pm a, \epsilon, \mu_1)}{\frac{1}{2}\rho \bar{u}^2_{\epsilon,\mu_1}} \tag{2.69a}$$

$$f_0 = \frac{4\tau_{12}(\pm a, 0, 0)}{\frac{1}{2}\rho \bar{u}^2_0} \tag{2.69b}$$

where $\tau_{12}(\pm a, \epsilon, \mu_1)$ is the shear stress at the walls located at $y = \pm a$, i.e.,

$$\tau_{12}(\pm a, \epsilon, \mu_1) = \mu_0 \left[\epsilon + \frac{1}{2}u'^2(\pm a, \epsilon, \mu_1)\right]^{-\frac{\alpha}{2}} u'(\pm a, \epsilon, \mu_1) - \mu_1 u''''(\pm a, \epsilon, \mu_1). \tag{2.70}$$

We begin by noting that if $u_0(y)$ is a solution of the boundary-value problem (2.56a,b) then so is $u_0(-y)$ and, thus, by uniqueness of solutions $u_0(y) = u_0(-y)$, $-a \le y \le a$; from this result it follows that $u_0'(y) = -u_0'(-y)$, so that $u_0'(0) = 0$. Moreover, with p_1, the constant pressure gradient, negative, a first integration of equation (2.56a) yields

$$\mu_0 \left[\frac{1}{2}u_0'^2(y)\right]^{-\frac{\alpha}{2}} u_0'(y) = -|p_1| y \tag{2.71}$$

where the constant of integration vanishes in view of the fact that $0 < a < 1$ and $u_0'(0) = 0$. From (2.71) it is immediate that

$$\begin{aligned} u_0'(y) &> 0, \quad y \in (-a, 0), \\ u_0'(y) &< 0, \quad y \in (0, a). \end{aligned} \tag{2.72}$$

In as much as $u_0'(y) \le 0$, for $y \in (0, a)$, we have $u_0'(y) = -\left|u_0'(y)\right|$, $0 \le y \le a$; therefore, if we set $C_\alpha = \mu_0 2^{\alpha/2}$, equation (2.71) becomes, on $(0, a)$,

$$\left|u_0'(y)\right|^{1-\alpha} = \frac{|p_1|}{C_\alpha} y, \quad 0 < y < a$$

or

$$\left|u_0'(y)\right| = C_\alpha y^{1/(1-\alpha)}, \quad 0 < y < a \tag{2.73}$$

with $C_\alpha = \left(\frac{|p_1|}{C_\alpha}\right)^{1/(1-\alpha)}$. We rewrite (2.73) as

$$u_0'(y) = -C_\alpha y^{1/(1-\alpha)}, \quad 0 < y < a \tag{2.74}$$

and integrate from a to y obtaining

$$u_0(y) = \frac{C_\alpha}{\bar{\gamma}+1}\left[a^{\bar{\gamma}+1} - y^{\bar{\gamma}+1}\right]; \quad \bar{\gamma} = \frac{1}{1-\alpha}. \tag{2.75}$$

In obtaining (2.75) we have used, of course, the boundary condition $u_0(a) = 0$. Substituting for $\bar{\gamma}$ in (2.75), and noting that $u_0(y) = u_0(-y)$, we find that

$$u_0(y) = d_\alpha\left[1 - \left(\frac{|y|}{a}\right)^{(2-\alpha)/(1-\alpha)}\right], \quad -a \le y \le a \tag{2.76}$$

where

$$\begin{aligned} d_\alpha &= C_\alpha\left(\frac{1-\alpha}{2-\alpha}\right)a^{(2-\alpha)/(1-\alpha)} \\ &= \left(\frac{|p_1|}{C_\alpha}\right)^{1/(1-\alpha)} a^{(2-\alpha)/(1-\alpha)}\left(\frac{1-\alpha}{2-\alpha}\right) \\ &= \left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{|p_1|a^{2-\alpha}}{\mu_0 2^{\alpha/2}}\right)^{1/(1-\alpha)}. \end{aligned} \tag{2.77}$$

It is clear, from (2.76), that

$$\max_{[-a,a]} u_0(y) \equiv u_0^{\max} = u_0(0) = d_\alpha. \tag{2.78}$$

By direct calculation, the mean velocity $\bar{u}_0$ associated with $u_0(y)$ is

$$\begin{aligned} \bar{u}_0 &= \frac{1}{2a}\int_{-a}^{a} u_0(y)\,dy \\ &= \frac{d_\alpha}{a}\int_0^a \left[1 - \left(\frac{\lambda}{a}\right)^\delta\right] d\lambda; \quad \delta = \frac{2-\alpha}{1-\alpha}. \end{aligned} \tag{2.79}$$

Carrying out the integration in (2.79), we are led to

$$\bar{u}_0 - \left(\frac{2-\alpha}{3-2\alpha}\right)d_\alpha \equiv \left(\frac{2-\alpha}{3-2\alpha}\right)u_0^{\max} \tag{2.80}$$

or, in view of (2.77),

$$\bar{u}_0 = \left(\frac{1-\alpha}{3-2\alpha}\right)\left(\frac{|p_1|a^{2-\alpha}}{\mu_0 2^{\alpha/2}}\right)^{1/(1-\alpha)}. \tag{2.81}$$

We note that

$$\bar{u}_0\Big|_{\alpha=0} = \frac{1}{3}\frac{|p_1|a^2}{\mu_0} \tag{2.82}$$

which is the classical result associated with Navier–Stokes. From (2.78) and (2.80) it follows that

$$\lim_{\alpha\to 1-} u_0^{\max} = \lim_{\alpha\to 1-} d_\alpha = \lim_{\alpha\to 1-} \bar{u}_0. \tag{2.83}$$

However, in view of (2.77),

$$\lim_{\alpha\to 1^-} d_\alpha = \lim_{\alpha\to 1^-}\left[\left(\frac{1-\alpha}{2-\alpha}\right)\left\{\frac{|p_1|a^{2-\alpha}}{\mu_0 2^{\alpha/2}}\right\}^{1/(1-\alpha)}\right]$$

from which it is clear that the critical quantity in computing $\lim\limits_{\alpha\to 1-} u_0^{\max}$ is

$$e_\alpha = \frac{|p_1|a^{2-\alpha}}{\mu_0 2^{\alpha/2}}. \tag{2.84}$$

Specifically, if $e_\alpha > 1$, for α sufficiently close to 1, then $u_0^{\max} \to \infty$ as $\alpha \to 1^-$. Suppose that we set $V = \bar{u}_0|_{\alpha=0}$ in (2.65), so that $|p_1| = \rho V^2/a$, then

$$e_\alpha^{1/(1-\alpha)} = \left[\frac{V^2}{\nu_0 2^{\alpha/2}}\right]^{1/(1-\alpha)} > \left[\frac{V^2}{\nu_0\sqrt{2}}\right]^{1/(1-\alpha)} \tag{2.85}$$

while

$$\lim_{\alpha\to 1^-} R_0^{(\alpha)} = \frac{V^2}{\nu_0}.$$

Thus,

$$u_0^{\max} \to \infty, \text{ as } \alpha \to 1^- \tag{2.86a}$$

provided

$$\lim_{\alpha\to 1^-} R_0^{(\alpha)} > \sqrt{2}. \tag{2.86b}$$

To emphasize further the role of the criteria (2.86b) and its connection with the status of the parameter α in situations involving small physical viscosity (i.e., α close to 1), we compute the $\lim\limits_{\alpha\to 1^-} u_0'(a)$. From (2.76) and (2.80), we have

$$u_0(y) = \left(\frac{3-2\alpha}{2-\alpha}\right)\bar{u}_0\left[1-\left(\frac{y}{a}\right)^{(2-\alpha)/(1-\alpha)}\right], \quad 0 \le y \le a \tag{2.87}$$

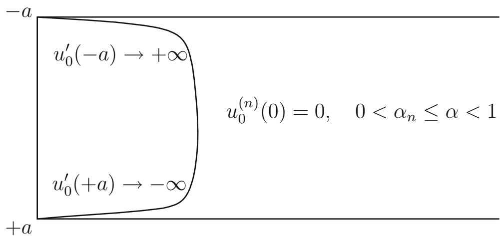

Fig. 2.1 Non-Newtonian velocity profile, $0 < \alpha < 1$

so that

$$u_0'(y) = -\left(\frac{3-2\alpha}{1-\alpha}\right)\bar{u}_0\left(\frac{y}{a}\right)^{1/(1-\alpha)} \tag{2.88}$$

and, thus,

$$\frac{1}{(3-2\alpha)}u_0'(a) = \frac{1}{(1-\alpha)}\bar{u}_0. \tag{2.89}$$

Therefore, by virtue of (2.81) and (2.84),

$$\begin{aligned} \lim_{\alpha\to 1^-} u_0'(a) &= -\lim_{\alpha\to 1^-}\left[\frac{1}{(1-\alpha)}\bar{u}_0\right] \\ &= -\lim_{\alpha\to 1^-}\left[\frac{1}{(3-2\alpha)}e_\alpha^{1/(1-\alpha)}\right] \\ &= -\lim_{\alpha\to 1^-} e_\alpha^{1/(1-\alpha)} = -\infty \end{aligned} \tag{2.90}$$

if $\lim\limits_{\alpha\to 1^-} R_0^{(\alpha)} > \sqrt{2}$; under these same conditions

$$\lim_{\alpha\to 1^-} u_0'(-a) = \lim_{\alpha\to 1^-}\left[-u_0'(a)\right] = +\infty \tag{2.91}$$

so that if $\lim\limits_{\alpha\to 1^-} R_0^{(\alpha)} > \sqrt{2}$, and α is close to 1, the velocity profile assumes the form indicated in Fig. 2.1, above.

In fact, not only is $u_0'(0) = 0$, but the rapid flattening of the profile, depicted in Fig. 2.1, with respect to the axis $y = 0$, as $\alpha \to 1^-$, is easily demonstrated as follows: from (2.88), and the companion result for $-a \leq y \leq 0$,

$$u_0''(y) = \frac{(3-2\alpha)}{(1-\alpha)^2}\bar{u}_0\left(\frac{|y|}{a}\right)^{\alpha/(1-\alpha)} \tag{2.92}$$

so that $u_0''(0) = 0$, for all α, $0 < \alpha < 1$. Then

$$u_0'''(y) = -\alpha \frac{(3-2\alpha)}{(1-\alpha)^3} \bar{u}_0 \left(\frac{|y|}{a} \right)^{(2\alpha-1)/(1-\alpha)} \tag{2.93}$$

so that $u_0'''(0) = 0$, for $\alpha > \dfrac{1}{2}$. A further computation shows that $u_0''''(0) = 0$, for $\alpha > \dfrac{2}{3}$, and it is clear that by an induction argument we may show that $u_0^{(n)}(0) = 0$, $\alpha > \alpha_n$, for some α_n sufficiently close to 1.

From (2.81) it follows directly that the volume flow Q_0, as given by (2.67b), is

$$Q_0 = 2a \left(\frac{1-\alpha}{3-2\alpha} \right) \left[\frac{|p_1| a^{2-\alpha}}{\mu_0 2^{\alpha/2}} \right]^{1/(1-\alpha)}. \tag{2.94}$$

Also, from (2.81), it is a simple matter to compute that

$$|p_1| = \frac{\mu_0 2^{\alpha/2}}{a^{2-\alpha}} \left(\frac{3-2\alpha}{1-\alpha} \right) \bar{u}_0^{1-\alpha}. \tag{2.95}$$

Now, by (2.70), $\tau_{12}(-a, 0, 0) = |p_1| a$, or

$$\tau_{12}(-a, 0, 0) = \frac{\mu_0 2^{\alpha/2}}{a^{1-\alpha}} \left(\frac{3-2\alpha}{1-\alpha} \right) \bar{u}_0^{1-\alpha} \tag{2.96}$$

so that, by (2.69b), the friction factor is

$$f_0 = \frac{\nu_0 2^{3+(\alpha/2)}}{a^{1-\alpha}} \left(\frac{3-2\alpha}{1-\alpha} \right) \bar{u}_0^{-(\alpha+1)}. \tag{2.97}$$

Also, in view of (2.95), and the fact that $a > 1$,

$$\frac{\mu_0 \bar{u}_0^{1-\alpha}}{a^2} < |p_1| < \frac{3\sqrt{2}}{a(1-\alpha)} \mu_0 \bar{u}_0^{1-\alpha} \tag{2.98}$$

an estimate which should be useful in experimentally approximating the values of the constitutive parameters μ_0, α based on careful measurements of the magnitude of the pressure gradient p_1 and the mean velocity $\bar{u}_0$.

2.3.3 *Estimates for the Poiseuille Flow $u(y; \epsilon, \mu_1)$ when $\epsilon, \mu_1 \neq 0$*

In this section we provide those precise qualitative estimates, for the unique solution $u(y; \epsilon, \mu_1)$ of the nonlinear boundary-value problem (2.55a,b), which allow us to compare $u(y;\epsilon,\mu_1)$ with the solution $u_0(y)$ of (2.56a,b) and which are missing from the analysis in Sect. 2.2; the results obtained in this subsection render meaningful the content of results such as (2.97), (2.98) for the flow $u_0(y)$. We begin by introducing the following notation: we set

$$w(y;\epsilon,\mu_1) = u'(y;\epsilon,\mu_1), \tag{2.99a}$$

$$\hat{w}(y;\epsilon) = w(y;\epsilon,0), \tag{2.99b}$$

$$z(y;\epsilon,\mu_1) = \epsilon + |w(y;\epsilon,\mu_1)|^2, \tag{2.99c}$$

$$\hat{z}(y;\epsilon) = z(y;\epsilon,0) = \epsilon + |\hat{w}(y;\epsilon)|^2 \tag{2.99d}$$

and

$$\gamma(w) = (\epsilon + |w|^2)^{-\alpha/2} w = z^{-\alpha/2} w. \tag{2.100}$$

When the interpretation is obvious, we will supress the explicit dependence on y and write, e.g., $\hat{z}(\epsilon) = \hat{z}(y;\epsilon)$. We also note that $\hat{w}(y;0) = u'(y;0,0) \equiv u_0'(y)$ which is given explicitly by (2.88). Our goal is to estimate the difference $u(y;\epsilon,\mu_1) - u(y;0,0) = u(y;\epsilon,\mu_1) - u_0(y)$, from both below and above and to then use the resulting bounds to estimate quantities such as Q_{ϵ,μ_1}, the volume flow associated with the unique solution of the boundary-value problem (2.55a,b). The bounds for $u(y;\epsilon,\mu) - u_0(y)$ on $[-a,a]$ will be obtained from a series of lemmas which culminate in Theorem 2.3; the first of these results is as follows:

Lemma 2.1. *Let $u(y;\epsilon,\mu_1)$ be the unique classical solution of* (2.55a,b) *and $u_0(y)$ the unique solution of* (2.56a,b). *Then $\exists K_\alpha > 0$, depending only on α, such that on* $[-a,a]$

$$\left|u'(y;\epsilon,0) - u_0'(y)\right| < (1 + K_\alpha)\sqrt{\epsilon}. \tag{2.101}$$

Proof. We set $\mu_1 = 0$ in (2.55a,b), divide through by μ_0 and set $p_1^* = p_1/\mu_0$; then $u(y;\epsilon,0)$ is the solution of the boundary-value problem

$$\left[\left(\epsilon + |\hat{w}(y;\epsilon)|^2\right)^{-\alpha/2} \hat{w}(y;\epsilon)\right]' = p_1^*, \quad -a < y < a, \tag{2.102a}$$

$$u(\pm a;\epsilon,0) = 0 \tag{2.102b}$$

where we have used the definitions in (2.99a,b). Now, if $u(y;\epsilon,\mu_1)$ is a solution of (2.55a,b), then so is $u(-y;\epsilon,\mu_1)$ for any $\epsilon,\mu_1 \geq 0$; by uniqueness of solutions to the boundary-value problem we must have $u(y;\epsilon,\mu_1) = u(-y;\epsilon,\mu_1)$ from which it follows that

$$u'(y;\epsilon,\mu_1) = -u'(-y;\epsilon,\mu_1), \quad -a < y < a \tag{2.103}$$

and, therefore, $u'(0;\epsilon,\mu_1) = 0$, for all $\epsilon, \mu_1 \geq 0$. Integration of equation (2.102a) leads, therefore, to

$$\left(\epsilon + |\hat{w}(y;\epsilon)|^2\right)^{-\alpha/2} \hat{w}(y;\epsilon) = p_1^* y, \quad -a < y < a \tag{2.104}$$

as $\hat{w}(0;\epsilon) = u'(0;\epsilon,0) = 0$. It then follows from (2.104) that

$$\hat{w}(y;\epsilon) \neq 0, \quad \forall \epsilon \geq 0, \quad y \neq 0 \tag{2.105}$$

as $0 < \alpha < 1$. Squaring both sides of (2.104) and using the definition (2.99d), we obtain

$$\hat{z}(y;\epsilon)^{-\alpha} \hat{w}^2(y;\epsilon) = p_1^{*2} y^2, \quad -a < y < a. \tag{2.106}$$

We rewrite (2.106) in the form [recall that $\hat{z}(\epsilon) = \hat{z}(y;\epsilon)$, $-a \leq y \leq a$]

$$\hat{z}(\epsilon)^{-\alpha}[\hat{z}(\epsilon) - \epsilon] = p_1^{*2} y^2$$

or

$$\hat{z}(\epsilon)^{1-\alpha} - \epsilon \hat{z}(\epsilon)^{-\alpha} = p_1^{*2} y^2, \quad -a < y < a. \tag{2.107}$$

If we now differentiate (2.107) with respect to ϵ we obtain, after a simple calculation,

$$\hat{z}(\epsilon)^{-\alpha}\left[(1-\alpha)\hat{z}_\epsilon - 1 + \alpha\epsilon\hat{z}(\epsilon)^{-1}\hat{z}_\epsilon\right] = 0 \tag{2.108}$$

where $\hat{z}_\epsilon = \dfrac{\partial}{\partial\epsilon}\hat{z}(y;\epsilon)$. We now restrict our attention to the set of all $y \in (-a, 0)$. As

$$\hat{z}(\epsilon) = \epsilon + \left|u'(y;\epsilon,0)\right|^2 \neq 0, \quad \forall \epsilon \geq 0, \quad y \in (-a,0) \tag{2.109}$$

it follows from (2.108) that

$$(1-\alpha)\hat{z}_\epsilon - 1 + \alpha\epsilon\hat{z}(\epsilon)^{-1}\hat{z}_\epsilon = 0, \quad \epsilon \geq 0, \quad y \in (-a,0) \tag{2.110}$$

in which case we find that

$$\hat{z}_\epsilon = -\frac{\hat{z}(\epsilon)}{(1-\alpha)\hat{z}(\epsilon) + \alpha\epsilon}, \quad \epsilon \geq 0, \quad y \in (-a,0). \tag{2.111}$$

As a direct consequence of (2.111) we see that

$$0 \le \hat{z}_\epsilon(y;\epsilon,0) \le \frac{1}{1-\alpha}, \quad \epsilon \ge 0, \quad y \in (-a,0). \tag{2.112}$$

Now, for $y \in (-a,0)$ we may write that

$$\hat{z}(y;\epsilon) = \hat{z}(y;0) + \int_0^\epsilon \hat{z}_\lambda(y;\lambda)\,d\lambda. \tag{2.113}$$

Combining (2.112) and (2.113), we then have

$$0 \le \hat{z}(y;\epsilon) - \hat{z}(y;0) \le \frac{\epsilon}{1-\alpha}; \quad \epsilon \ge 0,\ y \in [-a,0) \tag{2.114}$$

where we have used the continuity of $u'(y;\epsilon,\mu)$ to extend the result to $y = -a$. However,

$$\hat{z}(0;0) = |\hat{w}(0;0)|^2 = u'^2(0;0,0) = 0 \tag{2.115a}$$

and

$$\begin{aligned} \hat{z}(0,\epsilon) &= \epsilon + |\hat{w}(0;\epsilon)|^2 \\ &= \epsilon + \left|u'(0;\epsilon,0)\right|^2 \\ &= \epsilon \end{aligned} \tag{2.115b}$$

so $\hat{z}(0;\epsilon) - \hat{z}(0;0) = \epsilon < \dfrac{\epsilon}{1-\alpha}$, for $0 < \alpha < 1$, and, thus, equation (2.114) also holds at $y = 0$. Now, (2.114) is equivalent to

$$0 \le \left[\epsilon + \hat{w}^2(y;\epsilon)\right] - \hat{w}^2(y;0) \le \frac{\epsilon}{1-\alpha}, \quad y \in [-a,0]$$

or

$$-\epsilon \le \hat{w}^2(y;\epsilon) - \hat{w}^2(y;0) \le \frac{\alpha\epsilon}{1-\alpha}, \quad y \in [-a,0] \tag{2.116}$$

which, in turn, yields the two estimates

$$\hat{w}^2(y;0) - \epsilon \le \hat{w}^2(y;\epsilon), \tag{2.117a}$$

$$\hat{w}^2(y;\epsilon) \le \hat{w}^2(y;0) + \frac{\alpha\epsilon}{1-\alpha} \tag{2.117b}$$

on $[-a,0]$. Consider the set of all $y \in [-a,0]$ such that $\hat{w}(y;0) \ge \sqrt{\epsilon}$ for fixed $\epsilon > 0$; for y in this set it follows from (2.117b) that

$$0 \le \hat{w}^2(y;\epsilon) \le \hat{w}^2(y;0) + \frac{\alpha\epsilon}{1-\alpha}, \quad \begin{cases} y \in [-a,0] \\ \hat{w}(y;0) \ge \sqrt{\epsilon} \end{cases} \tag{2.118}$$

with the upper bound holding, of course, on all of $[-a,0]$. Therefore, for all $y \in [-a,0]$, such that $\hat{w}(y;0) \ge \sqrt{\epsilon}$,

$$0 \le \hat{w}(y;\epsilon) \le \hat{w}(y;0) + \sqrt{\frac{\alpha\epsilon}{1-\alpha}} \tag{2.119}$$

and we have used the fact that (2.104), with $p_1^* < 0$, implies that $\hat{w}(y;\epsilon) > 0$, $\forall\epsilon \ne 0$, $y \in [-a,0)$. Now, suppose that $y \in [-a,0]$ but $\hat{w}(y;0) < \sqrt{\epsilon}$; then by (2.117b) we have

$$\hat{w}^2(y;\epsilon) < \epsilon + \frac{\alpha\epsilon}{1-\alpha} = K_\alpha^2\epsilon \tag{2.120}$$

with $K_\alpha^2 = 1/(1-\alpha)$. Thus, if $y \in [-a,0]$ and $\hat{w}(y;0) < \sqrt{\epsilon}$ then

$$\hat{w}(y;\epsilon) < K_\alpha\sqrt{\epsilon} \tag{2.121}$$

and

$$\begin{aligned} |\hat{w}(y;\epsilon) - \hat{w}(y;0)| &\le \hat{w}(y;\epsilon) + \hat{w}(y;0) \\ &< (1+K_\alpha)\sqrt{\epsilon} \end{aligned}$$

or

$$\hat{w}(y;0) - (1+K_\alpha)\sqrt{\epsilon} < \hat{w}(y;\epsilon) < \hat{w}(y;0) + (1+K_\alpha)\sqrt{\epsilon} \tag{2.122}$$

for all $y \in [-a,0]$ such that $\hat{w}(y;0) < \sqrt{\epsilon}$. However,

$$K_\alpha = \sqrt{1 + \frac{\alpha}{1-\alpha}} > \sqrt{\frac{\alpha}{1-\alpha}}$$

so a comparison of (2.119) and (2.122) shows that (2.122) holds for all $y \in [-a,0]$. Using the definitions of $\hat{w}(y;\epsilon)$, $\hat{w}(y;0)$ we may rewrite (2.122) as

$$\left|u'(y;\epsilon,0) - u_0'(y)\right| < (1+K_\alpha)\sqrt{\epsilon}, \quad y \in [-a,0] \tag{2.123}$$

where $K_\alpha = \dfrac{1}{\sqrt{1-\alpha}}$. Replacing, y by $-y$, for $y \in [0,a]$, we see that (2.123) holds for all y, $-a \le y \le a$, as both $u'(y;\epsilon,0)$ and $u_0'(y)$ are odd functions on $[-a,a]$; this establishes the validity of the estimate (2.101) and concludes the proof of the lemma. □

Lemma 2.1 enables us to compare $u(y;\epsilon,0)$ with $u_0(y)$ on $[-a,a]$; our next set of lemmas are aimed at enabling us to compare $u(y;\epsilon,\mu_1)$ with $u(y;\epsilon,0)$, the first of these being stated as follows:

Lemma 2.2. *Let $u(y;\epsilon,\mu_1)$ be the unique classical solution of* (2.55a,b) *and set $t(y;\epsilon,\mu_1) = u'''(y;\epsilon,\mu_1)$. Then $\exists C_+, C_- > 0$, both independent of ϵ and μ, such that*

$$t(y;\epsilon,\mu_1) \le C_+, \quad y \in [-a,0], \tag{2.124a}$$

$$t(y;\epsilon,\mu_1) \ge -C_-, \quad y \in [0,a]. \tag{2.124b}$$

Proof. We will establish only (2.124a), which is all that is needed in the sequel: the proof of (2.124b) follows in an entirely analogous fashion. We begin by recalling that $u(y;\epsilon,\mu)$ and $u(y;\epsilon,0)$ are, respectively, the solutions of the nonlinear ordinary differential equations

$$\left\{\left[\epsilon + w^2(y;\epsilon,\mu_1)\right]^{-\alpha/2} w(y;\epsilon,\mu_1)\right\}' - \mu_1^* w'''(y;\epsilon,\mu_1) = p_1^*, \tag{2.125a}$$

$$\left\{\left[\epsilon + \hat{w}^2(y;\epsilon)\right]^{-\alpha/2} \hat{w}(y;\epsilon)\right\}' = p_1^* \tag{2.125b}$$

subject to $u(\pm a;\epsilon,\mu_1) = u''(\pm a;\epsilon,\mu_1) = 0$ and $u(\pm a;\epsilon,0) = 0$, where $\mu_1^* = \mu_1/\mu_0$. Subtracting (2.125a) from (2.125b), and integrating with respect to y, we obtain

$$\frac{w(y;\epsilon,\mu)}{[\epsilon + w^2(y;\epsilon,\mu_1)]^{\alpha/2}} - \frac{\hat{w}(y;\epsilon)}{[\epsilon + \hat{w}^2(y;\epsilon)]^{\alpha/2}} = \mu_1^* w''(y;\epsilon,\mu_1). \tag{2.126}$$

For future reference we record here the following: first of all, as

$$u(-y;\epsilon,\mu_1) = u(y;\epsilon,\mu_1), \quad y \in [-a,a]$$

not only is $u'(y;\epsilon,\mu_1)$ an odd function on $[-a,a]$ but so is $u'''(y;\epsilon,\mu_1)$, while $u''(y;\epsilon,\mu_1)$ is an even function: in particular, $\forall \epsilon, \mu_1 \ge 0$, $u'''(0;\epsilon,\mu_1) = 0$. Next we observe that the use of definition (2.100) enables us to write (2.125a) in the form

$$\gamma\left[u'(y;\epsilon,\mu_1)\right]' - \mu_1^* u''''(y;\epsilon,\mu_1) = p_1^* \tag{2.127}$$

and that

$$\gamma'(w) = (\epsilon + w^2)^{-\alpha/2}\left[1 - \alpha w^2(\epsilon + w^2)^{-1}\right] \tag{2.128a}$$

$$\gamma''(w) = -w\left\{\frac{\alpha}{(\epsilon + w^2)^{(\alpha/2)+1}}\left[\frac{\epsilon + (1-\alpha)w^2}{\epsilon + w^2}\right] + \frac{2\alpha\epsilon}{(\epsilon + w^2)^{2+(\alpha/2)}}\right\} \tag{2.128b}$$

from which it follows, as $0 < \alpha < 1$, that $\gamma'(w) > 0$, $\forall \epsilon > 0$, while $\operatorname{sgn} \gamma''(w) = -\operatorname{sgn} w$. Now, from (2.127) with

$$s(y;\epsilon,\mu_1) = u''(y;\epsilon,\mu_1) \tag{2.129}$$

we have

$$\gamma'(u'(y;\epsilon,\mu_1))s(y;\epsilon,\mu_1) - \mu_1^* s''(y;\epsilon,\mu_1) = p_1^* < 0, \tag{2.130a}$$

$$s(-a;\epsilon,\mu_1) = s(a;\epsilon,\mu_1) = 0. \tag{2.130b}$$

Suppose that $s(y;\epsilon,\mu_1)$ takes a positive maximum at some $y_0 \in (-a,a)$, so that $s(y_0;\epsilon,\mu_1) > 0$. From (2.130a) we have

$$\gamma'\left[u'(y_0;\epsilon,\mu_1)\right]s(y_0;\epsilon,\mu_1) + |p_1^*| = \mu_1^* s''(y_0;\epsilon,\mu_1). \tag{2.131}$$

But $\gamma'(u'(y_0;\epsilon,\mu_1)) > 0$, while $s''(y_0;\epsilon,\mu_1) \le 0$, if y_0 is interior to $[-a,a]$ and s has a maximum there. Thus, $s(y;\epsilon,\mu_1)$ cannot achieve a positive maximum at a point $y_0 \in (-a,a)$ and any positive maximum of $s(y;\epsilon,\mu_1)$ must, therefore, occur at $y = \pm a$. In view of the boundary conditions (2.130b), it follows that there is no positive maximum for $s(y;\epsilon,\mu_1)$ anywhere on $[-a,a]$; thus, if the maximum of $s(y;\epsilon,\mu_1)$ occurs at an interior point $y_0 \in (-a,a)$ we must have $s(y_0;\epsilon,\mu_1) < 0$ in which case

$$s(y;\epsilon,\mu_1) \le s(y_0;\epsilon,\mu_1) < 0, \quad y \in (-a,a) \tag{2.132}$$

and the same result holds if the maximum occurs at $y = \pm a$, where s vanishes. By (2.131), $s(y;\epsilon,\mu_1)$ cannot have a zero maximum at interior point $y_0 \in (-a,a)$. Thus,

$$u''(y;\epsilon,\mu_1) < 0; \quad y \in (-a,a), \quad \epsilon,\mu_1 > 0 \tag{2.133}$$

which shows that the graph of $u(y;\epsilon,\mu_1)$ is concave (down) on $(-a,a)$. Now let $y \in (-a,\delta)$ for any $\delta \le a$. Then

$$\int_{-a}^{y} u'''(\lambda;\epsilon,\mu_1)\,d\lambda = u''(y;\epsilon,\mu_1) < 0 \tag{2.134}$$

and as y may be chosen arbitrarily close to $-a$ (and $u'''(y;\epsilon,\mu_1)$ is continuous in y on $(-a,a)$) it follows that

$$u'''(-a;\epsilon,\mu_1) < 0 \tag{2.135a}$$

$$u'''(a;\epsilon,\mu_1) > 0 \tag{2.135b}$$

since $u'''(y;\epsilon,\mu_1)$ is an odd function of y on $(-a,a)$. From the definitions of $s(y;\epsilon,\mu_1)$ and $t(y;\epsilon,\mu_1)$,

$$t(y;\epsilon,\mu_1) = s'(y;\epsilon,\mu_1), \quad y \in (-a,a). \tag{2.136}$$

Therefore, if we differentiate (2.131) with respect to y we readily obtain

$$\gamma'[u'(y;\epsilon,\mu_1)]t(y;\epsilon,\mu_1) + \gamma''[u'(y;\epsilon,\mu_1)][u''(y;\epsilon,\mu_1)]^2 - \mu_1^* t''(y;\epsilon,\mu_1) = 0. \tag{2.137}$$

The calculation leading to (2.137) may be validated by the following elementary argument: in Sect. 2.2 it was demonstrated that the boundary-value problem given by (2.130a,b), subject to the additional constraint $u(\pm a,\epsilon,\mu_1) = 0$, has a unique classical solution, i.e. a solution in $C^4(-a,a)$; in light of this observation, and the definition of γ,

$$s''(y;\epsilon,\mu_1) = \frac{1}{\mu_1^*}\left\{|p_1^*| + \gamma'\left[u'(y;\epsilon,\mu_1)\right]s(y;\epsilon,\mu_1)\right\}$$

is continuously differentiable in y on $(-a,a)$ and (2.137) holds. Repetition of this argument shows that the unique classical solution of the boundary-value problem is, in fact, in $C^\alpha(-a,a)$.

We now return to (2.137) and assume that $t(y;\epsilon,\mu)$ achieves a positive maximum at $y_0 \in (-a,a)$ so that $t''(y_0;\epsilon,\mu_1) \le 0$; then, by (2.137) it must be true that

$$\gamma'[u'(y_0;\epsilon,\mu_1)]t(y_0;\epsilon,\mu_1) + \gamma''[u'(y_0;\epsilon,\mu_1)][u''[y_0;\epsilon,\mu_1)]^2 \le 0. \tag{2.138}$$

However, by (2.128b)

$$\operatorname{sgn}\gamma''[u'(y_0;\epsilon,\mu_1)] = -\operatorname{sgn}u'(y_0;\epsilon,\mu_1). \tag{2.139}$$

But $u''(y;\epsilon,\mu_1) < 0$, $y \in (-a,a)$, while $u'(0;\epsilon,\mu_1) = 0$, so $u'(y;\epsilon,\mu_1) < 0$ for $y \in (0,a)$. Thus, if $y_0 \in (0,a)$, then by (2.139) we must have $\gamma''[u'(y_0;\epsilon,\mu_1)] > 0$, contradicting (2.138). This means, of course, that if $t(y;\epsilon,\mu_1)$ achieves a positive maximum at $y_0 \in (-a,a)$ then, in fact, $y_0 \in (-a,0)$; note that $t(0;\epsilon,\mu) = 0$ as $u'''(y;\epsilon,\mu_1)$ is odd on $(-a,a)$. At such a $y_0 \in (-a,0)$ we will have, by virtue of (2.138),

$$t(y_0;\epsilon,\mu_1) \le \frac{-\gamma''[u'(y_0;\epsilon,\mu_1)][u''(y_0;\epsilon,\mu_1)]^2}{\gamma'[u'(y_0;\epsilon,\mu_1)]}. \tag{2.140}$$

Now, at y_0, $t'(y_0,;\epsilon,\mu_1) = s''(y_0;\epsilon,\mu_1) = 0$, in which case it follows from (2.130a) that

$$\gamma[u'(y_0;\epsilon,\mu_1)]u''(y_0;\epsilon,\mu_1) = p_1^* < 0 \tag{2.141}$$

so that

$$u''(y_0;\epsilon,\mu_1) = p_1^*/\gamma'[u'(y_0;\epsilon,\mu_1)]. \tag{2.142}$$

Substituting from (2.144) into (2.140), we obtain

$$t(y_0;\epsilon,\mu_1) \leq \frac{-\gamma''[u'(y_0;\epsilon,\mu_1)]}{\gamma'^3[u'(y_0;\epsilon,\mu_1)]} p_1^{*2}. \tag{2.143}$$

However, by the pointwise bound established in Sect. 2.2, which precedes (2.46), we have – with $v_{\mu_1}(y) = u'(y;\epsilon,\mu_1)$ – the existence of $C_1 > 0$ such that

$$\max_{[-a,a]} \left|u'(y;\epsilon,\mu)\right| \leq C_1. \tag{2.144}$$

It now follows from (2.143) and (2.135a) that $\exists C_+ > 0$, independent of both ϵ and μ_1, such that

$$t(y;\epsilon,\mu_1) \leq t(y_0;\epsilon,\mu_1) \leq C_+, \quad y \in [-a,0]. \tag{2.145}$$

If $t(y;\epsilon,\mu_1) \leq 0$ on $(-a,0)$, so that no positive maximum exists on $[-a,0]$, then certainly (2.145) holds $\forall C_+ > 0$. An analogous argument, which begins with the assumption that $t(y;\epsilon,\mu_1)$ has a negative minimum on $(-a,a)$ can be used, as above, to establish the existence of a $C_- > 0$, independent of both ϵ and μ, such that

$$t(y;\epsilon,\mu_1) \geq -C_-, \quad y \in [0,a] \tag{2.146}$$

but we omit the details. □

The next lemma provides us with an upper bound for $u(y;\epsilon,\mu_1) - u(y;\epsilon,0)$; we have, specifically, the following result:

Lemma 2.3. *Let $u(y;\epsilon,\mu_1)$ be the unique classical solution of* (2.55a,b). *Then for all $\epsilon,\mu_1 > 0$, and all $y \in [-a,a]$,*

$$u(y;\epsilon,\mu_1) - u(y;\epsilon,0) \leq \frac{aC_+}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^* \tag{2.147}$$

where C_+, C_1, independent of both ϵ and μ, are the positive constants appearing, respectively, in (2.124a) *and* (2.144).

Proof. From (2.126) and the definitions of $\gamma(\cdot)$ and $t(y;\epsilon,\mu)$ we have

$$\gamma[w(y;\epsilon,\mu_1)] - \gamma[\hat{w}(y;\epsilon)] = \mu_1^* t(y;\epsilon,\mu_1). \tag{2.148}$$

However,

$$\gamma(w) - \gamma(\hat{w}) = \gamma'(\bar{w})(w - \hat{w}) \tag{2.149}$$

with

$$\hat{w}(y;\epsilon) \leq \bar{w}(y;\epsilon,\mu_1) \leq w(y;\epsilon,\mu_1)$$

for each fixed $y \in [-a, a]$, $\epsilon, \mu_1 > 0$. Thus,

$$w(y;\epsilon,\mu_1) - \hat{w}(y;\epsilon) = \frac{\mu_1^* t(y;\epsilon,\mu_1)}{\gamma'[\bar{w}(y;\epsilon,\mu_1)]} \tag{2.150}$$

From (2.128a),

$$\gamma'(w) = \frac{\epsilon + (1-\alpha)w^2}{(\epsilon + w^2)^{1+(\alpha/2)}}$$

so

$$\begin{aligned}\frac{1}{\gamma'(\bar{w})} = \frac{(\epsilon + \bar{w}^2)^{1+(\alpha/2)}}{\epsilon + (1-\alpha)\bar{w}^2} < \frac{(\epsilon + \bar{w}^2)^{1+(\alpha/2)}}{(1-\alpha)\epsilon + (1-\alpha)\bar{w}^2} &= \frac{1}{1-\alpha}(\epsilon + \bar{w}^2)^{\alpha/2} \\ &\leq \frac{1}{1-\alpha}\left(\sqrt{\epsilon} + |\bar{w}|\right)^{\alpha} \leq \frac{1}{1-\alpha}\left(\sqrt{\epsilon} + C_1\right)^{\alpha}\end{aligned}$$

by virtue of (2.144) and (2.149). Employing this last estimate for $[\gamma'(\bar{w})]^{-1}$, as well as (2.124a), in (2.150) we are led to the upper bound

$$w(y;\epsilon,\mu_1) - \hat{w}(y;\epsilon) \leq \frac{\mu_1^* C_+}{1-\alpha}\left(\sqrt{\epsilon} + C_1\right)^{\alpha}, \quad y \in [-a, 0). \tag{2.151}$$

Choosing $y \in (-a, 0]$, and integrating both sides of (2.151) from $-a$ to y, we have, in view of the definitions of $w(y;\epsilon,\mu_1)$ and $\hat{w}(y;\epsilon)$, and the fact that $u(-a;\epsilon,\mu_1) = 0$, for $\epsilon > 0$ and $\mu_1 \geq 0$,

$$u(y;\epsilon,\mu_1) - u(y;\epsilon,0) \leq \frac{aC_+}{1-\alpha}\left(\sqrt{\epsilon} + C_1\right)^{\alpha}\mu_1^*, \quad y \in [-a, 0]. \tag{2.152}$$

However, $u(y;\epsilon,\mu_1) = u(-y;\epsilon,\mu_1)$, for $y \in [-a, a]$, so we see that the upper bound in (2.152) is, in fact, valid for all y, $-a \leq y \leq a$. □

The next to the last lemma in this sequence is

Lemma 2.4. *Let $u(y;\epsilon,\mu_1)$ be the unique classical solution of* (2.55a,b). *Then for $-a \leq y \leq a$,*

$$|u(y;\epsilon,\mu_1) - u(y;\epsilon,0)| \le \frac{\sqrt{aC_2}}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*1/2} \tag{2.153}$$

with $C_1 > 0$ *the constant appearing in* (2.144)*, and* $C_2 > 0$ *the constant appearing in the estimate* (2.46)*.*

Proof. By virtue of (2.150), it follows that for $y \in [-a, 0]$,

$$\left|\int_{-a}^{y}[w(\lambda;\epsilon,\mu_1) - \hat{w}(\lambda;\epsilon)]\,d\lambda\right| = \mu_1^*\left|\int_{-a}^{y}\frac{t(\lambda;\epsilon,\mu_1)}{\gamma'(\bar{w}(\lambda;\epsilon,\mu_1)]}\,d\lambda\right| \tag{2.154}$$

from which we obtain

$$|u(y;\epsilon,\mu_1) - u(y;\epsilon,0)| \le \frac{\mu_1^*}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\int_{-a}^{y}|t(\lambda;\epsilon,\mu_1)|\,d\lambda \tag{2.155}$$

by again bounding $[\gamma'(\bar{w})]^{-1}$ from above and using the definitions of w and $\hat{w}$. However,

$$\begin{aligned}\mu_1^*\int_{-a}^{y}|t|\,d\lambda &= \mu_1^{*1/2}\int_{-a}^{y}\mu_1^{*1/2}|t|\,d\lambda \\ &\le \mu_1^{*1/2}\left(\int_{-a}^{y}d\lambda\right)^{1/2}\left(\int_{-a}^{y}\mu_1^*|t|^2 d\lambda\right)^{1/2} \\ &\le \sqrt{a\mu_1^*}\left(\int_{-a}^{a}\mu_1^*|t|^2\,d\lambda\right)^{1/2}.\end{aligned}$$

However, by virtue of the estimate (2.46) of Sect. 2.2, there exists a positive constant, independent of both ϵ and μ_1, which we will denote by C_2, such that

$$\int_{-a}^{a}\mu_1^*|t|^2\,d\lambda \le C_2$$

and we are led to the bound

$$\mu_1^*\int_{-a}^{y}|t|\,d\lambda \le (a\mu_1^* C_2)^{1/2}. \tag{2.156}$$

Use of the bound (2.156) in the estimate (2.155) now yields the estimate (2.153) for $y \in [-a, 0]$ and the fact that $u(y;\epsilon,\mu_1)$ is an even function of y on $[-a, a]$, for all $\epsilon, \mu_1 \ge 0$, then establishes the validity of the estimate in (2.153) for all y, $-a \le y \le a$. □

Our final lemma in this section is merely a synthesis of Lemmas 2.1–2.4, namely,

Lemma 2.5. *Let* $u(y;\epsilon,\mu_1)$ *be the unique classical solution of the problem* (2.55a,b)*. Then,* $\exists C_+, C_1, C_2$*, all positive and independent of both* ϵ *and* μ_1*, such* *that for all* y, $-a \le y \le a$,

$$\frac{-\sqrt{aC_2}}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*\,1/2} \le u(y;\epsilon,\mu_1)-u(y;\epsilon,0) \le \frac{aC_+}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*}. \tag{2.157}$$

We are now in a position to state and prove the basic result of this section, i.e., we have

Theorem 2.3. *If* $u(y;\epsilon,\mu_1)$ *is the unique classical solution of the boundary-value problem* (2.55a,b)*, while* $\mu_0(y)$ *is the corresponding solution of the boundary-value problem* (2.56a,b)*, then* $\exists C_+, C_1, C_2$*, all positive and independent of* ϵ *and* μ_1*, such that for all* y, $-a \le y \le a$*, we have, with* $K_\alpha = (1-\alpha)^{-1/2}$, $0 < \alpha < 1$,

$$\begin{aligned}-(1+K_\alpha)a\sqrt{\epsilon}-\frac{\sqrt{aC_2}}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*\,1/2} &\le u(y;\epsilon,\mu_1)-\mu_0(y)\\ &\le (1+K_\alpha)a\sqrt{\epsilon}+\frac{aC_+}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*}.\end{aligned} \tag{2.158}$$

Proof. By virtue of Lemma 2.1, we have

$$-(1+K_\alpha)\sqrt{\epsilon} < u'(y;\epsilon,0)-u_0'(y) < (1+K_\alpha)\sqrt{\epsilon} \tag{2.159}$$

for all $y \in [-a,a]$, where $K_\alpha = (1-\alpha)^{-1/2}$. By integrating (2.159) from $-a$ to y, for $y \in [-a,0)$, we find that

$$-(1+K_\alpha)a\sqrt{\epsilon} < u(y;\epsilon,0)-u_0(y) < (1+K_\alpha)a\sqrt{\epsilon} \tag{2.160}$$

with this last result holding for all $y \in [-a,a]$, as $u(y;\epsilon,0)$, $u_0(y)$ are both even functions of y. Since

$$u(y;\epsilon,\mu_1)-u_0(y) = [u(y;\epsilon,\mu_1)-u(y;\epsilon,0)] + [u(y;\epsilon,0)-u_0(y)]$$

the Theorem 2.3 now follows by combining (2.157) and (2.159). □

As a direct consequence of the estimates in Theorem 2.3, we have the following bounds for the difference of the net volume flows Q_{ϵ,μ_1} and Q_0, and the mean velocities $\bar{u}_{\epsilon,\mu_1}$ and $\bar{u}_0$:

Theorem 2.4. *Under the same conditions as those which prevail in Theorem 2.3 the difference* $\bar{u}_{\epsilon,\mu_1}-\bar{u}_0$ *of the mean velocities also satisfies the estimate* (2.158)*, while the difference* $Q_{\epsilon,\mu_1}-Q_0$ *of the volume flows satisfies*

$$\begin{aligned}-2(1+K_\alpha)a^2\sqrt{\epsilon}-\frac{2a^{3/2}C_2^{1/2}}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*\,1/2} &\le Q_{\epsilon,\mu_1}-Q_0\\ &\le 2(1+K_\alpha)a^2\sqrt{\epsilon}+\frac{2a^2C_+}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*}.\end{aligned} \tag{2.161}$$

Remarks. The proof of Theorem 2.4 is a direct consequence of the estimates in (2.158) and the definitions of the mean velocities and volume flows. Moreover, in view of (2.94) and (2.161) we may exhibit the explicit bounds

$$Q_{\epsilon,\mu_1} \geq 2a\left(\frac{1-\alpha}{3-2\alpha}\right)\left[\frac{|p_1|a^{2-\alpha}}{\mu_0 2^{\alpha/2}}\right]^{1/(1-\alpha)} - 2(1+K_\alpha)a^2\sqrt{\epsilon} - \frac{2a^{3/2}C_2^{1/2}}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^\alpha\left(\frac{\mu_1}{\mu_0}\right)^{1/2} \tag{2.162a}$$

and

$$Q_{\epsilon,\mu} \leq 2a\left(\frac{1-\alpha}{3-2\alpha}\right)\left[\frac{|p_1|a^{2-\alpha}}{\mu_0 2^{\alpha/2}}\right]^{1/(1-\alpha)} + 2(1+K_\alpha)a^2\sqrt{\epsilon} - \frac{2a^2C_+}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^\alpha\left(\frac{\mu_1}{\mu_0}\right). \tag{2.162b}$$

Similar estimates may be developed for the friction factor f_{ϵ,μ_1} in (2.69a), by employing the bounds for $\bar{u}_{\epsilon,\mu_1}$, if we observe that (2.55a) is equivalent to

$$\frac{\partial}{\partial y}\tau_{12}(y;\epsilon,\mu_1) = p_1 \tag{2.163}$$

so that for all $\epsilon, \mu_1 \geq 0$, $\tau_{12}(\pm a;\epsilon,\mu_1) = \pm|p_1|a$.

2.4 Uniqueness of Steady Poiseuille Flow in the Class of Equilibrium Flows Between Parallel Plates

2.4.1 Introduction

In Sect. 2.2 we considered the problem of existence and uniqueness for steady Poiseuille flow of an incompressible, bipolar, viscous fluid in a parallel-wall channel. In rectangular Cartesian coordinates the flow assumes the form $\boldsymbol{v}^p = (u(y), 0, 0)$ and satisfies the nonlinear boundary-value problem (2.55a,b) where the channel walls are located at $y = \pm a$. The existence of a unique solution u of (2.55a,b) was established in the set W_M, for $M > 0$ sufficiently large, W_M as given by (2.12), with $B_M(0)$ the ball of radius M in $H_0^{\frac{3}{2}+\delta}(-a,a)$, $0 < \delta < \frac{1}{2}$. In this section we will consider the broader problem of uniqueness for steady, bipolar, viscous flows in the domain Ω_a specified by (2.8). It will be more convenient in this section to return to the subscript notation for coordinates and vector components. Therefore, we will write for Ω_a,

$$\Omega_a = \{(x_1,x_2,x_3) \mid x_2 \in [-a,a],\ -\infty < x_1, x_3 < \infty\}. \tag{2.164}$$

In the domain Ω_a, a steady, bipolar, viscous flow $\boldsymbol{v}$ (without an external forcing function) will satisfy

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nabla \cdot (2\mu \boldsymbol{e}) - 2\mu_1 \nabla \cdot (\Delta \boldsymbol{e}) \tag{2.165a}$$

$$\operatorname{div} \boldsymbol{v} = \boldsymbol{0} \tag{2.165b}$$

where $\mu = \mu(|\boldsymbol{e}|)$ represents the nonlinear viscosity

$$\mu = \mu_0(\epsilon + e_{kl}e_{kl})^{-\alpha/2}. \tag{2.165c}$$

In addition to (2.165a,b), $\boldsymbol{v}$ must satisfy, on $\partial\Omega_a$,

$$\boldsymbol{v}(x_1, \pm a, x_3) = \boldsymbol{0}, \quad \tau_{i22}(\boldsymbol{v})\big|_{x_2=\pm a} = 0, \quad i = 1, 2, 3 \tag{2.166}$$

where the higher-order boundary conditions in (2.166) follow from the stipulation that

$$\tau_{ijk}\nu_j\nu_k - \tau_{jkl}\nu_j\nu_k\nu_l\nu_i = 0, \quad i = 1, 2, 3$$

on $\partial\Omega_a$, coupled with the fact that the exterior unit normal is given by $\boldsymbol{\nu} = (0, \pm 1, 0)$. For the steady Poiseuille velocity field in Ω_a we will write

$$\boldsymbol{v}^p = (u(x_2; \epsilon, \mu_1), 0, 0), \quad -a \leq x_2 \leq a \tag{2.167}$$

where $u(x_2; \epsilon, \mu_1) \equiv u(x_2)$ satisfies

$$\mu_0 \left[\left(\epsilon + \frac{1}{2} u'^2(x_2) \right)^{-\alpha/2} u'(x_2) \right]' - \mu_1 u''''(x_2) = p_1, \quad -a < x_2 < a, \tag{2.168a}$$

$$u(\pm a) = u''(\pm a) = 0 \tag{2.168b}$$

with $p_1 = \dfrac{\partial p}{\partial x_2}$ the constant pressure gradient. At this point, we know that there exists at least one solution of the nonlinear boundary-value problem (2.165a,b,c), (2.166), such that

$$\boldsymbol{v} - \boldsymbol{v}^p \in \boldsymbol{H}^4(\Omega_a) \tag{2.169}$$

namely, $\boldsymbol{v} = \boldsymbol{v}^p$.

Our goal in this section will be to show that, under specific restrictions on the constitutive parameters ϵ, μ_0, μ_1, and α, the plate separation $2a$, and the constant pressure gradient p_1 associated with the problem (2.168a,b), $\boldsymbol{v} = \boldsymbol{v}^p$ is the unique solution of the boundary-value problem (2.165a,b,c), (2.166) in the domain Ω_a

which satisfies the regularity condition (2.169); to this end we set $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}^p$ and examine some of the consequences of the condition $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$ in the next subsection.

2.4.2 The Condition $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$

In Sect. 2.4.1 we set $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}^p$ where $\boldsymbol{v}$ and $\boldsymbol{v}^p$ are, respectively, (1) any solution of (2.165a,b,c), (2.166) and (2) the vector field (2.167) which is determined by the unique, classical, solution of (2.168a,b). In this subsection we use the conventional notation

$$\boldsymbol{D}^{\boldsymbol{\beta}}\boldsymbol{w} = \frac{\partial^{|\boldsymbol{\beta}|}\boldsymbol{w}}{\partial^{\beta_1}x_1\partial^{\beta_2}x_2\partial^{\beta_3}x_3}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$, $\beta_i \geq 0$, $i = 1, 2, 3$, and $|\boldsymbol{\beta}| = \beta_1 + \beta_2 + \beta_3$. The condition $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$ then reads

$$\sum_{|\boldsymbol{\beta}|\leq 4} \int_{\Omega_a} \left|\left|\boldsymbol{D}^{\boldsymbol{\beta}}\boldsymbol{w}\right|\right|^2 d\boldsymbol{x} < \infty$$

which is equivalent to

$$\sum_{i=1}^{3} \sum_{|\boldsymbol{\beta}|\leq 4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-a}^{a} \left|\boldsymbol{D}^{\boldsymbol{\beta}}w_i(x_1, x_2, x_3)\right|^2 dx_2\right) dx_1\, dx_3 < \infty \tag{2.170}$$

Setting

$$W_i^{\boldsymbol{\beta}}(x_1, x_3) = \int_{-a}^{a} \left|\boldsymbol{D}^{\boldsymbol{\beta}}w_i(x_1, x_2, x_3)\right|^2 dx_2 \tag{2.171}$$

we may rewrite (2.170) in the form

$$\sum_{i=1}^{3} \sum_{|\boldsymbol{\beta}|\leq 4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_i^{\boldsymbol{\beta}}(x_1, x_3)dx_1\, dx_3 < \infty. \tag{2.172}$$

The finiteness of the integrals displayed in (2.172) now implies (see Fig. 2.2) that there exists a sequence $\{S_n\}$ of smooth surfaces in x_1, x_2, x_3 space intersecting, for each n, the planes at $x_2 = \pm a$, in a sufficiently smooth fashion, along which $x_1^2 + x_3^2 \to \infty$, as $n \to \infty$, and such that

$$W_i^{\boldsymbol{\beta}}(x_1, x_3)\big|_{\bar{S}_n} \to 0, \text{ as } n \to \infty \tag{2.173}$$

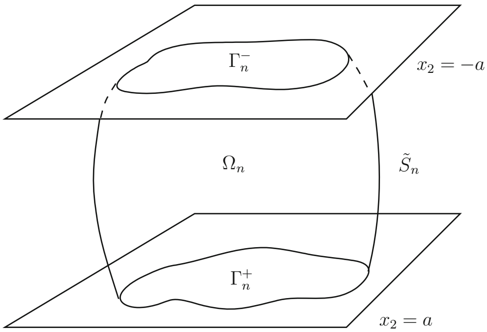

Fig. 2.2 The domain Ω_n in Lemma 2.6

for $|\boldsymbol{\beta}| \le 4$, $1 \le i \le 3$, where $\bar{S}_n$ is the set of all points (x_1, x_2, x_3) lying on S_n such that $-a < x_2 < a$. If we define $\Gamma_n^{\pm}$ to be the sets of points lying on the planes $x_2 = \pm a$ which are bounded, respectively, by the curves of intersection of S_n with the planes at $x_2 = \pm a$, then by the terminology "sufficiently smooth", in the definition of S_n, we mean simply that the bounded domain $\Omega_n \subseteq R^3$ with boundary

$$\partial\Omega_n = \bar{S}_n \cup \Gamma_n^+ \cup \Gamma_n^- \tag{2.174}$$

admits of application of the divergence theorem. By virtue of the definition (2.171) of $W_i^{\boldsymbol{\beta}}$, the criterion (2.173), and the Sobolev embedding lemma (see Appendix A), it follows that for $|\boldsymbol{\beta}| \le 3$,

$$\max_{[-a,a]} \left| \boldsymbol{D}^{\boldsymbol{\beta}} w_i(x_1, x_2, x_3) \right|_{\bar{S}_n} \to 0, \text{ as } n \to \infty \tag{2.175}$$

for each $i = 1, 2, 3$. The radiation conditions expressed by (2.175) are the principal consequences of the restriction $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$, which will be of use to us in the following subsections; among the results which follow from this restriction is the following lemma of Poincaré type:

Lemma 2.6. *Let* $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$ *with* $\boldsymbol{w}(\pm a) = 0$*; then,*

$$\int_{\Omega_a} ||\boldsymbol{w}||^2 d\boldsymbol{x} \le (2a^2 + \theta)^2 \int_{\Omega_a} \left|\left|\nabla^2 \boldsymbol{w}\right|\right|^2 d\boldsymbol{x} \tag{2.176}$$

for any $\theta > 0$*, where* $||\cdot||$ *is the Euclidean norm on* $\mathbb{R}^3$.

Proof. We set $\boldsymbol{w} = (w_1, w_2, w_3)$ and let w represent any of the w_i, $i = 1, 2$, or 3. As

$$w(x_1, x_2, x_3) = \int_{-a}^{x_2} \frac{\partial w}{\partial x_2}(x_1, \tau, x_3)\, d\tau$$

we have

$$w^2(x_1, x_2, x_3) \leq 2a \int_{-a}^{a} \left(\frac{\partial w}{\partial x_2}\right)^2 dx_2$$

and, therefore,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-a}^{a} w^2(x_1, x_2, x_3) dx_2\, dx_1\, dx_2 &\leq 4a^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-a}^{a} \left(\frac{\partial w}{\partial x_2}\right)^2 dx_2\, dx_1\, dx_3 \\ &\leq 4a^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-a}^{a} ||\nabla w||^2 dx_2\, dx_1\, dx_3.\end{aligned} \tag{2.177}$$

Now we consider $\int_{\Omega_a} ||\nabla w||^2\, d\boldsymbol{x}$ where Ω_n is the bounded domain in $\mathbb{R}^3$ which is bounded by the smooth surface $\partial\Omega_n$ of (2.174); we compute that

$$\begin{aligned}\int_{\Omega_n} ||\nabla w||^2\, d\boldsymbol{x} &= \int_{\Omega_n} \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_i}\, d\boldsymbol{x} = \int_{\Omega_n} \frac{\partial}{\partial x_i}\left(w\frac{\partial w}{\partial x_i}\right) d\boldsymbol{x} - \int_{\Omega_n} w\nabla^2 w\, d\boldsymbol{x} \\ &= \int_{\partial\Omega_n} w\frac{\partial w}{\partial x_i}\nu_{i(n)} d\sigma_n - \int_{\Omega_n} w\nabla^2 w\, d\boldsymbol{x} \\ &= -\int_{\Gamma_n^+} w\frac{\partial w}{\partial x_1} d\boldsymbol{a}^+ + \int_{\Gamma_n^-} w\frac{\partial w}{\partial x_1} d\boldsymbol{a}^- \\ &\quad + \int_{\tilde{S}_n} w\frac{\partial w}{\partial x_i}\tilde{\nu}_{i(n)} d\tilde{\sigma}_n - \int_{\Omega_n} w\nabla^2 w\, d\boldsymbol{x}\end{aligned} \tag{2.178}$$

where $\boldsymbol{\nu}(n)$ is the exterior unit normal to $\partial\Omega_n$, $\tilde{\boldsymbol{\nu}}_{(n)}$ the exterior unit normal to $\tilde{S}_n$, $d\boldsymbol{a}^{\pm}$ the infinitesimal surface elements in the domains $\Gamma_n^{\pm}$, located in the planes at $x_2 = \pm a$, $d\sigma_n$ the infinitesimal surface element on $\partial\Omega_n$, and $d\tilde{\sigma}_n$ the infinitesimal surface element on $\tilde{S}_n$. In as much as $w(\pm a) = 0$, w vanishes on $\Gamma_n^{\pm}$, for each n, and (2.178) reduces to

$$\int_{\Omega_n} ||\nabla w||^2 d\boldsymbol{x} = \int_{\tilde{S}_n} w\frac{\partial w}{\partial x_i}\tilde{\nu}_{i(n)} d\tilde{\sigma}_n - \int_{\Omega_n} w\nabla^2 w\, d\boldsymbol{x}. \tag{2.179}$$

Letting $n \to \infty$ in (2.179), and employing the radiation condition (2.175), we obtain

$$\int_{\Omega_a} ||\nabla w||^2\, d\boldsymbol{x} = \int_{\Omega_a} (-w)\nabla^2 w\, d\boldsymbol{x} \leq \frac{1}{2\lambda}\int_{\Omega_a} w^2 d\boldsymbol{x} + \frac{\lambda}{2}\int_{\Omega_a} (\nabla^2 w)^2\, d\boldsymbol{x} \tag{2.180}$$

for any $\lambda > 0$. Combining (2.180) with (2.177) we have, therefore, the estimate

$$\frac{1}{4a^2}\int_{\Omega_a} w^2\, d\boldsymbol{x} \leq \frac{1}{2\lambda}\int_{\Omega_a} w^2\, d\boldsymbol{x} + \frac{\lambda}{2}\int_{\Omega} (\nabla^2 w)^2\, d\boldsymbol{x}$$

or, for any $\lambda > 2a^2$,

$$\int_{\Omega_a} w^2 \, d\boldsymbol{x} \leq \frac{\lambda}{2\left(\frac{1}{4a^2} - \frac{1}{2\lambda}\right)} \int_{\Omega_a} (\nabla^2 w)^2 \, d\boldsymbol{x}. \tag{2.181}$$

The lemma now follows if we write down (2.181) for each of w_1, w_2, and w_3, sum the resulting three estimates, and take $\lambda = 2a^2 + \theta$, with θ any positive constant.

□

2.4.3 Key Lemmas for Nonlinear Viscosity and the Poiseuille Flow in Ω_a

Prior to stating and proving the uniqueness theorem for steady channel flow of a bipolar, viscous fluid in Sect. 2.4.4, we will establish, in this subsection, two key lemmas; one of these relates directly to the structure of the nonlinear viscosity (2.165c) while the other is based on the structure of the Poiseuille flow field in Ω_a. The first result is the following inequality for vectors in $\mathbb{R}^n$:

Lemma 2.7. *Let* $\boldsymbol{u} = (u_1, \ldots, u_n)$*,* $\boldsymbol{v} = (v_1, \ldots, v_n)$*, and suppose that* $\epsilon > 0$ *and* $0 < \alpha < 1$*; then*

$$\sum_{i=1}^{n} \left(\frac{u_i}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} - \frac{v_i}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} \right) (u_i - v_i) \geq 0 \tag{2.182}$$

Proof. We let σ stand for the sum on the left-hand side of the inequality in (2.182); then,

$$\begin{aligned}
\sigma &= \frac{||\boldsymbol{u}||^2}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} + \frac{||\boldsymbol{v}||^2}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} - \boldsymbol{u} \cdot \boldsymbol{v} \left[\frac{1}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} + \frac{1}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} \right] \\
&\geq \frac{||\boldsymbol{u}||^2}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} + \frac{||\boldsymbol{v}||^2}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} - ||\boldsymbol{u}||||\boldsymbol{v}|| \left[\frac{1}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} + \frac{1}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} \right] \\
&= \left(\frac{||\boldsymbol{u}||}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} - \frac{||\boldsymbol{v}||}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} \right) (||\boldsymbol{u}|| - ||\boldsymbol{v}||) \,.
\end{aligned} \tag{2.183}$$

We now set, for $s \in R^1$,

$$\gamma(s) = s(\epsilon + s^2)^{-\alpha/2}, \quad 0 < \alpha < 1 \tag{2.184a}$$

then

$$\gamma'(s) = (\epsilon + s^2)^{-(\alpha/2+1)}(\epsilon + (1-\alpha)s^2) \tag{2.184b}$$

so that $\gamma'(s) > 0$, in which case for $s_1 \geq s_2$,

$$(\gamma(s_1) - \gamma(s)2))(s_1 - s_2) \geq 0$$

and the Lemma 2.7 follows from the last inequality in (2.183). □

The second lemma in this subsection depends on the qualitative behavior of solutions to the nonlinear boundary-value problem (2.168a,b) as well as on the structure of the solutions to the associated non-Newtonian problem (2.56a,b), which we rewrite here as

$$\mu_0 \left[\left(\frac{1}{2} u_0'(x_2) \right)^{-\alpha/2} u_0'(x_2) \right]' = p_1, \quad -a < x_2 < a \tag{2.185a}$$

$$u_0(\pm a) = 0. \tag{2.185b}$$

Lemma 2.8. *Let $\boldsymbol{v}^p$ be defined by* (2.167) *with $u(x_2; \epsilon, \mu_1)$, $-a \leq x_2 \leq a$, the unique solution of* (2.168a,b). *Let $\boldsymbol{e}^p = \frac{1}{2}(\nabla \boldsymbol{v}^p + (\nabla \boldsymbol{v}^p)^t)$ be the associated rate of deformation tensor. Then, for any vector field $\boldsymbol{w}(\cdot) \in \boldsymbol{L}^2(\Omega_a)$,*

$$-\int_{\Omega_a} \boldsymbol{w} \cdot \boldsymbol{e}^p \cdot \boldsymbol{w}\, d\boldsymbol{x} \leq \Gamma \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x} \tag{2.186}$$

where

$$\Gamma = \Gamma(a, p_1; \mu_0, \mu_1, \alpha, \epsilon) \equiv a \left(\frac{|p_1|a}{\mu_0 2^{\alpha/2}} \right)^{1/(1-\alpha)} + \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-\alpha}}\right)\sqrt{\epsilon}. \tag{2.187}$$

Proof. We let $\boldsymbol{x} \in \Omega_a$ and refer the rate of deformation tensor $\boldsymbol{e}^p$ to its principal axes at $\boldsymbol{x}$. Then at $\boldsymbol{x}$

$$\boldsymbol{w} \cdot \boldsymbol{e}^p \cdot \boldsymbol{w} = \boldsymbol{e}^p_{ij} w_i w_j \geq ||\boldsymbol{w}||^2 \cdot \min[e^p_{11}, e^p_{22}, e^p_{33}] \tag{2.188}$$

where the e^p_{ii}, $i = 1, 2, 3$, are the eigenvalues of $\boldsymbol{e}^p$. As

$$\operatorname{div} \boldsymbol{v}^p = \operatorname{tr} \boldsymbol{e}^p = e^p_{11} + e^p_{22} + e^p_{33} = 0 \tag{2.189}$$

at least one of the eigenvalues of $\boldsymbol{e}^p$ must be negative at $\boldsymbol{x}$. We denote the largest negative eigenvalue of $\boldsymbol{e}^p$ at $\boldsymbol{x} \in \Omega_a$ by $-|\lambda_{\boldsymbol{e}^p}(\boldsymbol{x})|$ so that, at $\boldsymbol{x}$,

$$\boldsymbol{w} \cdot \boldsymbol{e}^p \cdot \boldsymbol{w} \geq -|\lambda_{\boldsymbol{e}^p}(\boldsymbol{x})|||\boldsymbol{w}||^2, \quad \boldsymbol{x} \in \Omega_a. \tag{2.190}$$

However, $\boldsymbol{v}^p$ and, hence, $\boldsymbol{e}^p$ and $\lambda_{\boldsymbol{e}^p}$ depend only on x_2 (and not on either x_1 or x_3) so $\lambda_{\boldsymbol{e}^p}(\boldsymbol{x}) = \bar{\lambda}_{\boldsymbol{e}^p}(x_2)$, $-a \le x_2 \le a$. Setting, therefore

$$|\lambda| = \max_{-a \le x_2 \le a} \left|\bar{\lambda}_{\boldsymbol{e}^p}(x_2)\right| \tag{2.191}$$

we have

$$\boldsymbol{w}(\boldsymbol{x}) \cdot \boldsymbol{e}^p(\boldsymbol{x}) \cdot \boldsymbol{w}(\boldsymbol{x}) \ge -|\lambda| ||\boldsymbol{w}(\boldsymbol{x})||^2, \quad \boldsymbol{x} \in \Omega_a \tag{2.192}$$

in which case

$$-\int_{\Omega_a} \boldsymbol{w} \cdot \boldsymbol{e}^p \cdot \boldsymbol{w}\, d\boldsymbol{x} \le |\lambda| \int_{\Omega_a} ||\boldsymbol{w}(x)||^2\, d\boldsymbol{x}. \tag{2.193}$$

Now, consider the tensor field

$$\nabla \boldsymbol{v}^p = \begin{pmatrix} 0 & u'(x_2;\epsilon,\mu_1) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which yields the rate of deformation tensor

$$\boldsymbol{e}^p = \frac{1}{2} \begin{pmatrix} 0 & u'(x_2;\epsilon,\mu_1) & 0 \\ u'(x_2;\epsilon,\mu_1) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.194}$$

Therefore,

$$\det(\boldsymbol{e}^p - \eta \boldsymbol{I}) = -\eta \left(\eta^2 - \frac{1}{4} u'^2(x_2;\epsilon,\mu_1) \right) \tag{2.195}$$

so that the eigenvalues of $\boldsymbol{e}^p$ at any $\boldsymbol{x} \in \Omega_a$ are given by $\eta_1 = 0$, $\eta_{2,3} = \pm u'(x_2;\epsilon,\mu_1)$. We now avail ourselves of the results in Sect. 2.3.3. As $u'(x_2;\epsilon,\mu_1) > 0$, $-a < x_2 < 0$, and $u'(x_2;\epsilon,\mu_1) < 0$, $0 < x_2 < a$,

$$\begin{aligned} |\lambda| &= \frac{1}{2} \max_{[-a,a]} \left| u'(x_2;\epsilon,\mu_1) \right| \\ &= \frac{1}{2} \max_{[-a,a]} u'(x_2,\epsilon,\mu_1). \end{aligned} \tag{2.196}$$

But, $u''(x_2;\epsilon,\mu_1) < 0$, for $x_2 \in (-a,a)$, $\epsilon, \mu_1 > 0$ so, in fact,

$$|\lambda| = \frac{1}{2} u'(-a;\epsilon,\mu_1). \tag{2.197}$$

Now, if we define, once again, $\gamma(s)$, $s \in R^1$, by (2.184a), then (2.168a) may be written in the form

$$\mu_0 \gamma(u')' - \mu_1 u'''' = p_1. \tag{2.198}$$

Writing (2.198) down for the cases $\mu_1 > 0$, and $\mu_1 = 0$, subtracting the resulting equations, and then integrating with respect to x_2, we find that

$$\gamma(u'(x_2; \epsilon, \mu_1)) - \gamma(u'(x_2; \epsilon, 0)) = \mu_1^* u'''(x_2; \epsilon, \mu_1) \tag{2.199}$$

where $\mu_1^* = (\mu_1/\mu_0)$. Setting $x_2 = -a$ in (2.199), and using the fact that $u'''(-a; \epsilon, \mu_1) < 0$, (all derivatives at $x_2 = -a$ are, of course, the usual right-handed derivatives) we find that

$$\gamma(u'(-a; \epsilon, \mu_1)) < \gamma(u'(-a; \epsilon, 0)); \quad \epsilon, \mu_1 > 0. \tag{2.200}$$

In view of (2.184b), $\gamma'(s) > 0$, $\forall s \in R^1$, so it follows from (2.200) that

$$u'(-a; \epsilon, \mu_1) < u'(-a; \epsilon, 0), \quad \epsilon, \mu_1 > 0 \tag{2.201}$$

and, thus, by (2.197)

$$|\lambda| < \frac{1}{2} u'(-a; \epsilon, 0), \quad \epsilon, \mu_1 > 0. \tag{2.202}$$

But, in light of (2.123), $\forall x_2 \in [-a, 0]$ and $\epsilon > 0$,

$$u'(x_2; \epsilon, 0) < u_0'(x_2) + \left(1 + \frac{1}{\sqrt{1-\alpha}}\right)\sqrt{\epsilon}. \tag{2.203}$$

A direct calculation based on (2.76) and (2.77), with y replaced by x_2, produces

$$u_0'(-a) = a\left(\frac{|p_1|a}{u_0 2^{\alpha/2}}\right)^{1/(1-\alpha)}, \quad 0 < \alpha < 1 \tag{2.204}$$

so that, by virtue of (2.202)–(2.204),

$$|\lambda| < \frac{a}{2}\left(\frac{|p_1|a}{\mu_0^0 2^{\alpha/2}}\right)^{1/(1-\alpha)} + \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-\alpha}}\right)\sqrt{\epsilon}. \tag{2.205}$$

The desired conclusion of Lemma 2.8, namely, the estimate (2.186), with Γ given by (2.181), now follows directly from (2.193) and (2.205). □

2.4.4 Uniqueness of Solutions

Having established Lemmas 2.6–2.8, in the previous subsections, we are now in a position to state and prove the main result of this section, namely,

Theorem 2.5. *For $\mu_1 > 0$ sufficiently large $\boldsymbol{v}^p$, as defined by* (2.167)*,* (2.168a,b)*, is the unique solution of the nonlinear boundary-value problem* (2.165a,b)*,* (2.166) *satisfying* (2.169)*.*

Proof. For simplicity we will begin by replacing 2μ and $2\mu_1$, respectively, in (2.165a) by μ and μ_1 where $\mu = \mu(\boldsymbol{v})$ is given by (2.165c). We already know that $\boldsymbol{v}^p$ is a solution of (2.165a,b), (2.166) which obviously, satisfies (2.169); suppose that $\boldsymbol{v}(\boldsymbol{x})$ is any other solution. Then

$$\begin{aligned} \boldsymbol{v}\cdot\nabla\boldsymbol{v} &= -\nabla p + \nabla\cdot(\mu\boldsymbol{e}) - \mu_1\nabla\cdot(\Delta\boldsymbol{e}), \\ \boldsymbol{v}^p\cdot\nabla\boldsymbol{v}^p &= -\nabla p^p + \nabla\cdot(\mu_p\boldsymbol{e}^p) - \mu_1\nabla\cdot(\Delta\boldsymbol{e}^p) \end{aligned} \tag{2.206}$$

where $\boldsymbol{e}^p = \dfrac{1}{2}(\nabla\boldsymbol{v}^p + (\nabla\boldsymbol{v}^p)^t)$, and $\mu_p = \mu(\boldsymbol{e}^p)$, so that $\mu_p = \mu_0(\epsilon + e^p_{kl}e^p_{kl})^{-\alpha/2}$; also $\nabla p^p = (p_1, 0, 0)$. Of course $\boldsymbol{v}^p\cdot\nabla\boldsymbol{v}^p = 0$, and the second equation in (2.206) just reduces to (2.168a), but we shall find it convenient for our present purposes to leave it in the form in which we have written it. If we subtract the second equation in (2.206) from the first, and set

$$\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}^p \text{ and } P = p - p^p \tag{2.207}$$

then we obtain

$$(\boldsymbol{v}^p+\boldsymbol{w})\cdot\nabla(\boldsymbol{v}^p+\boldsymbol{w}) - \boldsymbol{v}^p\cdot\nabla\boldsymbol{v}^p = -\nabla P + \nabla\cdot(\mu\boldsymbol{e} - \mu_p\boldsymbol{e}^p) - \mu_1\nabla\cdot\Delta(\boldsymbol{e}-\boldsymbol{e}^p) \tag{2.208}$$

the above result holding throughout Ω_a. Expanding the left-hand side of (2.208), and then integrating the resulting expression over Ω_n (refer to Fig. 2.2), we obtain

$$\begin{aligned} \int_{\Omega_n} \boldsymbol{v}^p\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{w}\,d\boldsymbol{x} + \int_{\Omega_n} \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{w}\,d\boldsymbol{x} + \int_{\Omega_n} \boldsymbol{w}\cdot\nabla\boldsymbol{v}^p\cdot\boldsymbol{w}\,d\boldsymbol{x} &= -\int_{\Omega_n} \nabla p\cdot\boldsymbol{w}\,d\boldsymbol{x} \\ + \int_{\Omega_n} \nabla\cdot(\mu\boldsymbol{e} - \mu_0\boldsymbol{e}^p)\cdot\boldsymbol{w}\,d\boldsymbol{x} - \mu_1\int_{\Omega_n} \nabla\cdot\Delta(\boldsymbol{e}-\boldsymbol{e}^p)\cdot\boldsymbol{w}\,d\boldsymbol{x}. \end{aligned} \tag{2.209}$$

We now proceed to study each of the integrals over Ω_n, which are displayed in (2.209), in the limit as $n\to\infty$. First of all,

$$\begin{aligned} \int_{\Omega_n} \boldsymbol{v}^p\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{w}\,d\boldsymbol{x} &= \int_{\Omega_n} v^p_j\frac{\partial w_i}{\partial x_j}w_i\,d\boldsymbol{x} = \frac{1}{2}\int_{\Omega_n} v^p_i\frac{\partial}{\partial x_j}(w_i\,w_i)\,d\boldsymbol{x} \\ &= \frac{1}{2}\int_{\partial\Omega_n} v^p_j w_i w_i \nu_{j(n)} d\sigma_n - \frac{1}{2}\int_{\Omega_n} w_i w_i\frac{\partial v^p_j}{\partial x_j}\,d\boldsymbol{x}. \end{aligned}$$

But div $\boldsymbol{v}^p = 0$, while $\boldsymbol{w}$ vanishes on both $\Gamma_n^{\pm}$, so

$$\int_{\Omega_n} \boldsymbol{v}^p \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{w}\, d\boldsymbol{x} = \frac{1}{2} \int_{\tilde{S}_n} v_j^p w_i w_i \tilde{\nu}_{j(n)}\, d\tilde{\sigma}_n. \tag{2.210}$$

As $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$ by (2.169), it follows from (2.175) that $w_i\big|_{\tilde{S}_n} \to 0$, as $n \to \infty$, uniformly for $-a \leq x_2 \leq a$. Thus

$$\int_{\Omega_n} \boldsymbol{v}^p \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{w}\, d\boldsymbol{x} \to 0, \text{ as } n \to \infty. \tag{2.211}$$

Next,

$$\begin{aligned} \int_{\Omega_n} \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{w}\, d\boldsymbol{x} &= \int_{\Omega_n} w_i \frac{\partial w_i}{\partial x_j} w_j\, d\boldsymbol{x} = \frac{1}{2} \int_{\Omega_n} w_j \frac{\partial}{\partial x_j}(w_i w_i)\, d\boldsymbol{x} \\ &= \frac{1}{2} \int_{\partial\Omega_n} w_i w_i w_j \nu_{j(n)} d\sigma_n - \frac{1}{2} \int_{\Omega_n} w_i w_i \frac{\partial w_j}{\partial x_j}\, d\boldsymbol{x}. \end{aligned}$$

But $\nabla \cdot \boldsymbol{w} = 0$, with $\boldsymbol{w}$ vanishing on $\Gamma_n^{\pm}$, so

$$\int_{\Omega_n} \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{w}\, d\boldsymbol{x} = \frac{1}{2} \int_{\tilde{S}_n} w_i w_i w_j \tilde{\nu}_{j(n)} d\tilde{\sigma}_n \tag{2.212}$$

in which case, just as in (2.210),

$$\int_{\Omega_n} \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{w}\, d\boldsymbol{x} \to 0, \text{ as } n \to \infty. \tag{2.213}$$

Continuing, we have,

$$\begin{aligned} \int_{\Omega_n} \nabla P \cdot \boldsymbol{w}\, d\boldsymbol{x} &= \int_{\Omega_n} \nabla \cdot (P\boldsymbol{w})\, d\boldsymbol{x} \text{ (as div } \boldsymbol{w} = 0) \\ &= \int_{\partial\Omega_n} P\boldsymbol{w} \cdot \boldsymbol{\nu}_{(n)}\, d\sigma_n \\ &= \int_{\tilde{S}_n} P\boldsymbol{w} \cdot \tilde{\boldsymbol{\nu}}_{(n)}\, d\tilde{\sigma}_n. \end{aligned}$$

Thus, if P is bounded on Ω_a, then by (2.175)

$$\int_{\Omega_n} \nabla P \cdot \boldsymbol{w}\, d\boldsymbol{x} \to 0, \text{ as } n \to \infty \tag{2.214}$$

We now turn our attention to those integrals over Ω_n in (2.209) which do not vanish, in the limit, as $n \to \infty$. By Lemma 2.8

$$\lim_{n\to\infty} \int_{\Omega_n} \boldsymbol{w} \cdot \nabla \boldsymbol{v}^p \cdot \boldsymbol{w}\, d\boldsymbol{x} = \int_{\Omega_a} \boldsymbol{w} \cdot \nabla \boldsymbol{v}^p \cdot \boldsymbol{w}\, d\boldsymbol{x} \geq -\Gamma \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x} \tag{2.215}$$

with Γ given by (2.187). Also,

$$\begin{aligned}\int_{\Omega_n} \nabla \cdot (\mu \boldsymbol{e} - \mu_p \boldsymbol{e}^p) \cdot \boldsymbol{w}\, d\boldsymbol{x} &= \int_{\Omega_n} \frac{\partial}{\partial x_j}(\mu e_{ij} - \mu_p e_{ij}^p)(v_i - v_i^p)\, d\boldsymbol{x} \\ &= \int_{\partial\Omega_n} (\mu e_{ij} - \mu_p e_{ij}^p)(v_i - v_i^p)\nu_{j(n)}\, d\sigma_n \\ &\quad - \int_{\Omega_n} (\mu e_{ij} - \mu_0 e_{ij}^p) \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i^p}{\partial x_j}\right) d\boldsymbol{x} \\ &= \int_{\tilde{S}_n} (\mu e_{ij} - \mu_0 e_{ij}^p) w_i \tilde{\nu}_{j(n)}\, d\tilde{\sigma}_n \\ &\quad - \int_{\Omega_n} (\mu e_{ij} - \mu_0 e_{ij}^0) \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i^p}{\partial x_j}\right) d\boldsymbol{x}\end{aligned}$$

as $w_i = v_i - v_i^p$ vanishes on both $\Gamma_n^{\pm}$. The regularity requirement (2.169) now implies that

$$\mu_{ij} = \mu(\boldsymbol{v}) e_{ij}(\boldsymbol{v}) - \mu_p(\boldsymbol{v}^p) e_{ij}^p(\boldsymbol{v}^p) \tag{2.216}$$

is, for each pair i, j, $1 \leq i, j \leq 3$, bounded on Ω_a while, as previously, $w_i\big|_{\tilde{S}_n} \to 0$ as $n \to \infty$. Thus, letting $n \to \infty$ in the last of the equalities preceding (2.216) we find that

$$\begin{aligned}\int_{\Omega_a} \nabla \cdot (\mu \boldsymbol{e} - \mu_p \boldsymbol{e}^p) \cdot \boldsymbol{w}\, d\boldsymbol{x} &= -\int_{\Omega_a} (\mu e_{ij} - \mu_0 e_{ij}^p) \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i^p}{\partial x_j}\right) d\boldsymbol{x} \\ &= -\int_{\Omega_a} (\mu e_{ij} - \mu_p e_{ij}^p)(e_{ij} - e_{ij}^p)\, d\boldsymbol{x}\end{aligned} \tag{2.217}$$

where the last result follows from the symmetry of e_{ij}, e_{ij}^p in their indices. Finally,

$$\begin{aligned}-\mu_1 \int_{\Omega_n} \nabla \cdot \Delta(\boldsymbol{e} - \boldsymbol{e}^p) \cdot \boldsymbol{w}\, d\boldsymbol{x} &= -\mu_1 \int_{\Omega_n} \frac{\partial}{\partial x_j} \left[\frac{\partial^2}{\partial x_k^2}(e_{ij} - e_{ij}^p)\right] w_i\, d\boldsymbol{x} \\ &= -\mu_1 \int_{\partial\Omega_n} \frac{\partial^2}{\partial x_k^2}(e_{ij} - e_{ij}^p) w_i \nu_{j(n)}\, d\sigma_n\end{aligned}$$

$$
\begin{aligned}
&+ \mu_1 \int_{\Omega_n} \frac{\partial w_i}{\partial x_j} \frac{\partial^2}{\partial x_k^2}(e_{ij} - e_{ij}^p)\, d\boldsymbol{x} \\
&= -\mu_1 \int_{\partial\Omega_n} \nabla^2 (e_{ij} - e_{ij}^p) w_i \nu_{j(n)}\, d\sigma_n \\
&\quad + \mu_1 \int_{\partial\Omega_n} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\nu_{k(n)} d\sigma_n \\
&\quad + \mu_1 \int_{\Omega_n} \frac{\partial^2 w_i}{\partial x_j \partial x_k} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\, d\boldsymbol{x}.
\end{aligned}
$$

For the first boundary integral in this last set of identities, i.e., for

$$
\int_{\partial\Omega_n} \nabla^2 (e_{ij} - e_{ij}^p) w_i \nu_{j(n)}\, d\sigma_n
$$

we have $w_i\big|_{\Gamma_n} = 0$, $i = 1, 2, 3$ while $\nabla^2(e_{ij} - e_{ij}^p)$ is bounded on Ω_a, $1 \le i, j \le 3$ and $w_i\big|_{\tilde{S}_n} \to 0$, as $n \to \infty$, $i = 1, 2, 3$, so

$$
\int_{\partial\Omega_n} \nabla^2 (e_{ij} - e_{ij}^p) w_i \nu_{j(n)} d\sigma_n \to 0, \text{ as } n \to \infty. \tag{2.218}
$$

For the second boundary integral in the last identity following (2.217) we have

$$
\begin{aligned}
\int_{\partial\Omega_n} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\nu_{k(n)}\, d\sigma_n &= \int_{\Gamma_n^+} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\nu_{k(n)}\, da^+ \\
&\quad + \int_{\Gamma_n^-} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^0)\nu_{k(n)}\, da^- \\
&\quad + \int_{\tilde{S}_n} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\tilde{\nu}_{k(n)}\, d\tilde{\sigma}_n.
\end{aligned} \tag{2.219}
$$

Although $\nu_{k(n)}\big|_{\Gamma_n}\pm = \pm\delta_{k2}$ for all n, we need not avail ourselves of that fact here; instead we note that we may write

$$
\begin{aligned}
\int_{\Gamma_n^\pm} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\nu_{k(n)} da^\pm &= \int_{\Gamma_n^\pm} \frac{\partial}{\partial x_k}(e_{ij}(\boldsymbol{v}) - e_{ij}^p(\boldsymbol{v}^p)) e_{ij}(\boldsymbol{w})\nu_{k(n)} da^\pm \\
&= \int_{\Gamma_n^\pm} \frac{\partial}{\partial x_k}(e_{ij}(\boldsymbol{w})) e_{ij}(\boldsymbol{w})\nu_{k(n)} da^\pm \\
&= \int_{\Gamma_n^\pm} \tau_{ijk}(\boldsymbol{w}) e_{ij}(\boldsymbol{w})\nu_{k(n)} da^\pm = 0
\end{aligned} \tag{2.220}
$$

by virtue of Lemma B.3 and the second set of boundary conditions in (2.166). The result of the calculation in (2.220) is to reduce the boundary integral in (2.219) to

$$\int_{\partial\Omega_n} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e^p_{ij})\nu_{k(n)} d\sigma_n = \int_{\tilde{S}_n} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e^p_{ij})\tilde{\nu}_{k(n)} d\tilde{\sigma}_n. \tag{2.221}$$

However, $\frac{\partial}{\partial x_k}(e_{ij} - e^p_{ij})$ is bounded on Ω_a, for $1 \le i, j, k \le 3$, while $\left.\frac{\partial w_i}{\partial x_j}\right|_{\tilde{S}_n} \to 0$, as $n \to \infty$, for $1 \le i, j \le 3$ (all by virtue of the regularity condition (2.169). Therefore,

$$\int_{\partial\Omega_n} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e^p_{ij})\nu_{k(n)} d\sigma_n \to 0, \text{ as } n \to \infty. \tag{2.222}$$

Employing (2.217) and (2.218) in the last of the equations following (2.217), and letting $n \to \infty$, there results:

$$\begin{aligned} -\mu_1 \int_{\Omega_a} \nabla \cdot \Delta(\boldsymbol{e} - \boldsymbol{e}^p) \cdot \boldsymbol{w}\, d\boldsymbol{x} &= -\mu_1 \int_{\Omega_a} \frac{\partial^2 w_i}{\partial x_j \partial x_k} \frac{\partial}{\partial x_k}(e_{ij} - e^p_{ij})\, d\boldsymbol{x} \\ &= -\mu_1 \int_{\Omega_a} \frac{\partial^2 w_i}{\partial x_j \partial x_k} \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{w})\, d\boldsymbol{x} \\ &= -\mu_1 \int_{\Omega_a} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\, d\boldsymbol{x}. \end{aligned} \tag{2.223}$$

If we now let $n \to \infty$ in (2.209), and make use of the results in (2.211), (2.213), (2.214), (2.215), (2.217) and (2.223), we obtain the estimate

$$-\Gamma \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x} + \int_{\Omega_a} (\mu e_{ij} - \mu_0 e^p_{ij})(e_{ij} - e^p_{ij})\, d\boldsymbol{x} \le -\mu_1 \int_{\Omega_a} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\, d\boldsymbol{x}. \tag{2.224}$$

It requires, however, only a trivial extension of Lemma 2.7 to conclude that at each fixed $\boldsymbol{x} \in \Omega_a$, for $0 < \alpha < 1$,

$$\left[\frac{e_{ij}(\boldsymbol{v})}{(\epsilon + e_{kl}(\boldsymbol{v})e_{kl}(\boldsymbol{v}))^{\alpha/2}} - \frac{e_{ij}(\boldsymbol{v}^p)}{(\epsilon + e_{kl}(\boldsymbol{v}^p)e_{kl}(\boldsymbol{v}^p))^{\alpha/2}}\right] \cdot (e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{v}^p)) \ge 0 \tag{2.225}$$

and, of course, this leads us from (2.224) to

$$\mu_1 \int_{\Omega_a} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\, d\boldsymbol{x} \le \Gamma \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x}. \tag{2.226}$$

However, for $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$ satisfying the boundary conditions (2.166),

$$\int_{\Omega_a} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{x})}{\partial x_k}\, d\boldsymbol{x} \geq \frac{1}{2} \int_{\Omega_a} \frac{\partial^2 w_i}{\partial x_j \partial x_k} \frac{\partial^2 w_i}{\partial x_j \partial x_k}\, d\boldsymbol{x} \tag{2.227a}$$

while

$$\int_{\Omega_a} \frac{\partial^2 w_i}{\partial x_j \partial x_k} \frac{\partial^2 w_i}{\partial x_j \partial x_k}\, d\boldsymbol{x} \geq \frac{1}{(2a^2+\theta)^2} \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x} \tag{2.227b}$$

for any $\theta > 0$, the last inequality following by the Poincaré-type estimate (2.176) of Lemma 2.6. Combining (2.226) with (2.227a,b) now produces

$$\left(\frac{\mu_1}{2(2a^2+\theta)^2} - \Gamma\right) \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x} \leq 0 \tag{2.228}$$

from which it is immediate that $\boldsymbol{w} = 0$, a.e. in $\boldsymbol{L}^2(\Omega_a)$, if $\mu_1 > 2(2a^2+\theta)^2\Gamma$. □

2.5 Existence and Asymptotic Stability of Time-Dependent Poiseuille Flows

2.5.1 Introduction

In this section our attention will be focused on the natural counterpart to the boundary-value problem (2.55a,b), i.e., we consider time-dependent Poiseuille flow in the domain $\Omega_a \times [0, T)$, $T > 0$, of the form

$$\boldsymbol{v} = (u(y, t; \epsilon, \mu_1), 0, 0) \tag{2.229}$$

which satisfies the initial-boundary value problem

$$\rho \dot{u} = -p' + \mu_0 \left[(\epsilon + u'^2)^{-\alpha/2} u'\right]' - \mu_1 u'''', \tag{2.230a}$$

$$u(\pm a, t; \epsilon, \mu_1) = u''(\pm a, t; \epsilon, \mu_1) = 0, \quad t \in [0, T), \tag{2.230b}$$

$$u(y, 0; \epsilon, \mu_1) = u_0(y; \epsilon, \mu_1) \tag{2.230c}$$

where we have once again written $v_1 = u$ and $y = x_2$, with $\dot{u} = \dfrac{\partial u}{\partial t}$, $u' = \dfrac{\partial u}{\partial y}$, etc. In certain places we may write u_t for $\dfrac{\partial u}{\partial t}$ and, without loss of generality, we will set $\rho = 1$. The initial data function u_0 is assumed to be of class $C^2(-a, a)$ while the pressure distribution $p(y, t)$ is taken as being prescribed, with $p \in C^{1,0}(y, t)$ for $y \in (-a, a)$, $t > 0$.

Our goals in this section are twofold: first of all, to establish the existence of a unique solution to the initial-boundary value problem (2.230a,b,c) which is of class $C^{4,1}$ on $(-a, a) \times [0, T)$, for any $T > 0$, and, secondly, to prove that the unique equilibrium solution of the boundary-value problem (2.55a,b) is stable (in fact, asymptotically stable) in a sense which will be made precise in Sect. 2.5.4.

2.5.2 Some Preliminary Estimates for an Associated Parabolic Problem

We begin with some considerations related to the solutions of the following linear parabolic initial-boundary value problem for $w = w(y, t)$ on $(-a, a) \times [0, T)$:

$$\dot{w} = -\mu_1 w^{(4)} + f(y, t), \tag{2.231a}$$

$$w(\pm a, t) = w_{xx}(\pm a, t) = 0, \quad t > 0, \tag{2.231b}$$

$$w(y, 0) = w_0(y), \quad y \in [-a, a] \tag{2.231c}$$

where $w^{(4)} \equiv w''''$, $w_0(\cdot) \in C^2(-a, a)$, $f \in C^{0,0}(y, t)$ for $(y, t) \in (-a, a) \times [0, T)$, and $\mu_1 > 0$. It is a direct consequence of standard results for linear parabolic initial boundary-value problems (specifically, e.g., Sects. 9, 10 of part 2 in [Fr]) that under the hypotheses delineated above, there exists a unique solution $w(y, t)$ of (2.231a,b,c) on $[-a, a] \times [0, T)$, for any $T > 0$, which is of class $C^{4,1}(y, t)$; in order to carry out the analysis in Sect. 2.5.3, we are interested in deriving certain *a priori* estimates which are satisfied by the unique solution $w(x, t)$ of (2.231a,b,c). Our first step consists of multiplying (2.231a) through by $w(y, \tau)$, $(y, \tau) \in (-a, a) \times [0, T)$, $t < T$, and integrating over $(-a, a) \times [0, t)$; we obtain

$$\frac{1}{2}\int_{-a}^{a} w^2(x, t)\, dx + \mu_1 \int_0^t \int_{-a}^{a} w^{(4)} w\, dx\, d\tau = \frac{1}{2}\int_{-a}^{a} w^2(x, 0)\, dx + \int_0^t \int_{-a}^{a} w \cdot f\, dx\, d\tau \tag{2.232}$$

However, in view of the boundary conditions (2.231b), two successive integrations by parts yield

$$\int_0^t \int_{-a}^{a} w^{(4)} w\, dx\, d\tau = \int_0^t \int_{-a}^{a} w_{xx}^2\, dx\, d\tau \tag{2.233}$$

and, therefore,

$$\int_{-a}^{a} w^2(x, t)\, dx + 2\mu_1 \int_0^t \int_{-a}^{a} w_{xx}^2 dx\, d\tau = 2\int_0^t \int_{-a}^{a} w \cdot f\, dx\, d\tau + \int_{-a}^{a} w^2(x, 0)\, dx. \tag{2.234}$$

Next, we multiply (2.231a) by $\dot{w}(y, \tau)$ and, again, integrate over $[-a, a] \times [0, t)$ so as to obtain

$$\int_0^t \int_{-a}^a w^2\, dx\, d\tau = \int_0^t \int_{-a}^a f \cdot \dot{w}\, dx\, d\tau - \mu_1 \int_0^t \int_{-a}^a w^{(4)} \dot{w}\, dx\, d\tau. \tag{2.235}$$

Applying two successive integrations by parts to the second integral on the right-hand side of (2.235) yields, in view of (2.231b),

$$\begin{aligned}\int_0^t \int_{-a}^a w^{(4)} \dot{w}\, dx\, d\tau &= \int_0^t \int_{-a}^a w_{xx} \dot{w}_{xx}\, dx\, d\tau \\ &= \frac{1}{2} \int_{-a}^a w_{xx}^2(x, t)\, dx - \frac{1}{2} \int_{-a}^a w_{xx}^2(x, 0)\, dx\end{aligned} \tag{2.236}$$

so that

$$\begin{aligned}&2 \int_0^t \int_{-a}^a w^2\, dx\, d\tau + \mu_1 \int_{-a}^a w_{xx}^2(x, t)\, dx \\ &\qquad = 2 \int_0^t \int_{-a}^a f \cdot \dot{w}\, dx\, d\tau + \mu_1 \int_{-a}^a w_{xx}^2(x, 0)\, dx.\end{aligned} \tag{2.237}$$

Now, if $w(y, t)$ is a solution of (2.231a,b,c), then so is $w(-y, t)$; by uniqueness of solutions, therefore, $w(y, t) = w(-y, t)$, for $y \in (-a, a)$, $t > 0$. It then follows that $w_y(y, t) = -w_y(-y, t)$ for $y \in (-a, a)$, $t > 0$, in which case $w_y(0, t) = 0$ for $t > 0$. In view of this observation, elementary calculations show that $\exists k(a) > 0$ (in fact, is is a simple exercise to show that we may take $k(a) = a^4/4$) such that

$$\int_{-a}^a w^2(x, t)\, dx \le k(a) \int_{-a}^a w_{xx}^2(x, t)\, dx, \quad t > 0, \tag{2.238a}$$

$$\int_0^t \int_{-a}^a w^2(x, \tau)\, dx\, d\tau \le k(a) \int_0^t \int_{-a}^a w_{xx}^2(x, \tau)\, dx\, d\tau. \tag{2.238b}$$

Returning to (2.234) we have, by (2.238b), for any $\delta > 0$,

$$\begin{aligned}2 \int_0^t \int_{-a}^a f \cdot w\, dx\, d\tau &\le \delta \int_0^t f^2 dx\, d\tau + \frac{1}{\delta} \int_0^t \int_{-a}^a w^2\, dx\, d\tau \\ &\le \delta \int_0^t \int_{-a}^a f^2\, dx\, d\tau + \frac{k(a)}{\delta} \int_0^t \int_{-a}^a w_{xx}^2\, dx\, d\tau\end{aligned} \tag{2.239}$$

so employing this last estimate in (2.234) yields

$$\begin{aligned}&\int_{-a}^a w^2(x, t)\, dx + \left(2\mu_1 - \frac{k(a)}{\delta}\right) \int_0^t \int_{-a}^a w_{xx}^2\, dx\, d\tau \\ &\qquad \le \delta \int_0^t \int_{-a}^a f^2\, dx\, d\tau + \int_{-a}^a w^2(x, 0)\, dx\end{aligned} \tag{2.240}$$

where we assume, of course, that δ has been chosen so large that $\delta > k(a)/2\mu_1$. In a similar fashion

$$2\int_0^t \int_{-a}^a f \cdot \dot{w}\, dx\, d\tau \le \int_0^t \int_{-a}^a f^2\, dx\, d\tau + \int_0^t \int_{-a}^a \dot{w}^2\, dx\, d\tau \tag{2.241}$$

and use of this estimate in (2.237) leads us to

$$\begin{aligned}\int_0^t \int_{-a}^a \dot{w}^2\, dx\, d\tau + \mu_1 \int_{-a}^a w_{xx}^2(x,t)\, dx \\ \le \int_0^t \int_{-a}^a f^2\, dx\, d\tau + \mu_1 \int_{-a}^a w_{xx}^2(x,0)\, dx.\end{aligned} \tag{2.242}$$

By virtue of the estimates (2.240), (2.242) we see that the unique solution of (2.231a,b,c) satisfies

$$w(\cdot,t) \in W^{2,2}(-a,a), \quad t > 0, \tag{2.243a}$$

$$\dot{w} \in L^2((-a,a)\times[0,t)), \quad t > 0, \tag{2.243b}$$

$$w \in H^2((-a,a)\times[0,t)), \quad t > 0. \tag{2.243c}$$

2.5.3 Existence of Weak Solutions

In this section we will employ an iteration scheme to establish the existence of an appropriately defined weak solution to the following initial-boundary value problem on $[-a,a]\times[0,T)$, $T > 0$, for the function $v = v(y,t)$:

$$\dot{v} = -\mu_1 v^{(4)} + \bar{\gamma}(v')' + g(y,t), \quad (y,t) \in (-a,a)\times[0,T), \tag{2.244a}$$

$$v(\pm a,t) = v_{xx}(\pm a,t) = 0, \quad t > 0, \tag{2.244b}$$

$$v(y,0) = v_0(y). \tag{2.244c}$$

In (2.244a,b,c), for $s \in R^1$

$$\bar{\gamma}(s) = \mu_0(\epsilon + s^2)^{-\alpha/2} s \tag{2.245}$$

and we take $g \in C^{0,0}(y,t)$ for $(y,t) \in [-a,a]\times[0,T)$. The initial-boundary value problem (2.230a,b,c) may be directly identified with (2.244a,b,c) if we fix $\epsilon > 0$, $\mu_1 > 0$, take $\rho = 1$, $u(y,t;\epsilon,\mu) = v(y,t)$, and $\dfrac{\partial p}{\partial y}(y,t) = g(y,t)$. We make the following definition rrelative to the problem (2.244a,b,c):

Definition 2.1. A function $v(y,t)$ defined on $Q_T = [-a,a] \times [0,T)$, is a weak solution of the initial-boundary value problem (2.244a,b,c) provided that $\dot{v} \in L^2(Q_T)$, $v'' \in L^2(Q_T)$, while v satisfies (2.244b,c) and, for every test function $\phi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} \phi\dot{v}\,dx\,dt + \mu_1 \iint_{Q_T} \phi'' v''\,dx\,dt = \iint_{Q_T} \phi\bar{\gamma}'(w')w''\,dx\,dt + \iint_{Q_T} \phi g\,dx\,dt. \tag{2.246}$$

Prior to introducing the iteration scheme associated with (2.244a,b,c) we again note that (as a direct consequence of Sects. 9, 10, part 2, of [Fr]) the linear parabolic equation

$$\dot{v} = -\mu_1 v^{(4)} + h(y,t)v_{yy} + g(y,t) \tag{2.247}$$

subject to the initial and boundary data (2.244b,c), possesses, for $h \in C^{0,0}(y,t)$, on $[-a,a] \times [0,T)$, a unique classical solution $v \in C^{4,1}(y,t)$ on $(-a,a) \times [0,T)$.

Consider now the iteration scheme defined by solutions of the initial-boundary value problem

$$\dot{w}_n = -\mu_1 w_n^{(4)} + \bar{\gamma}'(w'_{n-1})w''_n + g(y,t), \tag{2.248a}$$

$$w_n(\pm a,t) = w''_n(\pm a,t) = 0, \tag{2.248b}$$

$$w_n(y,0) = v_0(y) \tag{2.248c}$$

for $(y,t) \in [-a,a] \times [0,T)$, and each integer $n \geq 1$, where

$$w_0(y,t) = j(y,t), \quad (x,t) \in [-a,a] \times [0,T) \tag{2.248d}$$

is given, with $j \in C^{1,0}(y,t)$ for $(y,t) \in (-a,a) \times [0,T)$. Clearly, for each $n \geq 1$, $h_n(y,t) \equiv \bar{\gamma}'(w'_{n-1}(y,t))$ is continuous on $[-a,a] \times [0,T)$ so that (2.248a–d) possesses a unique solution $w_n(y,t)$ on $[-a,a] \times [0,T)$, with $w_n \in C^{4,1}(y,t)$ for $(y,t) \in (-a,a) \times [0,T)$; for this solution $w_n(y,t)$ we may state the following:

Lemma 2.9. *Let $w_n(y,t)$, for $n \geq 1$, be the unique classical solution of (2.248a,b,c) on $[-a,a] \times [0,T)$, $T > 0$, subject to (2.248d). Then $\exists k(T) > 0$ such that, for $t \leq T$,*

$$||w''_n||_{L^\infty([0,t);L^2(-a,a))} \leq k^{1/2}(T), \tag{2.249a}$$

$$||\dot{w}_n||_{L^2([0,t)\times(-a,a))} \leq \mu_1^{1/2} k^{1/2}(T), \tag{2.249b}$$

$$||w''_n||_{L^2([0,t)\times(-a,a))} \leq \frac{1}{\sqrt{2}}\mu_1^{1/2}\epsilon^{\alpha/2}k^{1/2}(T). \tag{2.249c}$$

Proof. If we set

$$f_n(y,t) = g(y,t) + \bar{\gamma}'(w'_{n-1})w''_n \tag{2.250}$$

then (2.248a) assumes the form

$$\dot{w}_n = -\mu_1 w_n^{(4)} + f_n(y,t), \quad n \geq 1 \tag{2.251}$$

and the *a priori* estimates associated with the parabolic initial-boundary value problem (2.231a,b,c) apply to the unique solution $w_n(y,t)$, $n \geq 1$ of (2.251), (2.248b,c) subject to the choice of the initial iterate as per (2.248d). In order to implement the aforementioned estimates we note that by virtue of (2.245), for $s \in R^1$,

$$\bar{\gamma}'(s) = \frac{\epsilon + (1-\alpha)s^2}{(\epsilon + s^2)^{1+(\alpha/2)}} > 0 \tag{2.252}$$

and that $\bar{\gamma}'$ is an even function of $s \in R^1$. Furthermore, a straightforward calculation yields

$$\bar{\gamma}''(s) = -s\left\{\frac{\alpha}{(\epsilon + s^2)^{1+(\alpha/2)}}\left[\frac{\epsilon + (1-\alpha)s^2}{\epsilon + s^2}\right] + \frac{2\alpha\epsilon}{(\epsilon + s^2)^{2+(\alpha/2)}}\right\} \tag{2.253}$$

so that $\bar{\gamma}''(s) < 0$, for $s > 0$. It thus follows that

$$\max_{s \in R^1} \bar{\gamma}'(s) = \bar{\gamma}'(0) = \epsilon^{-\alpha/2}. \tag{2.254}$$

Now, in view of (2.250) and (2.254),

$$\begin{aligned}\int_0^t \int_{-a}^a f_n^2\,dx\,d\tau &\leq 2\int_0^t \int_{-a}^a g^2(x,\tau)\,dx\,d\tau \\ &\quad + 2\int_0^t \int_{-a}^a \left(\bar{\gamma}'(w'_{n-1})\right)^2 (w''_n)^2\,dx\,d\tau \\ &\leq 2\int_0^t \int_{-a}^a g^2(x,\tau)\,dx\,d\tau + 2\epsilon^{-\alpha}\int_0^t \int_{-a}^a (w''_n)^2\,dx\,d\tau.\end{aligned} \tag{2.255}$$

We set, for $n \geq 1$,

$$b_n(t) = \int_0^t \int_{-a}^a (w''_n(x,\tau))^2\,dx\,d\tau \tag{2.256}$$

so that $b_n(0) = 0$, while

$$b_n(t) = \int_{-a}^a w''^2_n(x,t)\,dx \tag{2.257}$$

and by (2.248c)

$$\dot{b}_n(0) = \int_{-a}^{a} w_n''^2(x,0)\,dx = \int_{-a}^{a} v_0''^2(x)\,dx. \tag{2.258}$$

Applying the estimate (2.242) with $w \to w_n$ and $f \to f_n$ we have

$$\int_0^t \int_{-a}^{a} \dot{w}_n^2\,dx\,d\tau + \mu_1 \int_{-a}^{a} w_n''^2(x,t)\,dx \le \int_0^t \int_{-a}^{a} f_n^2\,dx\,d\tau + \mu_1 \int_{-a}^{a} w_n''^2(x,0)\,dx. \tag{2.259}$$

If we drop the first integral on the left-hand side of (2.259), and employ (2.255)–(2.258), we are led to the differential inequality

$$\mu \dot{b}_n(t) \le C(T) + 2\epsilon^{-\alpha} b_n(t) + \mu_1 \dot{b}_n(0) \tag{2.260}$$

for $t \le T$, where

$$C(T) \equiv 2\int_0^T \int_{-a}^{a} g^2(x,\tau)\,dx\,d\tau. \tag{2.261}$$

Setting, for $n \ge 1$,

$$\begin{aligned} d_n(T) &= \frac{1}{\mu_1} C(T) + \dot{b}_n(0) \\ &= \frac{1}{\mu_1} C(T) + \int_{-a}^{a} v_0''^2(x)\,dx \equiv d(T) \end{aligned} \tag{2.262}$$

we see that (2.260) can be rewritten in the form

$$\dot{b}_n(t) \le d(T) + \left(\frac{2}{\mu_1 \epsilon^{\alpha}}\right) b_n(t) \tag{2.263}$$

which, by use of the integrating factor $\exp\left(-\dfrac{2}{\mu_1 \epsilon^{\alpha}} t\right)$, yields the estimate

$$b_n(t) \le \frac{\mu_1 \epsilon^{\alpha}}{2} d(T) \left[\exp\left(\frac{2t}{\mu_1 \epsilon^{\alpha}}\right) - 1\right]. \tag{2.264}$$

However, substitution of (2.264) back into (2.263) then produces the bound

$$\dot{b}_n(t) \le d(T) \exp\left[\frac{2}{\mu_1 \epsilon^{\alpha}} t\right], \quad t \le T. \tag{2.265}$$

We now return to the estimate (2.259) which, when coupled with (2.255)–(2.258) and (2.261), we can use to deduce the inequality

$$\int_0^t \int_{-a}^a \dot{w}_n^2 \, dx \, d\tau + \mu_1 \dot{b}_n(t) \le C(T) + 2\epsilon^{-\alpha} b_n(t) + \mu_1 \dot{b}_n(0). \tag{2.266}$$

Employing the estimate (2.264) for $b_n(t)$ on the right-hand side of (2.266) we now find, in succession, that for $t \ge T$,

$$\begin{aligned}\int_0^t \int_{-a}^a \dot{w}_n^2 \, dx \, d\tau + \mu_1 \dot{b}_n(t) &\le C(T) + \mu_1 \dot{b}_n(0) \\ &\quad + \mu d(T) \left[\exp\left(\frac{2t}{\mu_1 \epsilon^\alpha} \right) - 1 \right] \\ &= (C(T) + \mu \dot{b}_n(0)) \exp\left(\frac{2t}{\mu_1 \epsilon^\alpha} \right) \\ &= \mu d(T) \exp\left(\frac{2t}{\mu_1 \epsilon^\alpha} \right).\end{aligned} \tag{2.267}$$

If we set

$$k(T) \equiv d(T) \exp\{(2/\mu_1 \epsilon^\alpha) T\} \tag{2.268}$$

then directly from (2.265) and (2.267) we have, for $t \le T$, and $n \ge 1$,

$$\int_{-a}^a w_n''^2(x,t) \, dx \le k(T), \tag{2.269a}$$

$$\int_0^t \int_{-a}^a \dot{w}_n^2(x,\tau) \, dx \, d\tau \le \mu_1 k(T) \tag{2.269b}$$

while, in view of (2.264)

$$\int_0^t \int_{-a}^a w_n''^2(x,\tau) \, dx \, d\tau \le \frac{1}{2} \mu_1 \epsilon^\alpha k(T) \tag{2.269c}$$

for $t \le T$ and $n \ge 1$. The statements embodied in (2.269a,b,c) are, of course, equivalent to (2.249a,b,c). □

We are now ready to state and prove the main result in this section, namely,

Theorem 2.6. *Let $w_n(y,t)$ be the unique classical solution, for $n \ge 1$, of* (2.248a,b,c) *on $[-a,a] \times [0,T)$, $T > 0$, which is subject to* (2.248d). *Then, as $n \to \infty$, $\{w_n\}$ converges to a weak solution of the initial-boundary value problem* (2.244a,b,c).

Proof. As a consequence of (2.249b,c), the boundary conditions (2.248d), and a straightforward interpolation argument, it follows that $\exists C > 0$ such that for $t \le T$, and $n \ge 1$,

$$\begin{aligned} ||w_n'||_{H^{1/2}([0,t)\times(-a,a))} &\le C\left(||\dot{w}||_{L^2([0,t)\times(-a,a))} + ||w_n''||_{L^2([0,t)\times(-a,a))}\right) \\ &\le \left(1 + \frac{1}{\sqrt{2}}\epsilon^{\alpha/2}\right) C\mu_1^{1/2}k^{1/2}(T). \end{aligned} \tag{2.270}$$

We now set $Q_T = [-a, a] \times [0, T)$. As a direct result of the bounds (2.249b,c) and (2.270) it follows that (by picking, if necessary, subsequences of previously chosen subsequences) we may single out a subsequence of $\{w_n\}$, which we will also denote by $\{w_n\}$, such that $w_n \to w$ weakly in $L^2(Q_T)$ and, hence, also in the sense of distributions, while

$$w_n'' \to w'', \text{ weakly in } L^2(Q_T), \tag{2.271a}$$

$$\dot{w}_n \to \dot{w}, \text{ weakly in } L^2(Q_T), \tag{2.271b}$$

$$w_n' \to w', \text{ weakly in } H^{1/2}(Q_T). \tag{2.271c}$$

However, by the compactness of the embedding $H^{1/2}(Q_T) \hookrightarrow L^2(Q_T)$ (see Appendix A) it follows from (2.271c) that

$$\{w_n' \to w' \text{ strongly in } L^2(Q_T)\} \Rightarrow \{w_n' \to w' \text{ a.e. in } Q_T\}. \tag{2.272}$$

Now, let $\phi(y, t)$ be a test function, i.e., suppose that $\phi \in C_0^\infty(Q_T)$; we multiply (2.248a) through by ϕ and integrate over $[-a, a] \times [0, T)$ obtaining, after two integrations by parts,

$$\begin{aligned} &\iint_{Q_T} \phi\dot{w}_n\, dx\, dt + \mu_1 \iint_{Q_T} \phi'' w_n''\, dx\, dt \\ &\qquad = \iint_{Q_T} \phi\bar{\gamma}'(w_{n-1}')w_n''\, dx\, dt + \iint_{Q_T} \phi g\, dx\, dt. \end{aligned} \tag{2.273}$$

As $\bar{\gamma}'$ is clearly continuous, by (2.272) we have that $\bar{\gamma}'(w_{n-1}') \to \bar{\gamma}'(w')$, a.e. in Q_T, while $\bar{\gamma}' > 0$ with $\bar{\gamma}'(w_{n-1}') \le \bar{\gamma}'(0)$, $\forall n \ge 1$. Thus,

$$\bar{\gamma}'(w_{n-1}') \to \bar{\gamma}'(w'), \text{ strongly in } L^2(Q_T). \tag{2.274}$$

Therefore, by the dominated convergence theorem,

$$\iint_{Q_T} \phi\bar{\gamma}'(w_{n-1}')w_n'' dx\, dt \to \iint_{Q_T} \phi\bar{\gamma}'(w')w''\, dx\, dt \tag{2.275}$$

as $n \to \infty$. Extracting the limit in (2.273) as $n \to \infty$, we find that, as a consequence of (2.271a,b) and (2.275), $w(y,t)$ satisfies

$$\iint_{Q_T} \phi \dot{w}\, dx\, dt + \mu_1 \iint_{Q_T} \phi'' w''\, dx\, dt \\ = \iint_{Q_T} \phi \bar{\gamma}'(w') w''\, dx\, dt + \iint_{Q_T} \phi g\, dx\, dt. \tag{2.276}$$

Clearly w satisfies, as well, the boundary conditions (2.244b) and the initial condition (2.244c), which completes the proof of Theorem 2.6. □

Remarks. The regularity of the weak solution to (2.244a,b,c) now follows by standard arguments (e.g., see [Fr]). It is, in fact, easily established that $v \in C^4(y,t)$ on $(-a,a) \times [0,T)$ for any $T > 0$.

2.5.4 Uniqueness and Stability of Solutions

In the last subsection we established the existence of a weak solution $v(y,t)$ to the initial-boundary value problem (2.244a,b,c), indicating that, in fact, $v \in C^{4,1}(y,t)$ on $(-a,a) \times [0,T)$, $T > 0$, so that the weak solution is actually a classical solution of our problem. In this section we prove that any classical solution of (2.244a,b,c) is unique and that the (unique) equilibrium Poiseuille flow (2.9), with $u(y;\epsilon,\mu_1)$ being the solution of the boundary-value problem (2.55a,b), is linearly asympotitically stable as well as asymptotically stable, within the class of all flows in $\Omega_a \times [0,T)$, $T > 0$, of the Poiseuille type (2.10). We begin with the following result:

Theorem 2.7. *If $u(y,t)$, $v(y,t)$ are any two classical solutions of the initial boundary-value problem (2.244a,b,c), i.e., two solutions in $C^{4,1}(y,t)$, and $w(y,t) = v(y,t) - u(y,t)$, then for any $t > 0$, $||w(\cdot,t)||_{L^2(-a,a)} = 0$.*

Proof. As in the hypothesis of the theorem we set $w = v - u$ in which case $w(y,t)$ clearly satisfies

$$\dot{w} + \mu_1 w^{(4)} = \bar{\gamma}(w' + u')' - \bar{\gamma}(u')'. \tag{2.277}$$

Multiplying (2.277) through by $w(y,t)$, integrating over $[-a,a]$, and then integrating by parts, twice in succession, the resulting integral $\int_{-a}^{a} w w^{(4)}\, dx$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w^2\, dx + \mu_1 \int_{-a}^{a} w_{xx}^2\, dx = \int_{-a}^{a} \left(\bar{\gamma}(w' + u') - \bar{\gamma}(u')\right)' w\, dx \tag{2.278}$$

where we have used the fact that $\forall t > 0$, $w(\pm a, t) = w_{xx}(\pm a, t) = 0$. Integrating the integral on the right-hand side of (2.278) by parts, we are led from (2.278) to

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w^2\,dx + \mu_1 \int_{-a}^{a} w_{xx}^2\,dx = -\int_{-a}^{a}\left(\bar{\gamma}(w' + u') - \bar{\gamma}(u')\right) w'\,dx$$

or

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w^2\,dx + \mu_1 \int_{-a}^{a} w_{xx}^2\,dx = -\int_{-a}^{a}\left[\bar{\gamma}(v') - \bar{\gamma}(u')\right](v' - u')\,dx. \quad (2.279)$$

However,

$$(\bar{\gamma}(p) - \bar{\gamma}(q))(p - q) \geq 0, \quad \forall p, q \in R^1$$

by virtue of the monotonicity of $\bar{\gamma}$, i.e. (2.252). From (2.279), therefore, we obtain

$$\int_{-a}^{a} w^2(x,t)\,dx \leq \int_{-a}^{a} w^2(x,0)\,dx = 0 \quad (2.280)$$

as $u(y,0) = v(y,0) \equiv v_0(y)$, $\forall y \in [-a, a]$. □

Now, if we make use of (2.252), we see that (2.244a) has the equivalent form

$$\dot{v} = -\mu_1 v^{(4)} + \left[\frac{\epsilon + (1-\alpha)v'^2}{(\epsilon + v'^2)^{1+(\alpha/2)}}\right] v'' + g(y,t) \quad (2.281)$$

from which it is apparent that if $v(y,t)$ is a classical solution of (2.244a,b,c) on $[-a, a] \times [0, T)$, $T > 0$, then so is $v(-y, t)$. In view of the uniqueness theorem proven above we then have $v(y,t) = v(-y,t)$, $y \in [-a, a]$, $t > 0$, from which it follows that $v'(0,t) = 0$, $t > 0$. The Poincaré-type estimate of (2.238a) with $k(a) = a^4/4$, applies, therefore, to v as does the usual estimate

$$\int_{-a}^{a} v^2(x,t)\,dx \leq 4a^2 \int_{-a}^{a} v_x^2(x,t)\,dx. \quad (2.282)$$

From here on we will again denote the unique solution of (2.55a,b) by $u(y)$. Thus, $u(y)$ satisfies, for $y \in (-a, a)$,

$$\bar{\gamma}(u'(y))' - \mu_1 u''''(y) = p_1 \quad (2.283a)$$

with

$$u(\pm a) = u''(\pm a) = 0. \quad (2.283b)$$

Also, let $v_\delta(y,t)$ be the unique classical solution of (2.244a,b,c) with $g(y,t) = -\dfrac{\partial p(y,t)}{\partial y} \equiv -p_1$, $\forall t > 0$, and $v_0(y) = u(y) + \delta f(y)$, with $\delta > 0$ and $||f||_{L^\infty(-a,a)} \le K$, for some $K > 0$, i.e., $v_\delta(y,t)$ satisfies

$$\dot{v}_\delta = -\mu_1 v_\delta^{(4)} + \bar{\gamma}(v_\delta')' - p_1, \quad y \in (-a,a),\ t > 0, \tag{2.284a}$$

$$v_\delta(\pm a,t) = v_\delta''(\pm a,t) = 0, \quad t > 0, \tag{2.284b}$$

$$v_\delta(y,0) = u(y) + \delta f(y), \quad y \in [-a,a]. \tag{2.284c}$$

Now, if $\delta = 0$, then the unique solution of (2.284a,b,c) is clearly $v_0(y,t) = u(y)$, $y \in (-a,a)$, $\forall t > 0$. Our goal is to study the behavior of the unique classical solution $v_\delta(y,t)$, $y \in (-a,a)$, as $t \to \infty$. To this end we set

$$w_\delta(y,t) = v_\delta(y,t) - u(y); \quad y \in [-a,a],\ t > 0 \tag{2.285}$$

in which case w_δ is easily seen to satisfy

$$\dot{w}_\delta + \mu_1 w_\delta^{(4)} = \bar{\gamma}(v_\delta')' - \bar{\gamma}(u')', \quad y \in (-a,a),\ t > 0, \tag{2.286a}$$

$$w_\delta(\pm a,t) = w_\delta''(\pm a,t) = 0, \quad t > 0, \tag{2.286b}$$

$$w_\delta(y,0) = \delta f(y), \quad y \in [-a,a]. \tag{2.286c}$$

We first look at the problem of linear asymptotic stability of the equilibrium solution u. We write that, to within terms of order $O(\big|\big|w_\delta'\big|\big|^2_{L^\infty(-a,a)})$,

$$\bar{\gamma}(v_\delta') \equiv \bar{\gamma}(w_\delta' + u') = \bar{\gamma}(u') + \bar{\gamma}'(u')w_\delta' \tag{2.287}$$

in which case the linearized equation associated with (2.286a) is just

$$\dot{w}_\delta + \mu_1 w_\delta^{(4)} = \left(\bar{\gamma}'(u')w_\delta'\right)' \tag{2.288}$$

where

$$\bar{\gamma}'(u') = \frac{\epsilon + (1-\alpha)u'^2}{(\epsilon + u'^2)^{1+(\alpha/2)}}. \tag{2.289}$$

We then have the following result concerning the linearized asymptotic stability of the (equilibrium) plane Poiseuille solution $u(y)$:

Theorem 2.8. *Let u be the unique solution of the boundary-value problem (2.283a,b) and w_δ the unique solution of the linearized initial-boundary value problem (2.288), (2.286b,c). Then for any $\delta > 0$, and $f \in L^2(-a,a)$, $||w_\delta(\cdot,t)||_{L^2(-a,a)}$ decays to zero, exponentially, as $t \to \infty$.*

Proof. Multiplying (2.288) through by $w_\delta(y,t)$, integrating over $(-a,a)$, and then integrating the term involving μ_1 by parts, in the familiar fashion, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w_\delta(x,t)\,dx + \mu_1\int_{-a}^{a}(w_\delta''(x,t))^2\,dx &= \int_{-a}^{a}(\bar{\gamma}'(u')w_\delta')'w_\delta\,dx \\ &= -\int_{-a}^{a}\bar{\gamma}'(u')w_\delta'^2\,dx.\end{aligned} \tag{2.290}$$

From (2.289)

$$\begin{aligned}\bar{\gamma}'(u') &= (\epsilon+u'^2)^{-\alpha/2}\left[1-\alpha\frac{u'^2}{\epsilon+u'^2}\right] \\ &\geq (1-\alpha)/(\epsilon+u'^2)^{\alpha/2}.\end{aligned} \tag{2.291}$$

But, from the analysis in Sect. 2.2, we know that $\exists C'>0$, independent of ϵ and μ_1, such that $\left|\left|u'\right|\right|_{L^\infty(-a,a)} \leq C'$, in which case (2.291) implies that

$$\bar{\gamma}'(u'(y)) \geq (1-\alpha)/(\epsilon+C'^2)^{\alpha/2}, \quad y\in[-a,a]. \tag{2.292}$$

Employing the lower bound (2.292) in (2.290) now yields the estimate

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq \frac{-(1-\alpha)}{(\epsilon+C'^2)^{\alpha/2}}\int_{-a}^{a} w_\delta'^2(x,t)\,dx \tag{2.293}$$

which is valid for all $\mu_1>0$, any $\epsilon>0$, and $t>0$. However, by virtue of (2.282), which holds if $v(\pm a)=0$ (without regard to whether the condition $u'(0)=0$ is satisfied),

$$\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq 4a^2\int_{-a}^{a} w_\delta'^2(x,t)\,dx, \quad \forall t>0. \tag{2.294}$$

Combining (2.293) and (2.294), we see that, for $t>0$,

$$\frac{d}{dt}\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq \frac{-(1-\alpha)}{2a^2(\epsilon+C'^2)^{\alpha/2}}\int_{-a}^{a} w_\delta^2(x,t)\,dx \tag{2.295}$$

from which it follows by (2.286c) that, for $t>0$,

$$\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq \delta^2\int_{-a}^{a} f^2(x)\,dx\cdot\exp(-\eta t) \tag{2.296}$$

where $\eta>0$ is given by

$$\eta = (1-\alpha)/2a^2(\epsilon+C'^2)^{\alpha/2}. \tag{2.297}$$

Irrespective of the magnitude of the perturbation δf, therefore, $||w_\delta(\cdot,t)||_{L^2(-a,a)} \to 0$, as $t \to \infty$, and the proof of Theorem 2.8 is complete. □

Because of the monotonicity property associated with $\bar{\gamma}(\cdot)$, and the Poincaré type inequality (2.283a), we can actually do much better than the linearized asymptotic stability result expressed by Theorem 2.8 above; in fact we may state the following:

Theorem 2.9. *Let u be the unique solution of* (2.283a,b)*, v_δ the unique solution of* (2.284a,b,c)*, and set $w_\delta = v_\delta - u$, so that w_δ satisfies* (2.286a,b,c)*. Then, for any $\delta > 0$, and $f \in L^2(-a,a)$, $||w_\delta(\cdot,t)||_{L^2(-a,a)}$ decays to zero, exponentially, as $t \to \infty$.*

Proof. We multiply (2.286a) by w_δ, integrate over $[-a,a]$, and effectuate the required integrations by parts employing (2.286b), so as to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w_\delta^2(x,t)\,dx + \mu_1\int_{-a}^{a} w_\delta''^2(x,t)\,dt = -\int_{-a}^{a}\left[\bar{\gamma}(v_\delta') - \bar{\gamma}(u')\right][v_\delta' - u']\,dx. \tag{2.298}$$

Using the monotonicity property of $\bar{\gamma}$, i.e., $\forall\, p,q \in R^1$, $(\bar{\gamma}(p) - \bar{\gamma}(q))(p-q) \geq 0$, we obtain from (2.298)

$$\frac{d}{dt}\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq 2\mu_1\int_{-a}^{a} w_\delta''^2(x,t)\,dx. \tag{2.299}$$

However, $v_\delta(y,t) = v_\delta(-y,t)$, and $u(y) = u(-y)$, for $t > 0$, $y \in [-a,a]$, so $w_\delta(y,t) = w_\delta(-y,t)$, for $t > 0$, $y \in [-a,a]$. Thus, $w_\delta'(0,t) = 0$, $\forall t \geq 0$, while $w_\delta(\pm a,t) = 0$, $\forall t > 0$. The Poincaré type inequality (2.238a) applies, therefore, to w_δ and its use in (2.299) produces

$$\frac{d}{dt}\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq -\frac{\mu_1}{2a^4}\int_{-a}^{a} w_\delta^2(x,t)\,dx, \quad t > 0. \tag{2.300}$$

Integration of (2.300) yields

$$\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq \delta^2\left(\int_{-a}^{a} f^2(x)\,dx\right)\exp\left[\frac{-\mu_1 t}{2a^4}\right], \quad t > 0 \tag{2.301}$$

and Theorem 2.9 has been established. □

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