Chapter 2
Review

This chapter is an introduction to some aspects of twistor geometry and the twistor space description of space-time conformal field theories. In particular, we will focus on the twistor space description of $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory. The material is not original. I hope only to provide a presentation of the material that is oriented towards recent applications to scattering amplitudes and Wilson loops. For a recent review with similar aspirations see reference [1]. Some additional useful resources on twistor theory are [2, 3]

2.1 Twistor Theory

2.1.1 Complexified Compactified Minkowski Space

Conformal field theories in four dimensions are naturally defined on the conformal compactification of Minkowski space, whose complexification will be denoted by $\mathbb{CM}^\sharp$, and on which the complexified conformal group, SL(4, $\mathbb{C}$), naturally acts. The analytic continuation to complex space-time coordinates is important throughout this thesis. The reason is that the space-time coordinates will actually describe the four-momenta of scattering states, and the techniques that we use depend on the analytic properties of scattering amplitudes as functions of complex momenta. When reality conditions are important we will state this explicitly.

Complexified compactified Minkowski space, $\mathbb{CM}^\sharp$, has a beautiful description as a complex quadric surface in the complex projective space $\mathbb{CP}^5$. Introducing six homogeneous coordinates on $\mathbb{CP}^5$ transforming in the anti-symmetric tensor representation of SL(4, $\mathbb{C}$),

$$X^{IJ} = -X^{JI} \qquad I, J = 1, \ldots, 4, \tag{2.1}$$

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the quadric surface is

$$X \cdot X := X_{IJ}\, X^{IJ} = 0. \tag{2.2}$$

The anti-symmetric pair of indices has been lowered using the canonical anti-symmetric tensor ϵ_{IJKL} of the complexified conformal group SL(4, $\mathbb{C}$),

$$X_{IJ} = \frac{1}{2}\epsilon_{IJKL}X^{KL}. \tag{2.3}$$

The conformal structure of complexified compactified Minkowski space $\mathbb{CM}^\sharp$ is then inherited from the anti-symmetric tensor ϵ_{IJKL}: two points X and Y are null separated if and only if

$$X \cdot Y = 0. \tag{2.4}$$

Allowing all possible Y, the above equation defines the tangent plane to the quadric at X, and hence its intersection with the quadric is the complex null cone of X.

2.1.2 Twistor Space

Twistor space is the complex projective space $\mathbb{CP}^3$, whose four complex homogeneous coordinates $Z^I \sim rZ^I$ transform in the fundamental representation of the complexified conformal group SL(4, $\mathbb{C}$). Strictly speaking, the complexified conformal group is the group of complex projective linear transformations PGL(4, $\mathbb{C}$) = SL(4, $\mathbb{C}$)/$\mathbb{Z}_4$ acting on $\mathbb{CP}^3$.

The relationship between twistor space geometry and space-time conformal structure is encapsulated in the so-called 'incidence relation'. Given a point in complexified compactified space-time X, the incidence relation is

$$X_{IJ}\, Z^J = 0. \tag{2.5}$$

The condition $X \cdot X = 0$ implies that the skew tensor X^{IJ} has rank two, so there is only one independent equation up to complex rescalings. Thus the incidence relation defines a holomorphically embedded line in twistor space, $X = \mathbb{CP}^1$. The skew tensor X^{IJ} can be reconstructed from the line by forming

$$X^{IJ} = A^{[I}B^{J]} \tag{2.6}$$

where A^I and B^I are any two points on the line X. Conversely, any complex projective line in twistor space can be expressed as above, thus defining a point in $\mathbb{CM}^\sharp$. Hence points in space-time are in one-to-one correspondence with complex projective lines in twistor space Fig. 2.1.

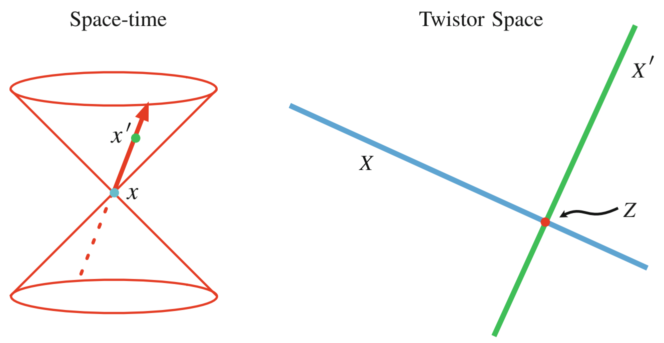

Fig. 2.1 Points in space-time correspond to lines in twistor space. Two space-time points are null separated if and only if their corresponding twistor lines intersect

A fundamental property of this correspondence is that two space-time points are null separated if and only if the corresponding twistor lines intersect intersect. To see this, suppose that there is a simultaneous solution of the equations

$$X_{IJ}\,Z^J = 0 \qquad Y_{IJ}\,Z^J = 0 \tag{2.7}$$

then

$$X^{IJ} = Z^{[I}A^{J]} \qquad Y^{IJ} = Z^{[I}B^{J]} \tag{2.8}$$

for some A^I and B^J and consequently $X \cdot Y = 0$. The converse statement is similarly straightforward. The complex structure of twistor space is therefore equivalent to the conformal structure of compactified space-time.

For a given twistor Z^I, the general solution of the incidence relation is

$$X^{IJ} = X_0^{IJ} + Y^{[I}Z^{J]}, \tag{2.9}$$

where X_0 is any particular solution and Y^I is an arbitrary twistor. The coordinates Y^I thus parametrize a complex two-dimensional plane in $\mathbb{CM}^\sharp$ all of whose tangent vectors are mutually orthogonal and null. This is called an 'α-plane'. Thus points in twistor space correspond to α-planes in space-time.

2.1.3 *The Infinity Twistor*

For many purposes, it is convenient to break manifest conformal symmetry and introduce a local affine coordinate system on Minkowski space. This requires choosing a point at infinity, represented by a simple anti-symmetric tensor I^{IJ}. The points in $\mathbb{CM}^\sharp$ lying in the surface

$$\mathcal{I} = \left\{X \in \mathbb{CM}^\sharp,\ I \cdot X = 0\right\} \tag{2.10}$$

define the light-cone of the point at infinity. Removing these points from compactified complexified space-time, we obtain complexified Minkowski space $\mathbb{CM} = \mathbb{CM}^\sharp \setminus \mathcal{I}$, which is isomorphic to $\mathbb{C}^4$. The point at infinity I^{IJ} defines a metric on the remaining $\mathbb{CM}$ defined by the formula

$$g(X, Y) = \frac{X \cdot Y}{(I \cdot X)\,(I \cdot Y)} \tag{2.11}$$

and breaking the complexified conformal symmetry SL(4, $\mathbb{C}$) down to the complexified Poincare group.

The fundamental indices of SL(4, $\mathbb{C}$) can now be decomposed into components transforming in spinor representations of the complexified Lorentz subgroup SL(2, $\mathbb{C}$)$\times$ SL(2, $\mathbb{C}$) in the following way,

$$Z^I = (\lambda_\alpha, \mu^{\dot\alpha}). \tag{2.12}$$

The point at infinity can be represented by the following simple matrices

$$I^{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix} \qquad I_{IJ} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.13}$$

and then we can introduce affine coordinates $\{x^{\alpha\dot\alpha}\}$ on $\mathbb{CM}$ by

$$X^{IJ} = \begin{pmatrix} \epsilon_{\alpha\beta} & -i x_\alpha{}^{\dot\beta} \\ i x^{\dot\alpha}{}_\beta & -\frac{1}{2} x^2 \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}. \tag{2.14}$$

They are related to the standard affine coordinates $\{x^\mu\}$ on $\mathbb{C}^4$ through the linear relationship $x^{\alpha\dot\alpha} = x^\mu \sigma_\mu^{\alpha\dot\alpha}$. The metric (2.11) on complexified Minkowski space then becomes the standard metric on $\mathbb{C}^4$.

The twistors with vanishing primary components, $\lambda_\alpha = 0$, are incident on the light-cone at infinity $\mathcal{I}$, and removing these points from twistor space, we obtain a correspondence between lines in $\mathbb{CP}^3 - \{\lambda_\alpha = 0\}$ and points in $\mathbb{CM}$. The all important incidence relation describing the correspondence becomes

$$\mu^{\dot\alpha} = -i x^{\alpha\dot\alpha} \lambda_\alpha \tag{2.15}$$

and clearly defines a complex line in twistor space provided that $\lambda_\alpha \neq 0$. In the other direction, the general solution of the incidence relation for a given twistor is

$$x^{\alpha\dot\alpha} = x_0^{\alpha\dot\alpha} + \lambda^\alpha \sigma^{\dot\alpha} \tag{2.16}$$

where $x_0^{\alpha\dot{\alpha}}$ is one particular solution and $\sigma^{\dot{\alpha}}$ is an arbitrary spinor. This expression defines a totally null α-plane in affine coordinates.

Finally, we note that infinitessimal conformal transformations are now realized by the following generators

$$
\begin{aligned}
P_{\alpha\dot{\alpha}} &= \lambda_\alpha \frac{\partial}{\partial \mu^{\dot{\alpha}}} \\
J_{\alpha\beta} &= \frac{1}{2}\left(\lambda_\alpha \frac{\partial}{\partial \lambda^\beta} + \lambda_\beta \frac{\partial}{\partial \lambda^\alpha}\right) \qquad & J_{\dot{\alpha}\dot{\beta}} &= \frac{1}{2}\left(\mu_{\dot{\alpha}} \frac{\partial}{\partial \mu^{\dot{\beta}}} + \mu_{\dot{\beta}} \frac{\partial}{\partial \mu^{\dot{\alpha}}}\right) \\
D &= \frac{1}{2}\left(\lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} - \mu^{\dot{\alpha}} \frac{\partial}{\partial \mu^{\dot{\alpha}}}\right) \qquad & K^{\alpha\dot{\alpha}} &= \mu^{\dot{\alpha}} \frac{\partial}{\partial \lambda_\alpha}
\end{aligned}
\tag{2.17}
$$

in terms of the spinor components. However, we note that by following the integral curves of the above special conformal generator $K^{\alpha\dot{\alpha}}$, points at infinity can be reached at finite parameter values. In this sense, a complete realisation of finite conformal transformations requires the compactified space $\mathbb{CM}^\sharp$.

2.1.4 Reality Conditions

We now consider the reality condition appropriate for real Minkowski space, $\mathbb{M}$. The conformal group is now locally isomorphic to the real form SU(2, 2) preserving the pseudo-hermitian metric

$$
g_{I\bar{J}} Z^I \bar{Z}^{\bar{J}} = \lambda_\alpha \bar{\mu}^\alpha + \mu^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}. \tag{2.18}
$$

This partitions twistor space into three components

$$
\mathbb{PT}^+ := \{Z \cdot \bar{Z} > 0\} \qquad \mathbb{PN} := \{Z \cdot \bar{Z} = 0\} \qquad \mathbb{PT}^- := \{Z \cdot \bar{Z} < 0\}. \tag{2.19}
$$

For lines X lying entirely inside null twistor space $\mathbb{PN}$, the incidence relation and its complex conjugate imply that

$$
0 = i(x - x^\dagger)^{\alpha\dot{\alpha}} \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} \tag{2.20}
$$

for all spinors λ^α. This condition is possible if and only if $x^{\alpha\dot{\alpha}}$ is hermitian and hence describes a real point in Minkowski space, $\mathbb{M}$. The line representing the point at infinity I is automatically inside $\mathbb{PN}$.

Consider now the α-plane associated to a point $Z^I \in \mathbb{PN}$. It is straightforward to show that the α-plane then necessarily contains a real point and conversely, that the existence of such a real point implies that $Z^I \in \mathbb{PN}$. The α-plane then intersects real Minkowski space in the null geodesic

$$x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + r\lambda^\alpha\bar{\lambda}^{\dot{\alpha}} \tag{2.21}$$

where $x_0^{\alpha\dot{\alpha}} = \bar{\mu}^\alpha \mu^{\dot{\alpha}} / i\bar{\mu}^\alpha \lambda_\alpha$ and $r \in \mathbb{R}$, and furthermore, each null geodesic uniquely determines an α-plane. Thus, the points in null twistor space $\mathbb{PN}$ are in one-to-one correspondence with null geodesics in real Minkowski space $\mathbb{M}$.

2.1.5 The Penrose Transform

In this subsection, we describe the twistor space formulation of some free conformal field theories. Solutions of massless free field equations, for example,

$$\partial^{\alpha_1\dot{\alpha}_1}\phi_{\alpha_1\cdots\alpha_n} = 0 \qquad \text{or} \qquad \partial^{\alpha_1\dot{\alpha}_1}\phi_{\dot{\alpha}_1\cdots\dot{\alpha}_n} = 0 \tag{2.22}$$

are in one-to-one correspondence with cohomology classes on certain regions of twistor space. The concrete correspondence is provided by the Penrose transform [4, 5]. Here we proceed by example.

Let us begin by constructing solutions of the scalar wave equation

$$\Box\phi(x) = 0. \tag{2.23}$$

The starting point for the construction is an (0, 1)-form $\tilde{\phi}$ on some open region $U \subset \mathbb{CP}^3$, valued in sections of the line bundle $\mathcal{O}(-2)$. This means that $\tilde{\phi}$ has weight -2 in the homogeneous coordinates Z^I. It is important to consider an open region $U \subset \mathbb{CP}^3$, since there are no globally holomorphic sections of the line bundle $\mathcal{O}(-2)$. We can now pull $\tilde{\phi}$ back to the spinor bundle using the incidence relations and integrate over the fibres

$$\phi(x) = \int \langle\lambda\, \mathrm{d}\lambda\rangle \wedge \tilde{\phi}\,(\lambda_\alpha, ix^{\alpha\dot{\alpha}}\lambda_\alpha). \tag{2.24}$$

This expression defines a scalar field configuration on an open region of space-time corresponding to twistor lines inside $U \subset \mathbb{CP}^3$. The above integral expression is called the Penrose transform.

We now make two important observations. Firstly, the integral is invariant under the transformation

$$\tilde{\phi} \longrightarrow \tilde{\phi} + \bar{\partial}\,\tilde{\alpha} \tag{2.25}$$

where $\tilde{\alpha}$ is any section on $\mathcal{O}(-2)$ and hence depends only on the cohomology class in $H^{0,1}_{\bar{\partial}}(U, \mathcal{O}(-2))$ represented by $\tilde{\phi}$. Secondly, acting underneath the integral sign with the operator $\Box = \partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}$ the resulting expression vanishes since $\lambda^\alpha\lambda_\alpha = 0$.

Thus we have obtained a scalar field configuration obeying the scalar wave equation from a cohomology class $H^{0,1}_{\bar\partial}(U, \mathcal{O}(-2))$.

Let us consider a concrete example. A simple cohomology representative is the so called elementary state [6]

$$\tilde\phi(Z) = \frac{I^{IJ} P_I Q_J}{P_I Z^I} \bar\partial \frac{1}{Q_J Z^J}, \tag{2.26}$$

which is holomorphic away from the complex two-planes defined by $P := \{P_I Z^I = 0\}$ and $Q := \{Q_I Z^I = 0\}$. The Penrose transform is straightforwardly evaluated with the result

$$\phi(x) = \frac{1}{(x-y)^2}, \tag{2.27}$$

where the space-time point $y^{\alpha\dot\alpha} = p^\alpha q^{\dot\alpha} - q^\alpha p^{\dot\alpha}$ corresponds to the line formed by the intersection of the two-planes in twistor space, $Y = P \cap Q$. The Penrose transform is well-defined when the poles at the intersections $X \cap P$ and $X \cap Q$ do not coincide, or equivalently when the twistor lines X and Y are skew. If the lines X and Y intersect, then $(x-y)^2 = 0$ and the solution becomes singular.

This simple example provides some insight into the commonly used open regions for twistor space cohomology classes. Suppose that the intersection $P \cap Q$ lies entirely in $\mathbb{PT}^-$. Then the spacetime field configuration $\phi(x)$ is certainly non-singular on all lines $X \subset \mathbb{PT}^+$, or equivalently at spacetime points $x \in \mathbb{CM}^+$ that are future time-like. Such cohomology classes that are defined on the open region $U = \mathbb{PT}^+$ and lead to space-time field configurations that are non-singular on $\mathbb{CM}^+$ are called positive frequency fields. The analogous construction applies for negative frequency fields. It should be noted, however, that plane wave solutions are non-normalisable and cannot be accomodated in the present scheme.

In general, the Penrose transform provides an isomorphism between cohomology classes $H^{0,1}_{\bar\partial}(U, \mathcal{O}(2h-2))$ on some open region of twistor space $U \subset \mathbb{CP}^3$ and solutions of the zero-rest-mass field equations of helicity h. When the helicity $h \leq 0$ the integral transform becomes

$$\phi_{\alpha_1 \ldots \alpha_h}(x) = \int_X \langle \lambda \mathrm{d}\lambda \rangle \lambda_{\alpha_1} \ldots \lambda_{\alpha_{|h|}} \, \tilde\phi(\lambda_\alpha, i x^{\alpha\dot\alpha} \lambda_\alpha), \tag{2.28}$$

whereas when $h > 0$ we have

$$\phi_{\dot\alpha_1 \ldots \dot\alpha_h}(x) = \int_X \langle \lambda \mathrm{d}\lambda \rangle \frac{\partial^h}{\partial \mu^{\dot\alpha_1} \ldots \mu^{\alpha_h}} \, \tilde\phi(\lambda_\alpha, i x^{\alpha\dot\alpha} \lambda_\alpha). \tag{2.29}$$

In the following, we are mainly interested the description of non-abelian gauge fields, which require a more sophisticated framework in twistor space.

2.1.6 Self–Dual Yang–Mills Theory

We now consider the twistor space formulation of classical self-dual Yang-Mills theory. As above, we choose to work with Dolbeaut cohomology classes since this allows an off-shell formulation of the theory. This is essential when we come to constructing the twistor action and quantum mechanical perturbation theory in twistor space [7].

Our starting point is a smooth complex vector bundle of rank N

$$E \longrightarrow U \subset \mathbb{CP}^3 \tag{2.30}$$

with $c_1(E) = 0$ and equipped with an almost complex structure $\bar{D} = \bar{\partial} + a$. Later we will impose that the bundle be holomorphic on the open region $U \subset \mathbb{CP}^3$. However, it is important that we can construct a holomorphic frame on any line $X \subset U$ without imposing this condition.

The bundle is automatically holomorphic on pulling back to any line $X \subset U$ since $\bar{D}^2|_X = 0$ for dimensional reasons. It is also topologically trivial since our assumption $c_1(E) = 0$ implies also that $c_1(E|_X) = 0$. Hence, the bundle $E|_X$ is necessarily a sum of line bundles

$$E|_X = \bigoplus_i \mathcal{O}(d_i) \qquad \text{where} \qquad \sum_i d_i = 0 \tag{2.31}$$

and is holomorphically trivial if and only if $d_i = 0$ individually. In perturbation theory, we will always expand around a holomorphically trivial bundle, and small perturbations cannot change the discreet decomposition. Thus we can assume the bundle $E|_X$ to be holomorphically trivial.

This means we can find a smooth gauge transformation $h_X(\lambda, \bar{\lambda})$ such that

$$h_X^{-1} \circ (\bar{\partial} + a)\big|_X \circ h_X \;=\; \bar{\partial}\big|_X \,. \tag{2.32}$$

and under which covariantly holomorphic objects become simply holomorphic. Hence $h_X(\lambda, \bar{\lambda})$ is said to define a holomorphic frame for the bundle $E|_X$. The holomorphic frame itself obeys

$$(\bar{\partial} + a)\big|_X \; h_X = 0 \tag{2.33}$$

and hence is defined up to

$$h_X(\lambda, \bar{\lambda}) \to h_X(\lambda, \bar{\lambda})\, g(X) \tag{2.34}$$

where the gauge transformation $g(X)$ is globally holomorphic on the Riemann sphere and hence constant. We emphasize that the holomorphic frame can be constructed, at least in perturbation theory, without imposing that the bundle be holomorphic on the whole open region $U \subset \mathbb{CP}^3$. This observation will allow us to construct certain observables off-shell in twistor space.

In order to construct a space-time gauge bundle, no further conditions are required. Since the bundle $E|_X$ is automatically holomorphically trivial, we can always find N linearly independent globally holomorphic sections, which are unique up to constant GL(N) transformations. The space of such holomorphic sections $\Gamma(X, E|_X) \cong \mathbb{C}^{\mathrm{N}}$ thus provides the fibres of a complex vector bundle on an open region of space-time $\mathbb{CM}$ with complexified gauge group GL(N).

However, in order to construct a connection on the space-time gauge bundle, we must impose further conditions on the almost complex structure $\bar{D}$. We now pull the operator $\bar{D}$ back to the spin bundle using the projection map $p : \mathbb{S} \to U$ inherited from the incidence relations. The components are

$$\bar{D}_\lambda = \bar{\partial}_\lambda + a_\lambda \quad \bar{D}_{\dot{\alpha}} = \lambda^\alpha \partial_{\alpha\dot{\alpha}} + a_{\dot{\alpha}} \tag{2.35}$$

where the operators $\bar{\partial}_\lambda$ and $\lambda^\alpha \partial_{\alpha\dot{\alpha}}$ annihilate functions on the spin bundle $f(\lambda_\alpha, ix^{\alpha\dot{\alpha}}\lambda_\alpha)$ that are pulled back from holomorphic functions on U. They are extended to covariant operators using the components

$$a_\lambda = \bar{\partial}_\lambda \lrcorner\, p^* a \quad a_{\dot{\alpha}} = \lambda^\alpha \partial_{\alpha\dot{\alpha}} \lrcorner\, p^* a. \tag{2.36}$$

In this notation, the holomorphic frame obeys the equation $\bar{D}_\lambda\, h = 0$.

We have already mentioned that $\left[\bar{D}_\lambda, \bar{D}_\lambda\right] = 0$ for dimensional reasons. In particular, this allowed the construction of the holomorphic frame. In order to construct a space-time connection, we impose the further condition

$$\left[\bar{D}_\lambda, \bar{D}_{\dot{\alpha}}\right] = 0. \tag{2.37}$$

An immediate consequence, using the holomorphic frame condition $\bar{D}_\lambda h = 0$ twice, is that

$$\begin{aligned} \bar{\partial}_\lambda \left(h^{-1} \bar{D}_{\dot{\alpha}}\, h\right) &= h^{-1} \bar{D}_\lambda\, \bar{D}_{\dot{\alpha}}\, h \\ &= h^{-1}\left[\bar{D}_\lambda, \bar{D}_{\dot{\alpha}}\right] h \\ &= 0. \end{aligned} \tag{2.38}$$

Thus the combination $h^{-1} \bar{D}_{\dot{\alpha}}\, h$ is holomorphic along the fibres of the spin bundle and has weight one, and therefore must take the form

$$h^{-1} \bar{D}_{\dot{\alpha}}\, h = \lambda^\alpha\, A_{\alpha\dot{\alpha}}(x) \tag{2.39}$$

for some space-time field $A_{\alpha\dot{\alpha}}$. Under the gauge transformation (2.34) preserving the holomorphic frame we have $A_{\alpha\dot{\alpha}} \to g^{-1}(\partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}})g$, and hence we have obtained a gauge connection on the space-time complex vector bundle.

Finally, we impose the remaining condition

$$\left[\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\right] = 0 \tag{2.40}$$

meaning that the twistor bundle is now holomorphic on U. This equation implies that the curvature of the space-time connection $A_{\alpha\dot{\alpha}}$ is flat on restriction to any α-plane. This condition can equivalently be expressed

$$\begin{aligned}\lambda^{\alpha}\lambda^{\beta}\left[D_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}\right] &= \left[\lambda^{\alpha} D_{\alpha\dot{\alpha}}, \lambda^{\beta} D_{\beta\dot{\beta}}\right] \\ &= \left[h^{-1}\bar{D}_{\dot{\alpha}}h, h^{-1}\bar{D}_{\dot{\beta}}h\right] \\ &= h^{-1}\left[\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\right]h \\ &= 0.\end{aligned} \tag{2.41}$$

for any λ_{α}. Thus decomposing the space-time curvature into irreducible components

$$\left[D_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}\right] = \epsilon_{\alpha\beta} F_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}} F_{\alpha\beta} \tag{2.42}$$

we find that the self-dual curvature vanishes, $F_{\alpha\beta} = 0$. The non-vanishing component of the curvature is constructed via the non-linear Penrose transform

$$F_{\dot{\alpha}\dot{\beta}}(x) = \int_X \langle\lambda d\lambda\rangle\, h^{-1} \frac{\partial^2 a}{\partial\mu^{\dot{\alpha}}\partial\mu^{\dot{\beta}}}\, h \tag{2.43}$$

and obeys the equation of motion

$$D^{\alpha\dot{\alpha}} F_{\dot{\alpha}\dot{\beta}} = 0. \tag{2.44}$$

In summary, we have shown that given a holomorphic vector bundle $E \to U$ on an open region of twistor space $U \subset \mathbb{CP}^3$ that is topologically trivial $c_1(E) = 0$, we can construct a classical solutions of self-dual Yang-Mills theory on a corresponding region of space-time. It can be shown that this correspondence is in fact one-to-one [7].

2.2 $\mathcal{N} = 4$ Test Supersymmetric Yang–Mills Theory

2.2.1 Supertwistor Space

Let us now consider the $\mathcal{N} = 4$ supersymmetric extension of the above construction for self-dual Yang-Mills theory. Twistor space is now extended to the complex projective superspace $\mathbb{CP}^{3|4}$ by including four fermionic coordinates. Concretely, it is defined by the homogeneous coordinates

$$\mathcal{Z}^I = (\lambda_\alpha, \mu^{\dot\alpha}, \chi^a) \tag{2.45}$$

and the equivalence relation $\mathcal{Z}^I \sim r\,\mathcal{Z}^I$ for any $r \in \mathbb{C}^*$. The fermionic coordinates χ^a transform in the fundamental representation of an SU(4) R-symmetry. It will be important that supertwistor space is a Calabi-Yau supermanifold, meaning that it has a canonical top holomorphic form,

$$D^{3|4}\mathcal{Z} = \frac{1}{4!}\epsilon_{IJKL} Z^I dZ^J dZ^K dZ^L \; d^4\chi \tag{2.46}$$

a feature that is unique to maximal supersymmetry.

There is now a relationship between twistor space and the chiral superspace $\mathbb{CM}^{4|8}$ with coordinates $\left(x^{\alpha\dot\alpha}, \theta^{\alpha a}\right)$, where again the fermionic coordinates transform in the fundamental of SU(4). The correspondence is encapsulated by the extended incidence relations

$$\mu^{\dot\alpha} = i\, x^{\alpha\dot\alpha}\lambda_\alpha \qquad \chi^a = \theta^{\alpha a}\lambda_\alpha. \tag{2.47}$$

Given a point $\left(x^{\alpha\dot\alpha}, \theta^{\alpha a}\right)$ in chiral superspace, the incidence relations define a holomorphically embedded complex projective line in twistor space, $X \cong \mathbb{CP}^1$. Furthermore, as for the bosonic correspondence, two points in chiral superspace are null separated, that is

$$(x_1 - x_2)^2 = 0 \quad (x_1 - x_2) \cdot (\theta_1 - \theta_2) = 0 \tag{2.48}$$

if and only if the corresponding twistor lines intersect. Thus, the complex structure of supertwistor space determines and is determined by the superconformal structure of chiral superspace.

The construction of α-planes requires some additional explanation compared to the purely bosonic case. Given a point in supertwistor space $\mathcal{Z}^I$, the general solution of the incidence relations is now

$$x^{\alpha\dot\alpha} = x_0^{\alpha\dot\alpha} + \lambda^\alpha\tilde\lambda^{\dot\alpha} \qquad \theta^{\alpha a} = \theta_0^{\alpha a} + \lambda^\alpha\eta^a \tag{2.49}$$

where $(\tilde\lambda^{\dot\alpha}, \eta^a)$ are parameters labelling the solutions. The parameters span a superplane $\mathbb{C}^{2|4}$ inside chiral superspace $\mathbb{CM}^{4|8}$. This plane is completely null, meaning that all tangent vectors are orthogonal and null, in the sense of Eq. (2.48). These are the α-planes.

Twistor space is an important tool for studying superconformal field theories because it a carries a particularly natural action of the superconformal group. For maximal $\mathcal{N} = 4$ supersymmetry we have an infinitessimal action of the general linear supergroup GL(4|4, $\mathbb{C}$) by

$$J^I{}_J = Z^I \frac{\partial}{\partial Z^J}. \tag{2.50}$$

In order to generate the superconformal group SL(4|4, $\mathbb{C}$) we must ensure the generators have vanishing supertrace by removing a component proportional to $(-1)^I \delta^I{}_J\, b$ where $b = (-1)^K J^K{}_K$. In the following, we almost always consider homogeneous functions of the coordinates Z^I of weight zero, in which case $h = 0$ and we recover the action of PSL(4|4, $\mathbb{C}$) on supertwistor space $\mathbb{CP}^{3|4}$. In terms of spinor components, the generators (2.50) are

$$P_{\alpha\dot\alpha} = \lambda_\alpha \frac{\partial}{\partial \mu^{\dot\alpha}} \qquad J_{\alpha\beta} = \frac{1}{2}\left(\lambda_\alpha \frac{\partial}{\partial \lambda^\beta} + \lambda_\beta \frac{\partial}{\partial \lambda^\alpha}\right) \qquad J_{\dot\alpha\dot\beta} = \frac{1}{2}\left(\mu_{\dot\alpha} \frac{\partial}{\partial \mu^{\dot\beta}} + \mu_{\dot\beta} \frac{\partial}{\partial \mu^{\dot\alpha}}\right)$$
$$K^{\alpha\dot\alpha} = \mu^{\dot\alpha} \frac{\partial}{\partial \lambda_\alpha} \qquad D = \frac{1}{2}\left(\lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} - \mu^{\dot\alpha} \frac{\partial}{\partial \mu^{\dot\alpha}}\right) \qquad R^a_b = \chi^a \frac{\partial}{\partial \chi^b}, \tag{2.51}$$

and

$$Q_{\alpha a} = \lambda_\alpha \frac{\partial}{\partial \chi^a} \qquad \widetilde{Q}^a_{\dot\alpha} = \chi^a \frac{\partial}{\partial \mu^{\dot\alpha}}$$
$$S^{\alpha a} = \chi^a \frac{\partial}{\partial \lambda_\alpha} \qquad \widetilde{S}^{\dot\alpha}_a = \mu^{\dot\alpha} \frac{\partial}{\partial \chi^a}. \tag{2.52}$$

Finally, the real form relevent for lorentzian signature, PSU(2, 2|4), consists of those transformations that preserve the same pseudo-hermitian metric (2.18) as in the bosonic case.

2.2.2 The Self-Dual Theory

Let us now construct the self-dual sector of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory from the twistor space perspective. The argument is an extension of the purely bosonic case presented above. We introduce a smooth complex vector bundle of rank N

$$E \longrightarrow U \subset \mathbb{CP}^{3|4}, \tag{2.53}$$

with vanishing first Chern class $c_1(E) = 0$ and an almost complex structure $\bar{D} = \bar\partial + \mathcal{A}$. In the supersymmetric case, it is important that the partial connection is a one-form with components in the bosonic directions of supertwistor space only, that is we have $\mathcal{A} = \mathcal{A}_{\bar I}\, d\bar{Z}^{\bar I}$.

In perturbation theory, the bundle is again automatically holomorphic and holomorphically trivial once pulled back to any line $X \subset U$. Hence we can construct a holomorphic frame $\mathrm{H}(X, \lambda, \bar\lambda)$ that obeys

$$\bar{D}_\lambda\, \mathrm{H} = 0 \tag{2.54}$$

and depends smoothly on the line X, or equivalently, on the coordinates $\left(x^{\alpha\dot{\alpha}}, \theta^{\alpha a}\right)$. Furthermore, we can find N linearly independent globally holomorphic sections, which are unique up to constant GL(N) transformations. The space of such holomorphic sections $\Gamma(X, E|_X) \cong \mathbb{C}^{\mathrm{N}}$ now form the fibres of a complex vector bundle on a region of chiral superspace $\mathbb{CM}^{4|8}$, with complexified gauge group GL(N).

In order to construct a superconnection on the chiral superspace, we must impose further conditions on the almost complex structure. Pulling back to the spin bundle, the components of $\bar{D}$ are

$$\begin{aligned} \bar{D}_\lambda &= \bar{\partial}_\lambda + \mathcal{A}_\lambda \\ \bar{D}_{\dot{\alpha}} &= \lambda^\alpha \partial_{\alpha\dot{\alpha}} + \mathcal{A}_{\dot{\alpha}} \\ \bar{D}_a &= \lambda^\alpha \partial_{\alpha a} \end{aligned} \tag{2.55}$$

where $\mathcal{A}_\lambda$ and $\mathcal{A}_{\dot{\alpha}}$ are defined exactly as in the bosonic case. Again, the curvature $\left[\bar{D}_\lambda, \bar{D}_\lambda\right] = 0$ for dimensional reasons and, to construct a space-time superconnection, we must impose the further conditions

$$\begin{aligned} \left[\bar{D}_\lambda, \bar{D}_{\dot{\alpha}}\right] &= 0 \\ \left[\bar{D}_\lambda, \bar{D}_a\right] &= 0. \end{aligned} \tag{2.56}$$

These conditions imply that the combinations $\mathrm{H}^{-1} D_{\dot{\alpha}}\, \mathrm{H}$ and $\mathrm{H}^{-1} \bar{D}_a \mathrm{H}$ are holomorphic in λ_α with weight one, and hence

$$\begin{aligned} \mathrm{H}^{-1} \bar{D}_{\dot{\alpha}}\, \mathrm{H} &= \lambda^\alpha\, \mathcal{A}_{\alpha\dot{\alpha}}(x, \theta) \\ \mathrm{H}^{-1} \bar{D}_a\, \mathrm{H} &= \lambda^\alpha\, \mathcal{A}_{\alpha a}(x, \theta) \end{aligned} \tag{2.57}$$

where the space-time fields $(\mathcal{A}_{\alpha\dot{\alpha}}, \mathcal{A}_{\alpha a})$ transform as a superconnection under super gauge transformations that preserve the holomorphic frame. They allow the construction of space-time super-covariant derivatives $(\nabla_{\alpha\dot{\alpha}}, \nabla_{\alpha a})$.

An important difference compared to the bosonic case is that the conditions impose space-time equations of motion for the scalars and fermions. The reason is that the superconnection we constructed automatically obeys the integrability constraints

$$\begin{aligned} \lambda^\alpha \lambda^\beta \left[\, \nabla_{\alpha\dot{\alpha}}\, , \nabla_{\beta b}\, \right] &= 0 \\ \lambda^\alpha \lambda^\beta \left\{\, \nabla_{a\alpha}\, , \nabla_{\beta b}\, \right\} &= 0. \end{aligned} \tag{2.58}$$

These integrability constraints are absent in bosonic theory. In the supersymmetric theory, they imply some equations of motion for the scalar and fermion partners of the gluon. Therefore, the space-time superconnection cannot be constructed completely off-shell, causing subtle technical problems for the quantum theory when a regulator is introduced—the consequences of this are further discussed in Chap. 6. This is in fact a well-known problem with standard superspace approaches to space-time gauge theories with extended supersymmetry and can be remedied by harmonic superspace methods [8, 9].

Finally, we impose the remaining condition

$$\left[\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\right] = 0 \tag{2.59}$$

meaning that the twistor bundle is now holomorphic on the whole of U. This implies that the curvature of the space-time superconnection $\mathcal{A}_{\alpha\dot{\alpha}}$ obeys

$$\lambda^{\alpha}\lambda^{\beta}\left[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}\right] = 0. \tag{2.60}$$

or equivalently that the corresponding supercurvature is self-dual, $\mathcal{F}_{\alpha\beta} = 0$. These are the remaining equations of motion of the self-dual theory. Thus, combining Eqs. (2.58) and (2.60), the equations of motion of the self-dual theory are equivalent to the statement that the space-time superconnection is flat when projected onto α-planes.

Since the above construction is rather abstract, let us now expand in components fields and derive the corresponding space-time equations of motion. We begin by expanding the partial connection in powers of the fermionic components

$$\begin{aligned}\mathcal{A}(Z,\chi) = a(Z,\bar{Z}) + \chi^{a}\tilde{\gamma}_{a}(Z,\bar{Z}) + \frac{1}{2!}\chi^{a}\chi^{b}\phi_{ab}(Z,\bar{Z}) \\ + \frac{1}{3!}\epsilon_{abcd}\chi^{a}\chi^{b}\chi^{c}\gamma^{d}(Z,\bar{Z}) + \chi^{1}\chi^{2}\chi^{3}\chi^{4}g(Z,\bar{Z}).\end{aligned} \tag{2.61}$$

The condition that the bundle be holomorphic

$$\bar{\partial}\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0, \tag{2.62}$$

implies the following equations for the component fields

$$\begin{aligned}&\bar{\partial} + a\wedge a = 0\\ &\bar{D}\gamma_{a} = 0\\ &\bar{D}\phi^{ab} - \phi^{a}\wedge\phi^{b} = 0\\ &\bar{D}\,\bar{\gamma}^{a} - \frac{1}{2}\varepsilon^{abcd}\left(\gamma_{b}\wedge\phi_{cd} + \phi_{cd}\wedge\gamma_{b}\right) = 0\\ &\bar{D}g + \gamma_{a}\wedge\bar{\gamma}^{a} + \frac{1}{4}\varepsilon^{abcd}\phi_{ab}\wedge\phi_{cd},\end{aligned} \tag{2.63}$$

where $\bar{D} = \bar{\partial} + a$ is the bosonic part of the twistor superfield $\mathcal{A}$.

The space-time component fields are now constructed from the components of the partial connection $\mathcal{A}$ by the Penrose transform. The key ingredient in the transform is the bosonic part of the holomorphic frame $h(x, \lambda, \bar{\lambda})$ obtained by discarding higher order terms in the fermionic expansion

$$H(x, \theta, \lambda, \bar{\lambda}) = h(x, \lambda, \bar{\lambda}) + \cdots . \tag{2.64}$$

The space-time component fields are then constructed by the following Penrose transforms

$$\begin{aligned}
F_{\dot{\alpha}\dot{\beta}}(x) &= \int_X \langle \lambda \mathrm{d}\lambda \rangle \; h^{-1}(x, \lambda) \; \frac{\partial^2 a}{\partial \mu^{\dot{\alpha}} \mu^{\dot{\beta}}}(\lambda) \; h(x, \lambda) \\
\tilde{\psi}_{\dot{\alpha} a}(x) &= \int_X \langle \lambda \mathrm{d}\lambda \rangle \; h^{-1}(x, \lambda) \; \frac{\partial \tilde{\gamma}_a}{\partial \mu^{\dot{\alpha}}}(\lambda) \; h(x, \lambda) \\
\tilde{\phi}_{ab}(x) &= \int_X \langle \lambda \mathrm{d}\lambda \rangle \; h^{-1}(x, \lambda) \; \tilde{\phi}_{ab}(\lambda) \; h(x, \lambda) \\
\psi_\alpha{}^a(x) &= \int_X \langle \lambda \mathrm{d}\lambda \rangle \; h^{-1}(x, \lambda) \; \lambda_\alpha \, \gamma^a(\lambda) \; h(x, \lambda) \\
G_{\alpha\beta}(x) &= \int_X \langle \lambda \mathrm{d}\lambda \rangle \; h^{-1}(x, \lambda) \; \lambda_\alpha \, \lambda_\beta \, g(\lambda) \; h(x, \lambda).
\end{aligned} \tag{2.65}$$

and transform in the adjoint representation of the complexified gauge group $\mathrm{GL}(\mathrm{N}, \mathbb{C})$ under bosonic gauge transformations. The auxilliary field $G_{\alpha\beta}$ is non-dynamical. The remaining dynamical fields are the gauge field $A_{\alpha\dot{\alpha}}$, complex Weyl fermions ψ^a_α in the fundamental representation **4**, and scalars ϕ_{ab} in the antisymmetric tensor **6** of the R-symmetry group $\mathrm{SU}(4)_R$. These fields form the $\mathcal{N} = 4$ supermultiplet.

The component form of the holomorphic condition (2.63) and the definition of the bosonic space-time connection, $h^{-1} D_{\dot{\alpha}} h = \lambda^\alpha A_{\alpha\dot{\alpha}}$, then imply the following equations of motion for the space-time fields

$$\begin{aligned}
F_{\alpha\beta} &= 0 \\
D^{\alpha\dot{\alpha}} \bar{\psi}_{\dot{\alpha} a} &= 0 \\
\Box \phi_{ab} - \left\{ \phi^{\dot{\alpha}}_a , \phi_{\dot{\alpha} b} \right\} &= 0 \\
D^{\alpha\dot{\alpha}} \psi^a_\alpha + \mathrm{i} \left[\phi^{ab}, \phi^{\dot{\alpha}}_b \right] &= 0 \\
D^{\alpha\dot{\alpha}} G_{\alpha\beta} + \mathrm{i} \left\{ \psi^a_\beta , \bar{\psi}^{\dot{\alpha}}_a \right\} + D_{\beta\dot{\alpha}} \left[\phi^{ab}, \phi_{ab} \right] &= 0 \, .
\end{aligned} \tag{2.66}$$

These are precisely the field equations for self-dual $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, obtained from the space-time action [10, 11],

$$\mathcal{L}_1 = \mathrm{tr} \Big\{ -\frac{1}{4} G^{\alpha\beta} F_{\alpha\beta} + i \bar{\psi}_{\dot{\alpha} a} D^{\dot{\alpha}\alpha} \psi^a_\alpha + \frac{1}{4} D_{\alpha\dot{\alpha}} \phi^{ab} D^{\alpha\dot{\alpha}} \phi_{ab} + \bar{\psi}_{\dot{\alpha} a} \big[\phi^{ab}, \bar{\psi}^{\dot{\alpha}}_b \big] \Big\} \, . \tag{2.67}$$

2.2.3 The Complete Theory

We formulate the complete $\mathcal{N} = 4$ supersymmetric Yang-Mills theory as an expansion around the self-dual sector of the theory, which had a particularly simple construction in twistor space. The additional interaction lagrangian is

$$\mathcal{L}_2 = g^2 \mathrm{tr}\Big\{ -\frac{1}{8} G^{\alpha\beta} G_{\alpha\beta} - \psi^{\alpha a}\big[\bar{\phi}_{ab}, \psi^b_\alpha\big] + \frac{1}{8}\big[\phi^{ab}, \phi^{cd}\big]\big[\bar{\phi}_{ab}, \bar{\phi}_{cd}\big]\Big\} \tag{2.68}$$

and leads to the equations of motion

$$\begin{aligned}
&F_{\alpha\beta} = g^2\, G_{\alpha\beta} \\
&D^{\alpha\dot{\alpha}}\, \bar{\psi}_{\dot{\alpha}a} = g^2\, \mathrm{i}\, [\phi_{ab}, \psi^{\alpha a}] \\
&\Box\phi_{ab} - \left\{\phi^{\dot{\alpha}}_a, \phi_{\dot{\alpha}b}\right\} = \frac{g^2}{2}\epsilon_{abcd}\left\{\psi^{\alpha c}, \psi^d_\alpha\right\} + g^2\left[\phi_{ac}, \left[\phi_{bd}, \phi^{cd}\right]\right] \\
&D^{\alpha\dot{\alpha}}\psi^a_\alpha + \mathrm{i}\left[\phi^{ab}, \bar{\psi}^{\dot{\alpha}}_b\right] = 0 \\
&D^{\alpha\dot{\alpha}} G_{\alpha\beta} + \mathrm{i}\left\{\psi^a_\beta, \phi^{\dot{\alpha}}_a\right\} + D_{\beta\dot{\alpha}}\left[\phi^{ab}, \phi_{ab}\right] = 0.
\end{aligned} \tag{2.69}$$

The corrections to the self-dual sector all appear proportional to the squared coupling g^2 on the right-hand side.

Let us concentrate on the relationship between the self-dual component of the curvature $F_{\alpha\beta}$ and the auxilliary field $G_{\alpha\beta}$. The components of the lagrangian involving the auxilliary field are

$$-\frac{1}{4}\mathrm{tr}\Big\{G^{\alpha\beta} F_{\alpha\beta} - \frac{1}{2} g^2 G^{\alpha\beta} G_{\alpha\beta}\Big\} \tag{2.70}$$

and the equations of motion $F_{\alpha\beta} = g^2 G_{\alpha\beta}$ reduce to the self-dual condition as $g^2 \to 0$. In this manner we are expanding around the self-dual sector. On the other hand, integrating out the auxilliary field we obtain the complete space-time action for maximally supersymmetric Yang-Mills theory,

$$\begin{aligned}
\mathcal{L} = \mathrm{tr}\Big\{ &-\frac{1}{4g^2} F^{\alpha\beta} F_{\alpha\beta} + i\bar{\psi}_{\dot{\alpha}a} D^{\dot{\alpha}\alpha}\psi^a_\alpha + \frac{1}{4} D_{\alpha\dot{\alpha}}\phi^{ab} D^{\alpha\dot{\alpha}}\phi_{ab} \\
&+ \bar{\psi}_{\dot{\alpha}a}\big[\phi^{ab}, \bar{\psi}^{\dot{\alpha}}_b\big] - g^2\psi^{\alpha a}\big[\bar{\phi}_{ab}, \psi^b_\alpha\big] + \frac{g^2}{8}\big[\phi^{ab}, \phi^{cd}\big]\big[\bar{\phi}_{ab}, \bar{\phi}_{cd}\big]\Big\}
\end{aligned} \tag{2.71}$$

This is indeed the unique, local, gauge invariant space-time action that is invariant under superconformal symmetries PSU(2,2|4).

2.2.4 The Twistor Action

We begin with the self-dual sector of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In the previous section, we explained that classical solutions are obtained from holomorphic vector bundles on supertwistor space, which are characterised by the vanishing of the (0, 2)-component of the curvature

$$\bar{\partial}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0. \tag{2.72}$$

Modulo complex gauge transformations. Thus holomorphic vector bundles on supertwistor space can be obtained as the stationary points of holomorphic Chern-Simons theory on twistor space,

$$S_1 = \frac{1}{2\pi} \int D^{3|4}\mathcal{Z} \wedge \mathrm{Tr}\left(\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right). \tag{2.73}$$

It is the unique gauge-invariant local action that depends only on the complex structure and the holomorphic volume form $D^{3|4}\mathcal{Z}$. The holomorphic Chern-Simons action was introduced in [12] and shown to reduce, in an appropriate gauge, to the self-dual action (2.67) in reference [13].

In order to expand the complete theory around the self-dual sector, we add further interaction terms to the holomorphic Chern-Simons theory. The interaction involves the logarithm of the determinant of the complex structure $(\bar{\partial} + \mathcal{A})|_X$ integrated over all lines inside null supertwistor space [13]. In the following, we will write this interaction term as

$$S_2 = g^2 \int \mathrm{d}^{4|8}X \;\log\det\, \bar{D}\big|_X\,, \tag{2.74}$$

although strictly speaking, one should subract the contribution $\log\det\,\bar{\partial}\big|_X$ from the trivial background connection. The determinant is actually a section of the determinant line bundle over the space of partial connections on the bundle $E|_X$ and picks up anomalous contributions under gauge transformations. However, the logarithm ensures that these contributions are additive and annihilated by the fermionic integral [13].

The determinant is not convenient for explicit calculations. However, the correction can be expanded in a power series in the twistor partial connection $\mathcal{A}$ leading to an infinite series of interaction terms (this expansion is derived in Chap. 5),

$$g^2 \sum_{n=2}^{\infty} \int \mathrm{d}^{4|8}X \int_{X^n} \frac{\mathrm{d}\rho_n \cdots \mathrm{d}\rho_1}{(\rho_1 - \rho_n)\cdots(\rho_2 - \rho_1)(\rho_1 - \rho_n)}\, \mathrm{Tr}\, \mathcal{A}(\rho_n)\cdots\mathcal{A}(\rho_1). \tag{2.75}$$

where each coordinate $\{\rho_1, \ldots, \rho_n\}$ is a complex coordinate on the line X. This expansion allows perturbative calculations to be performed straightforwardly in twistor space.

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