Curvature Measures, Isoperimetric Type Inequalities and Fully Nonlinear PDEs

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Abstract The notes consider two special fully nonlinear partial differential equations arising from geometric problems, one is of elliptic type and another is of parabolic type. The elliptic equation is associated to the problem of prescribing curvature measures, while an inverse mean curvature type of parabolic equation is introduced to prove the isoperimetric type inequalities for quermassintegrals of k-convex starshaped domains.

The material in the notes is compiled from the lectures given in the CIME Summer School in Cetraro, 2012. It treats some nonlinear elliptic and parabolic partial differential equations arising from geometric problems of hypersurfaces in $\mathbb{R}^{n+1}$. A curvature type of elliptic equation is used to solve the problem of prescribing curvature measures, which is a Minkowski type problem. An inverse mean curvature type of parabolic equation is employed for the proof of isoperimetric type inequalities for quermassintegrals of k-convex starshaped domains. Both types of equations are fully nonlinear, they belong to the category of general geometric fully nonlinear PDE.

The emphasis of the notes is the a priori estimates, which is the key in the theory of fully nonlinear PDE. These estimates are intend to be self-contained here, with minimal assumptions on basic knowledge in PDE and geometry, namely the standard maximum principles for linear elliptic and parabolic equations, the elementary formulas of Gauss, Codazzi and Weingarten for hypersurfaces in $\mathbb{R}^{n+1}$, and the curvature commutator identities. Two theorems we would use without proof for higher regularity are: the Evans-Krylov Theorem [11, 31] for uniformly fully nonlinear elliptic equations and the Krylov Theorem [31] for uniformly parabolic

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fully nonlinear PDE, since the proofs of these deep results would take considerable space.

The topics dealt in this notes are special samples of geometric nonlinear PDE. It is our hope they can serve as an introduction to the general theory of geometric analysis.

The notes are organized as follows. The curvature measures are introduced through the Steiner formula in differential geometric setting in Sect. 1, where the Steiner formula and the Minkowski identity are proved. As the geometric objects and the associated differential equations are involved the elementary symmetric functions, some important properties of these functions are collected in Sect. 2 with proofs, except the theory of hyperbolic polynomials of Garding which is put in the Appendix. Section 3 deals with the problem of prescribing curvature measures. A k-curvature fully nonlinear elliptic equation is set up there together with the a priori estimates of the solutions of the equation. Section 4 is devoted to the proof of the isoperimetric inequalities for quermassintegrals of k-convex star shaped domains, via parabolic approach. Again, the main part is the a priori estimates for the solutions of the corresponding parabolic equation. The literature comments appear at the end of the notes.

1 The Steiner Formula and Curvature Measures

Suppose Ω is a domain in $\mathbb{R}^{n+1}$, for each $x \in \mathbb{R}^{n+1}$, denote $p(\Omega, x)$ to be the set of the nearest points in Ω to x. Given any Borel set $\beta \in \mathfrak{B}(\mathbb{R}^{n+1})$, $\forall s > 0$, consider

$$A_s(\Omega, \beta) := \{x \in \mathbb{R}^{n+1} | 0 < d(\Omega, x) \leq s \text{ and } p(\Omega, x) \in \beta\}$$

which is the set of all points $x \in \mathbb{R}^{n+1}$ for which the distance $d(\Omega, x) \leq s$ and for which the nearest point $p(\Omega, x)$ belongs to β. If $\partial\Omega$ is smooth and β is open, for $s > 0$ small, one may write

$$A_s(\Omega, \beta) = \{X + t\nu(X) \quad | X \in \beta \cap M, 0 \leq t \leq s, \}$$

where $\nu(X)$ is the outer normal of M at X.

We assume the boundary of Ω, $M = \partial\Omega$, is C^2 (or smoother). Let

$$\kappa(X) = (\kappa_1(X), \cdots, \kappa_n(X))$$

be the principal curvatures of $X \in M$. To calculate the volume of $A_s(\Omega, \beta)$, pick any local orthonormal frame of M, so that the second fundamental form $(W_{ij}(X))$ of M at X is diagonal. As $(X + t\nu(X))_i = (1 + tW_{ii})X_i$, and $\nu(X)$ is orthogonal to X_i, the volume element at $X + t\nu(X)$ is simply

$$dV = (\prod_{i=1}^{n}(1 + tW_{ii}))d\mu_M dt = \sum_{i=0}^{n} \sigma_i(\kappa(X))t^i d\mu_M dt,$$

where $\sigma_i(\kappa)$ is the i-th elementary symmetric function of κ (see Definition (6)), and where $d\mu_g$ is the volume element with respect to the induced metric g of M in $\mathbb{R}^{n+1}$. Therefore,

$$V(A_s(\Omega,\beta)) = \int_0^s \int_{\beta\cap M} \sum_{i=0}^{n} \sigma_i(\kappa(X))t^i \, d\mu_M dt = \sum_{i=0}^{n} \Big(\int_{\beta\cap M} \sigma_i(\kappa(X)) d\mu_M\Big) \frac{s^{i+1}}{i+1}.$$

Set

$$\mathcal{C}_m(\Omega) = \sigma_{n-m}(\kappa) d\mu_M, \quad m = 0, 1, \cdots, n. \tag{1}$$

We have proved the Steiner formula,

$$V(A_s(\Omega,\beta)) = \sum_{m=0}^{n} \frac{s^{n+1-m}}{n+1-m} \mathcal{C}_m(\Omega,\beta), \tag{2}$$

for $\beta \in \mathfrak{B}(\mathbb{R}^{n+1})$ and $s > 0$.

In the context of classical convex geometry, the coefficients $\mathcal{C}_0(\Omega,\cdot), \cdots, \mathcal{C}_n(\Omega,\cdot)$ in (2) are called curvature measures of the convex body Ω. Formula (1) indicates that $\mathcal{C}_m(\Omega,\cdot)$ is well defined if $\partial\Omega$ is C^2 without convexity assumption. In general, $\mathcal{C}_m(\Omega)$ is a signed measure. The positivity of $\mathcal{C}_m(\Omega)$ for $0 \le m \le k$ is related to the notion of k-convexity (Definition 3.1).

The global quantities

$$V_{n-m}(\Omega) = C_{n,k} \int_M \sigma_m(\kappa) d\mu_M, \quad m = 0, 1, \cdots, n, \tag{3}$$

where $C_{n,k} = \frac{\sigma_k(1,\cdots,1)}{\sigma_{k-1}(1,\cdots,1)}$, are called the quermassintegrals of Ω in convex geometry, if Ω is convex. Again, we note that these quantities are well defined for general C^2 domain Ω without convexity condition.

It is clear that the curvature measures capture the geometry of M.

1. What are the relations between quermassintegrals?
2. How much information can we extract from the curvature measures?

These are the main questions we want to deal with in this notes. The first question has satisfactory answer when Ω is convex, which corresponds to the classical Alexandrov-Fenchel inequalities. Generalization of these inequalities to non-convex domains has gained much interest recently, but remains largely unsettled. We will focus on a class of non-convex star-shaped domains, where a clean result can be established. The second question can be answered in terms of the Minkowski type problem, the problem of prescribing curvature measures. It turns out there is an affirmative answer if we restrict ourselves to the class of non-convex star-shaped domains.

There is a different expression for $V_{n-m}(\Omega)$ involving the support function $u(X) = \langle X, \nu(X)\rangle$. The Minkowski identity states that $\forall k \geq 1$,

$$\int_M u\sigma_k(\kappa)d\mu_M = C_{n,k}\int_M \sigma_{k-1}(\kappa)d\mu_M, \tag{4}$$

By the Divergent theorem,

$$V_{n+1}(\Omega) = \frac{1}{n+1}\int_M ud\mu_M.$$

From (4), we may define

$$V_{(n+1)-k}(\Omega) = \int_M u\sigma_k(\kappa)d\mu_M, \tag{5}$$

for $k = 0, \cdots, n$. $V_{n+1}(\Omega)$ is multiple of the volume of Ω by a dimensional constant, $V_n(\Omega)$ is a multiple of the surface area of $\partial\Omega$ by another dimensional constant. In convex geometry, u is called the support function of Ω.

The Minkowski identity (4) can be verified using the fact that σ_k has divergent free structure (Lemma 2.1). Again, pick a local orthonormal frame on M, let $h = (W_{ij})$ be the second fundamental from and let $g^{-1}h = (h^i_j)$ be the Weingarten tensor. We compute

$$(\frac{|X|^2}{2})_{ij} = X_iX_j + X_{ij} = \delta_{ij} - \langle X, \nu(X)\rangle W_{ij} = \delta_{ij} - uW_{ij}.$$

Contracting with $\sigma_k^{ij} = \frac{\partial\sigma_k}{\partial h^i_j}(g^{-1}h)$ and integrating over M

$$\int_M \sigma_k^{ij}(\frac{|X|^2}{2})_{ij} = \int_M (\sum_i \sigma_k^{ij}\delta_{ij} - u\sigma_k^{ij}W_{ij}).$$

As

$$\sigma_k^{ij}\delta_{ij} = (n-k+1)\sigma_{k-1}, \quad \sigma_k^{ij}W_{ij} = k\sigma_k,$$

and by (8), we get

$$0 = (n-k+1)\int_M \sigma_{k-1}(g^{-1}h) - k\int_M u\sigma_k(g^{-1}h).$$

This is exactly the identity (4).

The Minkowski addition of two sets $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ is defined as

$$\Omega_1 + \Omega_2 = \{z = x + y | x \in \Omega_1, y \in \Omega_2\}.$$

The Minkowski addition is one of the basic operation in convex geometry. For general domain Ω, when $0 \leq s$ small, one may define

$$\Omega_s = \{z = x + y | x \in \Omega, y \in B_s\},$$

where B_s is the ball centered at the origin with radius s.

$$\Omega_s = \{X + t\nu(X) \quad |X \in \Omega, 0 \leq t \leq s.\}$$

If $M = \partial\Omega$ is smooth, the boundary $\partial\Omega_s = M_s$ is also smooth and can be written as

$$M_s = \{X + s\nu(X) \quad |X \in M.\}$$

Moreover, the normal of M_s at $X_s = X + s\nu(X)$ is the same as $\nu(X)$ for each $X \in M$. The support function of Ω_s is $u_s(X^s) = u(X) + s$. For any local orthonormal frame $e_1, \cdots, e_n$ on M such that $h = (W_{ij})$ is diagonal at the point, one may calculate the induced metric g_s on M^s

$$g_s = \sum_{i=1}^{n} (1 + h_i^i)^2 e_i \otimes e_i,$$

and the area element of M^s

$$d\mu_{M_s} = \det(I + sg^{-1}h) d\mu_M.$$

By the Minkowski identity, the volume of Ω_s can be computed as

$$\begin{aligned}
V(\Omega_s) &= \frac{1}{n+1} \int_M u_s \det(I + sg^{-1}h) d\mu_M \\
&= \frac{1}{n+1} \int_M \sum_{i=0}^{n} (u + s) s^i \sigma_i(g^{-1}h) d\mu_M \\
&= \frac{1}{n+1} \int_M \sum_{i=0}^{n} (us^i \sigma_i(g^{-1}h) + s^{i+1}) \sigma_i(g^{-1}h) d\mu_M \\
&= \frac{1}{n+1} \sum_{i=0}^{n} \frac{n+1}{i+1} s^{i+1} \int_M \sigma_i(g^{-1}h) d\mu_M + \frac{1}{n+1} \int_M u d\mu_M \\
&= \sum_{i=0}^{n+1} c_{n+1}^i t^{n+1-i} V_i(\Omega),
\end{aligned}$$

2 Some Properties of Elementary Symmetric Functions

The elementary symmetric functions appear naturally in the geometric quantities in the previous section. In order to carry on analysis, we need to understand properties of the elementary symmetric functions.

For $1 \leq k \leq n$, and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, the k-th elementary symmetric function is defined as

$$\sigma_k(\lambda) = \sum \lambda_{i_1} \dots \lambda_{i_k}, \tag{6}$$

where the sum is taken over all strictly increasing sequences $i_1, \dots, i_k$ of the indices from the set $\{1, \dots, n\}$. The definition can be extended to symmetric matrices. Denote $\lambda(W) = (\lambda_1(W), \dots, \lambda_n(W))$ to be the eigenvalues of the symmetric matrix W, set

$$\sigma_k(W) = \sigma_k(\lambda(W)).$$

It is convenient to set

$$\sigma_0(W) = 1, \quad \sigma_k(W) = 0, \quad \text{for } k > n.$$

It follows directly from the definition that, for any $n \times n$ symmetric matrix W, and $\forall t \in \mathbb{R}$,

$$\sigma_n(I + tW) = \det(I + tW) = \sum_{i=0}^{n} \sigma_i(W) t^i. \tag{7}$$

Conversely, (7) can also be used to define $\sigma_k(W)$, $\forall k = 0, \cdots, n$.

An important property of σ_k is the divergent free structure. Suppose M is a general Riemannian manifold of dimension n, W is a symmetric tensor on M. We call W is Codazzi if $DW = 0$. This property is equivalent to say that, for any local orthonormal frame $(e_1, \cdots, e_n)$ on M, write $W = (w_{ij})$, then $w_{ij,l} = \nabla_{e_l} w_{ij}$ is symmetric with respect to i, j, l. Some classical examples are

1. Second fundamental form h of any hypersurface in space form $N(c)$ with constant sectional curvature c, this follows from the Codazzi equation;
2. $W = \overline{\nabla}^2 v + cv$, $\forall v \in C^3(N(c))$.

Throughout the rest of the notes, we will use Einstein summation convention, unless it is otherwise indicated.

Below is the statement of divergent free structure of σ_k.

Lemma 2.1. *Suppose $e_1, \cdots, e_n$ is a local orthonormal frame on M, $W = (w_{ij})$ is a Codazzi tensor on M, then for each i,*

$$\sum_{j=1}^{n} \left(\frac{\partial \sigma_k}{\partial w_{ij}}\right)_j (W) = 0. \tag{8}$$

Proof. We first verify (8) for $k = n$. Denote C^{il} to be the cofactor of W, i.e.,

$$\frac{\partial \sigma_n}{\partial w_{il}} = C^{il}, \quad C^{il} w_{lj} = \det(W)\delta^i_j.$$

Differentiate above identity in e_m direction and contract with C^{jm},

$$C^{jm} C^{il}_m w_{lj} + C^{il} w_{lj,m} C^{jm} = \delta^i_j (\det(W))_m C^{jm}.$$

If $\det(W) \neq 0$ at the point, we get

$$C^{im}_m = C^{pq} C^{im} w_{pq,m} - C^{il} C^{jm} w_{lj,m} = C^{pq} C^{im} w_{pq,m} - C^{il} C^{jm} w_{jm,l} = 0.$$

If $\det(W) = 0$ at the point, we may approximate W by Codazzi tensor $\tilde{W} = W + tg$ where g is the metric tensor on M such that $\det(\tilde{W}) \neq 0$ for t small. Equation (8) is verified for the case $k = n$.

Observe that, for $t \in \mathbb{R}$,

$$\sigma_n(\tilde{W}) = \sum_{m=0}^{n} t^m \sigma_{n-m}(W).$$

Apply (8) for the case $k = n$,

$$\sum_{m=0}^{n} t^m \sum_j \left(\frac{\partial \sigma_{n-m}}{\partial w_{ij}}(W)\right)_j = 0.$$

Since it is true for all $t \in \mathbb{R}$, we must have $\forall m$,

$$\sum_j \left(\frac{\partial \sigma_{n-m}}{\partial w_{ij}}(W)\right)_j = 0.$$

□

The following gives explicit algebraic formulas for $\sigma_k(W)$.

Proposition 2.2. *If $W = (W_{ij})$ is an $n \times n$ symmetric matrix, let $F(W) = \sigma_k(W)$ for $1 \leq k \leq n$. Then the following relations hold.*

$$\sigma_k(W) = \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1 \\ j_1,\dots,j_k=1}}^{n} \delta(i_1,\dots,i_k; j_1,\dots,j_k) W_{i_1 j_1} \cdots W_{i_k j_k},$$

$$F^{\alpha\beta} := \frac{\partial F}{\partial W_{\alpha\beta}}(W)$$

$$= \frac{1}{(k-1)!} \sum_{\substack{i_1,\dots,i_{k-1}=1\\ j_1,\dots,j_{k-1}=1}}^{n} \delta(\alpha, i_1, \dots, i_{k-1}; \beta, j_1, \dots, j_{k-1}) W_{i_1 j_1} \cdots W_{i_{k-1} j_{k-1}}$$

$$F^{ij,rs} := \frac{\partial^2 F}{\partial W_{ij} \partial W_{rs}}(W)$$

$$= \frac{1}{(k-2)!} \sum_{\substack{i_1,\dots,i_{k-2}=1\\ j_1,\dots,j_{k-2}=1}}^{n} \delta(i, r, i_1, \dots, i_{k-2}; j, s, j_1, \dots, j_{k-2}) W_{i_1 j_1} \cdots W_{i_{k-2} j_{k-2}},$$

where the Kronecker symbol $\delta(I; J)$ *for indices* $I = (i_1, \dots, i_m)$ *and* $J = (j_1, \dots, j_m)$ *is defined as*

$$\delta(I; J) = \begin{cases} 1, & \text{if } I \text{ is an even permutation of } J; \\ -1, & \text{if } I \text{ is an odd permutation of } J; \\ 0, & \text{otherwise.} \end{cases}$$

$$\sum_{i,j,m} \sigma_k^{ij}(W) W_{im} W_{mj} = \sigma_1(W)\sigma_k(W) - (k+1)\sigma_{k+1}(W).$$

Proof. The first identity follows from (7) by equalized the coefficient in front of t^k. The second and third identities follow from the first identity. Notice that all the identity are invariant under orthornormal transformation. In particular, we may assume W is diagonal in the last identity. For $\lambda \in \mathbb{R}^n$, for any fixed $i \in \{1, \cdots, n\}$, denote $(\lambda|i) \in \mathbb{R}^n$ with i-th component of λ replaced by 0. Differentiation of (6) yields

$$\frac{\partial \sigma_k(\lambda)}{\partial \lambda_i} = \sigma_{k-1}(\lambda|i). \tag{9}$$

Again it can read off from (6),

$$\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i). \tag{10}$$

Thus,

$$\lambda_i \sigma_k(\lambda|i) = \lambda_i (\sigma_k(\lambda) - \lambda_i \sigma_{k-1}(\lambda|i) = \lambda_i \sigma_k(\lambda) - \lambda_i^2 \sigma_{k-1}(\lambda|i).$$

Using homogeneity of σ_{k+1}, the last identity in the proposition follows from the above by summing up over i. $\square$

Definition 2.3. For $1 \le k \le n$, let Γ_k is a cone in $\mathbb{R}^n$ determined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \quad \sigma_1(\lambda) > 0, \dots, \sigma_k(\lambda) > 0\}.$$

A $n \times n$ symmetric matrix W is called belong to Γ_k is $\lambda(W) \in \Gamma_k$.

Let $W^1, \cdots, W^n$ be $n \times n$ symmetric matrices, define $\sigma_n(W^1, \dots, W^n)$ to be the coefficient in front of the factor $t_1 \cdots t_n$ of the polynomial $\det(t_1 W^1 + \cdots + t_n W^n)$. It is called the mixed determinant of $W^1, \cdots, W^n$. In general, for $1 \le k \le n$, we define $\sigma_k(W^1, \dots, W^k) = \binom{n}{k}\sigma_n(W^1, \dots, W^k, I, \cdots, I)$, where the identity matrix I appears $(n-k)$ times. $\sigma_k(W^1, \dots, W^k)$ is called the complete polarization of the symmetric function σ_k.

The following Garding inequality plays important role in geometric PDE.

Lemma 2.4. *Γ_k is a convex cone. $\forall W^i \in \Gamma_k, i = 1, \dots, k$,*

$$\sigma_k^2(W^1, W^2, W^3, \cdots, W^k) \ge \sigma_k(W^1, W^1, W^3, \cdots, W^k)\sigma_k(W^2, W^2, W^3, \cdots, W^k), \tag{11}$$

equality hold if and only if W^1 and W^2 are proportional. And

$$\sigma_k(W^1, \cdots, W^k) \ge \sigma_k^{\frac{1}{k}}(W^1, \cdots, W^1) \cdots \sigma_k^{\frac{1}{k}}(W^k, \cdots, W^k), \tag{12}$$

the equality holds if and only if W^i, W^j are pairwise proportional.

Lemma 2.4 is a special case of Garding's theory of hyperbolic polynomials, which can be found in Appendix. The convexity of Γ_k follows from Proposition 5.2, (11) and (12) follow from Corollary 5.4 and Proposition 5.6 in Appendix.

Inequality (11) yields the Newton-MacLaurin inequality.

Lemma 2.5. *For $W \in \Gamma_k$,*

$$(n-k+1)(k+1)\sigma_{k-1}(W)\sigma_{k+1}(W) \le k(n-k)\sigma_k^2(W), \tag{13}$$

and

$$\sigma_{k+1}(W) \le c_{n,k}\sigma_k^{\frac{k+1}{k}}(W), \tag{14}$$

where $c_{n,k} = \frac{\sigma_{k+1}(I)}{\sigma_k^{\frac{k+1}{k}}}(I)$. The equality holds if and only if $W = cI$ for some $c > 0$.

Proof. If $\sigma_{k+1}(W) \le 0$, as $W \in \Gamma_k$, (13) is trivial. We may assume $\sigma_{k+1}(W) > 0$, so $W \in \Gamma_{k+1}$. Replace k by $k+1$ in (11), and set $W^1 = I$, $W^2 = \cdots = W^{k+1} = W \in \Gamma_k$, (13) follows from (11). The similar argument yields (14) using (12). $\square$

We remark that the Newton-MacLaurin inequality is valid for general symmetric matrix W (e.g., [28]).

The following lemma establish connection of σ_k with the ellipticity of Hessian and curvature equations.

Lemma 2.6. *Let $F = \sigma_k$, then the matrix $(\frac{\partial F}{\partial W_{ij}})$ is positive definite for $W \in \Gamma_k$. where W_{ij} are the entries of W. If $W \in \Gamma_k$, then $(W|i) \in \Gamma_{k-1}, \forall k = 0, 1, \cdots, n$, $i = 1, 2, \cdots, n$, where $(W|i)$ is the matrix with i-th column and i-th row deleted. Furthermore, if $W \in \Gamma_k$ and $\|W\| = \sqrt{\sum_{i,j} w_{ij}^2} \leq R$ for some $R > 0$, then there is $c_{n,k} > 0$ depending only on n, k, such that*

$$\frac{\sigma_k(W)}{R(1 + c_{n,k}\sigma_{k-1}^{\frac{1}{k-1}}(I))} I \leq (\frac{\partial F}{\partial W_{ij}}) \leq R^{k-1}\sigma_{k-1}(I)I. \tag{15}$$

Proof. Fix $W \in \Gamma_k$, for any positive definite matrix $A = (a_{ij})$, by Lemma 2.4,

$$0 < \sigma_k(W, \cdots, W, A) = \sum_{ij} \frac{\partial F}{\partial w_{ij}}(W)a_{ij}.$$

This implies the positivity of $(\frac{\partial F}{\partial W_{ij}})$. By Proposition 2.2 and the positivity of $(\frac{\partial F}{\partial W_{ij}})$, for each $l \leq k$, $W \in \Gamma_k$, and for any $i \in \{1, \cdots, n\}$,

$$0 < \frac{\partial \sigma_l}{\partial W_{ii}} = \sigma_{l-1}(W|i).$$

This yields $(W|i) \in \Gamma_{k-1}$.

To show (15), we only need to control $\frac{\partial \sigma_l}{\partial \lambda_i} = \sigma_{k-1}(\lambda|i)$, where $\lambda_i, i = 1, \cdots, n$ are the eigenvalues of W. By the assumption, and (14)

$$\begin{aligned} s \leq \sigma_k(W) &= \sigma_k(\lambda|i) + \lambda_i\sigma_{k-1}(\lambda|i) \\ &\leq \sigma_{k-1}(\lambda|i)(\lambda_i + c_{n,k}\sigma_{k-1}^{\frac{1}{k-1}}(\lambda|i)) \\ &\leq R(1 + c_{n,k}\sigma_{k-1}^{\frac{1}{k-1}}(I))\sigma_{k-1}(\lambda|i). \end{aligned}$$

this gives the lower bound in (15). The upper bound for $\sigma_{k-1}(\lambda|i)$ is trivial. □

We now switch to the quotient of elementary symmetric functions. Some of the concave properties of them will be used in crucial way in the a priori estimates in the rest of the sections.

Lemma 2.7. *For $0 \leq l < k \leq n$, let $F = (\frac{\sigma_k}{\sigma_l})^{\frac{1}{k-l}}$, then $(\frac{\partial F}{\partial w_{ij}})$ is positive definite for $W = (w_{ij}) \in \Gamma_k$. If $l = k - 1$, if $W \in \Gamma_k$ and $\|W\| = \sqrt{\sum_{i,j} w_{ij}^2} \leq R$ for some $R > 0$, then there is $c_{n,k} > 0$ depending only on n, k, such that*

$$\frac{F(W)}{R(1 + c_{n,k}\sigma_{k-1}^{\frac{1}{k-1}}(I))} I \leq (\frac{\partial F}{\partial w_{ij}}) \leq (n - k + 1)I. \tag{16}$$

Moreover, the function F is concave in Γ_{k-1}.

Proof. To simplify notation, define

$$Q_m = \frac{\sigma_m}{\sigma_{m-1}}.$$

For any $l < k$,

$$\frac{\sigma_k}{\sigma_l} = \prod_{j=1}^{k-l} Q_{l+j}. \tag{17}$$

As $Q_{l+j} > 0$ for $j = 1, \cdots, k-l$, for the first statement in lemma, we only need to check the positivity of $(\frac{\partial Q_{m+1}(W)}{\partial w_{ij}})$ for $W = (w_{ij}) \in \Gamma_k$ and for $m = l, \cdots, k-1$. By product rule,

$$\frac{\partial Q_{m+1}(W)}{\partial w_{ij}} = \frac{\sigma_m(W)\frac{\partial \sigma_{m+1}(W)}{\partial w_{ij}} - \sigma_{m+1}(W)\frac{\partial \sigma_m(W)}{\partial w_{ij}}}{\sigma_m^2(W)}.$$

By Proposition 2.2, the positivity of $(\frac{\partial \sigma_j(W)}{\partial w_{ij}})$ is invariant under orthonormal transformations, we only need to check the positivity of $\frac{\partial Q_{m+1}(\lambda)}{\partial \lambda_i}$ for $\lambda \in \Gamma_k$, $i \in \{1, \cdots, n\}$ and $m = l, \cdots, k-1$. Again,

$$\begin{aligned}
\frac{\partial Q_{m+1}(\lambda)}{\partial \lambda_i} &= \frac{\sigma_m(\lambda)\frac{\partial \sigma_{m+1}(\lambda)}{\partial \lambda_i} - \sigma_{m+1}(\lambda)\frac{\partial \sigma_m(\lambda)}{\partial \lambda_i}}{\sigma_m^2(\lambda)} \\
&= \frac{\sigma_m(\lambda)\sigma_m(\lambda|i) - \sigma_{m+1}(\lambda)\sigma_{m-1}(\lambda|i)}{\sigma_m^2(\lambda)} \\
&= \frac{\sigma_m(\lambda|i)\sigma_m(\lambda|i) - \sigma_{m+1}(\lambda|i)\sigma_{m-1}(\lambda|i)}{\sigma_m^2(\lambda)} \\
&\geq \frac{n}{(n-m)(m+1)}\frac{\sigma_m^2(\lambda|i)}{\sigma_m^2(\lambda)} \\
&> 0,
\end{aligned} \tag{18}$$

the Newton-MacLaurine inequality (13) is used in the last step as $(\lambda|i) \in \Gamma_{k-1}$ for each i. In particular, if $m = k-1$ and $W \in \Gamma_k$, for each i,

$$\begin{aligned}
0 < \frac{\partial Q_k(\lambda)}{\partial \lambda_i} &\leq \sum_i \frac{\partial Q_k(\lambda)}{\partial \lambda_i} \\
&\leq \sum_i \frac{\sigma_{k-1}(\lambda|i)}{\sigma_{k-1}(\lambda)} \\
&= n-k+1.
\end{aligned}$$

This provides the upper bound in (16). By (14)

$$\frac{\sigma_k(W)}{\sigma_{k-1}(W)} \le c_{n,k-1}\sigma_k^{\frac{1}{k}}(W).$$

For each $i = 1, \cdots, n$,

$$\frac{\sigma_{k-1}(\lambda|i)}{\sigma_{k-1}(\lambda)} = \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)}\frac{\sigma_{k-1}(\lambda|i)}{\sigma_k(\lambda)}.$$

Now the lower bound in (16) follows from (18) and (15).

Notice that if $f_1 > 0$ and $f_2 > 0$ are two concave function, for any $1 \ge \alpha \ge 0$, $f = f_1^\alpha f_2^{1-\alpha}$ is also concave. Hence, we only need to check the concavity of $\frac{\sigma_{m+1}}{\sigma_m}$ in Γ_{m+1}. In fact, we show $\frac{\sigma_{m+1}}{\sigma_m}$ in Γ_m.

$m = 0$ is trivial. For $m = 1$, there is a useful explicit formula. $\forall \lambda, \lambda \pm \xi \in \Gamma_1$, we have algebraic identity

$$2Q_2(\lambda) - Q_2(\lambda+\xi) - Q_2(\lambda-\xi) = \frac{(\sum_i(\xi_i\sigma_1(\lambda) - \lambda_i\sigma_1(\xi)))^2}{\sigma_1(\lambda)\sigma_1(\lambda+\xi)\sigma_1(\lambda-\xi)}.$$

This yields,

$$\frac{\partial^2 Q_2}{\partial^2\xi} = -\frac{(\sum_i(\xi_i\sigma_1(\lambda) - \lambda_i\sigma_1(\xi)))^2}{\sigma_1^3(\lambda)}$$

This gives the concavity of $\frac{\sigma_2}{\sigma_1}$ on Γ_1.

For $m > 1$, we use induction. For $\lambda \in \Gamma_m$, for each $i \in \{1, \cdots, n\}$ fixed, by (10) and Corollary 2.6,

$$\lambda_i + Q_m(\lambda|i) = \frac{\sigma_{m+1}(\lambda)}{\sigma_m(\lambda|i)} > 0.$$

Apply the last identity in Proposition 2.2,

$$\begin{aligned}(m+1)Q_m(\lambda) &= \sum_i(\lambda_i - \lambda_i^2\frac{\sigma_{m-1}(\lambda|i)}{\sigma_m(\lambda)})\\ &= \sum_i(\lambda_i - \lambda_i^2\frac{\sigma_{m-1}(\lambda|i)}{\sigma_m(\lambda|i)+\lambda_i\sigma_{m-1}(\lambda|i)})\\ &= \sum_i(\lambda_i - \frac{\lambda_i^2}{\lambda_i + Q_m(\lambda|i)}).\end{aligned}$$

For any $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, set $\lambda_{\epsilon^\pm} = \lambda \pm \epsilon\xi$. Take $\epsilon > 0$ small enough such that $\lambda_{\epsilon^\pm} \in \Gamma_m$, using the above identity for $\lambda, \lambda_{\epsilon^\pm}$, one compute

$$
\begin{aligned}
&(m+1)(2Q_{m+1}(\lambda)-Q_{m+1}(\lambda_{\epsilon}+)-Q_{m+1}(\lambda_{\epsilon}-))\\
=&\sum_i \Big(\frac{(\lambda_i+\epsilon\xi_i)^2}{Q_m(\lambda_{\epsilon}+|i)+\lambda_i+\epsilon\xi_i}+\frac{(\lambda_i-\epsilon\xi_i)^2}{Q_m(\lambda_{\epsilon}-|i)+\lambda_i-\epsilon\xi_i}\\
&-\frac{(2\lambda_i)^2}{Q_m(\lambda_{\epsilon}+|i)+Q_m(\lambda_{\epsilon}-|i)+2\lambda_i}\Big)\\
&+\sum_i\Big(\frac{(2\lambda_i)^2}{Q_m(\lambda_{\epsilon}+|i)+Q_m(\lambda_{\epsilon}-|i)+2\lambda_i}-\frac{2\lambda_i^2}{\lambda_i+Q_m(\lambda|i)}\Big)\\
=&\sum_i\frac{((\lambda_i+\epsilon\xi_i)Q_m(\lambda_{\epsilon}-)-(\lambda_i-\epsilon\xi_i)Q_m(\lambda_{\epsilon}+))^2}{(Q_m(\lambda_{\epsilon}+)+\lambda_i+\epsilon\xi_i)(Q_m(\lambda_{\epsilon}-)+\lambda_i-\epsilon\xi_i)(Q_m(\lambda_{\epsilon}+)+Q_m(\lambda_{\epsilon}-)+\epsilon\lambda_i)}\\
&-2\sum_i\lambda_i^2\frac{Q_m(\lambda_{\epsilon}+|i)+Q_m(\lambda_{\epsilon}-|i)-2Q_m(\lambda)}{(Q_m(\lambda_{\epsilon}+|i)+Q_m(\lambda_{\epsilon}-|i)+2\lambda_i)(\lambda_i+Q_m(\lambda|i))}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
-\frac{\partial^2 Q_{m+1}}{\partial^2\xi}&=\lim_{\epsilon\to 0}\frac{2Q_{m+1}(\lambda)-Q_{m+1}(\lambda_{\epsilon}+)-Q_{m+1}(\lambda_{\epsilon}-)}{\epsilon^2}\\
&\geq \lim_{\epsilon\to 0}-2\sum_i\lambda_i^2\frac{Q_m(\lambda_{\epsilon}+|i)+Q_m(\lambda_{\epsilon}-|i)-2Q_m(\lambda)}{\epsilon^2(Q_m(\lambda_{\epsilon}+|i)+Q_m(\lambda_{\epsilon}-|i)+2\lambda_i)(\lambda_i+Q_m(\lambda|i)}\\
&=-\sum_i\frac{\lambda_i^2(\frac{\partial^2 Q_m}{\partial\xi^2})(\lambda|i)}{(m+1)(Q_m(\lambda|i)+\lambda_i)^2}.
\end{aligned}
$$

As $(\lambda|i)\in\Gamma_{m-1}$, by induction hypothesis, $\frac{\partial^2 Q_m}{\partial\xi^2}(\lambda|i)\leq 0$. □

The following lemma will play key role for the problem of prescribing curvature measures.

Lemma 2.8. *Let $\alpha=\frac{1}{k-1}$, if $W\in\Gamma_k$ is a symmetric tensor on a Riemannian manifold M. For any local orthornormal frame $\{e_1,\cdots,e_n\}$, denote $W_{ij,s}=\nabla_{e_s}W_{ij}$. Then*

$$
(\sigma_k)^{ij,lm}W_{ij,s}W_{lm,s}\leq-\sigma_k\left[\frac{(\sigma_k)_s}{\sigma_k}-\frac{(\sigma_1)_s}{\sigma_1}\right]\left[(\alpha-1)\frac{(\sigma_k)_s}{\sigma_k}-(\alpha+1)\frac{(\sigma_1)_s}{\sigma_1}\right]. \quad (19)
$$

Proof. By the concavity of $\left(\frac{\sigma_k}{\sigma_1}\right)^{\frac{1}{k-1}}(W)$, we have

$$
0\geq\frac{\partial^2}{\partial W_{ij}\partial W_{lm}}\Big(\left(\frac{\sigma_k}{\sigma_1}\right)^{\frac{1}{k-1}}\Big)W_{ij,s}W_{lm,s}. \quad (20)
$$

Denote $\alpha = \frac{1}{k-1}$. Direct computations yield,

$$0 \geq \frac{\partial^2}{\partial W_{ij}\partial W_{lm}}\left(\frac{\sigma_k}{\sigma_1}\right)^{\alpha} \cdot W_{ij,s}W_{lm,s}$$

$$\begin{aligned} &= \alpha\left(\frac{\sigma_k}{\sigma_1}\right)^{\alpha}\left[\frac{(\sigma_k)^{ij,lm}}{\sigma_k} + \frac{(\alpha-1)(\sigma_k)^{ij}(\sigma_k)^{lm}}{\sigma_k^2}\right. \\ &\quad \left. - \frac{2\alpha(\sigma_k)^{ij}(\sigma_1)^{lm}}{\sigma_k\sigma_1} + \frac{(\alpha+1)(\sigma_1)^{ij}(\sigma_1)^{lm}}{\sigma_1^2}\right]W_{ij,s}W_{lm,s} \end{aligned} \tag{21}$$

Equivalently,

$$\begin{aligned} \frac{(\sigma_k)^{ij,lm}W_{ij,s}W_{lm,s}}{\sigma_k} &\leq -\left[\frac{(\alpha-1)(\sigma_k)^{ij}(\sigma_k)^{lm}}{\sigma_k^2} - \frac{2\alpha(\sigma_k)^{ij}(\sigma_1)^{lm}}{\sigma_k\sigma_1}\right. \\ &\quad \left. + \frac{(\alpha+1)(\sigma_1)^{ij}(\sigma_1)^{lm}}{\sigma_1^2}\right]W_{ij,s}W_{lm,s} \\ &\leq -\left[\frac{(\sigma_k)_s}{\sigma_k} - \frac{(\sigma_1)_s}{\sigma_1}\right]\left[(\alpha-1)\frac{(\sigma_k)_s}{\sigma_k} - (\alpha+1)\frac{(\sigma_1)_s}{\sigma_1}\right] \end{aligned} \tag{22}$$

□

Note in Lemma 2.8, one may replace σ_k by any positive function F with the property that $(\frac{F}{\sigma_1})^{\alpha}$ is concave for some $\alpha > 0$. The following is a corollary of Lemma 2.8.

Corollary 2.9. *If* $\frac{(\sigma_1)_s}{\sigma_1} = \frac{(\sigma_k)_s}{\sigma_k} - r$ *for some* r*,*

$$(\sigma_k)^{ij,lm}W_{ij;s}W_{ij;s} \leq \max\left\{2r(\sigma_k)_s - \frac{k}{k-1}r^2\sigma_k, 0\right\}. \tag{23}$$

3 Prescribing Curvature Measures

Assume $\Omega \subset \mathbb{R}^{n+1}$ is a bounded star-shaped domain with respect to the origin. We may parametrize $M = \partial\Omega$ over $\mathbb{S}^n$ by positive radial function ρ Due to the parametrization, the prescribe curvature measure problem for this class of domains can be reduced to a curvature type nonlinear partial differential equation of ρ on $\mathbb{S}^n$. We want to establish the existence theorems of prescribing general $(n-k)$-th curvature measure problem with $k > 0$ on bounded C^2 star-shaped domains. When $k = n$, the prescribing curvature measure $\mathcal{C}_0$ is the Alexandrov problem corresponding to a Monge-Ampère type equation on $\mathbb{S}^n$, which won't be treated here.

In order to make the problem in proper PDE setting, we need to impose some geometric condition on $\partial\Omega$.

Definition 3.1. A domain Ω is called k-convex if its principal curvature vector $\kappa(x) = (\kappa_1, \cdots, \kappa_n) \in \Gamma_k$ at every point $x \in \partial\Omega$.

For each star-shaped domain Ω with $M = \partial\Omega$, express M as a radial graph of $\mathbb{S}^n$,

$$\begin{aligned} R_M : \mathbb{S}^n &\longrightarrow M \\ z &\longmapsto \rho(z)z. \end{aligned}$$

From (1) the $(n-k)$-th curvature measure on each Borel set β in $\mathbb{S}^n$ can be defined as

$$\mathcal{C}_k(M, \beta) := \int_{R_M(\beta)} \sigma_k(\kappa) d\mu_g.$$

The precise statement of the problem for prescribing $(n-k)$-th curvature measure is: *given a positive function $f \in C^2(\mathbb{S}^n)$, find a closed hypersurface M as a radial graph over $\mathbb{S}^n$, such that $\mathcal{C}_{n-k}(M, \beta) = \int_\beta f d\mu$ for every Borel set β in $\mathbb{S}^n$, where $d\mu$ is the standard volume element on $\mathbb{S}^n$.*

For the C^2 graph M on $\mathbb{S}^n$, denote the induced metric to be g and the density function is $\sqrt{\det g}$. Then

$$\mathcal{C}_{n-k}(M, \beta) = \int_{R_M(\beta)} \sigma_k d\mu_g = \int_\beta \sigma_k \sqrt{\det g} d\mathbb{S}^n. \tag{24}$$

We now write down the local expressions of the induced metric, support function u, second fundamental form and Weingarten curvatures in terms of positive function ρ and its derivatives $\nabla\rho$, $\nabla^2\rho$. Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame on $\mathbb{S}^n$, and denote e_{ij} the standard spherical metric with respect to this frame (which is the identity matrix). We use $\overline{\nabla}$ as the gradient operator with respect to standard metric on $\mathbb{S}^n$. To simplify notation, for any function v on $\mathbb{S}^n$, we will write $\overline{\nabla}_{e_i} v = v_i$ as covariant derivative with respect to e_i on $\mathbb{S}^n$ in this subsection, if there is no confusion. From the radial parametrization $X(x) = \rho(x)x$,

$$\begin{aligned} X_i &= \rho_i x + \rho e_i, \\ X_{ij} &= \rho_{ij} x + \rho_i e_j + \rho_j e_i + \rho(e_i)_j = \rho_{ij} x + \rho_i e_j + \rho_j e_i - \rho e_{ij} x. \end{aligned}$$

The following identities can be read off from the above.

$$
\begin{aligned}
\nu &= \frac{\rho x - \overline{\nabla}\rho}{\sqrt{\rho^2+|\overline{\rho}|^2}}\\
u &= \frac{\rho^2}{\sqrt{\rho^2+|\overline{\nabla}\rho|^2}}\\
g_{ij} &= \rho^2\delta_{ij} + \rho_i\rho_j\\
g^{ij} &= \frac{1}{\rho^2}(\delta^{ij} - \frac{\rho_i\rho_j}{\rho^2+|\overline{\nabla}\rho|^2})\\
h_{ij} &= (\sqrt{\rho^2+|\overline{\nabla}\rho|^2})^{-1}(-\rho\overline{\nabla}_i\overline{\nabla}_j\rho + 2\rho_i\rho_j + \rho^2 e_{ij})\\
h^i_j &= \frac{1}{\rho^2\sqrt{\rho^2+|\overline{\nabla}\rho|^2}}(e^{ik} - \frac{\rho_i\rho_k}{\rho^2+|\overline{\nabla}\rho|^2})(-\rho\overline{\nabla}_k\overline{\nabla}_j\rho + 2\rho_k\rho_j + \rho^2 e_{kj}).
\end{aligned}
\tag{25}
$$

From (25),

$$\sqrt{\det g} = \rho^{n-1}\sqrt{\rho^2 + |\overline{\nabla}\rho|^2}.$$

The prescribing $(n-k)$-th curvature measure problem can be deduced to the following curvature equation on $\mathbb{S}^n$:

$$\sigma_k(\kappa_1, \cdots, \kappa_n) = \sigma_k(h^i_j) = \frac{f}{\rho^{n-1}\sqrt{\rho^2+|\overline{\nabla}\rho|^2}}, \tag{26}$$

where $f > 0$ is the given function on $\mathbb{S}^n$. A solution of (26) is called admissible if $\kappa(X) \in \Gamma_k$ at each point $X \in M$. We note that any positive C^2 function ρ on $\mathbb{S}^n$ satisfying (26) is automatically an admissible solution. Since the principal curvatures at a maximum point of ρ are positive, solution is admissible at this point. As Γ_k and $\mathbb{S}^n$ are connected, and $\kappa(X)$ varies continuously, the fact of $\sigma_k(\kappa(X)) > 0$ implies solution is admissible at each point of M.

The following is the statement of solvability of the problem of the prescribing curvature measures.

Theorem 3.2. *Let $n \geq 2$ and $1 \leq k \leq n-1$. Suppose $f \in C^2(\mathbb{S}^n)$ and $f > 0$. Then there exists a unique k-convex star-shaped hypersurface $M \in C^{3,\alpha}$, $\forall\alpha \in (0,1)$ such that it satisfies (26). Moreover, there is a constant C depending only on $k, n, \|f\|_{C^{1,1}}, \|1/f\|_{C^0}$, and α such that,*

$$\|\rho\|_{C^{3,\alpha}} \leq C. \tag{27}$$

The rest of the section is devoted to the proof of Theorem 3.2. The main task will be the a priori estimates for solutions of (26). We will use the radial parametrization on $\mathbb{S}^n$ for the estimates up to C^1. Then we will work directly on M for the curvature estimates, which is equivalent to C^2 estimates.

It will be convenient to introduce a new variable $\gamma = \log\rho$. Set

$$\omega := \sqrt{1 + |\overline{\nabla}\gamma|^2}.$$

The unit outward normal and support function can be expressed as $\nu = \frac{1}{\omega}(1, -\gamma_1, \cdots, -\gamma_n)$ and $u = \frac{e^{\gamma}}{\omega}$ respectively. Moreover,

$$\begin{aligned} g_{ij} &= e^{2\gamma}(\delta_{ij} + \gamma_i\gamma_j), \\ g^{ij} &= e^{-2\gamma}(e^{ij} - \tfrac{\gamma_i\gamma_j}{\omega^2}) \\ h_{ij} &= \tfrac{e^{\gamma}}{\omega}(-\gamma_{ij} + \gamma_i\gamma_j + e_{ij}) \\ h^i_j &= \tfrac{e^{-\gamma}}{\omega}(e^{ik} - \tfrac{\gamma_i\gamma_k}{\omega^2})(-\gamma_{kj} + \gamma_k\gamma_j + e_{kj}). \end{aligned} \tag{28}$$

Notice that the Weingarten tensor in (28) is in general not symmetric with respect local lo orthonormal frames $(e_1, \cdots, e_n)$ on $\mathbb{S}^n$, even though it is symmetric with respect to local orthonormal frames on M. We observe that the symmetric matrix $(e^{ij} - \frac{\gamma_i\gamma_j}{\omega^2})$ has an obvious square root S. That is,

$$S = (S_{ij}) = (e_{ij} - \frac{\gamma_i\gamma_j}{\omega(\omega+1)}), \quad (e^{ij} - \frac{\gamma_i\gamma_j}{\omega^2}) = S^2. \tag{29}$$

S can be used to symmetrize the Weingarten tensor. The eigenvalues of (h^i_j) is the same as eigenvalues of $\frac{e^{-\gamma}}{\omega}B$, with B defined as

$$\begin{aligned} B =: (b_{ij}) &= S(-\gamma_{lm} + \gamma_l\gamma_m + e_{lm})S \\ &= (-\gamma_{ij} + \delta_{ij} + \frac{\sum_l(\gamma_i\gamma_{lj} + \gamma_j\gamma_{il})\gamma_l}{\omega(\omega+1)} - \frac{\gamma_i\gamma_j\sum_{l,m}\gamma_l\gamma_{lm}\gamma_m}{\omega^2(1+\omega)^2}). \end{aligned} \tag{30}$$

Curvature equation (26) can be rewritten as

$$\frac{e^{(n-k)\gamma}}{\omega^{k-1}}\sigma_k(B) = f. \tag{31}$$

As B is a function in $\overline{\nabla}^2\gamma, \overline{\nabla}\gamma$ only, it is independent of γ. Set

$$\tilde{F}(\overline{\nabla}^2\gamma, \overline{\nabla}\gamma) = -\sigma_k(B). \tag{32}$$

Denote $\sigma_k^{ij}(B) = \frac{\partial\sigma_k}{\partial b_{ij}}$, we compute

$$(\tilde{F}^{ij}) = (\frac{\partial\tilde{F}}{\partial\gamma_{ij}}) = S(\sigma_k^{ij}(B))S. \tag{33}$$

Since S in (29) is positive definite, we have $(\frac{\partial\tilde{F}}{\partial\gamma_{ij}}) > 0$.

3.1 Uniqueness and C^1-Estimates

Lemma 3.3. *Let $1 \le k < n$. Let L denote the linearized operator at a solution ρ of (26), if v satisfies $L(v) = 0$ on $\mathbb{S}^n$, then $v \equiv 0$ on $\mathbb{S}^n$. Moreover, suppose ρ, $\tilde\rho$ are two solutions of (26) and $\lambda(\rho_i) \in \Gamma_k$, for $i = 1, 2$. Then $\rho_1 \equiv \rho_2$.*

Proof. (31) can be put in the form of

$$\frac{e^{(n-k)\gamma}}{\omega^{k-1}} \tilde F(\overline\nabla^2 \gamma, \overline\nabla \gamma) = -f. \tag{34}$$

The linearized operator at γ is

$$L(v) = \frac{e^{(n-k)\gamma}}{\omega^{k-1}} \tilde F^{ij} v_{ij} + \sum_l b_l v_l - (n-k) f v,$$

for some function $b_l, l = 1, \cdots, n$. The first statement in lemma follows immediately from the maximum principle.

Suppose $\gamma = \log\rho$ and $\tilde\gamma = \log\tilde\rho$ are two solutions of (31), denote $\tilde\omega = \sqrt{1 + |\overline\nabla\tilde\gamma|^2}$ and $\tilde B$ to be the corresponding tensor B in (30) with γ replaced by $\tilde\gamma$. For $t \in [0, 1]$, set

$$\gamma^t = t\gamma + (1-t)\tilde\gamma, \quad \omega_t = \sqrt{1 + |\overline\nabla\gamma^t|^2}, \qquad B^t = tB + (1-t)\tilde B.$$

Set $v = \gamma - \tilde\gamma$, as $B^t \in \Gamma_k$,

$$\begin{aligned}
0 &= \frac{e^{(n-k)\gamma}}{\omega^{k-1}} F(B) - \frac{e^{(n-k)\tilde\gamma}}{\tilde\omega^{k-1}} F(\tilde B)\\
&= \int_0^1 \frac{d}{dt}\Big(\frac{e^{(n-k)\gamma^t}}{\omega_t^{k-1}} F(B^t)\Big) dt\\
&= \int_0^1 (n-k)\Big(\frac{e^{(n-k)\gamma^t}}{\omega_t^{k-1}} F(B^t)\Big) dt + \int_0^1 \Big(\frac{e^{(n-k)\gamma^t}}{\omega_t^{k-1}} F^{ij}(B^t)\Big) dt (b_{ij} - \tilde b_{ij}) + mod(\overline\nabla v).
\end{aligned}$$

Write $S = (S^i_j)$, and observe that S only involves $\overline\nabla\gamma, \overline\nabla^2\gamma$ (and so is $\tilde S$), by the Mean Value Theorem,

$$B - \tilde B = -S(\overline\nabla^2 v)S + mod(\overline\nabla v),$$

and

$$0 = \Big(\int_0^1 (n-k)\Big(\frac{e^{(n-k)\gamma^t}}{\omega_t^{k-1}} \tilde F(B^t)\Big) dt\Big) v - \int_0^1 \Big(\frac{e^{(n-k)\gamma^t}}{\omega_t^{k-1}} F^{ij}(B^t)\Big) dt\, S_i^\alpha S_j^\beta v_{\alpha\beta} + mod(\overline\nabla v).$$

Since $(\int_0^1(\frac{e^{(n-k)\gamma^t}}{\omega_t^{k-1}}F^{ij}(B^t))dt)S^{i\alpha}S^{\beta j}) > 0$, $\int_0^1(n-k)(\frac{e^{(n-k)\gamma^t}}{\omega_t^{k-1}}\tilde{F}(B^t))dt > 0$, v satisfies the following elliptic equation,

$$a^{ij}(x)v_{ij}(x) + b^k(x)v_k(x) + c(x)v(x) = 0, \quad \forall x \in \mathbb{S}^n,$$

with $c(x) < 0$ for all $x \in \mathbb{S}^n$. The maximum principle yields $v \equiv 0$. That is $\rho = \tilde{\rho}$. $\square$

It is useful to write down some differential identities for general C^1 symmetric function F. $F(W)$ is symmetric if it is invariant under orthonormal transformation. With B is defined in (30), set $\tilde{F}(\overline{\nabla}^2\gamma, \overline{\nabla}\gamma) = -F(B)$. Define $F^{ij} = \frac{\partial F}{\partial b_{ij}}$, $\tilde{F}^{ij} = \frac{\partial \tilde{F}}{\partial \gamma_{ij}}$. It follows from (30) that

$$(\tilde{F}^{ij}) = S(F^{ij})S. \tag{35}$$

Lemma 3.4. *For any C^1 symmetric function $F(B)$, set $\phi = \frac{|\overline{\nabla}\gamma|^2}{2}$, then there exist c_m depending on $(\overline{\nabla}^2\gamma, \overline{\nabla}\gamma, F)$, such that*

$$\tilde{F}^{ij}\phi_{ij} = \sum_m c_m\phi_m - \sum_l \gamma_l(F(B))_l + F^{ij}(\delta_{ij}|\overline{\nabla}\gamma|^2 - \gamma_j\gamma_i + \delta_{ij}\gamma_{ii}^2). \tag{36}$$

Proof. By (30),

$$\begin{aligned}
\phi_{ij} &= \sum_l(\gamma_l\gamma_{lij} + \gamma_{li}\gamma_{lj}) \\
&= \sum_l\big(\gamma_l(\gamma_{lij} + \delta_{li}\gamma_j - \gamma_j\delta_{il}) + \gamma_{li}\gamma_{lj}\big) \\
&= \sum_l\big(\gamma_l(\gamma_{ijl} + \delta_{ij}\gamma_l - \gamma_j\delta_{il}) + \gamma_{li}\gamma_{lj}\big) \\
&= \sum_l \gamma_l(-b_{ijl} + (\frac{\gamma_i\phi_j + \gamma_j\phi_i}{\omega(\omega+1)} - \frac{\gamma_i\gamma_j\sum_m\gamma_m\phi_m}{\omega^2(1+\omega)^2})_l) \\
&\quad + \delta_{ij}|\overline{\nabla}\gamma|^2 - \gamma_j\gamma_i + \delta_{ij}\gamma_{ii}^2 \\
&= \sum_l \gamma_l(-b_{ijl} + (\frac{\gamma_i\phi_{lj} + \gamma_j\phi_{li}}{\omega(\omega+1)} - \frac{\gamma_i\gamma_j\sum_m\gamma_m\phi_{ml}}{\omega^2(1+\omega)^2})) \\
&\quad + \delta_{ij}|\overline{\nabla}\gamma|^2 - \gamma_j\gamma_i + \delta_{ij}\gamma_{ii}^2 + c_{ij}^m\phi_m,
\end{aligned}$$

where we used the fact that tensor $A_{ij} := \gamma_{ij} + \gamma e_{ij}$ is Codazzi for any function $\gamma \in C^3(\mathbb{S}^n)$. We rewrite above identity as

$$\phi_{ij} = \sum_l \Big(\frac{\gamma_i\gamma_l\phi_{lj} + \gamma_l\gamma_j\phi_{li}}{\omega(\omega+1)} - \frac{\gamma_i\gamma_j \sum_{m,l}\gamma_l\gamma_m\phi_{ml}}{\omega^2(1+\omega)^2}\Big)c_{ij}^m\phi_m$$
$$+\delta_{ij}|\overline{\nabla}\gamma|^2 - \gamma_j\gamma_i + \delta_{ij}\gamma_{ii}^2 - \sum_l \gamma_l b_{ijl},$$

or equivalently

$$S\overline{\nabla}^2\phi S - (c_{ij}^m\phi_m) = |\overline{\nabla}\gamma|^2 I - (\gamma_i\gamma_j) + (\overline{\nabla}^2\gamma)^2 - (\sum_l \gamma_l b_{ijl}).$$

Set $c_m = \sum_{ij} F^{ij}c_{ij}^m$, contracting above identity with F^{ij}, it follows from (35),

$$\begin{aligned}\tilde{F}^{ij}\phi_{ij} - \sum_m c_m\phi_m &= -\sum_l F^{ij}(B)\gamma_l b_{ijl} + F^{ij}(\delta_{ij}|\overline{\nabla}\gamma|^2 - \gamma_j\gamma_i + \delta_{ij}\gamma_{ii}^2)\\ &= -\sum_l \gamma_l(F(B))_l + F^{ij}(\delta_{ij}|\overline{\nabla}\gamma|^2 - \gamma_j\gamma_i + \delta_{ij}\gamma_{ii}^2).\end{aligned}$$

□

Proposition 3.5. *If M satisfies (26), then*

$$\Big(\frac{\min_{\mathbb{S}^n} f}{C_n^k}\Big)^{\frac{1}{n-k}} \le \min_{\mathbb{S}^n}|X| \le \max_{\mathbb{S}^n}|X| \le \Big(\frac{\max_{\mathbb{S}^n} f}{C_n^k}\Big)^{\frac{1}{n-k}}.$$

Moreover, there exits a constant C depending only on n, k, $\min_{\mathbb{S}^n} f$, $|f|_{C^1}$ such that

$$\max_{\mathbb{S}^n}|\overline{\nabla}\rho| \le C.$$

Proof. (γ_{ij}) is semi-negative definite at maximum point of ρ and $\overline{\nabla}\gamma = 0$. By (31),

$$f = \frac{e^{(n-k)\gamma}}{\omega^{k-1}}\sigma_k(B) = e^{(n-k)\gamma}\sigma_k(B) \ge e^{(n-k)\gamma}.$$

This yields an upper bound of γ. A lower bound of γ follows similarly, as (γ_{ij}) is semi-positive definite at any minimum point of ρ.

To obtain an upper bound for $|\overline{\nabla}\rho|$ is now equivalent to obtain an upper bound of $\phi = \frac{|\overline{\nabla}\gamma|^2}{2}$. Suppose $p \in \mathbb{S}^n$ is a maximum point of ϕ. At p,

$$\overline{\nabla}|\overline{\nabla}\gamma|^2 = 0, \quad \overline{\nabla}\omega = 0, \quad B = (-\gamma_{ij} + \delta_{ij}). \tag{37}$$

It follows from (36) with $F(B) = \sigma_k(B)$, at p,

$$\begin{aligned}
0 &\geq \sum_{ij} F^{ij}\phi_{ij} \\
&= -\sum_l \gamma_l(\sigma_k(B))_l + \sum_{ij} \sigma_k^{ij}(\delta_{ij}|\overline{\nabla}\gamma|^2 - \gamma_i\gamma_j + \delta_{ij}\gamma_{ii}^2) \\
&\geq -\sum_l \gamma_l(e^{-(n-k)\gamma}\omega^{k-1}f)_l \\
&= \big((n-k)|\overline{\nabla}\gamma|^2 f - \overline{\nabla}\gamma\cdot\overline{\nabla}f\big)e^{(k-n)\gamma}\omega^{k-1} \\
&\geq c(|\overline{\nabla}\gamma|^2 - C|\overline{\nabla}\gamma|)e^{(k-n)\gamma}\omega^{k-1}, \qquad (38)
\end{aligned}$$

where $c \geq \delta, C \leq \frac{1}{\delta}$ are two positive constants with δ depending only on $n, k, \inf f, |\overline{\nabla}f|$. The gradient estimate follows from (38). □

3.2 C^2-Estimates and the Existence

We precede to prove C^2 a priori estimates, this is equivalent to obtain curvature estimate for M due to C^1 estimates we have already obtained. For this purpose, it is convenient to work directly on induced metric g on $M \subset \mathbb{R}^{n+1}$. For $X \in M$, choose local orthonormal frame $\{e_1, \cdots, e_n\}$ on M, and $\nu = e_{n+1}$ is the unit outer normal of the hypersurface, such that $\{e_1, \cdots, e_{n+1}\}$ of $\mathbb{R}^{n+1}$ is a local orthonormal frame in $\mathbb{R}^{n+1}$. We use lower indices to denote covariant derivatives with respect to the induced metric.

The second fundamental form is the symmetric $(2,0)$-tensor given by the matrix $\{h_{ij}\}$, and we denote the Weingarten tensor $\{h_i^j\} = \{g^{jl}h_{li}\}$,

$$h_{ij} = \langle \partial_i X, \partial_j \nu\rangle. \qquad (39)$$

We have the following identities,

$$\begin{aligned}
X_{ij} &= -h_{ij}\nu \quad \text{(Gauss formula)} \\
(\nu)_i &= h_i^j X_j \quad \text{(Weigarten equation)} \\
h_{ijk} &= h_{ikj} \quad \text{(Codazzi formula)} \\
R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk} \quad \text{(Gauss equation)},
\end{aligned} \qquad (40)$$

where R_{ijkl} is the $(4,0)$-Riemannian curvature tensor. We also have

$$\begin{aligned}
h_{ijkl} &= h_{ijlk} + h_{mj}R_{imlk} + h_{im}R_{jmlk} \\
&= h_{klij} + (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} + (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}.
\end{aligned} \qquad (41)$$

Since $\{e_1, \cdots, e_n\}$ is an orthonormal frame on M, $g_{ij} = \delta_{ij}$, $h_{ij} = h^i_j$. The principal curvatures $(\kappa_1, \cdots, \kappa_n)$ are the eigenvalues of the second fundamental form with respect to the metric which satisfy

$$\det(h_{ij} - \kappa g_{ij}) = 0.$$

The curvature equation (26) on $\mathbb{S}^n$ can also be equivalently expressed as a curvature equation on M,

$$\sigma_k(\kappa_1, \cdots, \kappa_n)(X) = \frac{u(X)}{|X|^{n+1}} f\Big(\frac{X}{|X|}\Big), \quad \forall X \in M. \tag{42}$$

Proposition 3.6. *For* $1 < k < n$, *let* $F \equiv \sigma_k = \Phi u$ *and denote* $H \equiv \sigma_1$, *then at a maximum point of* $\frac{H}{u}$,

$$\begin{aligned} F^{ij}\big(\tfrac{H}{u}\big)_{ij} = \tfrac{1}{u}[\Phi_{ss}u + 2\Phi_s u_s] - \big(\tfrac{H}{u}\big)\Phi_l\langle X, X_l\rangle - (k-1)\big(\tfrac{H}{u}\big)\Phi \\ +(k-1)\phi|A|^2 - \tfrac{1}{u}F^{ij;ml}h_{ij;s}h_{ml;s}, \end{aligned} \tag{43}$$

where A *denotes the second fundamental form.*

Proof. By definition, $u = \langle X, \nu\rangle$. Compute the first and second order covariant derivatives, we have

$$\begin{aligned} u_s &= h_{sl}\langle X, X_l\rangle \\ u_{ij} &= h_{ij;l}\langle X, X_l\rangle + h_{ij} - (h^2)_{ij}u \end{aligned} \tag{44}$$

Also since (h_{ij}) is Codazzi, by Ricci identity and Gauss equation,

$$\begin{aligned} h_{ij;kl} &= h_{kl;ij} + (h_{lk}h_{im} - h_{lm}h_{ik})h_{mj} + (h_{lj}h_{im} - h_{lm}h_{ij})h_{mk} \\ F^{ij}h_{ij;st} &= F_{st} - F^{ij;ml}h_{ml;s}h_{ij;t}. \end{aligned} \tag{45}$$

At any maximum point $P \in M^n$ of $\frac{H}{u}$, $\big(\frac{H}{u}\big)_i(P) = 0$. At P,

$$\begin{aligned} F^{ij}\big(\tfrac{H}{u}\big)_{ij} &= F^{ij}\Big[\tfrac{H_{ij}}{u} - \tfrac{u_j}{u}\big(\tfrac{H}{u}\big)_i - \tfrac{u_i}{u}\big(\tfrac{H}{u}\big)_j - \big(\tfrac{H}{u}\big)\tfrac{u_{ij}}{u}\Big] \\ &= \tfrac{1}{u}F^{ij}H_{ij} - \tfrac{1}{u}\big(\tfrac{H}{u}\big)F^{ij}u_{ij}. \end{aligned} \tag{46}$$

Apply formulas (44) and (45),

$$
\begin{aligned}
\frac{1}{u}F^{ij}H_{ij} &= \frac{1}{u}F^{ij}h_{ss;ij}\\
&= \frac{1}{u}F^{ij}\big[h_{ij;ss} + (h_{ij}h_{sm} - h_{jm}h_{si})h_{ms} + (h_{js}h_{sm} - h_{jm}h_{ss})h_{mi}\big]\\
&= \frac{1}{u}F^{ij}h_{ij;ss} + k\Phi|A|^2 - \frac{1}{u}F^{ij}(h^2)_{ij}H\\
&= \frac{1}{u}F_{ss} - \frac{1}{u}F^{ij;ml}h_{ij;s}h_{ml;s} + k\Phi|A|^2 - \big(\tfrac{H}{u}\big)F^{ij}(h^2)_{ij}\\
&= \frac{1}{u}[\Phi_{ss}u + 2\Phi_s u_s + \Phi u_{ss}] - \frac{1}{u}F^{ij;ml}h_{ij;s}h_{ml;s} + k\Phi|A|^2\\
&\quad -\big(\tfrac{H}{u}\big)F^{ij}(h^2)_{ij}\\
&= \frac{1}{u}[\Phi_{ss}u + 2\Phi_s u_s] + \frac{\Phi}{u}\big[H_l\langle X, X_l\rangle + H - |A|^2u\big]\\
&\quad -\frac{1}{u}F^{ij;ml}h_{ij;s}h_{ml;s} + k\Phi|A|^2 - \big(\tfrac{H}{u}\big)F^{ij}(h^2)_{ij}\\
&= \frac{1}{u}[\Phi_{ss}u + 2\Phi_s u_s] + \frac{\Phi}{u}H_l\langle X, X_l\rangle + \big(\tfrac{H}{u}\big)\Phi\\
&\quad -\frac{1}{u}F^{ij;ml}h_{ij;s}h_{ml;s} + (k-1)\phi|A|^2 - \big(\tfrac{H}{u}\big)F^{ij}(h^2)_{ij}.
\end{aligned}
\tag{47}
$$

We also compute

$$
\begin{aligned}
-\frac{1}{u}\big(\tfrac{H}{u}\big)F^{ij}u_{ij} &= -\frac{1}{u}\big(\tfrac{H}{u}\big)F^{ij}\Big[h_{ij;l}\langle X, X_l\rangle + h_{ij} - (h^2)_{ij}u\Big]\\
&= -\frac{1}{u}\big(\tfrac{H}{u}\big)F_l\langle X, X_l\rangle - k\phi\big(\tfrac{H}{u}\big) + \big(\tfrac{H}{u}\big)F^{ij}(h^2)_{ij}\\
&= -\frac{\Phi}{u}\big(\tfrac{H}{u}\big)u_l\langle X, X_l\rangle - \big(\tfrac{H}{u}\big)\Phi_l\langle X, X_l\rangle - k\Phi\big(\tfrac{H}{u}\big) + \big(\tfrac{H}{u}\big)F^{ij}(h^2)_{ij},
\end{aligned}
\tag{48}
$$

where $(h^2)_{ij} = h_{ik}h_{kj}$.

Adding up (47) and (48), and using the critical point condition, we obtain

$$
\begin{aligned}
F^{ij}\big(\tfrac{H}{u}\big)_{ij} &= \frac{1}{u}[\Phi_{ss}u + 2\Phi_s u_s] + \phi\big(\tfrac{H}{u}\big)_l\langle X, X_l\rangle - \big(\tfrac{H}{u}\big)\Phi_l\langle X, X_l\rangle\\
&\quad -(k-1)\big(\tfrac{H}{u}\big)\Phi - \frac{1}{u}F^{ij;ml}h_{ij;s}h_{ml;s} + (k-1)\Phi|A|^2\\
&= \frac{1}{u}[\Phi_{ss}u + 2\Phi_s u_s] - \big(\tfrac{H}{u}\big)\Phi_l\langle X, X_l\rangle - (k-1)\big(\tfrac{H}{u}\big)\Phi\\
&\quad -\frac{1}{u}F^{ij;ml}h_{ij;s}h_{ml;s} + (k-1)\Phi|A|^2,
\end{aligned}
\tag{49}
$$

(43) is verified. □

C^2 estimates can be established with the help of Proposition 3.6 and Corollary 2.9.

Lemma 3.7. *If M satisfies (42) for some $1 \le k \le n$, then there exists a constant C depending only on n, k, $\min_{S^n} f$, $|f|_{C^1}$, and $|f|_{C^2}$, such that*

$$
\max_M \sigma_1 \le C, \quad |\nabla^2\rho| \le C. \tag{50}
$$

Proof. We have already obtained the C^0 and C^1 estimates for ρ. For the case of $k = 1$, (42) is a mean curvature type equation which is of divergent form of quasilinear PDE. C^2 estimates follows from the classical quasilinear elliptic PDE theory. We work on $2 \le k \le n-1$ cases. When $k > 1$, the estimation of the curvature bound is equivalent to the estimation of mean curvature H (which yields C^2 bound on ρ). To see this, suppose mean curvature $H \le C$ is bounded from above. Since $\kappa \in \Gamma_k \subset \Gamma_2$, $(\kappa|i) \in \Gamma_1$. Hence, for each i,

$$C \ge H = \sigma_1(\kappa) = \kappa_i + \sigma_1(\kappa|i) \ge \kappa_i.$$

This give an upper bound of curvature. A lower bound follows from the fact $\sigma_1(\kappa) > 0$ and $\kappa_i \le C$ for each i.

As u is bounded from below and above, we only need to get an upper bound of $\frac{H}{u}$. Suppose $P \in M$ where $\frac{H}{u}$ achieves its maximum, it follows from (43)

$$\begin{aligned} 0 &\ge F^{ij}\big(\tfrac{H}{u}\big)_{ij} \\ &= \tfrac{1}{u}[\Phi_{ss}u + 2\phi_s u_s] - \big(\tfrac{H}{u}\big)\Phi_l\langle X, X_l\rangle - (k-1)\big(\tfrac{H}{u}\big)\Phi \\ &\quad -\tfrac{1}{u}F^{ij;ml}h_{ij;s}h_{ml;s} + (k-1)\Phi|A|^2. \end{aligned} \tag{51}$$

Recall $\Phi(X) = |X|^{-(n+1)} f(\frac{X}{|X|})$ and with C^0, C^1 estimates of $\rho = |X|$, we have the following estimates.

$$\begin{aligned} |\Phi_i|(P) &\le C(n, k, \min_{S^n} f, |f|_{C^1}) \\ |\Phi_{ii}|(P) &\le C(n, k, \min_{S^n} f, |f|_{C^1}, |f|_{C^2})\big(1 + |A|(P)\big) \end{aligned}$$

On the other hand, $|u_i| = |h^i_j \rho\rho_j| \le c_3|A|$. By (42),

$$\frac{\sigma_1}{u} = \frac{\sigma_1\phi}{\sigma_k}.$$

At a maximum point P of the test function $\frac{\sigma_1}{u}$, one has

$$\frac{(\sigma_1)_s}{\sigma_1} = \frac{(\sigma_k)_s}{\sigma_k} - \frac{\phi_s}{\phi}.$$

In Corollary 2.9, set $r = \frac{\phi_s}{\phi}(P)$, then

$$\begin{aligned} F^{ij;ml}h_{ij;s}h_{ml;s} &\le 2r(u\phi)_s - \tfrac{k}{k-1}r^2u\phi \\ &\le C_1(n, k, \min_{S^n} f, |f|_{C^1})|A| + C_2(n, k, \min_{S^n} f, |f|_{C^1}). \end{aligned}$$

With the above estimates, (51) can be simplified as

$$|A|^2(P) + c_4|A|(P) + c_5 \le 0, \tag{52}$$

where c_4 and c_5 are constants depending only on n, k, $\min_{S^n}\phi$, $|f|_{C^1}$, and $|f|_{C^2}$. Hence at P, $|A|(P)\le C$. In turn

$$\sigma_1(X)\le u(X)\frac{\sigma_1(P)}{u(P)}\le C,\quad \text{for any } X\in M.$$

This implies (50). □

We prove Theorem 3.2 using the method of continuity.

Proof. For any positive function $f\in C^2(\mathbb{S}^n)$, for $0\le t\le 1$ and $1\le k<n-1$, set

$$f_t(x)=[1-t+tf^{-\frac{1}{k}}(x)]^{-k}.$$

Consider the following family of equations for $0\le t\le 1$:

$$\sigma_k^{\frac{1}{k}}(\kappa_1,\cdots,\kappa_n)(x)=(f_t(x)\rho^{1-n}(\rho^2+|\nabla\rho|^2)^{-1/2})^{\frac{1}{k}},\quad \text{on } \mathbb{S}^n, \tag{53}$$

where $n\ge 2$. We want to find admissible solutions in the class of star-shaped hypersurfaces. Set

$$I=\{t\in[0,1]:\text{such that (53) is solvable.}\}$$

I is nonempty because $\rho=[C_n^k]^{-\frac{1}{n-2}}$ is a solution for $t=0$. By Lemmas 3.5, 3.7, 2.6 and 2.7, equation (53) is unform elliptic and concave, apply the Evans-Krylov theorem and the Schauder theorem, we have

$$\|\rho\|_{C^{3,\alpha}(\mathbb{S}^n)}\le C,$$

where C depends only on only on n, k, $\min_{S^n} f$, $\max_{S^n} f$, $|f|_{C^1}$, $|f|_{C^2}$ and α. The a priori estimates guarantee that I is closed. The openness comes from Lemma 3.3 and the inverse function theorem. This proves the existence part of the theorem. The uniqueness part of the theorem follows from Lemma 3.3. □

4 Isoperimetric Inequality for Quermassintegrals on Starshaped Domains

In this section, we use a geometric flow to establish isoperimetric inequalities for quermassintegrals of k-convex starshaped domains in $\mathbb{R}^{n+1}$.

Theorem 4.1. *Suppose* $1\le n-1$, *and suppose* Ω *is a* k*-convex starshaped domain in* $\mathbb{R}^{n+1}$, *then the following inequality holds,*

$$(V_{(n+1)-k}(\Omega))^{\frac{1}{n+1-k}}\le \mathbf{C}_{n,k}(V_{n-k}(\Omega))^{\frac{1}{n-k}}, \tag{54}$$

where

$$\mathbf{C}_{n,k} = \frac{(V_{(n+1)-k}(B))^{\frac{1}{n+1-k}}}{(V_{n-k}(B))^{\frac{1}{n-k}}},$$

B is the standard ball in $\mathbb{R}^{n+1}$. The equality holds if and only if Ω is a ball.

We consider the following normalized evolution equation on hypersurface M^n in $\mathbb{R}^{n+1}$.

$$\partial_t X = (\frac{1}{F(\kappa)} - ru)\nu, \tag{55}$$

where $F(\cdot, t)$ and $r(t)$ are to be determined, $u =< X, \nu >$ is the supporting function of the hypersurface.

We derive the evolution equations of various geometric quantities for the following general flow.

$$\partial_t X = f\nu. \tag{56}$$

Proposition 4.2. *Under flow (56), the following evolution equations hold.*

$$\begin{aligned}
\partial_t g_{ij} &= 2fh_{ij} \\
\partial_t \nu &= -\nabla f \\
\partial_t h_{ij} &= -\nabla_i \nabla_j f + f(h^2)_{ij} \\
\partial_t h^i_j &= -\nabla^i \nabla_j f - f(h^2)^i_j \\
\partial_t \sigma_k &= -\textstyle\sum_{ij} \sigma_k^{ij}(g^{-1}h) f_{ij} - f\big(\sigma_1(g^{-1}h)\sigma_k(g^{-1}h) - (k+1)\sigma_{k+1}(g^{-1}h)\big)
\end{aligned} \tag{57}$$

Proof. Pick any local coordinate chart $(x_1, \cdots, x_n)$ of M, denote $X_i = \frac{\partial X}{\partial x_i}$, $i = 1, \cdots, n$, as $\langle X_i, \nu\rangle = 0, \forall i$, by Weingarten equation (40),

$$\begin{aligned}
(g_{ij})_t &= \langle X_i, X_j\rangle_t \\
&= \langle X_{i,t}, X_j\rangle + \langle X_i, X_{j,t}\rangle \\
&= \langle X_{t,i}, X_j\rangle + \langle X_i, X_{t,j}\rangle \\
&= \langle (f\nu)_i, X_j\rangle + \langle X_i, (f\nu)_j\rangle \\
&= f\langle (\nu)_i, X_j\rangle + f\langle X_i, (\nu)_j\rangle \\
&= f\langle \sum_l h^l_i X_l, X_j\rangle + f\langle X_i, \sum_l h^l_j X_l\rangle \\
&= f\sum_l h^l_i g_{lj} + f\sum_l h^l_j g_{li} \\
&= 2fh_{ij}
\end{aligned}$$

Since ν is a unit vector field, ν_t has only tangential component. We only need to compute $\langle \nu_t, X_i\rangle$. As $\langle \nu, X_i\rangle \equiv 0$,

$$\langle \nu_t, X_i\rangle = -\langle \nu, X_{i,t}\rangle = -\langle \nu, (f\nu)_i\rangle = -\langle \nu, (f)_i\nu\rangle = -f_i.$$

This verifies the second identity in the proposition.

For the third identity, again using the fact ν is a unit vector field, by the second identity we just proved and the Gauss formula in (40),

$$\begin{aligned}
h_{ij,t} &= -\langle X_{ij}, \nu\rangle_t \\
&= -\langle X_{ij,t}, \nu\rangle - \langle X_{ij}, \nu_t\rangle \\
&= -\langle (f\nu)_{ij}, \nu\rangle + \langle h_{ij}\nu, \nabla f\rangle \\
&= -f_{ij} - f\langle \nu_{ij}, \nu\rangle \\
&= -f_{ij} - f\langle (h_i^l X_l)_j, \nu\rangle \\
&= -f_{ij} - f\langle (h_i^l)_j X_l \nu\rangle - f\langle h_i^l X_{lj}, \nu\rangle \\
&= -f_{ij} + f\langle h_i^l h_{lj}\nu, \nu\rangle \\
&= -f_{ij} + f h_i^l h_{lj}.
\end{aligned}$$

The fourth identity follows from the first and third, and the fact $g_t^{ij} = -g^{il}g^{mj}g_{lm,t}$. The final identity in the proposition follows from the fourth identity and Proposition 2.2. □

Corollary 4.3. *Under flow (55), where F is homogeneous of degree 1, then we have the following evolution equations.*

$$\begin{aligned}
\partial_t g_{ij} &= 2(\frac{1}{F} - ru)h_{ij} \\
\partial_t \nu &= -\nabla(\frac{1}{F} - ru) \\
\partial_t h_{ij} &= -\nabla_i\nabla_j(\frac{1}{F} - ru) + (\frac{1}{F} - ru)(h^2)_{ij} \\
\partial_t h_j^i &= -\nabla^i\nabla_j(\frac{1}{F} - ru) - (\frac{1}{F} - ru)(h^2)_j^i \\
\partial_t \sigma_k &= -\sum_{ij}\sigma^{ij}\nabla_j\nabla_i(\frac{1}{F} - ru) - (\frac{1}{F} - ru)\sigma_{k-1;i}\lambda_i^2 \\
\partial_t F &= -\dot{F}^{ij}\nabla^i\nabla_j(\frac{1}{F} - ru) - (\frac{1}{F} - ru)\dot{F}^{ij}(h^2)_j^i
\end{aligned} \tag{58}$$

Furthermore, the following heat type evolution equation for Weingarten map h_j^i is valid.

Proposition 4.4.

$$\begin{aligned}\partial_t h^i_j = {} & \frac{1}{F^2}\dot{F}^{kl}\nabla^k\nabla_l h^i_j + \frac{1}{F^2}\dot{F}(h^2)h^i_j + \frac{1}{F^2}\ddot{F}(\nabla h, \nabla h)\\ & -\tfrac{2}{F^3}\nabla^i F\nabla_j F - \frac{2}{F}(h^2)^i_j + r\nabla^i h^l_j < \nabla_l X, X > +rh^i_j.\end{aligned} \tag{59}$$

Proof. It follows from previous corollary, (41) and (44). □

4.1 Monotonicity Properties

We want to choose F and r in flow (55) such that the corresponding global geometric quantities are monotone along the flow. The Minkowski identity (4) plays key role here.

From identities in Corollary 4.3, for $1 \le l \le n-1$,

$$\begin{aligned}\partial_t \int_M \sigma_l d\mu_g &= \int_M \partial_t\sigma_l + \sigma_l\frac{1}{2}g^{ij}\partial_t g_{ij}\,d\mu_g\\ &= -\int_M (\frac{1}{F} - ru)\big(\sum_i \sigma_{l-1;i}\lambda_i^2 - \sigma_l\sigma_1\big)d\mu_g\\ &= (l+1)\int_M (\frac{1}{F} - ru)\sigma_{l+1}d\mu_g\\ &= (l+1)\left[\int_M \frac{1}{F}\sigma_{l+1}d\mu_g - r\int_M u\sigma_{l+1}d\mu_g\right]\\ &= (l+1)\left[\int_M \frac{1}{F}\sigma_{l+1}d\mu_g - rC_{n,l}\int_M \sigma_l d\mu_g\right],\end{aligned} \tag{60}$$

where $C_{n,l} = \frac{\sigma_{l+1}(I)}{\sigma_l(I)}$ is the constant in the Minkowski equality.

For the special case $l = n$ and for any f, by Proposition 4.2, along flow (56),

$$\begin{aligned}\partial_t \int_M \sigma_n d\mu_g &= \int_M \partial_t\sigma_n + \sigma_n\frac{1}{2}g^{ij}\partial_t g_{ij}\,d\mu_g\\ &= -\int_M f\big(\sum_i \sigma_{n-1;i}\lambda_i^2 - \sigma_n\sigma_1\big)d\mu_g\\ &= (l+1)\int_M f\big(\sigma_n\sigma_1 - \sigma_n\sigma_1\big)d\mu_g\\ &= 0\end{aligned} \tag{61}$$

That is, $V_0(\Omega)$ is a topological invariant. This gives topological obstruction for the problem of prescribing curvature measure $\mathcal{C}_0$.

From (60), if one wants to fix $\int_M \sigma_k d\mu_g$, one may choose $F = \frac{\sigma_k}{\sigma_{k-1}}$ in (55) and define r as

$$r(t) = \frac{\int_{M_t} \frac{\sigma_{k+1}\sigma_{k-1}}{\sigma_k} d\mu_g}{C_{n,k} \int_M \sigma_k d\mu_g}. \tag{62}$$

To be precise, we consider the normalized flow

$$\partial_t X = \left(\frac{\sigma_{k-1}}{\sigma_k} - ru\right)\nu, \tag{63}$$

The first step is to get an estimate on $r(t)$.

Lemma 4.5. *$r(t)$ is invariant under rescaling, and*

$$r(t) \leq \left(\frac{\sigma_{k-1}}{\sigma_k}\right)(I) = C_{n,k-1}, \tag{64}$$

equality holds if and only if M_t is the standard sphere.

Proof. The inequality follows directly from the Newton-MacLaurin inequality. If the equality holds, this means the Newton-MacLaurin inequality holds at every point of M_t. So M_t is umbilical at every point, it is a sphere. □

The following monotonicity property is crucial.

Proposition 4.6. *For any k-convex domain Ω, under flow equation (63), we have*

1. $\int_M \sigma_k d\mu_g$ *is a constant;*
2. $\int_M \sigma_{k-1} d\mu_g$ *is monotonically non-decreasing.*

Proof. By the choice of r and (60),

$$\partial_t \int_M \sigma_k d\mu_g = 0. \tag{65}$$

This proves the first part of the statement.

From (60),

$$\begin{aligned}
\partial_t \int_M \sigma_{k-1} d\mu_g &= k\Big[\int_M \frac{1}{F}\sigma_k d\mu_g - rC_{n,k-1}\int_M \sigma_{k-1} d\mu_g\Big] \\
&= k\int_M \left[\frac{1}{F}\frac{\sigma_k}{\sigma_{k-1}} - rC_{n,k-1}\right]\sigma_{k-1} d\mu_g \\
&= k\int_M \left[1 - \frac{\int_M \frac{\sigma_{k+1}\sigma_{k-1}}{\sigma_k} d\mu_g}{C_{n,k}\int_M \sigma_k d\mu_g} C_{n,k-1}\right]\sigma_{k-1} d\mu_g \\
&\geq k\int_M \left[1 - \frac{\sigma_{k+1}(I)\sigma_{k-1}(I)}{\sigma_k^2(I)}\frac{C_{n,k-1}}{C_{n,k}}\right]\sigma_{k-1} d\mu_g = 0,
\end{aligned} \tag{66}$$

where we used the Newton-MacLaurine inequality in the last step. □

We want to establish the following longtime existence and convergence of flow (63).

Theorem 4.7. *If Ω_0 is k-convex starshaped domain with smooth boundary M_0, flow (63) exists all time $t > 0$, it converges to a standard sphere centered at the origin.*

By a proper rescaling, we will assume $V_k(\Omega_0) = V_k(B)$ where B is the standard ball in $\mathbb{R}^{n+1}$.

The rest of the section is devoted to the proof of Theorem 4.7.

4.2 The Harnack Estimate

If M^n is starshaped, it can be parametrized as $X = \rho(x)x$, where $x \in S^n$. All the geometric information of the hypersurface except the parametrization are encoded in the function $\rho(x)$.

Write $\rho = |X(t)| = \rho(x(t), t)$, where X evolves according to

$$X_t = f\nu.$$

ρ satisfies

$$\frac{d\rho}{dt} = \rho_t + \rho_x \cdot x_t.$$

By (25),

$$\nu = \frac{\rho x - \overline{\nabla}\rho}{\sqrt{\rho^2 + |\overline{\nabla}\rho|^2}}.$$

We have,

$$f\frac{\rho x - \overline{\nabla}\rho}{\sqrt{\rho^2 + |\overline{\nabla}\rho|^2}} = \nu f = X_t = (\rho x)_t = (\rho_t + \rho_x \cdot x_t)x + \rho x_t. \tag{67}$$

Note that $x_t \perp x$, equalize the tangential components of $\mathbb{S}^n$ in (67),

$$x_t = -\frac{f\overline{\nabla}\rho}{\rho\sqrt{\rho^2 + |\overline{\nabla}\rho|^2}}.$$

Therefore,

$$\rho_x \cdot x_t = \overline{\nabla}\rho \cdot x_t = -\frac{f|\overline{\nabla}\rho|^2}{\rho\sqrt{\rho^2 + |\overline{\nabla}\rho|^2}}.$$

Put the above identity to (67), equalize the normal component of $\mathbb{S}^n$ in (67),

$$\rho_t = -\rho_x \cdot x_t + \frac{f\rho}{\sqrt{\rho^2 + |\overline{\nabla}\rho|^2}} = \frac{f\sqrt{\rho^2 + |\overline{\nabla}\rho|^2}}{\rho}.$$

In particular, if X satisfies (55), ρ satisfies

$$\partial_t \rho = \frac{\sqrt{\rho^2 + |\overline{\nabla}\rho|^2}}{\rho}\frac{1}{F} - r\rho. \tag{68}$$

Equation (68) is equivalent to (55) up to diffeomorphism, if we can prove that the starshapedness is preserved along the flow.

For the gradient estimate, we prefer to work on (68). As in the previous section dealing to the problem of prescribing curvature measure, let $\gamma \equiv \ln\rho$, and we choose a local orthonormal frame $\{e_i\}_{i=1}^n$ on S^n.

By the homogeneity of F,

$$\partial_t \gamma = \frac{\omega^2}{F(B)} - r, \tag{69}$$

where

$$\omega = \sqrt{1 + |\overline{\nabla}\gamma|^2}, \quad B = (-\gamma_{ij} + \delta_{ij} + \frac{\sum_l(\gamma_i\gamma_{lj} + \gamma_j\gamma_{il})\gamma_l}{\omega(\omega+1)} - \frac{\gamma_i\gamma_j\sum_{l,m}\gamma_l\gamma_{lm}\gamma_m}{\omega^2(1+\omega)^2}),$$

as defined in (30).

Proposition 4.8. *Let* $\phi = \frac{|\overline{\nabla}\gamma|^2}{2}$, *assume (69) preserves* $\kappa(t) \in \Gamma_k$,

$$\partial_t\phi = \mathcal{L}_{lj}\overline{\nabla}_l\overline{\nabla}_j\phi + W_k \cdot \overline{\nabla}_k\phi - \frac{\omega^2}{F^2(B)}\sum_{ij}\frac{\partial F}{\partial b_{ij}}(\delta_{ij}|\overline{\nabla}\gamma|^2 - \gamma_j\gamma_i + \delta_{ij}\gamma_{ii}^2). \tag{70}$$

where W_k *is a one-parameter family of vector fields depending on time, and* $\mathcal{L}_{ij}$ *is an elliptic operator defined as follows,*

$$\mathcal{L}_{ij} \equiv \tfrac{\omega^2}{F^2(B)}\tilde{F}^{ij}, \tag{71}$$

where $\tilde{F}^{ij}$ *defined as in (35). In consequence,* $\overline{\nabla}\gamma$ *is bounded from above independent of time* t.

Proof. $\kappa \in \Gamma_k$ is equivalent to $B \in \Gamma_k$, hence $F(B) > 0$. Rewrite the last equation in (69) as

$$F(B) = \frac{\omega^2}{\gamma_t + r}.$$

Proposition follows from Lemma 3.4 with a straightforward computation using identity (36). ☐

The following Harnack type gradient estimate is a directly consequence.

Corollary 4.9. *Let ρ be a positive solution to (68) on $S^n \times [0, T)$. Then there exists a constant C which depends on $\rho(\cdot, 0)$ but independent of t, such that at each time $t \in [0, T)$,*

$$\max_{S^n} \rho(\cdot, t) \leq C \cdot \min_{S^n} \rho(\cdot, t) \tag{72}$$

Proof. We prove the corollary for each fixed time $t_0 \in [0, T)$. Assume $\rho(\cdot, t_0)$ achieves maximum at x_+ and minimum at x_-, and let $\Gamma : [s_1, s_2] \longrightarrow M^n$ be a path joining x_- and x_+. We have

$$\begin{aligned} \log \frac{\rho(x_+, t_0)}{\rho(x_-, t_0)} &= \int_{s_1}^{s_2} \frac{d}{ds} [\log \rho(\Gamma(s), t_0)] ds \\ &= \int_{s_1}^{s_2} \frac{\overline{\nabla}\rho}{\rho} \cdot \frac{d\Gamma}{ds} ds \\ &\leq \int_{s_1}^{s_2} \overline{\nabla}\gamma \cdot \left|\frac{d\Gamma}{ds}\right| ds \\ &\leq \tilde{C} \int_{s_1}^{s_2} \left|\frac{d\Gamma}{ds}\right| ds. \end{aligned} \tag{73}$$

By taking Γ to be the shortest geodesic with constant speed 1 which joins x_- and x_+, we obtain $\int_{s_1}^{s_2} \left|\frac{d\Gamma}{ds}\right| ds = d(x_-, x_+) \leq \pi$. ☐

Lemma 4.10. *Suppose that $\rho > 0$ satisfies (68), then at any time $t_0 \geq 0$, if $x_0 \in \mathbb{S}^n$ is a minimum point of $\rho(x, t_0)$, then $\rho(x_0, t_0)_t \geq 0$, strict inequality holds unless $M(t_0)$ is a round unit sphere at the origin.*

Proof. The minimum point of $\rho(x, t_0)$ is the same as minimum point of $\gamma(x, t_0)$. By (69),

$$\gamma_t(x_0, t_0) = \frac{\omega^2(x_0, t_0)}{F(B(x_0, t_0))} - r(t_0).$$

As x_0 is a minimum point, $\overline{\nabla}\gamma(x_0, t_0) = 0$, so at (x_0, t_0), $\omega = 1$ and

$$B = (-\overline{\nabla}^2 \gamma + I) \leq I. \tag{74}$$

Hence,

$$F(B(x_0, t_0)) \leq F(I). \tag{75}$$

That is,

$$\frac{\omega^2(x_0,t_0)}{F(B(x_0,t_0))} \geq \frac{1}{F(I)}. \tag{76}$$

By Lemma 4.5, $r(t_0) < \dfrac{1}{F(I)}$ unless $M(t_0)$ is a round sphere (by normalization, it is a sphere of radius 1). We have

$$\gamma_t(x_0,t_0) = \frac{\omega^2(x_0,t_0)}{F(B(x_0,t_0))} - r(t_0) \geq \frac{1}{F(I)} - r(t_0) > 0,$$

unless $M(t_0)$ is a round sphere of radius 1. We claim if $\gamma_t(x_0,t_0) = 0$, this round sphere must centered at the origin. Suppose its center z is not the origin, we may assume $z = (0, \cdots, 0, s)$ for some $-1 < s < 0$. Now

$$\gamma(x,t_0) = \frac{1}{2}\log(1 + s^2 + 2sx_{n+1}).$$

The minimum point is $x_0 = (0, \cdots, 0, 1)$, it is easy to compute that

$$-\overline{\nabla}^2\gamma(x_0,t_0) = \frac{s}{(1+s)^2}I.$$

The strictly inequalities will occur in (74)–(76). Thus,

$$\gamma_t(x_0,t_0) = \frac{\omega^2(x_0,t_0)}{F(B(x_0,t_0))} - r(t_0) > \frac{1}{F(I)} - r(t_0) = 0.$$

contradiction. □

The following C^0 estimate is a direct consequence of Corollary 4.9 and Lemma 4.10.

Corollary 4.11. *Let ρ be a positive solution to (68) on $S^n \times [0,T)$. Then there exists a uniform positive constant C which does not depend on time t, such that for $\forall t \in [0,T)$,*

$$0 < \tfrac{1}{C} \leq \rho(x,t) \leq C, \tag{77}$$

for any point $(x,t) \in S^n \times [0,T)$. Moreover, $u(x,t) \geq c > 0$ for some constant c independent of t.

Proof. By Lemma 4.10, $\rho(x_0,t_0)_t > 0$ at any minimum point x_0 of $\rho(x,t_0)$, unless $M(t_0)$ is a round unit sphere centered at 0. That is, $\min_{x\in\mathbb{S}^n}\rho(x,t)$ is strictly increasing at t_0 unless $M(t_0)$ is a round sphere centered at 0. In any case,

$$\min_{x\in\mathbb{S}^n}\rho(x,t) \geq \min_{x\in\mathbb{S}^n}\rho(x,0). \tag{78}$$

An upper bound of ρ follows from the Harnack inequality (72).

The last statement in lemma follow from the identity

$$u = \frac{\rho^2}{\sqrt{\rho^2 + |\overline{\nabla}\rho|^2}}.$$

□

Since u is bound from below by a positive constant independent of t, flow (68) preserves the starshapedness. We want to show that Γ_k is also preserved along the flow. From the property of Γ_k, we only need to show $\sigma_k > 0$ is preserved. This is equivalent to show $F > 0$ is preserved.

Proposition 4.12. *There is $C > 0$, such that $\frac{1}{F} \leq C$.*

Proof. We consider function $G = \gamma_t + r$. We may rewrite (69) as

$$G = \frac{\omega^2}{F(B)} =: \overline{F}(\overline{\nabla}\gamma, \overline{\nabla}^2\gamma), \tag{79}$$

where $(\overline{F}^{ij}) = (\frac{\partial \overline{F}}{\partial \gamma_{ij}}) > 0$. Differentiate (79) in t variable, and notice that r is independent of x,

$$\begin{aligned} G_t &= \sum_{ij} \overline{F}^{ij}(\gamma_t)_{ij} + \sum_l \frac{\partial \overline{F}}{\partial \gamma_l}(\gamma_t)_l \\ &= \sum_{ij} \overline{F}^{ij} G_{ij} + \sum_l \frac{\partial \overline{F}}{\partial \gamma_l} G_l. \end{aligned}$$

G is bounded from above by the maximum principle. Since r is bounded, $\frac{1}{F(B)}$ is bounded. The boundedness of $\frac{1}{F}$ follows from C^0 and C^1 estimates. □

4.3 C^2 Estimates

Denote $\varphi \equiv \dfrac{1}{u}$, φ satisfies the following evolution equation.

Proposition 4.13. *Let ρ be a positive solution to (68) on $S^n \times [0, T)$. We have*

$$\partial_t \varphi = \frac{1}{F^2}\dot{F}^{ij}\nabla^i\nabla_j\varphi - \frac{\varphi}{F^2}\dot{F}(h^2) - \frac{2}{F^2\varphi}\dot{F}(\nabla\varphi, \nabla\varphi) + r\varphi + r\varphi^{-1}\nabla_l\varphi < X, \nabla_l X > . \tag{80}$$

Proof. We first write down the evolution equation of u using (55), (58) and (44). We work on local orthonormal frames on $M(t)$.

$$\begin{aligned}
u_t &= \langle X_t, \nu\rangle + \langle X, \nu_t\rangle \\
&= \frac{1}{F} - ru - \sum_l \langle X, X_l\rangle (\frac{1}{F} - ru)_l \\
&= \frac{1}{F} - ru + \sum_l \langle X, X_l\rangle (\frac{F^{ij}h_{ij,l}}{F^2} + ru_l) \\
&= \frac{1}{F} - ru + \frac{F^{ij}(u_{ij} - h_{ij} + (h^2)_{ij}u)}{F^2} + r\sum_l \langle X, X_l\rangle u_l \\
&= \frac{1}{F} - ru + \frac{F^{ij}u_{ij}}{F^2} - \frac{1}{F} + \frac{F^{ij}(h^2)_{ij}u}{F^2} + r\sum_l \langle X, X_l\rangle u_l
\end{aligned}$$

Proposition follows from above identity by inserting $u = \frac{1}{\varphi}$. □

Proposition 4.14. *Let ρ be a positive solution to (68) on $S^n \times [0,T)$. We have*

$$\begin{aligned}
\partial_t(\varphi h^i_j) = &\frac{1}{F^2}\dot{F}^{kl}\nabla^k\nabla_l(\varphi h^i_j)) - \frac{2\varphi}{F^3}\nabla^i F\nabla_j F + \frac{\varphi}{F^2}\ddot{F}^{kl,mn}\nabla^i h^k_l \nabla_j h^m_n \\
&-\tfrac{2}{F^2\varphi}\dot{F}^{kl}\nabla^k\varphi\nabla_l(\varphi h^i_j) + r\nabla_l(\varphi h^i_j) < \nabla_l X, X > \\
&-2\varphi\left[\tfrac{(h^2)^i_j}{F} - rh^i_j\right].
\end{aligned} \tag{81}$$

Proof. Proof follows from (59) and Proposition 4.13. □

Proposition 4.15. *Let ρ be a positive solution to (68) on $S^n \times [0,T)$ and let $\tilde{\kappa}(t) = \max\limits_{x\in M^n_t}(\kappa_1(x), \cdots, \kappa_n(x))$. Then for $t > 0$,*

$$\max_{M^n_t}\varphi\tilde{\kappa}(t) \le \max_{M^n_0}\varphi\tilde{\kappa}(0), \tag{82}$$

with the equality holds if and only if M_0 is a sphere centered at the origin. Since $\sigma_1(\kappa) > 0$, we have uniform curvature bounds.

Proof. Let x_0 be a point such that $h^1_1(x_0, t_0) = \kappa(t_0)$ for some direction e_1. By (81), and concavity of F,

$$(\varphi h^1_1(x_0, t_0))_t \le -2\varphi\left[\frac{(h^1_1)^2}{F(\kappa)} - rh^1_1\right].$$

At x_0, $h^1_1 = \tilde{\kappa}(t) \ge \kappa_i$ for all i. By the monotonicity, homogeneity of F and by Lemma 4.5,

$$\frac{h^1_1}{F(\kappa)} \ge \frac{1}{F(I)} \ge r. \tag{83}$$

We obtained at x_0, $(\varphi h^1_1(x_0, t_0))_t \le 0$.

We claim for any t_0, $(\varphi h^1_1(x_0,t))_t > 0$ unless $M(t_0)$ is the unit sphere centered at 0. Now suppose $(\varphi h^1_1(x_0,t))_t = 0$, all inequalities in (83) must be equalities. In particular,

$$r(t) = \frac{1}{F(I)}.$$

By Lemma 4.5 and normalization, $M(t_0)$ must be a sphere of radius 1. So $\kappa_1(x,t_0) = \cdots, \kappa_n(x,t_0) = 1, \forall x \in \mathbb{S}^n$ and we may use the standard spherical paramerization for $M(t_0)$. Suppose its center is $z \neq 0$, we may assume $z = (0,\cdots,0,s)$ for some $-1 < s < 0$. Now

$$u(x,t_0) = 1 + sx_{n+1}, \quad \varphi(x,t_0) = \frac{1}{1+sx_{n+1}}.$$

The minimum point is $x_0 = (0,\cdots,0,1)$, it follows from (81),

$$\partial_t(\varphi h^i_j) = \frac{1}{F^2}\dot{F}^{kl}\nabla^k\nabla_l(\varphi h^i_j) - 2\varphi\left[\frac{(h^2)^i_j}{F} - rh^i_j\right] = \frac{1}{F^2}\dot{F}^{kl}\nabla^k\nabla_l\varphi < 0,$$

contradiction. □

We now prove Theorem 4.7.

Proof. By C^2 estimates and Proposition 4.12, $\kappa \in \Gamma_k$ is preserved along flow (68). By Lemma 2.7, the equation is uniform parabolic. We may apply the Krylov Theorem [31] and the standard parabolic theory to conclude the longtime existence and regularity for the flow. To get exponential convergence, we use the uniform ellipticity of F. There is $c_0 > 0$ independent of t,

$$(\frac{\partial F(B)}{\partial b_{ij}})(x,t) \geq c_0 I, \quad \forall (x,t).$$

Thus, as $n \geq 2$,

$$\sum_i \frac{\partial F(B)}{\partial b_{ii}} \geq c_0 + \lambda_M(\frac{\partial F(B)}{\partial b_{ij}}),$$

where $\lambda_M(W)$ denoting the largest eigenvalue of W. By C^2 estimates, there is $\beta > 0$ independent of t such that

$$\frac{\omega^2}{F^2}\sum_{ij}\frac{\partial F(B)}{\partial b_{ij}}(\delta_{ij}|\overline{\nabla}\gamma|^2 - \gamma_i\gamma_j) \geq \beta|\overline{\nabla}\gamma|^2.$$

By Proposition 4.8,

$$\partial_t\Big(\frac{|\overline{\nabla}\gamma|^2}{2}\Big) \le \mathcal{L}_{lj}\overline{\nabla}_l\overline{\nabla}_j\Big(\frac{|\overline{\nabla}\gamma|^2}{2}\Big) + W_k\cdot\overline{\nabla}_k\Big(\frac{|\overline{\nabla}\gamma|^2}{2}\Big) - \beta|\overline{\nabla}\gamma|^2. \tag{84}$$

Set $Q = e^{\beta t}\frac{|\overline{\nabla}\gamma|^2}{2}$, Q satisfies differential inequality

$$\partial_t Q \le \mathcal{L}_{lj}\overline{\nabla}_l\overline{\nabla}_j Q + W_k\cdot\overline{\nabla}_k Q. \tag{85}$$

Therefore, Q is bounded from above independent of t. From there, we conclude $|\overline{\nabla}\gamma|^2 \to 0$ exponentially as $t\to\infty$. By our normalization, $\rho\to 1$ and $\overline{\nabla}\rho\to 0$ exponentially as $t\to\infty$.

For the exponential convergence of $\overline{\nabla}^m\rho$, apply integration by parts,

$$\int_{\mathbb{S}^n}|\overline{\nabla}^m\rho|^2 d\mu_{\mathbb{S}^n} \le C\Big(\int_{\mathbb{S}^n}|\overline{\nabla}^{m+1}\rho|^2 d\mu_{\mathbb{S}^n}\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{S}^n}|\overline{\nabla}^{m-1}\rho|^2 d\mu_{\mathbb{S}^n}\Big)^{\frac{1}{2}}.$$

By the a priori estimates, $\|\overline{\nabla}^{m+1}\rho\|_{L^\infty(\mathbb{S}^n)} \le c_m$ for some c_m independent of t. An induction argument yields that, for each $m\in\mathbb{N}^+$, there is $C_m>0, \beta_m>0$, such that

$$\|\overline{\nabla}^m\rho\|_{L^2(\mathbb{S}^n)} \le C_m e^{-\beta_m t}.$$

The Sobolev Lemma implies $\overline{\nabla}^m\rho \to 0$ exponentially and $t\to\infty$, for each $m\in\mathbb{N}^+$. □

We prove Theorem 4.1. In fact, the following is true.

Theorem 4.16. *Suppose Ω is a C^2 starshaped domain in $\mathbb{R}^{n+1}$. Assume $1\le k\le n-1$, that*

$$\kappa(x)\in\overline{\Gamma}_k = \{\lambda\in\mathbb{R}^n|\sigma_l(\lambda)\ge 0, \forall l = 1,\cdots,k.\},$$

then the following inequality holds,

$$(V_{(n+1)-k}(\Omega))^{\frac{1}{n+1-k}} \le \mathbf{C}_{n,k}(V_{n-k}(\Omega))^{\frac{1}{n-k}}, \tag{86}$$

where

$$\mathbf{C}_{n,k} = \frac{(V_{(n+1)-k}(B))^{\frac{1}{n+1-k}}}{(V_{n-k}(B))^{\frac{1}{n-k}}},$$

B is the standard ball in $\mathbb{R}^{n+1}$. The equality holds if and only if Ω is a ball.

Proof. Case 1. Ω is k-convex.

Inequality (86) follows directly from the above Proposition 4.6 and Theorem 4.7. We examine the equality case. Recall (66),

$$\begin{aligned}\partial_t \int_M \sigma_{k-1}(\kappa) d\mu_g &= k \int_M \left[1 - \frac{\int_M \frac{\sigma_{k+1}(\kappa)\sigma_{k-1}(\kappa)}{\sigma_k(\kappa)} d\mu_g}{C_{n,k} \int_M \sigma_k(\kappa) d\mu_g} C_{n,k-1}\right] \sigma_{k-1} d\mu_g \\ &\geq k \int_M \left[1 - \frac{\sigma_{k+1}(I)\sigma_{k-1}(I)}{\sigma_k^2(I)} \frac{C_{n,k-1}}{C_{n,k}}\right] \sigma_{k-1} d\mu_g = 0.\end{aligned} \tag{87}$$

At any time $t_0 \geq 0$, inequality is strict in (87) unless

$$\frac{\sigma_{k+1}(\kappa)\sigma_{k-1}(\kappa)}{\sigma_{k+1}(I)\sigma_{k-1}(I)\sigma_k(\kappa)} = \frac{\sigma_k(\kappa)}{\sigma_k^2(I)}, \quad \text{a.e. in } M(t_0).$$

That is the equality is the case in (13), this implies $M(t_0)$ is umbilical almost everywhere. As $M(t_0)$ is C^2, it is umbilical everywhere. $M(t_0)$ is a round sphere for each $t \geq t_0$. In particular, if equality is held in (86), then M is a sphere.

Case 2. General case.

We may approximate Ω by k-convex starshaped domains. The inequality follows from the approximation. We now treat the equality case. We first note that both $\int_M \sigma_k d\mu_g$ and $\int_M \sigma_{k-1} d\mu_g$ are positive, since there exists at least one elliptic point on an embedded compact hypersurface in Euclidean space and also the k-convexity condition. Suppose Ω is a weakly k-convex starshaped domain with equality in (86) attained. Let $M_+ = \{x \in M | \sigma_k(\kappa(x)) > 0\}$. M_+ is open and nonempty since M is compact and embedded in $\mathbb{R}^{n+1}$. We claim that M_+ is closed. This would imply $M = M_+$, so Ω is k-convex, by *Case 1*, we may conclude Ω is a standard ball.

We now prove that M_+ is closed. Pick any $\eta \in C_0^2(M_+)$ compactly supported in M_+. Let M_s be the hypersurface determined by position function $X_s = X + s\eta\nu$, where X is the support function of M and ν is the unit outernormal of M at X. Let Ω_s be the domain enclosed by M_s. It is easy to show M_s is k-convex starshaped when s is small enough. Define

$$\mathcal{I}_k(\Omega) = \frac{V_{(n+1)-k}^{\frac{1}{n+1-k}}(\Omega)}{V_{n-k}^{\frac{1}{n-k}}(\Omega)}. \tag{88}$$

Therefore $\mathcal{I}_k(\Omega_s) - \mathcal{I}_k(\Omega) \leq 0$ for s small, i.e.

$$\frac{d}{ds}\mathcal{I}_k(\Omega_s)|_{s=0} = 0.$$

Simple calculation yields

$$\frac{d}{ds}\int_{M_s} \sigma_l(\kappa_s) d\mu_{g_s}|_{s=0} = (l+1)\int_M \sigma_{l+1}(\kappa)\eta d\mu_g.$$

Therefore,

$$\frac{d}{ds}\mathcal{I}_k(\Omega_s)|_{s=0} = A\int_M (\sigma_{k+1}(\kappa) - c_1\sigma_k(\kappa))\eta d\mu_g = 0,$$

for some constant $A > 0$ with $c_1 = \frac{k(n-k)}{(k+1)(n-k+1)}\frac{1}{\mathcal{I}(B)^{n-k+1}(\int_M \sigma_k)^{\frac{1}{n-k}}} > 0$ and for all $\eta \in C_0^2(M_+)$. Thus,

$$\sigma_{k+1}(\kappa(x)) = c_1\sigma_k(\kappa(x)), \qquad \forall x \in M_+. \tag{89}$$

It follows from the Newton-MacLaurine inequality, there is a dimensional constant $\tilde{C}_{k,n}$ such that

$$\sigma_{k+1}(\kappa(x)) \le \tilde{C}_{k,n}\sigma_k^{1+1/k}(\kappa(x)), \quad \forall x \in M_+.$$

In view of (89), there is a positive constant c_2, such that

$$\sigma_k(\kappa(x)) \ge c_2 > 0, \qquad \forall x \in M_+, \tag{90}$$

where $c_2 = (\frac{c_1}{\tilde{C}_{k,n}})^k$ is a positive constant depending only on n, k, and Ω. (90) implies M_+ is closed. $\square$

5 Appendix

We present Garding's theory of hyperbolic polynomials here.

Definition 5.1. Let P be a homogeneous polynomial of degree m in a finite vector space V. For $\theta \in V$, P is called hyperbolic at θ if $P(\theta) \neq 0$ and the equation $P(x + t\theta) = 0$ (as a polynomial of $t \in \mathbb{C}$) has only real roots for every $x \in V$. We say P is complete if $P(x + ty) = P(x)$ for all x, t implies $y = 0$.

Proposition 5.2. *Suppose P is hyperbolic at θ, then the component Γ of θ in $\{x \in V; P(x) \neq 0\}$ is a convex cone, the zeros of $P(x + ty)$ as a polynomial in t are real $\forall x, y \in V$. The polynomial $\frac{P(x)}{P(\theta)}$ is real, and it is positive when $x \in \Gamma$. Furthermore, $(\frac{P(x)}{P(\theta)})^{\frac{1}{m}}$ is concave and homogeneous of degree 1 in Γ, equal to 0 on the boundary of Γ.*

Proof. We normalize $P(\theta) = 1$, then there exist $t_j \in \mathbb{R}$, $j = 1, \cdots, m$, such that

$$P(x + t\theta) = (t - t_1) \times \ldots \times (t - t_m).$$

In particular, $P(x) = (-t_1) \times \ldots \times (-t_m) \in \mathbb{R}$. Set

$$\Gamma_\theta = \{x \in V; P(x + t\theta) \neq 0, t \geq 0\}.$$

Γ_θ is open and $\theta \in \Gamma_\theta$ as $P(\theta + t\theta) = (1 + t)^m P(\theta)$ only has the zero $t = -1$. Notice that Γ_θ is also closed in $\{x \in V; P(x) \neq 0\}$. If $x \in \bar{\Gamma}_\theta$, then $P(x + t\theta) \neq 0$, when $t > 0$. Hence,

$$\Gamma_\theta = \{x \in \bar{\Gamma}_\theta, P(x) \neq 0\}.$$

If $x \in \Gamma_\theta$, then $x + t\theta \in \Gamma_\theta$ when $t > 0$. This implies that Γ_θ is connected, Therefore $\lambda x + \mu\theta \in \Gamma_\theta$ for all $\lambda > 0, \mu > 0$. That is, Γ_θ is star-shaped with respect to θ and $\Gamma_\theta = \Gamma$.

For $y \in \Gamma$ and $\delta > 0$ fixed,

$$E_{y,\delta} = \{x \in V; P(x + i\delta\theta + isy) \neq 0, Re(s) \geq 0\}$$

is open. If $s \neq 0$, $P(i\delta + isy) = (is)^m P(\frac{\delta\theta}{s} + y) = 0$, the hyperbolicity implies $s < 0$. That is, $0 \in E_{y,\delta}$. If $x \in \bar{E}_{y,\delta}$ and $Res > 0$, then Hurwitz' theorem implies $P(x + i\delta\theta + isy) \neq 0$. This is still true when $Re(s) = 0$ since $x + isy$ is real. Therefore, $E_{y,\delta}$ is both open and closed, and $E_{y,\delta} = V$. Thus,

$$P(x + i(\delta\theta + y)) \neq 0, \forall x \in \mathbb{R}^n, y \in \Gamma, \delta > 0.$$

For Γ is open, the above remains true for $\delta = 0$. Equation $P(x + ty) = 0$ has only real roots, for if $t = t_1 + it_2$ is a root with $t_2 \neq 0$ we would get $P(\frac{x+t_1 y}{t_2} + iy) = 0$. This means that y can play the role of θ, Γ is star-shaped with respect to every point in Γ. The convexity of Γ follows. We also have $P(y) > 0$ for all $y \in \Gamma$.

We now prove the concavity statement in the proposition. As $P(x + ty)$ has only real roots for $y \in \Gamma$, there are $t_j \in \mathbb{R}$, $j = 1, \ldots, m$,

$$P(x + ty) = P(y)(t - t_1) \times \ldots \times (t - t_m).$$

In turn,

$$P(sx + y) = P(y)(1 - st_1) \times \ldots (1 - st_m).$$

If $sx + y \in \Gamma$, we must have $1 - st_j > 0$ for every j. If $f(s) = logP(sx + y)$, then

$$f'(s) = -\sum \frac{t_j}{1 - st_j}, \quad f''(s) = -\sum \frac{t_j^2}{(1 - st_j)^2}.$$

Therefore, by Cauchy-Schwarz inequality,

$$m^2 e^{-\frac{f(s)}{m}} \frac{d^2(e^{\frac{f(s)}{m}})}{ds^2} = f'(s)^2 + mf''(s)$$
$$= (\sum \frac{t_j}{1-st_j})^2 - m\sum \frac{t_j^2}{(1-st_j)^2} \leq 0.$$

□

If P is a homogeneous polynomial of degree m. For $x^l = (x_1^l, \ldots, x_n^l) \in V$, $l = 1, \ldots, m$, we denote $< x^l, \frac{\partial}{\partial x} >= \sum_1^n x_j^l \frac{\partial}{\partial x_j}$ as a vector field. We define the complete polarization of P as

$$\tilde{P}(x^1, \ldots, x^m) = \frac{1}{m!} < x^1, \frac{\partial}{\partial x} > \ldots < x^m, \frac{\partial}{\partial x} > P(x).$$

It is a multilinear and symmetric in $x^1, \ldots, x^m \in V$, independent of x, and that

$$\tilde{P}(x, \ldots, x) = \frac{1}{m!} \frac{d^m}{dt^m} P(tx) = P(x), \forall x \in V.$$

And

$$P(t_1 x^1 + \ldots + t_m x^m) = m! t_1 \ldots t_m \tilde{P}(x^1, \ldots, x^m) + \ldots$$

where the dots denote terms not containing all the factors t_j.

Lemma 5.3. *If P is hyperbolic at θ and $m > 1$, then for any $y = (y_1, \ldots, y_n) \in \Gamma$,*

$$Q(x) = \sum_1^n y_j \frac{\partial}{\partial x_j} P(x)$$

is also hyperbolic at θ. In general, if $x^1, \ldots, x^l \in \Gamma$ for some $l < m$, then

$$\tilde{Q}_l(x) = \tilde{P}(x^1, \ldots, x^l, x, \ldots, x)$$

is hyperbolic at θ.

The proof is immediate by Rolle's theorem. Using polarization and Lemma 5.3, we list some of important examples of hyperbolic polynomials.

Corollary 5.4. *The following polynomials are hyperbolic.*

1. *The polynomial $P = (x_1)^2 - (x_2)^2 - \ldots - (x_n)^2$ is hyperbolic at $(1, 0, \ldots, 0)$.*
2. *The polynomial $P = x_1 \ldots x_n$ is complete hyperbolic at any θ with $P(\theta) \neq 0$. The positive cone Γ of P at $(1, \ldots, 1)$ is*

$$\Gamma = \{x = (x_1, \ldots, x_n); x_j > 0, \quad \forall j\}.$$

3. *In general the elementary symmetric function $\sigma_k(x)$ is complete hyperbolic at $(1,\ldots,1)$, the corresponding positive cone Γ_k is*

$$\Gamma_k = \{\sigma_l(x) > 0, \forall l \leq k\}.$$

4. *Let $\mathcal{S}$ denote set of all real $n \times n$ symmetric matrices. Then $\sigma_k(W), W \in \mathcal{S}$ is complete hyperbolic at the identity matrix, the corresponding positive cone is*

$$\Gamma_k = \{\sigma_l(W) > 0, \forall l \leq k\}.$$

5. *For $W^1,\ldots,W^l \in \Gamma_k$, $l < k$, then $Q_l(W) = \tilde{P}(W^1,\ldots,W^l,W,\ldots,W)$ is complete hyperbolic in Γ_k.*

Lemma 5.5. *Suppose P is a second order complete hyperbolic polynomial. Suppose both roots of $f(s) = P(sy + w)$ vanishing for some $y \in \Gamma$ and $w \in V$. Then, all the roots of $g(s) = P(sz + w)$ are vanishing for any $z \in \Gamma$.*

Proof. Since $P(y + tw) = P(y) \neq 0$ for all t, we must have $y + tw \in \Gamma$. By the convexity of Γ, we have $z + tw \in \Gamma$ for all t. So, $P(z + tw) \neq 0$. For any $z \in \Gamma$ and all t,

$$P(z)(1 + t\lambda_1)(1 + t\lambda_2) = P(z + tw) \neq 0,$$

λ_1, λ_2 are the roots of $P(sz + w)$. Since t is arbitrary, this gives $\lambda_1 = \lambda_2 = 0$. □

Lemma 2.4 is a special case of the following proposition.

Proposition 5.6. *Suppose P a homogenous polynomial of degree m, suppose it is hyperbolic at θ and $P(\theta) > 0$, then $\forall x^1,\ldots,x^m \in \Gamma$,*

$$\begin{aligned} P^2(x^1,x^2,x^3,\cdots,x^m) &\geq P(x^1,x^1,x^3,\cdots,x^m)P(x^2,x^2,x^3,\cdots,x^m) \\ P(x^1,\ldots,x^m) &\geq P(x^1)^{\frac{1}{m}}\ldots P(x^m)^{\frac{1}{m}}. \end{aligned} \tag{91}$$

If P is complete, the equality holds if and only if all x^j are pairwise proportional. This is also equivalent that for $x, y \in \Gamma$ not proportional, the function $h(t) = P(x + ty)^{\frac{1}{m}}$ is strictly concave in $t > 0$. If P is complete, then $\tilde{Q}_l(X) = \tilde{P}(x^1,\ldots,x^l,x,\ldots,x)$ is complete if $m - l \geq 2$ and $x^1,\ldots,x^l \in \Gamma$. In particular, $\tilde{P}(x^1,\ldots,x^m) > 0$ if $x^1 \in \bar{\Gamma}$ and $x^j \in \Gamma$ when $m \geq 2$.

Proof. Since $P^{\frac{1}{m}}(X)$ is concave in Γ, it follows that for any $x, y \in \Gamma$, $h(t) = P(x + ty)^{\frac{1}{m}}$ is concave in $t > 0$. So, $h^{"}(t) \leq 0$. A direct computation yields

$$h^{"}(0) = (m-1)(\tilde{P}(y,y,x,\ldots,x)P(X) - \tilde{P}(y,x,\ldots,x)^2)P(x)^{\frac{1}{m}-2}.$$

We get the inequality

$$\tilde{P}(y, y, x, \ldots, x)P(X) \le \tilde{P}(y, x, \ldots, x)^2.$$

In turn, it implies

$$\tilde{P}(y, x, \ldots, x)^m \ge P(y)P(x)^{m-1}.$$

We now apply induction argument. Take $y = x^1$ and assuming that (91) is already proved for hyperbolic polynomials of degree $m - 1$. Let $Q(x) = \tilde{P}(y, x, \ldots, x)$, we get

$$\begin{aligned}\tilde{P}(x^1, \ldots, x^m) &\ge (Q(x^2) \ldots Q(x^m))^{\frac{1}{(m-1)}} \\ &\ge (P(x^1)P(x^2)^{m-1} \ldots P(x^1)P(x^m)^{m-1})^{\frac{1}{m(m-1)}},\end{aligned}$$

which proves (91).

To prove the last statement in the proposition, it suffices to show that if $m \ge 3$, Q (defined above) is complete. suppose $Q(x) = Q(x + tz)$ for all x, t. In particular, $Q(y+tz) = Q(y)$. That means that $Q(ty+z) = Q(ty)$, so $P(ty+z) - P(ty) = a$ is independent of t. Since the zeros of $P(ty) + a = t^m P(y) + a$ must all be real, it follows that $a = 0$. This $P(y + sz) = P(y) \ne 0$ for all s, so it follows that $y + sz \in \Gamma$. Hence,

$$\frac{(sx + y + sz)}{(s + 1)} \in \Gamma, \forall x \in \Gamma, s > 0.$$

Letting $s \to \infty$, we conclude that $x + z \in \bar{\Gamma}$ for all $x \in \Gamma$. This implies $x + z \in \Gamma$. We can replace z by tz for any t, so $x + tz \in \Gamma$ for all t and $x \in \Gamma$. Thus $P(z + sx)$ can not have any zeros $\ne 0$, so $P(z + sx) = s^m P(x)$. That is $P(x + tz) = P(x)$ for all t and all $x \in \Gamma$. Since P is analytic, that means $P(x + tz) = P(x)$ for all t and all $x \in V$. By the completeness assumption on P, $z = 0$.

Finally, we discuss the equality case in (91). By the above, we may assume $m = 2$. If the equality holds, we have $P(y)P(x) = \tilde{P}(y, x)^2$. This implies the roots of the second order polynomial $p(t) = P(x+ty)$ are equal, i.e., $t_1 = t_2 = -\lambda \ne 0$. In turn, for all t,

$$P(y + (t + \lambda)^{-1}(x - \lambda y)) = (t + \lambda)^{-2} P(ty + x) = P(y).$$

That is both roots of the polynomial $f(s) = P(sy + (x - \lambda y))$ are vanishing.

From Lemma 5.5, we have $P(z + t(x - \lambda y)) = P(z)$ for all $z \in \Gamma$ and all t. Since Γ is open and P is analytic, $P(z + t(x - \lambda y)) = P(z)$ for all z and all t. By the completeness of P, $x - \lambda y = 0$. That is, x and y are proportional. $\square$

6 Notes

1. The definition of curvature measures in this notes follows from Federer [12], where he used Steiner's formula to define them for sets of positive reach. Alexandrov [3] initiated the problem of prescribing curvature measure $\mathcal{C}_0$, which he called the integral curvature. The problem of prescribing 0-th curvature measure is often referred as the Alexandrov problem in literature. It was Alexandrov who formulated the problem through radial parametrization. The existence and uniqueness of solutions were obtained by A.D. Alexandrov [3]. It can be deduced to a Monge-Ampére type equation on $\mathbb{S}^n$. For $n = 2$ the regularity of solutions of the Alexandrov problem in the elliptic case was proved by Pogorelov [37] and for higher dimensional cases, it was solved by Oliker [35]. The general regularity results (degenerate case) of the problem were obtained in [20]. The problem of prescribing general k-th curvature measures was settled for starshaped hypersurfaces recently in [27], though C^0 and C^1 estimates were obtained in [19] some time ago. The proof of Lemma 3.4 presented here is due to Junfang Li (Li, private notes (2012)), which can apply to more general curvature equations. Another proof of gradient estimate for (26) appeared in [25], there the question of when solution to (26) is discussed.
2. The presentation of theory of hyperbolic polynomials in Appendix basically follows the original paper of Garding [14]. Caffarelli-Nirenberg-Spruck [5] developed the study of k-Hessian equation in the category of Γ_k, followed by [6] for k-curvature equation. The proof of Lemma 2.7 is from [30], which in turn is inspired by Marcus and Lopes [32]. Lemma 2.8 was proved in [27]. Using $\frac{\kappa}{u}$ in C^2 estimates for k-curvature equation on star-shaped hypersurfaces was introduced in [6]. The complication for (26) is that the right hand side depends on $\nabla\rho$, the standard concavity of $\sigma_k^{\frac{1}{k}}$ is not sufficient in this case. C^2 estimate is still open for k-curvature equation on star-shaped k-convex hypersurfaces with general right hand side

$$\sigma_k(\kappa) = f(\nabla\rho(x), \rho(x), x), \quad x \in \mathbb{S}^n.$$

   In a recent work [26] established C^2 estimates for admissible solutions of above equation in the case $k = 2$ and for convex solutions for general k.
3. The classical isoperimetric inequalities for quermassintegrals of convex bodies are the consequence of the Alexandrov-Fenchel inequality [1, 2] in convex geometry. Trudinger was the first to consider such inequalities for k-convex domains in [40]. Theorem 4.1 was proved in [17]. The proof in [17] used un-normalized inverse mean curvature type flow for starshaped hypersurfaces studied by Gerhardt [15] and Urbas [41], where they established longtime existence and exponential convergence for a class of more general type of inverse mean curvature flow. In Sect. 3, we use normalized flow (63), which was initially devised in (Guan and Li, private notes) when they did not realize that the work of [15,41] would imply the monotonicity of the isoperimetric ratio $\mathcal{I}_k$ in (88). Flow

(63) considered here has an advantage that one can see how to design a flow to fit the monotonicity. Similar design was used previously in conformal geometry in [22, 23]. Junfang Li pointed out that, one may also pick $r(t) \equiv \frac{1}{F(I)}$ in (63), as in a recent paper [18]. With this choice of r, the proof of C^0 estimates for flow (63) can be simplified. The monotonicity in Proposition 4.6 is reversed as

$$\int_M \sigma_k d\mu_g \text{ is monotonically non-increasing; } \int_M \sigma_{k-1} d\mu_g \text{ is a constant.}$$

It is an open question if (54) is valid without the starshapedness condition. In the case $k = 1$, Huisken [29] verified the inequality replacing the star shapedness by the assumption that $\partial\Omega$ is outward-minimizing. Again, in the case $k = 1$, (54) was proved for general 1-convex domains in [7] for some constant **c** which is a not sharp. Under additional condition that Ω is $k + 1$-convex (without starshapedness assumption), inequality (54) is proved in [8] with some no-sharp constant **c**.

4. The normalized inverse mean curvature flow

$$X_t = (\frac{1}{H} - \frac{u}{n})\nu \tag{92}$$

preserves the surface area and increases the enclosed volume. This implies the isoperimetric inequality for mean convex star-shaped domain. The statement can be checked as below.

$$\begin{aligned} \tfrac{d}{dt}\int_M d\mu_g &= \int_M (\frac{1}{H} - \frac{u}{n})H d\mu \\ &= \tfrac{1}{n}\int_M (n - uH) d\mu \\ &= 0. \end{aligned} \tag{93}$$

The evolution of the volume $V(t)$ is

$$\begin{aligned} \tfrac{d}{dt}V &= \int_M (\frac{1}{H} - \frac{u}{n}) d\mu \\ &= \int_M \frac{1}{H} d\mu - \frac{n+1}{n} V \\ &\geq 0. \end{aligned} \tag{94}$$

where the last inequality comes from an inequality proved by Ros in [39], see formula (5) on page 449.

5. The prescribing measure problem is a counter part of the Christoffel-Minkowski problem, which is the problem of prescribing area measures for convex bodies. The Minkowski problem was considered by Minkowski in [33] in 1897. The differential geometric setting of the problem was solved in early 1950s by

Nirenberg [34] and Pogorelov [36] for $n = 2$. The solution of the Minkowski problem in higher dimension came much later in 1970s by Cheng-Yau [9] and Pogorelov [38]. The Minkowski problem is a special case ($k = n$) of the problem of prescribing general k-th ($1 \leq k \leq n$) area measures in convex geometry. At the other end ($k = 1$), it is the Christoffel problem. This case has been settled completely by Firey [13]. In general, the problem of prescribing k-th is termed the Christoffel-Minkowski problem. It is equivalent to solve the following equation

$$\sigma_k(u_{ij} + u\delta_{ij}) = \varphi \quad \text{on} \quad \mathbb{S}^n, \tag{95}$$

with convexity requirement $(u_{ij} + u\delta_{ij}) > 0$.
The intermediate Christoffel-Minkowski problem ($1 < k < n$) is still open, except for some special cases. There are also some sufficient conditions, we refer to [38] and [21]. The necessary and sufficient condition for the existence of admissible solutions of (95) is known (e.g., [24]). The main difficulty lies in the question of convexity for the admissible solutions (which in general are not *convex*) of (95).

6. The Minkowski problem can also be considered as a problem of prescribing the Gauss curvature on outernormals of convex hypersurfaces. The similar question was raised for other Weingarten curvature functions $\sigma_k(\kappa_1, \cdots, \kappa_n)$ for fixed $1 \leq k \leq n$ in [4] and [10]. The corresponding equation is

$$\frac{\sigma_n}{\sigma_{n-k}}(u_{ij} + u\delta_{ij}) = f \quad \text{on} \quad \mathbb{S}^n. \tag{96}$$

When $1 \leq k < n$, very little is known for this problem. No uniqueness result is known except the case $n = 2$ (e.g., see [4]). If the prescribed curvature function is invariant under an automorphic group G without fixed points, the problem is solvable [16].

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