Preface

This volume contains the notes from the CIME school "Fully Nonlinear PDEs in Real and Complex Geometry and Optics" held at Cetraro (Cosenza, Italy) during the week of July 9–13, 2012. The school consisted of four courses: *Extremal problems for quasiconformal mappings in space* by Luca Capogna, *Fully nonlinear equations in geometry* by Pengfei Guan, *Monge-Ampère type equations and geometric optics* by Cristian E. Gutiérrez, and *On the Levi Monge–Ampère equation* by Annamaria Montanari.

The purpose of the school was to present current areas of research, arising both in the theoretical and in the applied settings, that involve fully nonlinear partial differential equations. The solution to these problems requires the development of ad hoc techniques arising in the relevant geometrical framework, and in case of the applications, determined by the physical laws governing the studied phenomena. Precisely, the equations presented in the school come from conformal mapping theory, differential geometry, optics, and geometric theory of several complex variables.

The following is a quick overview of the contents of each course.

Luca Capogna's lectures provided an introduction to two vector-valued extremal L^∞-variational problems involving mappings. The first one concerns a classical problem in geometric function theory that first arose in a work of Grotzsch from 1928: among all orientation preserving quasiconformal homeomorphisms $w : \Omega \to \Omega'$ whose traces agree with a given mapping $u_0 : \partial\Omega \to \partial\Omega'$, find one which minimizes the functional $u \to \left\| \frac{|du|}{(\det du)^{\frac{1}{n}}} \right\|_\infty$. Variants of this problem occur when, instead of using boundary data, the class of competitors is defined in terms of a fixed homotopy class or by requesting that the traces map quasi-symmetrically $\partial\Omega$ into $\partial\Omega'$. The second problem goes back to two papers from 1934, one by Whitney and the other by MacShane, and leading to the recent theory of *absolutely minimal Lipschitz extensions*. That is, let $\Omega \subset \mathbb{R}^n$ be open, $F \subset \overline{\Omega}$ be a compact set, and let $g \in Lip(F, \mathbb{R}^m)$. Among all Lipschitz extensions of g from F to Ω, is there a canonical unique extension that in some sense has the smallest possible Lipschitz norm? The natural questions arising in connection with these problems are existence, uniqueness, and structure of the minimizers. The last few decades have seen intense activity from different communities of mathematicians in the study of both problems. However, at this time there does not seem to be much synergy and communication between these communities, both in terms of shared techniques used in the study of these problems and in terms of common point of views. One of the goals of Capogna's notes is to foster such synergies by outlining some of the common features in these problems.

Pengfei Guan's lectures considered nonlinear elliptic and parabolic partial differential equations arising from geometric problems for hypersurfaces in $\mathbb{R}^{n+1}$. The notes are an introduction to geometric analysis. A curvature-type elliptic equation is used to solve the problem of prescribing curvature measures, which is a Minkowski-type problem. Curvature measures are defined using the Steiner formula. An inverse mean curvature-type parabolic equation is employed for the proof of isoperimetric-type inequalities for quermassintegrals of k-convex star-shaped domains. Both types of equations are fully nonlinear geometric PDEs. The emphasis of Guan's notes is on a priori estimates, a key step in the theory of fully nonlinear PDEs. The presentation is self-contained and requires basic knowledge of PDEs and geometry, namely the standard maximum principles for linear elliptic and parabolic equations, the elementary formulas of Gauss, Codazzi, and Weingarten for hypersurfaces in $\mathbb{R}^{n+1}$, and the curvature commutator identities. Two basic and deep results are used without proofs: the Evans–Krylov theorem for uniformly fully nonlinear elliptic equations and Krylov's theorem for uniformly parabolic fully nonlinear PDEs.

Gutiérrez's course presented an introduction to Monge–Ampère-type equations and its applications to geometric optics. In general, these equations involve the Jacobian determinant of a map and arise in the mathematical description of numerous optical, acoustic, and electromagnetic applications, in particular, in lens and reflector antenna design. The geometric optics problems considered concern refraction, reflection, or both. A typical refraction problem that was considered in the lectures is the following: suppose we have two homogenous media I and II with different refractive indices, a light beam emanates from a point O, surrounded by medium I, and we seek an interface surface separating media I and II described by $\{\rho(x)x : x \in \Omega\}, \Omega \subset \mathbb{S}^{n+1}$, and such that all rays are refracted into either a set of prescribed directions or illuminate a given object lying on a surface, say on a plane in medium II. The first type of problem is called far field and the second near field. These two problems are of different mathematical nature: the first one is variational and the second is not. The input and output intensities of radiation are prescribed and so the problem of finding the interface surface is a typical inverse problem. Gutiérrez's notes explain how to solve these problems: in the far field case using mass transport, and in the near field case using the Minkowski method. The physical background underlying these problems is explained using Maxwell equations, and, as a consequence, a deduction of Fresnel formulas is presented. These formulas are finally applied to model the case when there is loss of energy due to internal reflection.

Annamaria Montanari's course focused on the Levi Monge–Ampère equation. The equation is related to notions of curvatures associated with pseudoconvexity and the Levi form, in a way similar to how the classical Gauss and mean curvatures are related to the convexity and the Hessian matrix. Given a nonnegative function k, the Levi Monge–Ampère equation for the graph of a function $u : \mathbb{R}^{2n+1} \to \mathbb{R}$ is

$$\det \Lambda = k(x; u)(1 + |Du|^2)^{\frac{n+1}{2}},$$

where Λ is the Levi form of the graph of u and Du is the Euclidean gradient of u. More generally, in her notes, Montanari considers elementary symmetric functions of the eigenvalues of the Levi form Λ and shows that these curvature equations contain geometric information about hypersurface considered. Next, the notes show that the curvature operator leads to a new class of second order fully nonlinear equations whose characteristic form, when computed on generalized pseudoconvex functions, is nonnegative definite with a kernel of dimension one. Thus, the equations are not elliptic at any point. However, they have the following redeeming feature: the missing ellipticity direction can be recovered by suitable commutation relations. Using this property, existence, uniqueness, and regularity of viscosity solutions of the Dirichlet problem for graphs with prescribed Levi curvature are proved. Basic notions and results from the theory of functions of several complex variables, and from the theory of viscosity solutions for fully nonlinear degenerate elliptic equations, are also described in the notes.

It is a great pleasure to thank the speakers for their very interesting lectures and for the useful lectures notes they have prepared for this volume. We also want to thank all the participants of the school for their interest in these subjects.

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