Chapter 2
The Effective Action Formalism for Cosmological Perturbations

2.1 Introduction

The standard model of cosmology uses General Relativity (GR) to describe gravitational interactions, an homogeneous/isotropic FRW metric to describe the geometry and matter content of cold dark matter (CDM)/photons/baryons to describe its constituents. Observations of the cosmic microwave background, supernovae, baryon acoustic oscillations, gravitational lensing and structure formation point to the existence of an additional component dubbed "dark energy", or a modification to gravity, which needs to be introduced to explain the the observed acceleration [1–5].

The simplest explanation is a cosmological constant, Λ, and the standard paradigm is the ΛCDM model. However, there is still considerable flexibility for the explanation to be something radically different. In general, we can model all possible theories as an extra "dark sector" component to the stress-energy-momentum tensor. The structure of the gravitational field equations means that this extra component can be used to model either "exotic matter" with an equation of state $P/\rho < -\frac{1}{3}$ or a modification to GR (i.e. modifying exactly how gravity responds to the presence of matter). Constructing viable models of modified gravity has become an important task with the discovery of the acceleration of the Universe; some modified gravity models may also be able to account for observations which otherwise require dark matter.

One way to model the dark sector is "Lagrangian engineering": write down ever more complicated new theories with a view of constraining their parameters and free functions to fit observation with the hope that self-accelerating solutions can be found. Theories where explicit forms of dark energy are written down also fall into this category. They include TeVeS [6, 7], Einstein-æther [8], Brans-Dicke [9], Horndeski [10–12] and $F(R)$ gravities [13, 14], quintessence [15, 16], k-essence [17, 18] and Gallileons [19]. This is by no means an exhaustive list, and we have made no mention of the plethora of higher dimensional theories. The reader is directed to the recent extensive review of modified gravity theories [20].

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Given this proliferation of modified gravity and dark energy models, it would be a good idea to construct a generic way of parameterising deviations from the GR+ΛCDM picture and various suggestions have been made [21–33] to do this for perturbations. This approach is called the "Parameterized-Post-Friedmannian" (PPF) framework, in analogy to the well established Parameterized-Post-Newtonian (PPN) framework which was invented for Solar System tests of General Relativity [34]. However, as we describe below, to date no generic approach has been proposed which has a physical basis.

In this thesis we describe a new way of parameterizing perturbations in the dark sector requiring, as an assumption, knowledge of the field content. We do not assume a specific Lagrangian density, but we are able to model the possible effects on observations by using an effective action to compute the possible perturbations to the gravitational field equations. This is done by limiting the action to terms which are quadratic in the perturbed field content which is sufficient to model linearized perturbations, and assuming that the spatial sections are isotropic.

Our theories will be completely general allowing for all possible degrees of freedom. Initially we do not impose reparametrization, or gauge, invariance. This is something which we would expect of a fundamental theory of dark energy, but not necessarily one for which the field content is just a coarse grained description. We will find that this can lead to an phenomological vector degree of freedom, ξ^{μ}. In the elastic dark energy theory [35–38], which can be used to describe the effects of a dark energy component composed of a topological defect lattice, ξ^{μ} represents a perturbation of the elastic medium from its equilibrium. We will see that the imposition of reparametrization invariance substantially reduces the number of free functions.

We note that many authors have consider possible dark energy theories which are effective Lagrangians in the traditional sense, that is, the terms in the Lagrangian represent an expansion of field operators which are suppressed at low energies [39–41]. Our approach is sufficiently similar to this approach to share the epitaph "effective action", but it is completely different in many ways. It is completely classical and is in no sense an expansion in energy scales. Moreover, it is just an effective action for the perturbations, and in no sense represents the full field theory of the dark energy.

2.2 Approaches to Parameterizing Dark Sector Perturbations

In this section we will provide a brief review of current approaches to studying generalized gravitational theories, concluding with a short discussion on the generalities of our approach.

2.2.1 Parameterized Post-Friedmannian Approach

A popular way to parameterize the dark sector takes an "observational" perspective. One can modify the equations governing the predictions of the Newtonian gravitational potential Φ and shear σ by introducing extra functions space and time into the relevant equations and then parametrizing these extra functions in an *ad hoc* fashion. Since it is possible to explicitly observe Φ and σ via the evolution structure and gravitational shear [23, 28, 29, 31, 42] (see also the more recent papers [43–45]), one can then compare them with the predictions of particular *ad hoc* choice and determine constraints on the deviation of a particular parameter from its value in General Relativity.

One way of doing this is by modifying the Poisson and gravitational slip equations, introducing two scale—and time-dependent functions, $Q = Q(k, a)$ and $R = R(k, a)$; k is the wavenumber in a Fourier expansion and a the scale factor. The Poisson and gravitational-slip equations then become

$$k^2\Phi = -4\pi G Q a^2 \rho \Delta, \qquad \Psi - R\Phi = -12\pi G Q a^2 \rho(1+w)\sigma, \tag{2.1}$$

where $\Delta \equiv \delta + 3H\theta(1+w)$ is the comoving density perturbation, $\delta \equiv \delta\rho/\rho$ the density contrast, θ the velocity divergence field, $w = P/\rho$ the equation of state and σ is the anisotropic stress. When these equations are derived in GR one finds that $Q(k, a) = R(k, a) = 1$. So if, by comparison to data, either of these parameters are shown to be inconsistent with unity, then deviations from GR can be established. In [32, 46] it was shown that the two functions Q, R are not necessarily independent: they can be linked by the perturbed Bianchi identity, depending on the structure of the underlying theory.

2.2.2 Generalized Gravitational Field Equations

Another way to investigate the dark sector takes a more theoretical standpoint, and is based on a more consistent modification of the governing field equations. The method stems from the fact that any modified gravity theory or model of dark energy can be encapsulated by writing the *generalized gravitational field equations*

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + U_{\mu\nu}, \tag{2.2}$$

where $G_{\mu\nu}$ is the Einstein tensor calculated from the spacetime metric, $T_{\mu\nu}$ is the energy-momentum tensor of all *known* species (radiation, Baryons, CDM etc) and $U_{\mu\nu}$ is a tensor which contains all *unknown* contributions to the gravitational field equations, which we call the *dark energy-momentum tensor* [24, 25, 27].

Because the Bianchi identity automatically holds for the Einstein tensor, $\nabla_\mu G^{\mu\nu} = 0$, in the standard case where the *known* and *unknown* sectors are decoupled (that is $\nabla_\mu T^{\mu\nu} = 0$) we have the conservation law

$$\nabla_\mu U^{\mu\nu} = 0. \tag{2.3}$$

This represents a constraint equation on the extra parameters and functions that may appear in a parameterization of the dark sector at the level of the background.

At perturbed order, the parameterization of $\delta U^{\mu\nu}$ is constrained by the perturbed conservation law

$$\delta(\nabla_\mu U^{\mu\nu}) = 0. \tag{2.4}$$

The shortcoming of this approach is that one must supply the components of $\delta U^{\mu\nu}$. Skordis [27] does this by expanding the components $\delta U^{\mu}{}_{\nu}$ in terms of pseudo derivative operators acting upon gauge invariant combinations of metric perturbations, by imposing the principles that (a) the field equations remain at most second order and (b) the equations are gauge-form invariant. A particular form of these components were considered in [27]:

$$-a^2\delta U^0{}_0 = \frac{1}{a}\mathcal{A}\hat{\Phi}, \quad -a^2\delta U^0{}_i = \nabla_i\left(\frac{1}{a^2}\mathcal{B}\hat{\Phi}\right), \quad a^2\delta U^i{}_i = \mathcal{C}_1\hat{\Phi} + \mathcal{C}_2\dot{\hat{\Phi}} + \mathcal{C}_3\hat{\Psi}, \tag{2.5a}$$

$$a^2\left[\delta U^i{}_j - \frac{1}{3}\delta^i_j\delta U^k{}_k\right] = (\nabla^i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2)(\mathcal{D}_1\hat{\Phi} + \mathcal{D}_2\dot{\hat{\Phi}} + \mathcal{D}_3\hat{\Psi}), \tag{2.5b}$$

where $\mathcal{O} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}_i, \mathcal{D}_i\}$ is a set of pseudo differential operators and $\{\hat{\Phi}, \hat{\Psi}\}$ are gauge invariant combinations of perturbed metric variables. The possible form that the elements of $\mathcal{O}$ can take is constrained by the perturbed Bianchi identity. For instance, it was shown that $\mathcal{C}_3 = \mathcal{D}_3 = 0$ is one of the sufficient consistency relations. A generalized version of this method can be found in [20, 32].

This scheme provides a way to compute and constrain observables without ever having to write down an explicit theory for the dark sector. There appears to be, however, a weakness in the current formulation of this strategy: there does not seem to be a physically obvious way to interpret the $\mathcal{O}$; for example, if one were to find that $\mathcal{C}_3 = 0$ is "required" for consistency with observational data, what does that impose physically upon the system? It is exactly this issue we address in this paper.

2.2.3 *Effective Action Approach*

The generalized gravitational field Eq. (2.2) can be constructed from an action

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\left[R + 16\pi G\mathcal{L}_\mathrm{m} - 2\mathcal{L}_\mathrm{d}\right]. \tag{2.6}$$

The matter Lagrangian $\mathcal{L}_\mathrm{m}$ contains all known matter fields (e.g. baryons, photons) and is used to construct the known energy momentum tensor $T^{\mu\nu}$, and the dark sector Lagrangian $\mathcal{L}_\mathrm{d}$ contains all "unknown" contributions to the gravitational sector, and will be used to construct the dark energy momentum tensor $U^{\mu\nu}$. One can define

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}(\sqrt{-g}\mathcal{L}_\mathrm{m}), \qquad U^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}(\sqrt{-g}\mathcal{L}_\mathrm{d}). \tag{2.7}$$

The dark sector Lagrangian may contain known fields in an unknown configuration or extra fields, but of course we do not know *a priori* what the dark sector Lagrangian density is.

Two simple examples are (i) a slowly-rolling minimally coupled scalar field parameterized by a potential, $V(\phi)$, and (ii) a modified gravity model parameterized by a free function of the Ricci scalar, $F(R)$. There are restrictions on the form of both of these functions to achieve acceleration, but once they have been applied there is still considerable freedom in the choices of $V(\phi)$ and $F(R)$ and wide ranges of behaviour of the expansion history, $a(t)$, can be arranged for particular choices of the functions. One would expect this to be the case in any self consistent dark energy model compatible with FRW metric and therefore it might seem reasonable to make the assumption that the dark stress-energy-momentum tensor $U_{\mu\nu} = \rho u_\mu u_\nu + P\gamma_{\mu\nu}$ where $w(a) = P/\rho$ is in 1–1 correspondence with $a(t)$. The important question, which we are concerned with, is how to parametrize the perturbations $\delta U^\mu{}_\nu$ in a general way based on some general physical principle. In this way our approach is similar to that discussed in Sect. 2.2.2

The overall ethos which we advocate is to write down an effective action, inspired by the approach that is taken in particle physics (see, e.g. [47]) where, for example, the most general modifications to the standard model are written down for a given field content that are compatible with some assumed symmetry/symmetries. Then all the free coefficients are constrained by experiment. In our case, we will specify the field content of the dark sector, for example, scalar or vector fields, and write down a general quadratic Lagrangian density for the perturbed field variables which is sufficient to generate equations of motion for linearized perturbations. We will also make the assumption that the spatial sections are isotropic which substantially reduces the number of free coefficients. See Fig. 2.1 for a schematic of our philosophy.

2.3 Formalism

2.3.1 Second Order Lagrangian

The underlying principle behind our method is to write down a effective Lagrangian for perturbed field variables. If our theory is constructed from a set of field variables $\{X^{(\mathrm{A})}\}$, then we write each field variable as a linearized perturbation about some background value,

$$X^{(\mathrm{A})} = \bar{X}^{(\mathrm{A})} + \delta X^{(\mathrm{A})}. \tag{2.8}$$

The action for the perturbed field variables $\{\delta X^{(\mathrm{A})}\}$ is computed by integrating a Lagrangian which is quadratic in the perturbed field variables. If there are "N" perturbed field variables, the effective Lagrangian for the perturbed field variables is given by

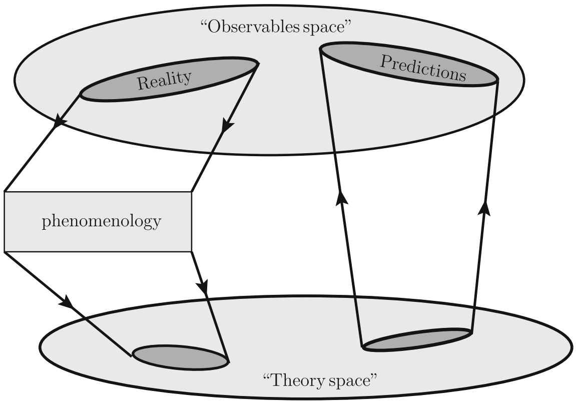

Fig. 2.1 Schematic depiction of our effective action approach. The "traditional" *modus operandi* is to pick a theory from "theory space" and obtain the possible observational predictions from that theory; this requires an educated guess as to what the theory that describes reality should be. The *modus operandi* of phenomenological field theory is to determine which points in theory space are consistent with observations of reality: this will automatically rule out theories which are incompatible with observations without ever needing to consider them. The focus of this thesis is to advocate the use of phenomenological field theory in the determination of the underlying gravitational theory to be used on cosmological scales

$$\mathcal{L}_{\{2\}}(\delta X^{(\mathrm{C})}) = \sum_{\mathrm{A}=1}^{\mathrm{N}} \sum_{\mathrm{B}=1}^{\mathrm{N}} \mathsf{G}_{\mathrm{AB}} \delta X^{(\mathrm{A})} \delta X^{(\mathrm{B})}, \tag{2.9}$$

where $\mathsf{G}_{\mathrm{AB}} = \mathsf{G}_{\mathrm{AB}}(\bar{X}^{(\mathrm{C})})$ is a set of arbitrary functions only depending on the background field variables; clearly, $\mathsf{G}_{\mathrm{AB}} = \mathsf{G}_{\mathrm{BA}}$. To obtain the equation of motion of the perturbed field variables $\{\delta X^{(\mathrm{A})}\}$ we must induce some variation in the $\{\delta X^{(\mathrm{A})}\}$ and subsequently demand that $\mathcal{L}_{\{2\}}$ is independent of these variations. If we vary the perturbed field variables with a variational operator $\hat{\delta}$,

$$\delta X^{(\mathrm{A})} \to \delta X^{(\mathrm{A})} + \hat{\delta}(\delta X^{(\mathrm{A})}), \tag{2.10}$$

then the effective Lagrangian will vary according to $\mathcal{L}_{\{2\}} \to \mathcal{L}_{\{2\}} + \hat{\delta}\mathcal{L}_{\{2\}}$, where

$$\hat{\delta}\mathcal{L}_{\{2\}} = 2 \sum_{\mathrm{A}=1}^{\mathrm{N}} \sum_{\mathrm{B}=1}^{\mathrm{N}} \mathsf{G}_{\mathrm{AB}} \delta X^{(\mathrm{A})} \hat{\delta}(\delta X^{(\mathrm{B})}). \tag{2.11}$$

The demand that the effective Lagrangian is independent of these variations is the statement that

$$\frac{\hat{\delta}}{\hat{\delta}(\delta X^{(\mathrm{B})})} \mathcal{L}_{\{2\}} = 0, \tag{2.12}$$

that is,

$$\sum_{\mathrm{A}=1}^{\mathrm{N}} \sum_{\mathrm{B}=1}^{\mathrm{N}} \mathsf{G}_{\mathrm{AB}} \delta X^{(\mathrm{A})} = 0. \tag{2.13}$$

These equations provide the equations of motion of the perturbed field variables. We will now show how to obtain the effective action for perturbations by directly perturbing the background action.

We will consider an action of the form

$$S = \int \mathrm{d}^4 x \sqrt{-g} \mathcal{L}, \tag{2.14}$$

where g is the determinant of the spacetime metric, $g_{\mu\nu}$, and $\mathcal{L}$ is the Lagrangian, which contains all fields in the theory. It will be useful to write the first and second variations of the action as

$$\delta S = \int \mathrm{d}^4 x \sqrt{-g} \Diamond \mathcal{L}, \qquad \delta^2 S = \int \mathrm{d}^4 x \sqrt{-g} \Diamond^2 \mathcal{L}, \tag{2.15}$$

where "$\Diamond$" is a useful measure-weighted pseudo-operator introduced in [48, 49], and is defined by

$$\Diamond^n \mathcal{L} \equiv \frac{1}{\sqrt{-g}} \delta^n (\sqrt{-g} \mathcal{L}). \tag{2.16}$$

We will only consider first perturbations of the field content of a theory. For the action (2.14) we can use the well known result

$$\frac{1}{\sqrt{-g}} \delta \sqrt{-g} = -\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} = +\frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu}, \tag{2.17}$$

to show that to quadratic order in the perturbations that the integrands in (2.15) are given by

$$\Diamond \mathcal{L} = \delta \mathcal{L} + \frac{1}{2} \mathcal{L} g^{\mu\nu} \delta g_{\mu\nu}, \tag{2.18a}$$

$$\Diamond^2 \mathcal{L} = \delta^2 \mathcal{L} + g^{\mu\nu} \delta g_{\mu\nu} \delta \mathcal{L} + \frac{1}{4} \mathcal{L} \left(g^{\mu\nu} g^{\alpha\beta} - 2 g^{\mu(\alpha} g^{\beta)\nu} \right) \delta g_{\mu\nu} \delta g_{\alpha\beta}. \tag{2.18b}$$

We treat the integrand of the second variation of the action, i.e. $\Diamond^2 \mathcal{L}$, as the effective Lagrangian, $\mathcal{L}_{\{2\}}$, for linearized perturbations,

$$\delta^2 S = \int \mathrm{d}^4 x \sqrt{-g} \Diamond^2 \mathcal{L} = \int \mathrm{d}^4 x \sqrt{-g}\, \mathcal{L}_{\{2\}} \tag{2.19}$$

It is called the *second order Lagrangian*. The final term of (2.18b) is an effective mass-term for the gravitational fluctuations $\delta g_{\mu\nu}$ which is always present even when the field which constitutes the dark sector does not vary, i.e. when $\delta\mathcal{L} = \delta^2\mathcal{L} = 0$.

Although we will be providing various explicit examples later on in the Thesis, we will briefly discuss how to write down $\mathcal{L}_{\{2\}}$ once the field content has been specified. If the field content is $\{X, Y\}$, then we write $\mathcal{L} = \mathcal{L}(X, Y)$, and then $\mathcal{L}_{\{2\}}$ is written down by writing all quadratic interactions of the perturbed fields with appropriate coefficients,

$$\mathcal{L}_{\{2\}} = A(t)\delta X\delta X + B(t)\delta X\delta Y + C(t)\delta Y\delta Y. \tag{2.20}$$

Notice that we have moved from having complete ignorance of how the fields X, Y combine to construct the Lagrangian density $\mathcal{L}$ to only requiring 3 "background" functions, $A(t), B(t), C(t)$ to be able to write $\mathcal{L}_{\{2\}}$ down. Typically, we would expect these functions to be specified in terms of the scale factor $a(t)$.

The theories we consider contribute to the gravitational field equations via the dark energy-momentum tensor, $U^{\mu\nu}$, which we define in the usual way, (2.7). The indices on the dark energy momentum tensor are symmetric by construction,

$$U_{\mu\nu} = U_{\nu\mu} = U_{(\mu\nu)}, \tag{2.21}$$

where tensor indices are symmetrised as $A_{(\mu\nu)} = \frac{1}{2}\big(A_{\mu\nu} + A_{\nu\mu}\big)$. The dark energy-momentum tensor above can be directly perturbed to give

$$\delta U^{\mu\nu} = -\frac{1}{2}\left[\sum_{\mathrm{A}}\left(\delta X^{(\mathrm{A})}\frac{1}{\sqrt{-g}}\frac{\delta}{\delta X^{(\mathrm{A})}}\frac{\delta}{\delta g_{\mu\nu}}(\sqrt{-g}\mathcal{L})\right) + U^{\mu\nu}g^{\alpha\beta}\delta g_{\alpha\beta}\right], \tag{2.22}$$

where $\{\delta X^{(\mathrm{A})}\}$ are the perturbed field variables. This can be written in a more succinct way by using the second order Lagrangian,

$$\delta U^{\mu\nu} = -\frac{1}{2}\left[4\frac{\partial(\mathcal{L}_{\{2\}})}{\partial(\delta g_{\mu\nu})} + U^{\mu\nu}g^{\alpha\beta}\delta g_{\alpha\beta}\right]. \tag{2.23}$$

Therefore, to obtain the gravitational contribution at perturbed order, due to our effective Lagrangian for perturbed field variables, one must compute the derivative of the second order Lagrangian with respect to the perturbed metric.

The equations of motion for a field X and its perturbation δX are found by regarding $\mathcal{L}$ and $\mathcal{L}_{\{2\}}$ as the relevant Lagrangian densities. Explicitly, the equations of motion for the field X and its perturbation, δX, are respectively given by

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)}\right) - \frac{\partial\mathcal{L}}{\partial X} = 0, \qquad \partial_\mu\left(\frac{\partial(\mathcal{L}_{\{2\}})}{\partial(\partial_\mu\delta X)}\right) - \frac{\partial(\mathcal{L}_{\{2\}})}{\partial\delta X} = 0. \tag{2.24}$$

The equations of motion governing the perturbation to the metric, $\delta g_{\mu\nu}$, are given by the perturbed gravitational field equations,

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} + \delta U_{\mu\nu}. \tag{2.25}$$

The perturbed conservation law for the dark energy-momentum tensor is

$$\delta(\nabla_\mu U^{\mu\nu}) = 0, \tag{2.26}$$

which can be written as

$$\nabla_\mu \delta U^{\mu\nu} + \frac{1}{2}\Big[U^{\mu\nu} g^{\alpha\beta} - U^{\alpha\beta} g^{\mu\nu} + 2 g^{\nu\beta} U^{\alpha\mu}\Big]\nabla_\mu \delta g_{\alpha\beta} = 0. \tag{2.27}$$

2.3.2 Isotropic $(3+1)$ *Decomposition*

One of the things it will be useful for us to do is to impose isotropy of spatial sections on the background spacetime. The motivation for doing this is that our goal is to study perturbations about an FRW background. After imposing isotropy we are able to use an isotropic $(3+1)$ decomposition to significantly simplify expressions. It is also possible to include anisotropic backgrounds as described in [37].

We foliate the 4D spacetime by 3D surfaces orthogonal to a time-like vector u_μ, which is normalized via

$$u^\mu u_\mu = -1. \tag{2.28a}$$

This induces an embedding of a 3D surface in a 4D space. The 4D metric is $g_{\mu\nu}$ and the 3D metric is $\gamma_{\mu\nu}$, and they are related by

$$\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu. \tag{2.28b}$$

The foliation implies that the time-like vector is orthogonal to the 3D metric,

$$u^\mu \gamma_{\mu\nu} = 0. \tag{2.28c}$$

The foliation induces a symmetric extrinsic curvature,

$$K_{\mu\nu} \equiv \nabla_\mu u_\nu, \tag{2.28d}$$

which is entirely spatial,

$$u^\mu K_{\mu\nu} = 0, \qquad K \equiv K^\mu{}_\mu = \gamma^{\mu\nu} K_{\mu\nu}. \tag{2.28e}$$

This can be used to deduce that

$$\nabla_{\mu}\gamma_{\alpha\beta} = 2K_{\mu(\alpha}u_{\beta)}, \qquad \nabla_{\mu}\gamma^{\mu\nu} = Ku^{\nu} \tag{2.28f}$$

A common application of the $(3+1)$ decomposition is to write down the only energy-momentum tensor compatible with the globally isotropic FRW metric,

$$T_{\mu\nu} = \rho u_{\mu}u_{\nu} + P\gamma_{\mu\nu}. \tag{2.29}$$

There are only two "coefficients" used in the decomposition of the energy-momentum tensor: the energy-density ρ and pressure P,

$$\rho = u^{\mu}u^{\nu}T_{\mu\nu}, \qquad P = \frac{1}{3}\gamma^{\mu\nu}T_{\mu\nu}. \tag{2.30}$$

Writing a tensor as a sum over combinations of u^{μ} and $\gamma_{\mu\nu}$ defines the isotropic $(3+1)$ decomposition. We will now show how to decompose tensors of higher rank. For example, an isotropic vector is completely decomposed as

$$A^{\mu} = Au^{\mu}, \tag{2.31}$$

where $A = A(t)$. Notice that before we imposed isotropy upon A^{μ} we would need 4 functions to specify all "free" components of A^{μ}; by imposing isotropy we have reduced the number of "free" functions from $4 \to 1$. A symmetric rank-2 isotropic tensor is completely decomposed as

$$B_{\mu\nu} = B_1 u_{\mu}u_{\nu} + B_2\gamma_{\mu\nu} = B_{\nu\mu}, \tag{2.32}$$

where $B_1 = B_1(t)$, $B_2 = B_2(t)$. The time-like part of $B_{\mu\nu}$ is B_1 and the space-like part is B_2. A rank-3 tensor symmetric in its second two indices is completely decomposed as

$$C_{\lambda\mu\nu} = C_1 u_{\lambda}\gamma_{\mu\nu} + C_2 u_{\lambda}u_{\mu}u_{\nu} + C_3\gamma_{\lambda(\mu}u_{\nu)} = C_{\lambda\nu\mu}. \tag{2.33}$$

This formalism can also be used to construct tensors which are entirely spatial. For example, a rank-4 tensor defined as

$$D_{\mu\nu\alpha\beta} = D_1\gamma_{\mu\nu}\gamma_{\alpha\beta} + D_2\gamma_{\mu(\alpha}\gamma_{\beta)\nu}, \tag{2.34}$$

is entirely spatial, a fact which is manifested by $u^{\mu}D_{\mu\nu\alpha\beta} = 0$, after one notes the symmetries in the indices $D_{\mu\nu\alpha\beta} = D_{(\mu\nu)(\alpha\beta)} = D_{\alpha\beta\mu\nu}$.

The coefficients which appear in an isotropic decomposition can only have time-like derivatives. For the coefficients B_1, B_2 in (2.32) we have

$$\nabla_{\mu}B_1 = -\dot{B}_1 u_{\mu}, \qquad \nabla_{\mu}B_2 = -\dot{B}_2 u_{\mu}, \tag{2.35}$$

where an overdot is used to denote differentiation in the direction of the time-like vector: $\dot{X} \equiv u^{\mu}\nabla_{\mu}X$.

2.3.3 Perturbation Theory

We will be making substantial use of perturbation theory in this paper, and so here we will take the time to concrete the notation and terminology we use. A large portion of the technology we are about to discuss was developed, amongst other things, to model relativistic elastic materials [35, 37, 50–60]; we will recapitulate the ideas and bring the technology into the language of perturbation theory to be used with a gravitational theory.

A quantity Q is perturbed about a background value, $\bar{Q}$, as $Q = \bar{Q} + \delta Q$. For example, the metric perturbed about a background $\bar{g}_{\mu\nu}$ is written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}. \tag{2.36}$$

It is important to realize that the operation of index raising and lowering does not commute with the variation. For example, $\delta g_{\mu\nu} = -g_{\mu(\alpha}g_{\beta)\nu}\delta g^{\alpha\beta}$ for the metric and $\delta(\nabla^{\mu}\phi) = g^{\mu\nu}\nabla_{\nu}\delta\phi + \delta g^{\mu\nu}\nabla_{\nu}\phi$ for the derivative of a scalar field ϕ.

Consider a quantity which is perturbed about some background value, $Q(t, \mathbf{x}) = \bar{Q}(t) + \delta Q(t, \mathbf{x})$. We can then employ two classes of coordinate system to follow the perturbation δQ through evolution; time evolution can be thought of as Lie-dragging a quantity along a time-like vector, u^{μ}, to "carve out" the world-line of the perturbation, i.e. operating on a quantity with $\pounds_u$. The first is where the density of the perturbations remains fixed (i.e. the coordinate system evolves to comove with the perturbations); this is a *Lagrangian* system. In the second, the coordinate system is fixed by some means (such as knowledge of the background geometry) and the density of the perturbations changes; this is an *Eulerian* system. We write perturbations in the Lagrangian system as δ_{L} and perturbations in the Eulerian system as δ_{E}. Evidently, a coordinate transformation can be used to transfer between the two systems, $x^{\mu} \rightarrow x^{\mu} + \xi^{\mu}$. The Eulerian and Lagrangian variations are linked by

$$\delta_{\mathrm{L}} = \delta_{\mathrm{E}} + \pounds_{\xi}, \tag{2.37}$$

where $\pounds_{\xi}$ is the Lie derivative along the gauge field ξ^{μ}. This setup is schematically depicted in Fig. 2.2.

We set the gauge field ξ^{μ} and time-like vector u_{μ} to be mutually orthogonal,

$$\xi^{\mu}u_{\mu} = 0. \tag{2.38}$$

This is because the time-like transformations which the component ξ^0 could induce are world-line preserving, and are redundant when u^{μ} is present (which is inherently

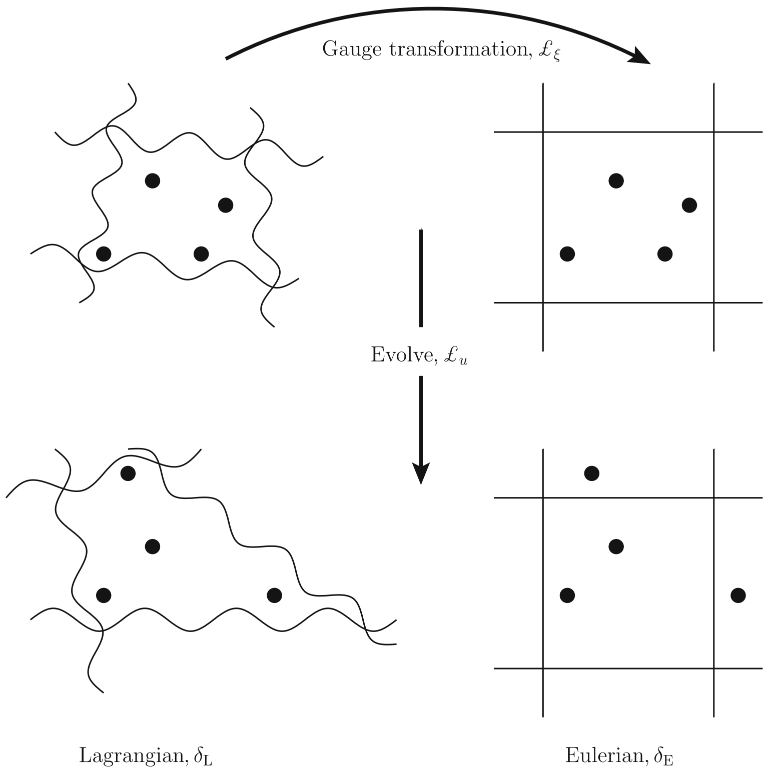

Fig. 2.2 Schematic view of the Eulerian and Lagrangian coordinate systems. The Lagrangian system can be said to be *comoving*, and the Eulerian system as being *fixed*. The Lagrangian system retains the density of a field, whereas the Eulerian system does not. This is schematically depicted by the "grid square" becoming deformed in the Lagrangian system on the *left*, to accommodate the movement of "particles" upon evolution in time. The grid square in the Eulerian system has remained fixed, meaning that the number of particles in a given square changes upon evolution. In cosmology we are perturbing against a *fixed* background: the FRW background, however calculations are often easier to perform in a comoving system. This means that physical relevance is taken from equations perturbed according to a Eulerian scheme

a world-line preserving evolution). See Fig. 2.3 for a schematic view illustrating this point. This is a choice which we discuss the ramifications of in Chap. 6.

There is an important question which arises: which perturbation scheme should we use to derive cosmologically relevant results, i.e. which δ should we use: δ_{E} or δ_{L}? In cosmological perturbation theory a quantity is perturbed from its value in a *fixed* (or known) background (such as its value in an FRW background). Therefore, equations should be perturbed relative to a fixed background, and so we should employ the Eulerian scheme.

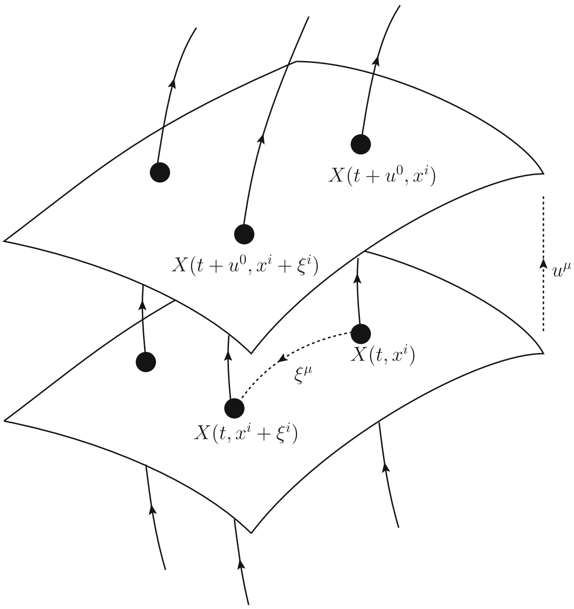

Fig. 2.3 Schematic view of the foliation and evolution, with three example world-lines drawn on, each piercing two 3D surfaces; u^μ is a time-like vector satisfying $u^\mu u_\mu = -1$. A quantity X on a surface with spacetime location (t, x^i) can be transformed into a quantity on the same surface but at a different location by transforming the coordinate on the surface, $x^i \to x^i + \xi^i$. This is a diffeomorphism which drags one world-line into another. If the time coordinate is transformed $t \to t + u^0$ then the quantity is evaluated on a different 3d surface, but on the same world-line. Thus, if we were to have a transformation $x^\mu \to x^\mu + \xi^\mu + \chi u^\mu$, where χ is an arbitrary scalar field, the time-like part of ξ^μ is redundant. Hence, we are free to set $\xi^\mu u_\mu = 0$, fixing the time-like part of the diffeomorphism field to be zero. So, we should have the interpretation that ξ^μ moves between world-lines and u^μ moves along world-lines

The equation of motion governing the metric perturbations is

$$\delta_{\rm E} G_{\mu\nu} = 8\pi G \delta_{\rm E} T_{\mu\nu} + \delta_{\rm E} U_{\mu\nu}, \tag{2.39}$$

and the perturbed conservation law that should be solved is the one evaluated in a Eulerian system,

$$\delta_{\rm E}\big(\nabla_\mu U^{\mu\nu}\big) = 0, \tag{2.40}$$

which can be written as

$$\nabla_\mu \delta_{\rm E} U^{\mu\nu} + \frac{1}{2}\Big[U^{\mu\nu} g^{\alpha\beta} - U^{\alpha\beta} g^{\mu\nu} + 2g^{\nu\beta} U^{\alpha\mu}\Big]\nabla_\mu \delta_{\rm E} g_{\alpha\beta} = 0. \tag{2.41}$$

If the Lagrangian variation of the dark energy-momentum tensor is the quantity that is supplied, (i.e. $\delta_{\rm L} U^{\mu\nu}$ is given), then one must be careful to use (2.37), to obtain the Eulerian perturbed quantity,

$$\delta_{\mathrm{E}} U^{\mu\nu} = \delta_{\mathrm{L}} U^{\mu\nu} - \xi^{\alpha} \nabla_{\alpha} U^{\mu\nu} + 2U^{\alpha(\mu} \nabla_{\alpha} \xi^{\nu)}. \tag{2.42}$$

Furthermore, to obtain the components of the mixed Eulerian perturbed dark energy-momentum tensor, one must use

$$\delta_{\mathrm{E}} U^{\mu}{}_{\nu} = g_{\alpha\nu} \delta_{\mathrm{E}} U^{\mu\alpha} + U^{\mu\alpha} \delta_{\mathrm{E}} g_{\nu\alpha}. \tag{2.43}$$

The Lagrangian and Eulerian perturbations of the metric are linked by

$$\delta_{\mathrm{E}} g_{\mu\nu} = \delta_{\mathrm{L}} g_{\mu\nu} - 2\nabla_{(\mu} \xi_{\nu)}. \tag{2.44}$$

For a vector field A^{μ} one finds that

$$\delta_{\mathrm{E}} A^{\mu} = \delta_{\mathrm{L}} A^{\mu} - \xi^{\alpha} \nabla_{\alpha} A^{\mu} + A^{\alpha} \nabla_{\alpha} \xi^{\mu}. \tag{2.45}$$

As final explicit example, the Eulerian and Lagrangian variations of a scalar field ϕ are linked via

$$\delta_{\mathrm{E}} \phi = \delta_{\mathrm{L}} \phi - \xi^{\mu} \nabla_{\mu} \phi. \tag{2.46}$$

An interesting lemma is that if $\nabla_{\mu}\phi \propto \dot{\phi} u_{\mu}$ then by (2.38) we find that the Eulerian and Lagrangian variations of a scalar field are identical, $\delta_{\mathrm{E}}\phi = \delta_{\mathrm{L}}\phi$. This means that a diffeomorphism does not change the perturbations of the scalar field; this is a consequence of the background field being homogeneous.

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