Preface

The problems we are going to study are all named: those after Kelvin, Boussinesq, and Cerruti, all three-dimensional, were solved explicitly during the second half of the nineteenth century (Cerruti's paper [7] appeared in 1882, long after the 1848 paper by Kelvin [18], the papers by Boussinesq [3, 4, 5, 6] between 1878 and 1892; in the same year 1892, a use of Boussinesq's solution in the guise of a Green kernel allowed for the solution of the Flamant Problem [9]). Two other problems in the same class were solved in the twentieth century: Melan's in 1932 [13]; Mindlin's, the three-dimensional version of Melan's, in 1936 and 1953 [14, 15].

All those problems were first dealt with by trying to solve the Navier equations for a displacement field having the representation constitutively implied by a tentative representation of the stress field in terms of a scalar potential. Such displacement field, in the absence of diffused volume loads, was the one induced by a *concentrated force* (or a *doublet*; see [14], Sect. 44 and 51–53; see also [4], Sect. 13.12).

When applied at an inner point, as in Kelvin's problem of a body occupying the whole space, a concentrated force was regarded as a special type of distance force. In modern terms, the Kelvin Problem can be rephrased as the problem of finding the Green function for the Navier equations in the whole space. A similar rephrasing fits both the Mindlin Problem [14, 15] and the Melan Problem [13], where a concentrated force is applied at an inner point of a half-space having dimension 3 and 2, respectively. Boussinesq, Cerruti, and Flamant considered instead concentrated forces applied to half-spaces, that is to say, they solved a boundary-value problem with a special assignment of tractions on the accessible part of the boundary. All these authors could not help to regard a concentrated force as an approximation: in the words of Love, the general problem under study was "[t]he problem… of the transmission into a solid body of force applied locally to a small part of its surface"; in short, "the problem of transmission of force" (see Love's "Historical Introduction", pp. 15 and 16 of [11]).

The original papers we quoted make an interesting and instructive reading, but generally not an easy one. Accounts at various levels of completeness and clarity are found in many textbooks, among which we mention, in addition to the quoted treatise by Love [11], those by Sokolnikoff [17], Malvern [12], Gladwell [10], Benvenuto [2], Davis and Selvadurai [8] (whose title inspired ours), Barber [1], and Sadd [16].

We propose a different approach where the basic equations of an elasticity problem—namely, balance, constitutive, and compatibility equations—are not combined into one partial differential equation for displacement, as Navier did, but instead used sequentially, in a heuristic fashion that is guided by the symmetries in the stress and displacement fields intrinsic to the problem at hand and justified a *posteriori* by appealing to the uniqueness theorem of classical linear elasticity.

Symmetries are the qualitative features of a problem that an educated intuition detects. When given an appropriate mathematical form, they dictate the choice of a priori representations for the unknown fields; such representations simplify substantially the solution process: think, for example, of the simplifications in solving St. Venant's problem induced by the a priori representation for the stress field he chose. In mechanics, the intuitive symmetries are those that the traction and displacement fields have, in duality, as a consequence of symmetries in the geometric, constitutive, and load, data; needless to say, symmetries in tractions entrain symmetries in stresses, while symmetries in strains are entrained by symmetries in displacements.

In this book, our discussion of each problem begins precisely with a careful examination of the prevailing symmetries. Here is the procedure we follow: we first look for a *general solution in terms* of stress of the balance equation; as a rule, what we find is a parametric family of solutions, among which we choose the only one turning out to be *geometrically compatible* when a linearly elastic and isotropic strain response to stress is postulated; accordingly, our choice is made by selecting the parameters so as to satisfy the *compatibility equation written in terms of stresses*. Next, we find the strain field from the stress field by a direct use of the constitutive equation; and, finally, we construct the displacement field by an explicit integration of the strain field.

It is important to realize that the strain and displacement fields we arrive at are those the applied loads induce in material bodies whose response to stress is modeled as it was almost invariably done in the nineteenth century, the golden age of classical elasticity. Now, the way most materials of geotechnical interest behave is far from being linearly elastic. Yet, it so happens that some, if not all, of the information embodied in the balanced stress states we determine do not depend on the material response, and hence play a direct and central role in designing, say, structure/foundation interactions.

In Chap. 1, our general plan of action is exemplified in an elementary one-dimensional case, where technical difficulties are minimal and yet the main conceptual ingredients are preserved. In higher dimensions, certain technical features are encountered that are different depending on whether a three- or two-dimensional version of the same problem is dealt with. This prompted us to begin by a preparatory Part I, consisting of two chapters. In Chap. 2, the basics of linear elasticity are reviewed, with special attention to the key features of classical plane elasticity. In Chap. 3, a quick account is given of curvilinear coordinates—a geometric tool that, when properly used, enhances the advantages of a preliminary inspection of symmetries—and of how differential operators are represented in non-Cartesian bases.

Part II, the bulk of this book, consists of three chapters, dedicated to, the Flamant, Boussinesq, and Kelvin Problems, respectively. In Chap. 4, the Flamant Problem is first dealt with in its simpler two-dimensional formulation; among other things, this leads to consideration of the Airy stress function. It is in this chapter that we individuate a number of significant examples of *concentrated contact interactions*, and we show how to give them an unambiguous mechanical status via the finite power expenditure they entrain. Chapter 5 begins with a discussion of a set of symmetry requirements for Boussinesq Problem that, although at first sight plausible, turn out to be geometrically inappropriate. It is then shown that, once the correct symmetries are stipulated, not only the solution is found, but also a number of related problems can be quickly solved by exploiting the superposition principle of linear elasticity, that is to say, by recognizing that the Boussinesq solution can serve as the Green function for those problems. In Chap. 6, we first show that solving Kelvin Problem by juxtaposition with seamless suture of two anti-mirror symmetric Boussinesq problems is possible only when the material is deemed incompressible; then, the two- and three-dimensional Kelvin problems are solved for compressible materials. In Chap. 7, the first of Part III, we tackle Melan's and Mindlin's problems (the former being the two-dimensional version of the latter) by the method of superposition and restriction introduced in Sect. 7.4 for their common one-dimensional version. Finally, in Chap. 8, we deal with a problem, Cerruti's, that has different symmetries and hence must be solved afresh.

Support information for various parts of the book are found in the final Appendix. In our intention, the matters surveyed there and in Part I can somehow clean up and complement the bag of knowledge of the audience we target, that is, *advanced undergraduate and graduate students in engineering and applied mathematics*. In consideration of the thorough discussion of physical motivations, the detailed presentation of heuristic arguments, and the unabridged treatment of the mathematical developments, we are convinced that teaching out of our book would be easy and rewarding. It seems to us that the book, although slim, is fairly well self-contained: the only prerequisites are a reasonable familiarity with linear algebra (in particular, manipulation of vectors and tensors) and with the usual differential operators of mathematical physics (gradient, divergence, curl, and laplacian); the few nonstandard notions are introduced with care.

Finally, to our knowledge, an equally exhaustive, compact, and consequential exposition of the classical problems listed in the subtitle is not found anywhere else. Thus, we hope our booklet will also serve as a reference for students in elasticity.

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