Chapter 2
Weak Self-Adjointness and Conservation Laws for a Family of Benjamin-Bona-Mahony-Burgers Equations

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Abstract Ibragimov introduced the concepts of self-adjoint and quasi-self-adjoint equations. Gandarias generalized these concepts and defined the concept of weak self-adjoint equations. In this paper we consider a family of Benjamin-Bona-Mahony-Burgers equations and we determine the subclass of equations which are self-adjoint, quasi-self-adjoint and weak self-adjoint. By using a general theorem on conservation laws proved by Ibragimov we obtain conservation laws for these equations.

Keywords Weak self-adjointness • Conservation laws

2.1 Introduction

Nonlinear PDEs that admit conservation laws arise in many disciplines of the applied sciences including physical chemistry, fluid mechanics, particle and quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magneto-hydro-dynamics, nonlinear optics, and the bio-sciences. Conservation laws are fundamental laws of physics. They maintain that a certain quantity, e.g. momentum, mass, or energy, will not change with time during physical processes.

In [16] (see also [15]) Ibragimov proved a general theorem on conservation laws for arbitrary differential equations which do not require the existence of Lagrangians. This new theorem is based on the concept of adjoint equations for nonlinear equations. There are many equations with physical significance which are not self-adjoint. Therefore one cannot eliminate the nonlocal variables from the conservation laws of these equations. Ibragimov in [15]) extended the

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concept of self-adjointness to quasi-self-adjointness. In [9] extended the concept of quasi-self-adjointness to weak-self-adjointness. Next, in [17] Ibragimov introduced a new concept: nonlinear self-adjointness.

Symmetry groups have several different applications in the context of nonlinear differential equations [3–5]. For example, they are used to obtain exact solutions and conservation laws of partial differential equations (PDEs) [8, 10]. The classical method for finding symmetry reductions of partial differential equations is the Lie group method [13, 18, 19]. The fundamental basis of this method is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. For PDEs with two independent variables a single group reduction transforms the PDE into an ordinary differential equation (ODE), which in general is easier to solve.

The Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$\Delta \equiv u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + (g(u))_x = 0, \tag{2.1}$$

where $u(x,t)$ represents the fluid velocity in the horizontal direction x, α is a positive constant, $\beta \in \mathrm{R}$ and $g(u)$ is a C^2-smooth nonlinear function appears in [11]. Equation (2.1) is the alternative regularized long-wave equation proposed by Peregrine [20] and Benjamin [2]. In [5, 6] we studied similarity reductions of the BBMB equation (2.1) and we derived a set of new solitons, kinks, antikinks, compactons, and Wadati solitons.

Wang et al. [21] introduced a method which is called the $\frac{G'}{G}$-expansion method to look for travelling wave solutions of nonlinear evolution equations. In [7] we found the functions $g(u) = u^m$ for which we can apply the $\frac{G'}{G}$-expansion method to (2.1). We obtained new travelling wave solutions which did not appear in [5, 6]. In [1] the $\frac{G'}{G}$-expansion method is used to establish travelling wave solutions for special form of the generalized (2.1) with $\alpha = 0$, $\beta = 1$, and $g(u) = \frac{u^2}{2}$. The solutions given in [1] were obtained by Bruzón and Gandarias in [7] and Kudryashov in [12].

The aim of this work is to determine, for (2.1), the subclasses of equations which are self-adjoint, quasi-self-adjoint, and weak self-adjoint. We also determine, by using the notation and techniques of the work [15, 16], some nontrivial conservation laws for (2.1). The paper is organized as follows. In Sect. 2.2 we determine the subclasses of equations of (2.1) which are self-adjoint, quasi-self-adjoint, and weak self-adjoint. In Sect. 2.3 we give the Lie symmetries of (2.1) equation obtained by Bruzón and Gandarias in [5–7]. In Sect. 2.4 we obtain some nontrivial conservation laws for (2.1). Finally, in Sect. 2.5 we give conclusions.

2.2 Determination of Self-Adjoint Equations

In [16] Ibragimov introduced a new theorem on conservation laws. The theorem is valid for any system of differential equations where the number of equations is equal to the number of dependent variables. The new theorem does not require existence of a Lagrangian and this theorem is based on a concept of an adjoint equation for nonlinear equations.

Definition 1. Consider an sth-order partial differential equation

$$F(x, u, u_{(1)}, \ldots, u_{(s)}) = 0 \tag{2.2}$$

with independent variables $x = (x^1, \ldots, x^n)$ and a dependent variable u, where $u_{(1)} = \{u_i\}$, $u_{(2)} = \{u_{ij}\}, \ldots$ denote the sets of the partial derivatives of the first, second, etc. orders, $u_i = \partial u/\partial x^i$, $u_{ij} = \partial^2 u/\partial x^i \partial x^j$. The adjoint equation to (2.2) is

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = 0, \tag{2.3}$$

with

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = \frac{\delta(vF)}{\delta u}, \tag{2.4}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}} \tag{2.5}$$

denotes the variational derivative (the Euler-Lagrange operator), and v is a new dependent variable. Here

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots$$

are the total differentiations.

Proposition 1. *Given the generalized BBMB equation (2.1), by applying definition (1), the adjoint equation to (2.1) is defined by*

$$F^* \equiv -\alpha u_{xx} - g_u u_x - \beta u_x + u_{txx} - u_t. \tag{2.6}$$

2.2.1 Weak Self-Adjoint Equations

We use the following definitions given in [15, 16].

Definition 2. Equation (2.2) is said to be **self-adjoint** if the equation obtained from the adjoint equation (2.3) by the substitution

$$v = u, \tag{2.7}$$

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)})$$

is identical to the original equation (2.2).

Definition 3. Equation (2.2) is said to be **quasi-self-adjoint** if the equation obtained from the adjoint equation (2.3) is equivalent to the original equation (2.2) upon the substitution

$$v = h(u), \tag{2.8}$$

with a certain function $h(u)$ such that $h'(u) \neq 0$.

And the following definition given in [9].

Definition 4. Equation (2.2) is said to be **weak self-adjoint** if the equation obtained from the adjoint equation (2.3) by the substitution

$$v = h(x, t, u), \tag{2.9}$$

such that $h_x(x, t, u) \neq 0$, $h_u(x, t, u) \neq 0$, is identical to the original equation, i.e.

$$F^* \mid_{v=h} = \lambda F. \tag{2.10}$$

Given the generalized BBMB equation (2.1) we apply definition (4). Taking into account the expression (2.6) and using (2.9) and its derivatives we rewrite (2.10)

$$\begin{aligned}
&u_t h_{uu} u_{xx} - \alpha h_u u_{xx} + h_{tu} u_{xx} - \alpha h_{xx} + u_t h_{uuu} u_x^2 - \alpha h_{uu} u_x^2 + h_{tuu} u_x^2 \\
&\quad -2\alpha h_{ux} u_x + 2u_t h_{uux} u_x + 2u_{tx} h_{uu} u_x - g_u h_u u_x - \beta h_u u_x + 2h_{tux} u_x \\
&\quad -g_u h_x - \beta h_x + u_t h_{uxx} + 2u_{tx} h_{ux} + u_{txx} h_u - u_t h_u + h_{txx} - h_t \\
&= \lambda(-\alpha\, u_{xx} + g_u\, u_x + \beta\, u_x - u_{txx} + u_t).
\end{aligned} \tag{2.11}$$

Comparing the coefficients for u_{txx}, we obtain $\lambda + h_u = 0$ and the following conditions must be satisfied:

$$\begin{aligned}
h_{uxx} &= 0, \\
h_{tu} - 2\,\alpha\, h_u &= 0, \\
2\, h_{tux} - 2\,\alpha\, h_{ux} &= 0, \\
h_{tuu} - \alpha\, h_{uu} &= 0, \\
h_{uuu} &= 0, \\
h_{ux} &= 0, \\
h_{uu} &= 0, \\
h_{uux} &= 0, \\
\alpha\, h_{xx} + g_u\, h_x + \beta\, h_x - h_{txx} + h_t &= 0.
\end{aligned} \tag{2.12}$$

Table 2.1 Weak self-adjoint equations (2.1)

Case_i	α	β	$g(u)$	h
1.	Arbitrary	Arbitrary	$k_3 - \dfrac{(k_2+\beta\,k_1)\,u}{k_1}$	$k_1\,x + k\,u + k_2\,t + k_3$
2.	0	Arbitrary	$k_3 - \dfrac{(k_2+\beta\,k_1)\,u}{k_1}$	$k_1\exp(-k\,t)$
3.	Arbitrary	Arbitrary	Arbitrary	C
4.	0	Arbitrary	Arbitrary	$k_1\,u + k_2$

Solving the system (2.12) we obtain that $h = k\,e^{2\alpha\,t}\,u + a(x,t)$ and α, β, $g(u)$ and $a(x,t)$ must satisfy the equation

$$2\,\alpha\,k\,e^{2\alpha\,t}\,u + a_x\,g_u + a_x\,\beta + \alpha\,a_{x\,x} - a_{t\,x\,x} + a_t = 0. \tag{2.13}$$

From (2.13) we obtain

- For $g(u) = k_3 - \dfrac{(k_2+\beta\,k_1)\,u}{k_1}$, with $k_1 \neq 0$ and α arbitrary constant
$$h = k_1\,x + k_2\,t + k_3.$$
- For $g(u) = k_3 - \dfrac{(k_2+\beta\,k_1)\,u}{k_1}$, with $k_1 \neq 0$ and $\alpha = 0$
$$h = k_1\,x + k\,u + k_2\,t + k_3.$$
- For α and β arbitrary constants and g arbitrary function
$$h = C, \quad \text{with C constant.}$$
- For $\alpha = 0$, β arbitrary constants and g arbitrary function
$$h = k_1\,u + k_2.$$

Consequently, we deduce that

Proposition 2. *Equation (2.1) is weak self-adjoint in cases given in Table 2.1. We remark that for $\alpha = 0$, β arbitrary constants and g arbitrary function equation (2.1) is self-adjoint. For α and β arbitrary constants and g arbitrary function (2.1) is quasi-self-adjoint with $h = C$.*

2.3 Classical Symmetries

To apply the Lie classical method to (2.1) we consider the one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$x^* = x + \epsilon\xi(x,t,u) + O(\epsilon^2), \tag{2.14}$$

$$t^* = t + \epsilon\tau(x,t,u) + O(\epsilon^2), \tag{2.15}$$

$$u^* = u + \epsilon\eta(x,t,u) + O(\epsilon^2), \tag{2.16}$$

where ϵ is the group parameter. We require that this transformation leaves invariant the set of solutions of (2.1). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$, and $\eta(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.17}$$

Having determined the infinitesimals, the symmetry variables are found by solving the characteristic equation which is equivalent to solving the invariant surface condition

$$\eta(x,t,u) - \xi(x,t,u)\frac{\partial u}{\partial x} - \tau(x,t,u)\frac{\partial u}{\partial t} = 0. \tag{2.18}$$

The set of solutions of (2.1) is invariant under the transformation (2.14)-(2.16) provided that

$$\mathrm{pr}^{(3)}\mathbf{v}(\Delta) = 0 \quad \text{when} \quad \Delta = 0,$$

where $\mathrm{pr}^{(3)}\mathbf{v}$ is the third prolongation of the vector field (2.17) given by

$$pr^{(3)}\mathbf{v} = \mathbf{v} + \sum_J \eta^J(x,t,u^{(3)})\frac{\partial}{\partial u_J}$$

where

$$\eta^J(x,t,u^{(3)}) = D_J(\eta - \xi u_x - \tau u_t) + \xi u_{Jx} + \eta u_{Jt},$$

with $J = (j_1,\ldots,j_k)$, $1 \le j_k \le 2$ y $1 \le k \le 3$. Hence we obtain the following ten determining equations for the infinitesimals:

$$\begin{aligned}
\tau_u &= 0,\\
\tau_x &= 0,\\
\xi_u &= 0,\\
\xi_t &= 0,\\
\eta_{uu} &= 0,\\
\alpha\tau_t + \eta_{tu} &= 0,\\
2\eta_{ux} - \xi_{xx} &= 0,\\
\eta_{uxx} - 2\xi_x &= 0,\\
\eta_x g_u - \alpha\eta_{xx} + \beta\eta_x - \eta_{txx} + \eta_t &= 0,\\
-\alpha\xi_{xx} - g_u\xi_x - \beta\xi_x - g_u\tau_t - \beta\tau_t - \eta g_{uu} + 2\alpha\eta_{ux} + 2\eta_{tux} &= 0.
\end{aligned} \tag{2.19}$$

From system (2.19) $\xi = \xi(x)$, $\tau = \tau(t)$ and $\eta = \gamma(x,t)u + \delta(x,t)$ where α, β, ξ, τ, γ, δ, and g satisfy

$$\begin{aligned} \gamma_t + \alpha\,\tau_t &= 0,\\ 2\,\gamma_x - \xi_{xx} &= 0,\\ \gamma_{xx} - 2\,\xi_x &= 0,\\ 2\,\alpha\,\gamma_x + 2\,\gamma_{tx} - g_{uu}\,u\,\gamma - \alpha\xi_{xx} - g_u\,\xi_x - \beta\xi_x - g_u\,\tau_t - \beta\,\tau_t - \delta\,g_{uu} &= 0,\\ -\alpha u\,\gamma_{xx} + g_u\,u\,\gamma_x + \beta\,u\,\gamma_x - u\,\gamma_{txx} + u\gamma_t + \delta_x\,g_u - \alpha\,\delta_{xx} + \beta\delta_x - \delta_{txx} + \delta_t &= 0. \end{aligned} \tag{2.20}$$

From (2.20) we obtain

$$\gamma = \frac{e^{-2x}}{8}\left((k_4 + 2\,k_3)\,e^{4\,x} + (4k_1 - 8\alpha\tau)\,e^{2x} - k_4 + 2k_3\right),$$
$$\xi = \frac{(k_4 + 2k_3)\;e^{2x}}{8} + \frac{(k_4 - 2k_3)\;e^{-2x}}{8} - \frac{k_4 - 4k_2}{4},$$

and α, β, τ, δ, and g are related by the following conditions:

$$\begin{aligned} &((g_u + \beta - 2\alpha)\,k_4 + (2g_u + 2\,\beta - 4\alpha)\,k_3)\,ue^{4x}\\ &\quad + \left(-4\alpha\tau_t u + \delta_x\,(4g_u + 4\beta) - 4\alpha\delta_{xx} - 4\delta_{txx} + 4\,\delta_t\right)\,e^{2\,x}\\ &\quad + ((g_u + \beta + 2\,\alpha)\;k_4 + (-2\,g_u - 2\,\beta - 4\,\alpha)\;k_3)\;u = 0, \end{aligned} \tag{2.21}$$

$$\begin{aligned} &((g_{uu}\,k_4 + 2\,g_{uu}\,k_3)\;u + (2\,g_u + 2\,\beta)\;k_4 + (4g_u + 4\beta)\,k_3)\,e^{4\,x}\\ &\quad + ((4g_{uu}k_1 - 8\alpha g_{uu}\tau)\,u + 8g_u\tau_t + 8\,\beta\,\tau_t + 8\,\delta g_{uu})\;e^{2\,x} + (2\,g_{uu}k_3 - g_{uu}k_4)\;u\\ &\quad + (-2g_u - 2\,\beta)\;k_4 + (4g_u + 4\,\beta)\;k_3 = 0. \end{aligned} \tag{2.22}$$

Solving system (2.21)-(2.22) we obtain that if g is an arbitrary function the only symmetries admitted by (2.1) are

$$\xi = k_1, \quad \tau = k_2, \quad \eta = 0. \tag{2.23}$$

The generators of this are $\mathbf{v}_1 = \dfrac{\partial}{\partial x}$ (corresponding to space translational invariance) and $\mathbf{v}_2 = \dfrac{\partial}{\partial t}$ (time translational invariance). In the following cases (2.1) has extra symmetries:

(i) If $\alpha = 0$, $g(u) = -\beta u + \frac{k}{a(n+1)}(au+b)^{n+1}$, $a \neq 0$,

$$\xi = k_1, \quad \tau = k_2 t + k_3, \quad \eta = -\frac{k_2}{an}(au+b).$$

Besides $\mathbf{v}_1$ and $\mathbf{v}_2$, we obtain the infinitesimal generator

$$\mathbf{v}_3 = t\partial_t - \frac{au+b}{an}\partial_u.$$

(ii) If $\alpha \neq 0$, $\beta \neq 0$ and $g(u) = au + b$,

$$\xi = k_1, \quad \tau = k_2, \quad \eta = \delta(x,t),$$

where δ satisfy

$$\alpha\delta_{xx} - g_u\delta_x - \beta\delta_x + \delta_{txx} - \delta_t = 0.$$

2.4 General Theorem on Conservation Laws

Much of the research on conservation laws centers around applications of Noether's theorem, which requires the existence of a Lagrangian. Anco and Bluman developed a procedure. The advantage of this procedure is that, in the Lagrangian case, it bypasses the actual formulation of the Lagrangian, and more importantly, it is applicable to non-Lagrangian systems.

Given a PDE (2.2) a conservation law for (2.2) is a relation of the form

$$\nabla \cdot \mathbf{C} = D_t(C^1) + D_x(C^2) = 0 \tag{2.24}$$

where $\mathbf{C} = (C^1, C^2)$ represents the conserved flux and density, respectively, and D_x, D_t denote the total derivative operators with respect to x and t, respectively. If (2.24) is a conservation law for (2.2), then it can be shown that there exists an operator λ such that

$$\nabla \cdot \mathbf{C} = \lambda(u)F$$

The operator λ is called the characteristic of the conservation law.

The conservation laws determined via Noether's theorem need to have a Lagrangian formulation. Noether's theorem connects conservation laws with variational symmetries with infinitesimal generators

We use the following theorem on conservation laws proved in [16]. Any Lie point, Lie-Bäcklund, or non-local symmetry

$$X = \xi^i(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial x^i} + \eta(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial u} \tag{2.25}$$

of Eq. (2.2) provides a conservation law $D_i(C^i) = 0$ for the simultaneous system (2.2), (2.3). The conserved vector is given by

$$C^i = \xi^i \mathcal{L} + W \left[\frac{\partial \mathcal{L}}{\partial u_i} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) - \cdots \right]$$
$$+ D_j(W) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) + \cdots \right] + D_j D_k(W) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}} - \cdots \right] + \cdots , \tag{2.26}$$

where W and $\mathcal{L}$ are defined as follows:

$$W = \eta - \xi^j u_j, \quad \mathcal{L} = v\, F\left(x, u, u_{(1)}, \ldots, u_{(s)}\right). \tag{2.27}$$

The proof is based on the following operator identity (N.H. Ibragimov, 1979):

$$X + D_i(\xi^i) = W \frac{\delta}{\delta u} + D_i \mathcal{N}^i, \tag{2.28}$$

where X is operator (2.25) taken in the prolonged form:

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i} + \zeta_{i_1 i_2} \frac{\partial}{\partial u_{i_1 i_2}} + \cdots ,$$
$$\zeta_i = D_i(\eta) - u_j D_i(\xi^j), \quad \zeta_{i_1 i_2} = D_{i_2}(\zeta_{i_1}) - u_{j i_1} D_{i_2}(\xi^j), \ \ldots .$$

For the expression of operator $\mathcal{N}^i$ and a discussion of the identity (2.28) in the general case of several dependent variables, see [14] (Sect. 8.4.4).

We will write the generators of a point transformation group admitted by (2.1) in the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}$$

by setting $t = x^1,\ x = x^2$. The conservation law will be written as (2.24)

Now we use the Ibragimov's Theorem on conservation laws to establish the conservation laws of (2.1). We have obtained that equation (2.1) is self-adjoint when it has the following form

$$u_t - u_{xxt} + \beta u_x + (g(u))_x = 0. \tag{2.29}$$

In this case, the formal Lagragian is

$$\mathcal{L} = v(u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + (g(u))_x).$$

For α and β arbitrary constants, $g(u)$ arbitrary function and $h = C$, (2.29) admits the generator $\mathbf{v}_1 + \mathbf{v}_2$. In this case we obtain trivial conservation laws.

Equation (2.29) admits the generator

$$\mathbf{v}_3 = t\partial t - \frac{1}{an}(au+b)\partial u,$$

and the normal form for this group is

$$W = -\frac{1}{an}(au+b) - t\, u_t.$$

The vector components are

$$\begin{aligned}
C^1 &= \frac{tu_t v_{xx}}{3} + \frac{uv_{xx}}{3n} + \frac{bv_{xx}}{3an} - \frac{u_x v_x}{3n} - \frac{tu_{tx} v_x}{3} + \frac{u_{xx} v}{3n} \\
&\quad + kt\,(au+b)^n\, u_x v - \frac{2tu_{txx} v}{3} - \frac{uv}{n} - \frac{bv}{an} \\
C^2 &= -\frac{tu_{tt} v_x}{3} - \frac{u_t v_x}{3n} - \frac{u_t v_x}{3} + \frac{2tu_t v_{tx}}{3} + \frac{2uv_{tx}}{3n} \\
&\quad + \frac{2bv_{tx}}{3an} - \frac{u_x v_t}{3n} - \frac{tu_{tx} v_t}{3} + \frac{2tu_{ttx} v}{3} + \frac{2u_{tx} v}{3n} \\
&\quad + \frac{2u_{tx} v}{3} - kt\,(au+b)^n\, u_t v - \frac{ku\,(au+b)^n\, v}{n} \\
&\quad - \frac{bk\,(au+b)^n\, v}{an}
\end{aligned} \tag{2.30}$$

Setting $v = u$ in (2.30)

$$\begin{aligned}
C^1 &= \frac{tu_t u_{xx}}{3} + \frac{2uu_{xx}}{3n} + \frac{bu_{xx}}{3an} - \frac{(u_x)^2}{3n} - \frac{tu_{tx} u_x}{3} \\
&\quad + ktu\,(au+b)^n\, u_x - \frac{2tuu_{txx}}{3} - \frac{u^2}{n} - \frac{bu}{an}, \\
C^2 &= -\frac{tu_{tt} u_x}{3} - \frac{2u_t u_x}{3n} - \frac{u_t u_x}{3} + \frac{2tuu_{ttx}}{3} + \frac{tu_t u_{tx}}{3} \\
&\quad + \frac{4uu_{tx}}{3n} + \frac{2uu_{tx}}{3} + \frac{2bu_{tx}}{3an} - ktu\,(au+b)^n\, u_t \\
&\quad - \frac{ku^2\,(au+b)^n}{n} - \frac{bku\,(au+b)^n}{an}.
\end{aligned} \tag{2.31}$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from C^1 to C^2 and obtain

$$\begin{aligned}
C^1 &= -\frac{(u_x)^2}{n} - \frac{u\,(a\,u+b)}{a\,n} \\
C^2 &= \frac{(2au+b)\,u_{tx}}{an} - \frac{k\,(au+b)^{n+1}\,(2a(n+1)u+bn)}{a^2 n\,(n+1)\,(n+2)}
\end{aligned} \tag{2.32}$$

For $g(u) = k_3 - \dfrac{(k_2 + \beta\, k_1)\, u}{k_1}$, with $k_1 \neq 0$, α arbitrary constant and $h = k_1\, x + k_2\, t + k_3$ (2.29) admits the generator $\mathbf{v}_1 + \mathbf{v}_2$. In this case, we do as before and we obtain

$$\begin{aligned} C^1 &= ({k_2}^2 + {k_1}^2)\, u \\ C^2 &= -\alpha\, ({k_2}^2 + {k_1}^2)\, u_x - ({k_2}^2 + {k_1}^2)\, u_{t\,x} - k_2 \left(\frac{{k_2}^2}{k_1} + k_1 \right) u \end{aligned} \tag{2.33}$$

For $g(u) = k_3 - \dfrac{(k_2 + \beta\, k_1)\, u}{k_1}$, with $k_1 \neq 0$ and $\alpha = 0$ and $h = k_1\, x + k\, u + k_2\, t + k_3$ (2.29) admits the generator $\mathbf{v}_1 + \mathbf{v}_2$. In this case, we proceed as before and we obtain the conservation law (2.33) with $\alpha = 0$.

2.5 Conclusions

In this work we have considered a generalized Benjamin-Bona-Mahony-Burgers equation (2.1). We have determined the subclasses of equations (2.1) which are self-adjoint, quasi-self-adjoint, and weak self-adjoint. By using a general theorem on conservation laws proved by Nail Ibragimov we found conservation laws for some of these partial differential equations without classical Lagrangians.

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