Chapter 2
Banach Contraction Principle and Its Generalizations

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2.1 Introduction

In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the "Banach Contraction Principle" (BCP) which is one of the most important results of analysis and considered as the main source of metric fixed point theory. It is the most widely applied fixed point result in many branches of mathematics because it requires the structure of complete metric space with contractive condition on the map which is easy to test in this setting. The BCP has been generalized in many different directions. In fact, there is vast amount of literature dealing with extensions/generalizations of this remarkable theorem. In this chapter, it is impossible to cover all the known extensions / generalizations of the BCP. However, an attempt is made to present some extensions of the BCP in which the conclusion is obtained under mild modified conditions and which play important role in the development of metric fixed point theory.

2.2 Contractions: Definition and Examples

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Definition 2.1. Let (X,d) be a metric space and let $f : X \to X$ be a mapping.

(a) A point $x \in X$ is called a *fixed point* of f if $x = f(x)$.
(b) f is called *contraction* if there exists a fixed constant $h < 1$ such that

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$$d(f(x),f(y)) \leq hd(x,y), \quad \text{for all } x,y \in X. \tag{2.1}$$

A contraction mapping is also known as Banach contraction. If we replace the inequality (2.1) with strict inequality and $h = 1$, then f is called *contractive* (or *strict contractive*). If (2.1) holds for $h = 1$, then f is called *nonexpansive*; and if (2.1) holds for fixed $h < \infty$, then f is called *Lipschitz continuous* . Clearly, for the mapping f, the following obvious implications hold:

contraction $\Rightarrow$ contractive $\Rightarrow$ nonexpansive $\Rightarrow$ Lipschitz continuous

Example 2.1. (a) Consider the usual metric space $(\mathbb{R},d)$. Define

$$f(x) = \frac{x}{a} + b, \quad \text{for all } x \in \mathbb{R}.$$

Then, f is contraction on $\mathbb{R}$ if $a > 1$ and the solution of the equation $x - f(x) = 0$ is $x = \frac{ab}{a-1}$.

(b) Consider the Euclidean metric space $(\mathbb{R}^2,d)$. Define

$$f(x,y) = \left(\frac{x}{a} + b, \frac{y}{c} + b\right), \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Then, f is contraction on $\mathbb{R}^2$ if $a,c > 1$. Now, solving the equation $f(x,y) = (x,y)$ for a fixed point, we get $x = \frac{ab}{a-1}$ and $y = \frac{cb}{c-1}$.

Using induction, one can easily get the following concerning iterates of a contraction mapping.

If f is a contraction mapping on a metric space (X,d) with contraction constant h, then f^n (where the superscript represents the nth iterate of f) is also a contraction on X with constant h^n.

Consider then the situation in which $f : X \to X$ is not necessarily a contraction mapping, but f^n is a contraction for some n.

Example 2.2. (a) Let $f : [0,2] \to [0,2]$ be defined by

$$f(x) = \begin{cases} 0, & x \in [0,1], \\ 1, & x \in (1,2]. \end{cases}$$

Then, $f^2(x) = 0$ for all $x \in [0,2]$, and so, f^2 is a contraction on $[0,2]$. Note that f is not continuous and thus not a contraction map.

(b) Define $f(x) = \cos x$ on $\mathbb{R}$. Then, f is not a contraction on $\mathbb{R}$. Indeed, suppose there exists $h \in (0,1)$ such that

$$\left|\frac{\cos x - \cos y}{x - y}\right| \leq h, \quad \text{for all } x \neq y.$$

Letting $y \to x$, we get $|\sin x| \le h$ for all x, which is false.
Note that the iterated function $f^2(x) = \cos(\cos x)$ satisfies

$$\left|\frac{d}{dx}(\cos(\cos x))\right| = |\sin(\cos x)\sin(x)| < \sin 1 < 1,$$

and thus, by the mean-value theorem, f^2 is a contraction on $\mathbb{R}$.

(c) Define $f(x) = \exp(-x)$ on $\mathbb{R}$. Then, f is not a contraction on $\mathbb{R}$. But, $f^2(x) = \exp(-\exp(-x))$ is a contraction on $\mathbb{R}$. Indeed,

$$\left|\frac{d}{dx}(\exp(-\exp(-x)))\right| = |\exp(-x-\exp(-x))| \le e^{-1} < 1,$$

and thus, the conclusion follows by the mean-value theorem.

For non-contractions, there are examples where f has a unique fixed point but an iterate of f does not.

Example 2.3. Define

$$f(x) = 1 - x, \quad \text{for all } x \in \mathbb{R}.$$

Then, f is not a contraction, has a unique fixed point, but note that

$$f^2(x) = x, \quad \text{for all } x \in \mathbb{R}$$

is rich with fixed points.

2.3 The Banach Contraction Principle with Some Applications

In this section, we will discuss the most basic fixed point theorem in analysis, known as the Banach Contraction Principle (BCP). It is due to Banach [11] and appeared in his Ph.D. thesis (1920, published in 1922). The BCP was first stated and proved by Banach for the contraction maps in the setting of complete normed linear spaces. At about the same time the concept of an abstract metric space was introduced by Hausdorff, which then provided the general framework for the principle for contraction mappings in a complete metric space. The BCP can be applied to mappings which are differentiable, or more generally, Lipschitz continuous. A number of articles with applications on the topic can also be found in [1, 7, 8, 29, 34, 46, 103].

Theorem 2.1 (Banach Contraction Principle). *Let (X,d) be a complete metric space, then each contraction map $f : X \to X$ has a unique fixed point.*

Proof. Let h be a contraction constant of the mapping f. We will explicitly construct a sequence converging to the fixed point. Let x_0 be an arbitrary but fixed element in X. Define a sequence of iterates $\{x_n\}$ in X by

$$x_n = f(x_{n-1}) \ \ (= f^n(x_0)), \quad \text{for all } n \geq 1. \tag{2.2}$$

Since f is a contraction, we have

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq hd(x_{n-1}, x_n), \quad \text{for any } n \geq 1.$$

Thus, we obtain

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1), \quad \text{for all } n \geq 1.$$

Hence, for any $m > n$, we have

$$d(x_n, x_m) \leq \left(h^n + h^{n+1} + \cdots + h^{m-1}\right) d(x_0, x_1) \leq \frac{h^n}{1-h} d(x_0, x_1).$$

We deduce that $\{x_n\}$ is Cauchy sequence in a complete space X. Let $x_n \to p \in X$. Now using the continuity of the map f, we get

$$p = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(p).$$

Finally, to show f has at most one fixed point in X, let p and q be fixed points of f. Then,

$$d(p, q) = d(f(p), f(q)) \leq hd(p, q).$$

Since $h < 1$, we must have $p = q$. □

The proof of the BCP yields the following useful information about the rate of convergence towards the fixed point.

Corollary 2.1. *Let f be a contraction mapping on a complete metric space (X, d) with contraction constant h and fixed point p. For any $x_0 \in X$, with f-iterates $\{x_n\}$, we have the estimates*

$$d(x_n, p) \leq \frac{h^n}{1-h} d(x_0, f(x_0)), \tag{2.3}$$

$$d(x_n, p) \leq hd(x_n, p), \tag{2.4}$$

$$d(x_n, p) \leq \frac{h}{1-h} d(x_{n-1}, x_n). \tag{2.5}$$

In fact, the three inequalities in Corollary 2.1 serve different purposes. The inequality (2.3) tells us, in terms of the distance between x_0 and $f(x_0) = x_1$, how many times we need to iterate f starting from x_0 to be certain that we are within a specified distance from the fixed point. This is an upper bound on how long we need to compute. It is called an a priori estimate. Inequality (2.4) shows that once we find a term by iteration within some desired distance of the fixed point, all further iterates will be within that distance. However, (2.4) is not so useful as an error estimate since both sides of (2.4) involve the unknown fixed point. The inequality (2.5) tells us, after each computation, how much closer we are to the fixed point in terms of the previous two iterations. This kind of estimate, called an a posteriori estimate, is very important because if two successive iterations are nearly equal, (2.5) guarantees that we are very close to the fixed point.

Corollary 2.2. *Let $f : X \to X$ be a contraction mapping on a complete metric space and $M \subseteq X$ be a closed subset such that $f(M) \subseteq M$. Then, the unique fixed point of f is in M.*

It may be the case that $f : X \to X$ is not a contraction on the whole space X, but rather a contraction on some neighborhood of a given point. In this case we have the following result:

Theorem 2.2. *Let (X,d) be a complete metric space and let $B_r(y) = \{x \in X : d(x,y) < r\}$, where $y \in X$ and $r > 0$. Let $f : B_r(y) \to X$ be a contraction map with contraction constant $h < 1$. Further, assume that*

$$d(y, f(y)) < r(1-h).$$

Then, f has a unique fixed point in $B_r(y)$.

Proof. Note that the uniform continuity of f allows us to extend f to a mapping defined on $\overline{B_r(y)}$ which is a contraction map having the same Lipschitz constant as the original map. We show that $f(\overline{B_r(y)}) \subseteq B_r(y)$. Let $x \in \overline{B_r(y)}$, then

$$d(y, f(x)) \le d(y, f(y)) + d(f(y), f(x)) < r(1-h) + hr = r,$$

and hence, $f : \overline{B_r(y)} \to B_r(y)$. Since $\overline{B_r(y)}$ is a complete metric space, using Theorem 2.1, f has a unique fixed point $p \in \overline{B_r(y)}$. Thus, $p \in B_r(y)$ because $p = f(p) \in B_r(y)$. □

Remark 2.1. If f is a contraction map on a complete metric space (X,d) with contraction constant h, then f^n is also a contraction on X with constant h^n and the unique fixed point of f is also the unique fixed point of any f^n.

We observed in Example 2.2 that in some situations a function is not a contraction but its iterate is a contraction map. To get the conclusion of the BCP for the original function, the following early example of an extension of BCP is due Caccioppoli [18].

Theorem 2.3. *Let (X,d) be a complete metric space and let $f : X \to X$ be a mapping such that for each $n \ge 1$, there exists a constant c_n such that*

$$d(f^n(x), f^n(y)) \leq c_n d(x,y), \quad \textit{for all } x,y \in X,$$

where $\sum_{n=1}^{\infty} c_n < \infty$. *Then, f has a unique fixed point.*

While, Bryant [17] extended BCP as follows.

Theorem 2.4. *Let* (X,d) *be a complete metric space and let* $f : X \to X$ *be a mapping such that for some positive integer n,* f^n *is contraction on* X. *Then, f has a unique fixed point.*

Proof. By BCP, f^n has a unique fixed point, say $x \in X$ with $f^n(x) = x$. Since

$$f^{n+1}(x) = f(f^n(x)) = f(x),$$

it follows that $f(x)$ is a fixed point of f^n, and thus, by the uniqueness of x, we have $f(x) = x$, that is, f has a fixed point. Since, the fixed point of f is necessarily a fixed point of f^n, so is unique. □

Remark 2.2. (a) Theorem 2.4 has importance in the scene that the mapping f is not even assumed to be continuous while the same result was appeared in the literature with continuity assumption on the mapping f, see, for example, [15, 30].
(b) In some applications, it is the case that the mapping f is a Lipschitz which is not necessarily a contraction, whereas some power of f is a contraction mapping.

Example 2.4. Consider the metric space $X = C[a,b]$, of continuous real-valued functions defined on the compact interval $[a,b]$. This is a Banach space with respect to the sup norm. Define $f : X \to X$ by

$$f(u)(t) = \int_a^t u(s)ds.$$

Then,

$$|f(u) - f(v)| \leq (b-a)\,|u-v|\,.$$

Now, we compute

$$f^2(u)(t) = \int_a^t (t-s)u(s)ds,$$

and inductively,

$$f^n(u)(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}u(s)ds.$$

Thus, we get

$$|f^n(u) - f^n(v)| \leq \frac{(b-a)^n}{n!}\,|u-v|\,.$$

Hence, f^n is a contraction map for all values of n for which $\frac{(b-a)^n}{n!} < 1$. It follows that f has the unique fixed point $u = 0$.

Remark 2.3. It was important in the proof of BCP that the contraction constant h be strictly less than 1. That gave us control over the rate of convergence of $f^n(x_0)$ to the fixed point since $h^n \to 0$ as $n \to \infty$. If we consider f is contractive mapping instead of a contraction, then we lose that control and indeed a fixed point need not exist.

Example 2.5. Let I be a closed interval in $\mathbb{R}$ and $f : I \to I$ be differentiable with $|f'(t)| < 1$ for all t. Then, the mean-value theorem implies $|f(x) - f(y)| < |x - y|$ for all $x \neq y$ in I. The following three functions satisfy this condition, where $I = [1, \infty)$ in the first case and $I = \mathbb{R}$ in the second and third cases:

$$f(x) = x + \frac{1}{x}; \;\; f(x) = \sqrt{x^2+1}; \;\; f(x) = \log(1 + exp(x)).$$

In each case, $f(x) > x$, so none of these functions has a fixed point in I.

Despite such examples, there is a fixed point theorem of Edelstein [30] when $d(f(x), f(y)) < d(x,y)$ for all $x \neq y$ provided the space is compact, which is not the case in the previous example.

Similarly, a nonexpansive mapping on a complete metric space need not has a fixed point. For instance, consider the translation operator by a nonzero vector in a Banach space, which is clearly a nonexpansive fixed point free mapping. On the other hand, a fixed point of a nonexpansive map need not be unique. For instance, consider an identity operator, which is obviously nonexpansive and rich with fixed points. Thus the fixed point theory of nonexpansive mappings is fundamentally different from that of the contraction mapping, and thus we shall not discuss in this chapter.

Remark 2.4. It is worth to mention that on the one hand, the BCP is very forceful and simple, and it became a classical tool in nonlinear analysis. But, on the other hand, Connell [26] gave an example of a metric space X such that X is not complete and every contraction on X has a fixed point. Thus, Theorem 2.1 cannot characterize the metric completeness of X which means the notion of contractions is too strong from this point of view.

A mapping f on a metric space (X,d) is called *Kannan mapping* if there exists $h \in [0, \frac{1}{2})$ such that

$$d(f(x), f(y)) \leq h \; \{d(x, Tx) + d(y, Ty)\}, \quad \text{for all } x, y \in X.$$

Kannan [44] proved that if X is complete, then every Kannan mapping has a fixed point. We note that Kannan fixed point theorem is not an extension of the BCP. In our opinion, Kannan theorem is also very important because Subrahmanyam [85] proved that Kannan theorem characterizes the metric completeness. That is, a metric space X is complete if and only if every Kannan mapping on X has a fixed point.

2.4 Some Other Extensions of BCP for Single-Valued Mappings

There have been numerous extensions of a milder form of BCP. In this section we present some of these.

The first generalization in this direction which received a significant importance is the following result of Rakotch [76].

Theorem 2.5. *Let (X,d) be a complete metric space, and suppose that $f : X \to X$ satisfies*

$$d(f(x), f(y)) \leq \eta(d(x,y))d(x,y), \quad \textit{for all } x, y \in X,$$

where η is a decreasing function on $\mathbb{R}^+$ to $[0,1)$. Then, f has a unique fixed point.

A variant of Rokotch's theorem has been given by Geraghty [33], in which the function η satisfies the simple condition that $\eta(t_n) \to 1 \Rightarrow t_n \to 0$.

In [14], Boyd and Wong obtained a more general result as follows.

Theorem 2.6 (Boyd–Wong Theorem). *Let (X,d) be a complete metric space, and suppose that $f : X \to X$ satisfies*

$$d(f(x), f(y)) \leq \psi(d(x,y)), \quad \textit{for all } x, y \in X,$$

where $\psi : \mathbb{R} \to [0,\infty)$ is upper semicontinuous from the right (that is, for any sequence $t_n \downarrow t \geq 0 \Rightarrow \limsup_{n\to\infty} \psi(t_n) \leq \psi(t)$) and satisfies $0 \leq \psi(t) < t$ for $t > 0$. Then, f has a unique fixed point.

Proof. Let x_0 be an arbitrary but fixed element in X. Define a sequence of iterates $\{x_n\}$ in X by

$$x_n = f(x_{n-1}) \ \ (= f^n(x_0)), \quad \text{for all } n \geq 1.$$

Set $a_n = d(x_{n-1}, x_n)$. Note that the sequence $\{a_n\}$ is monotone decreasing and bounded below, thus it is convergent and we let $\lim_{n\to\infty} a_n = a$. If $a > 0$, we obtain a contradiction. Indeed,

$$a_{n+1} \leq \psi(a_n),$$

and thus, $a \leq \psi(a)$, which contradicts to the property of ψ, and therefore, $\{a_n\}$ converges to 0. Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that this is not so. Then, there exists $\varepsilon > 0$ such that for any $k \in \mathbb{N}$, there is $m_k > n_k \geq k$ such that we have the relation

$$d_k = d(x_{m_k}, x_{n_k}) \geq \varepsilon.$$

Of course we can assume that $d(x_m, x_{m-1}) \leq \varepsilon$ and thus we have

$$d_k < d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \leq a_{m_k} + \varepsilon \leq a_k + \varepsilon,$$

which implies that $d_k \to 0$. On the other hand

$$\begin{aligned} d_k &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\ &\leq 2a_k + \psi(d_k). \end{aligned}$$

It follows that $\varepsilon \leq \psi(\varepsilon)$, a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $x_n \to z \in X$. Thus by the continuity of f, we get $f(z) = z$. Uniqueness of z follows from the contractive condition. □

Remark 2.5. In fact, two key steps involved in proving the existence of fixed point in each of the above results, showing that for given $x_0 \in X$, the iterative sequence $x_n = f^n(x_0) = f(x_{n-1})$ is a Cauchy sequence in the underlying spaces and then the continuity of the mapping guarantees the required fixed point.

Remark 2.6. In [14], Boyd and Wong observed that the upper semicontinuity condition of ψ can be dropped if the space X is metrically convex. While in [62], Matkowski further extended this result by assuming that ψ is continuous at 0 and that there exists a sequence $t_n \downarrow 0$ for which $\psi(t_n) < t_n$.

The following variant result is due to Matkowski [63], where the continuity condition on ψ is replaced with another suitable assumption.

Theorem 2.7. *Let (X, d) be a complete metric space, and suppose that $f : X \to X$ satisfies*

$$d(f(x), f(y)) \leq \psi(d(x,y)), \quad \textit{for all } x, y \in X,$$

where $\psi : (0,\infty) \to (0,\infty)$ is monotone nondecreasing and satisfies $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Then, f has a unique fixed point.

Proof. Let x_0 be an arbitrary but fixed element in X. Define a sequence of iterates $\{x_n\}$ in X by

$$x_n = f(x_{n-1}) \ \ (= f^n(x_0)), \quad \text{for all } n \geq 1.$$

Set $a_n = d(x_{n+1}, x_n)$. Note that

$$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \psi^n(d(x_1, x_0)) = 0.$$

Thus, $\lim_{n\to\infty} a_n = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence. Also note that for any $\varepsilon > 0$, $\psi(\varepsilon) < \varepsilon$. And since $\lim_{n\to\infty} a_n = 0$, so for $\varepsilon > 0$, we can choose n such that $a_n \leq \varepsilon - \psi(\varepsilon)$. Now, define

$$M = \{x \in X : d(x, x_n) \le \varepsilon\}.$$

Then for any $y \in M$, we have

$$\begin{aligned} d(f(y), x_n) &\le d(f(y), f(x_n)) + d(f(x_n), x_n) \\ &\le \psi(d(y, x_n)) + d(x_{n+1}, x_n) \\ &\le \psi(\varepsilon) + \varepsilon - \psi(\varepsilon) = \varepsilon. \end{aligned}$$

Thus $f(y) \in M$, that is; $f(M) \subseteq M$. It follows that $d(x_m, x_n) \le \varepsilon \ \forall m \ge n$. We obtain the conclusion by following the rest of the proof as in Theorem 2.6 □

Using somewhat different approach Meir and Keeler [64] extended Theorem 2.6 as follows.

Theorem 2.8. *Let (X, d) be a complete metric space, and suppose that $f : X \to X$ satisfies the condition: for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$,*

$$\varepsilon \le d(x, y) \le \varepsilon + \delta \Rightarrow d(f(x), f(y)) \le \varepsilon. \tag{2.6}$$

Then, f has a unique fixed point.

Clearly, the condition (2.6) implies that the mapping f is contractive. Thus, f is continuous and has a unique fixed point if it exists. Further, the condition (2.6) implies $d(x_n, x_{n+1})$ decreasing to zero. Finally, it is easy to show that $\{x_n\}$ is a Cauchy sequence in a complete metric space X by using the contrary technique.

In [21], Ćirić has obtained the following generalization of the BCP.

Theorem 2.9. *Let (X, d) be a complete metric space, and let $f : X \to X$ be a quasi-contraction, that is, for a fixed constant $h < 1$*

$$d(f(x), f(y)) \le h \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}, \text{ for all } x, y \in X. \tag{2.7}$$

Then, f has a unique fixed point.

It has been observed in [79] that Theorem 2.9 is also true if we replace (2.7) with the following equivalent contractive condition.

$$d(f(x), f(y)) \le h \max\left\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{1}{2}[d(x, f(y)) + d(y, f(x))]\right\}. \tag{2.8}$$

Several such type of contractive conditions have been studied by Rhoades [79], Jachymaski [39], and Ćirić [22].

Asymptotic fixed point theory involves assumptions about the iterates of the mapping in question. In fact the concept of asymptotic contractions is suggested in Theorem 2.3 which is the earliest extension of Banach contraction principle, (also see Theorem 2.4). For further historical comments, see, for example, [51].

Let Ψ denote the class of all mappings $\psi : [0,\infty) \to [0,\infty)$ such that ψ is continuous, and $\psi(t) < t$ for all $t > 0$.

Note that if (X,d) is any complete metric space and $f : X \to X$ is any mapping satisfying

$$d(f(x), f(y)) \le \psi(d(x,y)), \quad \text{for all } x, y \in X,$$

for any $\psi \in \Psi$, then by Theorem 2.6, f has a unique fixed point.

Using the Cantor's intersection theorem, Kirk [49] obtained the following asymptotic version of Theorem 2.6.

Theorem 2.10. *Let (X,d) be a complete metric space, and suppose that $f : X \to X$ satisfies*

$$d(f^n(x), f^n(y)) \le \psi_n(d(x,y)), \quad \textit{for all } x, y \in X,$$

where $\psi_n : [0,\infty) \to [0,\infty)$ are continuous and $\psi_n \to \psi \in \Psi$ uniformly. Further, assume that some orbit of f is bounded. Then, f has a unique fixed point.

See also [9, 41, 52] which are dealing with asymptotic version of Theorem 2.6. In [91] Suzuki obtained a result for asymptotic contractions of final type and claims his result is the final generalization in some sense.

In [4], Alber et al. suggested a generalization of BCP in the setting of Hilbert spaces and subsequently Rhoades [80] extended and improved their result to metric spaces.

Theorem 2.11. *Let (X,d) be a complete metric space, and suppose that $f : X \to X$ satisfies the following inequality*

$$d(f(x), f(y)) \le d(x,y) - \psi(d(x,y)), \quad \textit{for all } x, y \in X, \tag{2.9}$$

where $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$. Then, f has unique fixed point.

Note that if one takes $\psi(t) = (1-h)t$ where $0 < h < 1$, then the inequality (2.9) reduces to the inequality (2.1).

Remark 2.7. In the literature, a map $f : X \to X$ with inequality (2.9) is known as weakly contractive map. The function ψ involved in Theorem 2.11 known as alternating distance (also called control function), which was initially used in metric fixed point theory by Khan et al. [47]. This function and its generalizations have been used in fixed point problems in metric and probabilistic metric spaces, see, for example, [65, 67, 81, 82].

In [28], the following generalization has been appeared.

Theorem 2.12. *Let (X,d) be a complete metric space, and suppose that $f : X \to X$ satisfies the following inequality*

$$\phi(d(f(x),f(y))) \leq \phi(d(x,y)) - \psi(d(x,y)), \quad \textit{for all } x,y \in X,$$

where both the functions $\phi,\psi : [0,\infty) \to [0,\infty)$ are continuous and nondecreasing such that $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then, f has unique fixed point.

Recently, more general results in this direction have been appeared. For example, one of the results in [20] as follows.

Theorem 2.13. *Let (X,d) be a complete metric space, and suppose that $f : X \to X$ satisfies the following inequality*

$$\phi\big(d(f(x),f(y))\big) \leq \phi(m(x,y)) - \psi\left(\max\{d(x,y),d(y,f(y))\}\right),$$

where

$$m(x,y) = \max\left\{d(x,y),d(x,f(x)),d(y,f(y)),\frac{1}{2}\left[d(x,f(y))+d(y,f(x))\right]\right\},$$

for all $x,y \in X$, and $\phi,\psi : [0,\infty) \to [0,\infty)$ are functions such that ϕ is alternating distance and ψ is continuous with $\psi(t) = 0$ if and only if $t = 0$. Then, f has unique fixed point.

A direct consequence is the following result.

Corollary 2.3. *Let (X,d) be a complete metric space, and suppose that $f : X \to X$ satisfies the following inequality for all $x,y \in X$,*

$$\begin{aligned} &\phi(d(f^n(x),f^n(y))) \\ \leq\ &\phi\left(\max\left\{d(x,y),d(x,f^n(x)),d(y,f^n(y)),\frac{1}{2}\left[d(x,f^n(y))+d(y,f^n(x))\right]\right\}\right) \\ &-\psi(\max\{d(x,y),d(y,f^n(y))\}), \end{aligned}$$

where n is a positive integer and $\phi,\psi : [0,\infty) \to [0,\infty)$ are functions such that ϕ is alternating distance and ψ is continuous with $\psi(t) = 0$ if and only if $t = 0$. Then, f has unique fixed point.

Example 2.6. Let $X = \{0,1,2,3,\ldots\}$. Let $d : X \times X \to \mathbb{R}$ be given as

$$d(x,y) = \begin{cases} x+y, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then, (X,d) is a complete metric space. Define $\phi,\psi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t^2$ for all t, and

$$\psi(t) = \begin{cases} \frac{t^2}{2}, & \text{if } t \leq 1, \\ \frac{1}{2}, & \text{if } t > 1. \end{cases}$$

Let $f : X \to X$ be defined by

$$f(x) = \begin{cases} x-1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then, ϕ, ψ and f satisfy all the conditions of Theorem 2.13 and clearly 0 is the unique fixed of f.

Recently, Suzuki [88] established a new type of generalization of the BCP and characterizes the metric completeness of the underlying space.

Theorem 2.14. *Let (X,d) be a complete metric space, and suppose that $f : X \to X$. Define a nonincreasing function $\psi : [0,1) \to (1/2,1]$ by*

$$\psi(h) = \begin{cases} 1, & \textit{if } \ 0 \leq h \leq \frac{1}{2}(\sqrt{5}-1), \\ \frac{1-h}{h^2}, & \textit{if } \ \frac{1}{2}(\sqrt{5}-1) < h < \frac{1}{\sqrt{2}}, \\ \frac{1}{1+h}, & \textit{if } \ \frac{1}{\sqrt{2}} \leq h < 1. \end{cases}$$

Assume that there exists $h \in [0,1)$ such that

$$\psi(h)d(x,f(x)) \leq d(x,y) \ \Rightarrow \ d(f(x),f(y)) \leq hd(x,y), \quad \textit{for all } x,y \in X.$$

Then, f has a unique fixed point.

Proof. Since $\psi(h) \leq 1$, we get $\psi(h)d(x,f(x)) \leq d(x,f(x))$ for every $x \in X$. Note that

$$d(f(x),f^2(x)) \leq hd(x,f(x)) \ \ \forall x \in X.$$

Let x_0 be an arbitrary but fixed element in X. Define a sequence of iterates $\{x_n\}$ in X by

$$x_n = f(x_{n-1}) \ (= f^n(x_0)), \quad \text{for all } n \geq 1.$$

Then, we have $d(x_n,x_{n+1}) \leq h^n d(x_0,f(x_0))$, and thus $\sum_{n=1}^{\infty} d(x_n,x_{n+1}) < \infty$. By the usual arguments, we get $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some point $z \in X$. Then, for any $x \in X \setminus \{z\}$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n,z) \leq \frac{1}{3}d(x,z)$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Now, we have

$$
\begin{aligned}
\psi(h)d(x_n, f(x_n)) &\leq d(x_n, f(x_n) = d(x_n, x_{n+1}) \\
&\leq d(x_n, z) + d(x_{n+1}, z) \\
&\leq (2/3)d(x, z) = d(x, z) - d(x, z)/3 \\
&\leq d(x, z) - d(x_n, z) \leq d(x_n, x).
\end{aligned}
$$

Thus, by hypothesis

$$d(x_{n+1}, f(x)) \leq hd(x_n, x), \quad n \geq n_0,$$

and hence, we get

$$d(f(x), z) \leq hd(x, z), \quad \text{for all } x \in X \backslash \{z\}.$$

Now, for using the contrary arguments, we assume that $f^k(z) \neq z$ for all $k \in \mathbb{N}$. Note that

$$d(f^{k+1}(z), z) \leq h^k d(f(z), z), \quad k \in \mathbb{N}.$$

In this situation, each of the following three cases of ψ yields contradiction.

- $0 \leq h \leq \frac{1}{2}(\sqrt{5} - 1)$,
- $\frac{1}{2}(\sqrt{5} - 1) < h < \frac{1}{\sqrt{2}}$,
- $\frac{1}{\sqrt{2}} \leq h < 1$

(see for detail [88]). Therefore, in all the cases, there exists some $k \in \mathbb{N}$ such that $f^k(z) = z$. Since $\{f^n(z)\}$ is a Cauchy sequence, we get $f(z) = z$. The uniqueness of the fixed point follows from the fact that

$$d(f(x), z) \leq hd(x, z), \quad \text{for all } x \in X \backslash \{z\}.$$

Indeed, if $f(w) = w \neq z$, then $d(f(w), z) \leq hd(w, z)$ implies $d(w, z) < d(w, z)$, which is not possible. □

Obviously, the class of contraction mappings given in Theorem 2.14 contains the class of usual contractions. However, it has been observed in [88] that Suzuki's contractions and Kannan's contractions are independent but both types of contractions characterize the metric completeness of the underlying spaces.

Remark 2.8. Recently, a number of results appeared on the existence of a unique fixed point for a single-valued mapping f of a metric space (X, d) endowed with a partial order $\preceq$. Indeed, almost all such results deal with monotone mapping satisfying Banach contractive condition (2.1) with some restriction and for fixed $x_0 \in X$, $x_0 \preceq f(x_0)$ (or $f(x_0) \preceq x_0$). The first result in this direction was proved by Ran and Reurings [77], which is an analogue of the BCP in partially ordered sets. They also presented several applications to linear and nonlinear matrix equations. Subsequently, many fixed point results with interesting applications have appeared in this direction, see, for example, [35–38, 69–71, 74, 97] and others.

2.5 Caristi's Fixed Point Theorem

In 1976, Caristi [19] proved a wonderful fixed point theorem on complete metric spaces, which is related to the BCP and is equivalent to Ekland variational principle [31]. The Caristi's fixed point theorem has found many applications in nonlinear analysis, see [10, 16, 40, 45, 52, 93] for detail discussion.

Definition 2.2. (a) A real-valued function φ defined on X is said to be *lower semicontinuous* at x if for any sequence $\{x_n\} \subset X$, we have

$$x_n \to x \in X \quad \Rightarrow \quad \varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n).$$

(b) A single-valued self-mapping f on a metric space (X,d) is said to be *Caristi mapping* if there exists a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \quad \text{for all } x \in X. \tag{2.10}$$

Example 2.7. Each Banach contraction mapping f on a metric space (X,d) is a Caristi mapping with a function

$$\varphi(x) = \frac{1}{1-h} d(x, f(x)), \quad \text{for all } x \in X,$$

where h is a contraction constant.

Clearly, φ is a continuous real valued function on X and

$$\varphi(fx) \leq \frac{h}{1-h} d(x, fx) = h\varphi(x).$$

Note that for all $x \in X$,

$$d(x, fx) = (1-h)\varphi(x) = \varphi(x) - h\varphi(x) \leq \varphi(x) - \varphi(fx),$$

that is, f is a Caristi mapping.

Remark 2.9. In fact, the class of single-valued Caristi mappings is very large, including at least usual contractions, Ćirić contractive mappings and in particular Kannan mapping.

Theorem 2.15 (Caristi's Fixed Point Theorem). *Let (X,d) be a complete metric space. Then, each Caristi map $f : X \to X$ has a fixed point.*

Remark 2.10. The original proof of this results involved transfinite induction arguments. But, after the appearance of this remarkable theorem of Caristi, numerous papers were published on various proofs of this result. For example, see Wong [96], Penot [75], Siegel [83], and others. An elegant and direct proof of the Caristi's fixed point theorem is given in Deimling [29]. An elementary and straightforward approach is due to Brezis-Browder [16].

Remark 2.11. The key relation between Caristi's fixed point theorem and BCP was noted in Example 2.7. From the Caristi's fixed point theorem one cannot expect the all conclusions of the BCP. In the Caristi's fixed point theorem, the fixed point need not be unique and the sequence $\{f^n(x_0)\}$ need not even converge to a fixed point of f. Secondly, the map f satisfying (2.1) is continuous while the map f satisfying (2.10) is not necessarily continuous.

Remark 2.12. It is well known that the fixed point property for contraction mappings does not characterize metric completeness, see, for example, Suzuki and Takahashi [87]. However, some characterizations of metric completeness have been discussed by several authors. For example, Kirk [52] and Weston [95] proved that a metric space is complete if and only if it has the fixed point property for Caristi mappings. Moreover, Shiojiet et al. [84] proved that a metric space is complete if and only if it has the fixed point property for Kannan mappings. Thus, Kannan mappings and Caristi maps characterize metric completeness, while contraction mappings do not.

Regarding the problem of characterizations of metric completeness by means of contraction mappings, Suzuki and Takahashi [87] and independently Anisiu and Anisiu [5] proved that a convex subset Y of a normed space is complete if and only if every contraction $f : Y \to Y$ has a fixed point in Y. The most elegant result in this direction is due to Bessage [12] which states that if any mapping f on an arbitrary set X and each of its iterates f^n has a unique fixed point, then for each $h \in (0,1)$ there exists a metric d_h on X for which X is complete and f is a contraction mapping with contraction constant h. See [86] for more on Ekeland's variational principle and the equivalence between the Caristi's fixed point result and the completeness of metric spaces. Also, see [2, 3, 6–8, 61, 73].

2.6 Some Extensions of BCP Under Generalized Distances

In recent years, distances in metric have been introduced which generalize metrics and which have applications to obtaining the solutions of several new important problems in nonlinear analysis. The pioneering effort in this direction is papers of Kada et al. [43], Suzuki and Takahashi [87], Suzuki [89, 90], Lin and Du [59,60], and Ume [99] in metric spaces. In these papers, among other things, various distances are introduced, and relations between these distances with applications are established.

In [43], Kada et al. introduced a notion of w-distance on a metric space and using this notion, they improved the Caristi's fixed point theorem, Ekland variational principle, and Takahashi minimization theorem. Using the notion of w-distance, Suzuki and Takahashi [87] have introduced notions of single-valued and multivalued weakly contractive (in short, w-contractive) mappings and proved fixed point results for such mappings. Consequently, they generalized the Banach Contraction principle and Nadler's fixed point result [68].

Definition 2.3. A function $p : X \times X \to \mathbb{R}^+$ is called a *w-distance* on X if it satisfies the following for any $x,\ y,\ z \in X$:

(w_1) $p(x,z) \leq p(x,y) + p(y,z)$;
(w_2) a map $p(x,.) : X \to [0,\infty)$ is lower semicontinuous;
(w_3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$.

Note that, in general for $x,y \in X$, $p(x,y) \neq p(y,x)$ and not either of the implications $p(x,y) = 0 \Leftrightarrow x = y$ necessarily hold.

Example 2.8. (a) The metric d is a w-distance on X.
(b) Let $(Y, \|.\|)$ be a normed space. Then, the functions $p_1, p_2 : Y \times Y \to \mathbb{R}^+$ defined by

$$p_1(x,y) = \|y\| \quad \text{and} \quad p_2(x,y) = \|x\| + \|y\|, \quad \text{for all } x,y \in Y,$$

are w-distances.

For other examples and related results, see [43]. Here we state two useful lemmas. For further details, see [43, 60, 98].

Lemma 2.1. *Let (X,d) be a metric space and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to* 0*. Then, the following statements hold for every $x,y,z \in X$:*

(a) If $p(x_n,y) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$, in particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;
(b) If $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;
(c) If $p(x_n,x_m) \leq \alpha_n$ for any $n,m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;
(d) If $p(y,x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.2. *Let (X,d) be a metric space and let p be a w-distance on X. Let K be a closed subset of X. Suppose that there exists $u \in X$ such that $p(u,u) = 0$. Then, $p(u,K) = 0$ if and only if $u \in K$, where $p(u,K) = \inf_{y \in K} p(u,y)$.*

Using the concept of w-distance, Kada et al. [43] improved Caristi's fixed point theorem as follows:

Theorem 2.16. *Let (X,d) be a complete metric space and p be a w-distance on X. Then, each Caristi mapping f on X with respect to p has a fixed point $x_0 \in X$ and $p(x_0,x_0) = 0$.*

While Suzuki and Takahashi [87] improved the BCP as follows.

Theorem 2.17. *Let (X,d) be a complete metric space and p be a w-distance on X. Then, each contraction mapping f on X with respect to p has a unique fixed point $x_0 \in X$ and $p(x_0,x_0) = 0$.*

In [87], they also obtained a result on characterization of metric completeness.

Theorem 2.18. *Let (X,d) be a metric space. Then, X is complete if and only if every contraction mapping f on X with respect to p has a fixed point in X.*

Among others results, Ume [100] improved Theorem 2.9 for w-distance.

Theorem 2.19. *Let (X,d) be a complete metric space and let p be a w-distance on X. Let $f : X \to X$ be a mapping such that for a fixed constant $h < 1$ and for all $x, y \in X$,*

$$p(f(x), f(y)) \leq h \max\{p(x,y), p(x,fx), p(y,fy), p(x,fy), p(y,fx)\},$$

and $\inf\{p(x,u) + p(x,fx) : x \in X\} > 0$ *for every $u \in X$ with $u \neq f(u)$. Then, f has a unique fixed point $x_0 \in X$ and $p(x_0,x_0) = 0$.*

A number of fixed point results w. r. t w-distance have been appeared in the literature.

Generalizing the concept of w-distance, Suzuki [89] introduced the following notion of τ-distance on metric spaces.

Definition 2.4. A function $p : X \times X \to \mathbb{R}^+$ is said to be a *τ-distance* on X if it satisfies the following conditions for any $x,\ y,\ z \in X$:

(τ_1) $p(x,z) \leq p(x,y) + p(y,z)$;
(τ_2) $\eta(x,0) = 0$ and $\eta(x,t) \geq t$ for all $x \in X$ and $t \geq 0$, and η is concave and continuous in its second variable;
(τ_3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n,x_m)) : m \geq n\} = 0$ imply $p(u,x) \leq \liminf_n p(u,x_n)$ for all $u \in X$;
(τ_4) $\lim_n \sup\{p(x_n,y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n,t_n) = 0$ imply $\lim_n \eta(y_n,t_n) = 0$;
(τ_5) $\lim_n \eta(z_n, p(z_n,x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n,y_n)) = 0$ imply $\lim_n d(x_n,y_n) = 0$.

It has been observed in [89] that (τ_2) can be replaced by

(τ_2)$'$ $\inf\{\eta(x,t) : t \geq 0\} = 0$ for all $x \in X$, and η is nondecreasing in its second variable.

In general, a τ-distance p does not necessarily satisfy $p(x,x) = 0$. The metric d is a τ-distance on X. Each w-distance on a metric space X is also a τ- distance on X. Other examples and properties of τ-distance are given in [89].

Using the concept of τ-distance, Susuki [89] improved the BCP and Caristi's fixed point theorem as under:

Theorem 2.20. *Let (X,d) be a complete metric space and p be a τ-distance on X. Then, each contraction mapping f on X with respect to p has a unique fixed point $x_0 \in X$ and $p(x_0,x_0) = 0$.*

Theorem 2.21. *Let (X,d) be a complete metric space and p be a τ-distance on X. Then, each Caristi's mapping f on X with respect to p has a fixed point $x_0 \in X$ and $p(x_0,x_0) = 0$.*

Using this result, Suzuki [93] obtained generalized Caristi's fixed point theorem as follows.

Theorem 2.22. *Let (X,d) be a complete metric space, p be a τ-distance on X and let $g : X \to (0,\infty)$ be a function such that for some $r > 0$*

$$\sup\{g(x) : x \in X, \psi(x) \leq \inf_{z\in X} \psi(z) + r\} < \infty,$$

where $\psi : X \to (0,\infty)$ is a lower semicontinuous function. Let $f : X \to X$ be a map such that for each $x \in X$,

$$p(x, f(x)) \leq g(x)(\psi(x) - \psi(f(x))).$$

Then, there exists $x_o \in X$ such that $f(x_o) = x_o$ and $p(x_o,x_o) = 0$.

See also [90] for further results in this direction. Recently, Ume [99] introduced a new concept of a distance called u-distance, which generalizes w-distance, TataruŠs distance [94], and τ-distance. Some interesting fixed point results including BCP with respect to u-distance appeared in [13,99]. In the literature, some other distances have introduced, and among others results the BCP has been also studied with respect to these generalized distances see, for example, [3, 59, 72, 101].

2.7 Multivalued Versions of BCP

Investigations on the existence of fixed points for multivalued contraction mappings in the setting of metric spaces were initiated by Nadler in 1979. Using the concept of Hausdorff metric, he established multivalued version of the Banach contraction principle.

Let (X,d) be a metric space. We denote by 2^X the collection of all nonempty subsets of X, $Cl(X)$ the collection of all nonempty closed subsets of X, $CB(X)$ the collection of all nonempty closed bounded subsets of X, and H the *Hausdorff metric* on $CB(X)$, that is,

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}, \quad \text{for all } A,B \in CB(X),$$

where $d(x,A) = \inf_{y\in A} d(x,y)$. In the metric space $(CB(X),H)$, $\lim_{n\to\infty} A_n = A$ means that $\lim_{n\to\infty} H(A_n,A) = 0$. Let $A_1,A_2 \in CB(X)$. Then, for each $x \in A_1$ and $\varepsilon > 0$, there is $y \in A_2$ such that

$$d(x,y) \leq H(A_1,A_2) + \varepsilon.$$

Example 2.9. (a) Let $X = \mathbb{R}$, $A = [0,1]$, $B = [2,4]$. Then,

$$\sup_{a\in A} d(a,B) = 2, \quad \sup_{b\in B} d(b,A) = 3, \quad \text{and} \quad H(A,B) = 3.$$

(b) Let $A = B_r(a)$, $B = B_s(b)$, $a,b \in (X,d)$, $0 < r \le s$. Then, $H(A,B) = d(a,b) + s - r$.

Definition 2.5. Let (X,d) be a metric space and let $T : X \to 2^X$.

(a) An element $x \in X$ is called a *fixed point* of a multivalued mapping T if $x \in T(x)$. We denote $Fix(T) = \{x \in X : x \in T(x)\}$.
(b) A sequence $\{x_n\}$ in X is said to be an *iterative sequence* of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \in \mathbb{N}$.
(c) T is said to be a *contraction* [68] if for a fixed constant $h < 1$ and for each $x,y \in X$,

$$H(T(x),T(y)) \le h\, d(x,y). \tag{2.11}$$

Such a mapping T is also known as Nadler contraction.

Using the concept of Hausdorff metric, Nadler [68] proved the following theorem on the existence of fixed points for multivalued mappings, known as Nadler contraction principle (NCP).

Theorem 2.23 (Nadler' Fixed Point Theorem). *Let (X,d) be a complete metric space. Then, each contraction mapping $T : X \to CB(X)$ has a fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary fixed and let $x_1 \in T(x_0)$. Then, there exists $x_2 \in T(x_1)$ such that

$$d(x_1,x_2) \le H(T(x_0),T(x_1)) + h,$$

where $h < 1$ is a contraction constant. Continuing this iterative process, in general, there exists $x_{n+1} \in T(x_n)$ for each $n \in \mathbb{N}$ such that

$$d(x_n,x_{n+1}) \le H(T(x_{n-1}),T(x_n)) + h^n \le \cdots \le h^n d(x_0,x_1) + n\, h^n.$$

Thus, we have

$$\sum_{n=0}^{\infty} d(x_n,x_{n+1})) \le d(x_0,x_1)\left(\sum_{n=0}^{\infty} h^n\right) + \sum_{n=0}^{\infty} n\, h^n < \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence, and thus, there exists some $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now the continuity of T implies $\lim_{n\to\infty} H(T(x_n),T(x)) = 0$. Since $x_n \in T(x_{n-1})$, we have $\lim_{n\to\infty} d(x_n,T(x)) = 0$, which implies $x \in T(x)$ because $T(x)$ is closed. $\square$

Remark 2.13. (a) We observed in the above proof that using the property of the Hausdorff metric, we get an iterative sequence which is a Cauchy and converges to the fixed point of T.

(b) In contrast to its single-valued counterpart, fixed point in Theorem 2.23 need not be unique. Indeed, if X is bounded, then the map $T(x) = X$, for all $x \in X$ satisfies the conditions of Theorem 2.23.

Without using an iterative methods, a beautiful proof of Theorem 2.23 is given in [38] by Jachymski. The proof depends on the Axiom of choice and the Caristi's fixed point theorem.

Theorem 2.24. *Caristi's fixed point result (Theorem 2.15) yields NCP (Theorem 2.23).*

Proof. Let $T : X \to CB(X)$ be a Nadler contraction mapping with contraction constant h. Choose real α such that $h < \alpha < 1$. Let $x \in X$, then

$$\{y \in T(x) : \alpha d(x,y) \le d(x,T(x))\} \neq \emptyset.$$

By the axiom of choice, there is a map $f : X \to X$ such that $f(x) \in T(x)$ and $\alpha d(x, f(x)) \le d(x,T(x))$. Thus, we have

$$d(f(x),T(f(x))) \le H(T(x),T(f(X))) \le h(d(x, f(x))).$$

Note that

$$\begin{aligned} d(x, f(x)) &= \frac{1}{\alpha - h}\left(\alpha d(x, f(x)) - hd(x, f(x))\right) \\ &\le \frac{1}{\alpha - h}\left(d(x,T(x)) - d(f(x),T(f(x)))\right). \end{aligned}$$

Set $\varphi(x) = \frac{1}{\alpha-h} d(x,T(x))$. Then, we have

$$d(x, f(x)) \le \varphi(x) - \varphi(f(x)).$$

Also, note that for each $x, y \in X$, we get

$$|\varphi(x) - \varphi(y)| \le \frac{1}{\alpha - h}\{d(x,y) + H(T(x),T(y))\} \le \frac{h+1}{\alpha - h} d(x,y),$$

and thus, φ is continuous. Hence, by Theorem 2.15, there exists a fixed point in X. □

At the same time another proof of Theorem 2.23 has appeared in [42] without using an iterative technique.

In [66], Mizoguchi and Takahashi generalized Nadler's fixed point theorem as follows (which is also a partial answer to the problem proposed by Reich [78]).

Theorem 2.25. *Let* (X,d) *be a complete metric space and let* $T : X \to CB(X)$ *be a mapping such that for each* $x, y \in X$,

$$H(T(x),T(y)) \leq k(d(x,y))d(x,y),$$

where k *is a function from* $[0,\infty)$ *to* $[0,1)$ *satisfying* $\limsup_{r \to t^+} k(r) < 1$, *for every* $t \geq 0$. *Then,* T *has a fixed point.*

Remark 2.14. In the original statement, the domain of the function k is $(0,\infty)$. However both are equivalent because $d(x,y) = 0 \Rightarrow H(T(x),T(y)) = 0$. Also, note that the stronger condition assumed on k implies that $k(t) < h$ for some $0 < h < 1$. Thus with this condition, one may get that the map T is a contraction over a region for which $d(x,y)$ is sufficiently small.

Remark 2.15. In fact, the original proof of Theorem 2.25 is not simple. Alternative proofs appeared in [27, 81, 102]. The simplest alternative proof given in [92].

Proof. Define a function $\beta : [0,\infty) \to [0,1)$ by $\beta(t) = (k(t)+1)/2$. Then, we have

$$\limsup_{r \to t+0} \beta(r) < 1, \quad \text{for all } t \geq 0,$$

and for all $x, y \in X$ and $u \in T(x)$, there exists an element $v \in T(x)$ such that

$$d(u,v) \leq \beta(d(x,y))\, d(x,y).$$

Thus, we can define a sequence $\{x_n\}$in X such that for all integer $n \geq 1$, $x_{n+1} \in T(x_n)$ and

$$d(x_{n+1},x_{n+2}) \leq \beta(d(x_n,x_{n+1}))\, d(x_n,x_{n+1}).$$

For convenience, we put $a_n = d(x_n,x_{n+1})$. Hence, the sequence of nonnegative real numbers $\{a_n\}$ is non-increasing and thus converges to some nonnegative real number α. Note that there exist some $b \in [0,1)$ and $\varepsilon > 0$ such that $\beta(r) \leq b$ for all $r \in [\alpha, \alpha+\varepsilon]$. Now we can choose some integer $m \geq 1$ such that $m \leq a_n \leq \alpha+\varepsilon$ with $n \geq m$. Note that

$$a_{n+1} \leq \beta(a_n)\, a_n \leq b\, a_n,$$

and thus, we have

$$\sum_{n=1}^{\infty} a_n < \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence in the complete space X. Let $\{x_n\}$ converges to some $z \in X$. Note that

$$\begin{aligned} d(z,T(z)) &= \lim_{n\to\infty} d(x_{n+1},T(z)) \\ &\le \lim_{n\to\infty} H(T(x_n),T(z)) \\ &\le \lim_{n\to\infty} \beta(d(x_n,z))d(x_n,z) \\ &\le \lim_{n\to\infty} d(x_n,z) = 0. \end{aligned}$$

Since $T(z)$ is closed, we get $z \in T(z)$. □

Recently, Kikkawa and Suzuki [48] generalized Nadler's fixed point theorem and Theorem 2.14.

Theorem 2.26. *Define a strictly increasing function* η *from* $[0,1)$ *onto* $(\frac{1}{2},1]$ *by*

$$\eta(h) = \frac{1}{1+h}.$$

Let (X,d) *be a complete metric space and let* $T : X \to CB(X)$*. Assume that there exists fixed* $h \in [0,1)$ *such that*

$$\eta(h)d(x,T(x)) \le d(x,y) \Rightarrow H(T(x),T(y)) \le hd(x,y), \quad \text{for all } x,y \in X.$$

Then, T has a fixed point.

Proof. Take a real number k with $0 < h < k < 1$. Let $x_0 \in X$ be an arbitrary fixed and let $x_1 \in T(x_0)$. Then we have

$$\eta(h)d(x_0,T(x_0)) \le \eta(h)d(x_0,x_1) \le d(x_0,x_1).$$

From the hypothesis, it follows that

$$d(x_1,T(x_1)) \le H(T(x_0),T(x_1)) \le hd(x_0,x_1).$$

So, there exits $x_2 \in T(x_1)$ such that $d(x_1,x_2) \le kd(x_0,x_1)$. Continuing this process, we get a sequence $\{x_n\}$ in X such that $x_n \in T(x_{n-1})$ and $d(x_{n-1},x_n) \le kd(x_{n-2},x_{n-1})$. Thus, we have

$$\sum_{n=1}^{\infty} d(x_{n-1},x_n) \le \sum_{n=1}^{\infty} k^{n-1}d(x_0,x_n) < \infty,$$

and hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some point $z \in X$. We next show that

$$d(z,T(x)) \le hd(z,x), \quad \text{for all } x \in X \setminus \{z\}.$$

Since X is complete, $\{x_n\}$ converges to some point $z \in X$. Then, for any $x \in X\backslash z$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, z) \leq \frac{1}{3}d(x,z)$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Now we have

$$\begin{aligned}
\eta(h)d(x_n, T(x_n)) &\leq d(x_n, T(x_n)) \leq d(x_n, x_{n+1}) \\
&\leq d(x_n, z) + d(x_{n+1}, z) \\
&\leq (2/3)d(x,z) = d(x,z) - d(x,z)/3 \\
&\leq d(x,z) - d(x_n, z) \leq d(x_n, x).
\end{aligned}$$

Hence, $H(T(x_n, T(x)) \leq hd(x_n, x)$, it follows that $d(T(x), x_{n+1}) \leq hd(x, x_n)$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Thus, we get

$$d(T(x), z) \leq hd(x,z), \quad \text{for all } x \in X\backslash\{z\}.$$

We next prove that $H(T(x), T(z)) \leq hd(x,z)$ for all $x \in X$. If $d(x,z) = 0$, then we are done. So we assume that $d(x,z) > 0$. Then for every $n \in \mathbb{N}$, there exists $y_n \in T(x)$ such that

$$d(z, y_n) \leq d(z, T(x)) + \frac{1}{n}d(x,z).$$

For $n \in \mathbb{N}$, We have

$$\begin{aligned}
d(x, Tx) &\leq d(x, y_n) \leq d(x,z) + d(z, y_n) \\
&\leq d(x,z) + d(z, T(x)) + \frac{1}{n}d(x,z) \\
&\leq d(x,z) + hd(x,z) + \frac{1}{n}d(x,z) \\
&= \left(1 + h + \frac{1}{n}\right) d(x,z),
\end{aligned}$$

and hence, $\frac{1}{1+h}d(x, T(x)) \leq d(x,z)$. From the hypothesis, we have $H(T(x), T(z)) \leq hd(x,z)$. Finally, note that

$$d(z, T(z)) = \lim_{n\to\infty} d(x_{n+1}, T(z)) \leq \lim_{n\to\infty} H(T(x_n), T(z)) \leq \lim_{n\to\infty} hd(x_n, z) = 0,$$

and since $T(z)$ is closed, we get $z \in T(z)$. □

Many modifications and generalizations of Nadler's Theorem have been developed in successive years. In most cases, the nature of the Hausdorff metric is not used and the existence part of results can be proved without using the concept of a Hausdorff metric. For instance, recently, Feng and Liu [32] obtained some interesting fixed point results for multivalued mappings and extended Nadler's result as follows.

Theorem 2.27. *Let (X,d) be a complete metric space and let $T : X \to Cl(X)$ be a mapping such that for any fixed constants $h,b \in (0,1)$, $h < b$, and for each $x \in X$, there is $y \in T(x)$ satisfying the following conditions:*

$$b\, d(x,y) \leq d(x,T(x)),$$

and

$$d(y,T(y)) \leq h\, d(x,y).$$

Then, $Fix(T) \neq \emptyset$ provided the real-valued function g on X, $g(x) = d(x,T(x))$ is lower semicontinuous.

Proof. Let $x_0 \in X$ be an arbitrary fixed. Then by hypothesis, there exists $x_1 \in T(x_0)$ such that

$$bd(x_o,x_1) \leq d(x_0,T(x_0)) \quad and \quad d(x_1,T(x_1)) \leq hd(x_o,x_1).$$

Similarly, there is $x_2 \in T(x_1)$ satisfying

$$b\, d(x_1,x_2) \leq d(x_1,T(x_1)) \quad and \quad d(x_2,T(x_2)) \leq h\, d(x_1,x_2).$$

Continuing this process, we get a sequence $\{x_n\}$ in X satisfying $x_{n+1} \in T(x_n)$,

$$bd(x_n,x_{n+1}) \leq d(x_n,T(x_n)) \quad and \quad d(x_{n+1},T(x_{n+1})) \leq hd(x_n,x_{n+1}),$$

for all $n = 0,1,2,\ldots$. Thus, from the last two inequalities, we have for all $n = 0,1,2,\ldots$,

$$d(x_{n+1},x_{n+2}) \leq \frac{h}{b} d(x_n,x_{n+1}), \tag{2.12}$$

and

$$d(x_{n+1},T(x_{n+1})) \leq \frac{h}{b} d(x_n,T(x_n)). \tag{2.13}$$

Note that

$$d(x_{n+1},T(x_{n+1})) \leq d(x_n,T(x_n)).$$

Thus, $\{d(x_n,T(x_n))\}$ is a decreasing sequence. Further, for each $n \in \{0,1,2,\ldots\}$ we have

$$d(x_n,x_{n+1}) \leq \frac{h^n}{b^n} d(x_o,x_1),$$

and

$$d\left(x_n, T\left(x_n\right)\right) \leq \frac{h^n}{b^n} d\left(x_o, T\left(x_o\right)\right).$$

Since $h < b$, so we get that $d\left(x_n, T\left(x_n\right)\right) \rightarrow 0$ as $n \rightarrow \infty$. Set $a = \frac{h}{b}$. Then, for $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d\left(x_m, x_n\right) &\leq d\left(x_m, x_{m-1}\right) + d\left(x_{m-1}, x_{m-2}\right) + \cdots + d\left(x_{n+1}, x_n\right) \\ &\leq a^{m-1} d\left(x_o, x_1\right) + a^{m-2} d\left(x_o, x_1\right) + \cdots + a^n d\left(x_o, x_1\right) \\ &\leq \frac{a^n}{1-a} d\left(x_o, x_1\right), \end{aligned}$$

Due to $h < b$, we have $a^n \rightarrow 0$ as $n \rightarrow \infty$, and hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some point $x \in X$. We assert that x is a fixed point of T. Note that the sequence of nonnegative terms $\{f\left(x_n\right)\} = \{d\left(x_n, T\left(x_n\right)\right)\}$ is decreasing to 0, and since f is lower semicontinous, we have

$$0 \leq f\left(x\right) \leq \liminf_{n \rightarrow \infty} f\left(x_n\right) = 0,$$

which implies $f\left(x\right) = 0$. Hence, the closeness of $T\left(x\right)$ implies $x \in T\left(x\right)$. □

Remark 2.16. Theorem 2.27 generalizes Nadler's fixed point result. Indeed, if T satisfies the condition of Nadler's result, then the lower semicontinuity of function $f(x) = d(x, T(x))$ follows from the contraction condition. Further, since $T\left(x\right)$ is closed and bounded, so there exists $y \in T\left(x\right)$ such that

$$bd\left(x, y\right) \leq d\left(x, T\left(x\right)\right), \quad \text{for } b \in \left(0, 1\right).$$

Also,

$$d\left(y, T\left(y\right)\right) \leq H\left(T\left(x\right), T\left(y\right)\right) \leq hd\left(x, y\right), \quad \text{for } h \in \left(0, 1\right).$$

Thus, the existence of fixed point of T follows from the Theorem 2.27.

Klim and Wardowski [53] generalized Theorem 2.23 and Theorem 2.27. While in [23,24], Ćirić generalized all the above-mentioned fixed point results of this section.

On the other hand, mutivalued versions of the BCP with respect to generalized distances have appeared. Using w-distance, Suzuki and Takahashi [87] obtained multivalued version of the BCP which is an improved version of the Nadler's fixed point theorem (Theorem 2.23).

Theorem 2.28. *Let (X, d) be a complete metric space and let p be a w-distance on X. Let $T : X \rightarrow Cl(X)$ be a mapping such that for a fixed constant $h \in [0, 1)$ and for any $x, y \in X$, $u \in T(x)$, there is $v \in T(y)$ satisfying*

$$p(u,v) \le h\, p(x,y).$$

Then, there exists $x_0 \in X$ *such that* $x_0 \in T(x_0)$ *and* $p(x_0,x_0) = 0$.

Proof. Let $u_o \in X$ be fixed and let $u_1 \in T(u_o)$. Then, there exists $u_2 \in T(u_1)$ such that

$$p(u_1,u_2) \le hp(u_o,u_1).$$

Continuing this process, we have a sequence $\{u_n\}$ in X such that $u_{n+1} \in T(u_n)$ and

$$p(u_n,u_{n+1}) \le hp(u_{n-1},u_n), \quad \text{for every } n \in \mathbb{N}.$$

Thus, we have

$$p(u_n,u_{n+1}) \le hp(u_{n-1},u_n) \le h^2 p(u_{n-2},u_{n-1}) \le \cdots \le h^n p(u_o,u_1),$$

and hence, for any $n,m \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} p(u_n,u_m) &\le p(u_n,u_{n+1}) + p(u_{n+1},u_{n+2}) + \cdots + p(u_{m-1},u_m) \\ &\le \{h^n + h^{n+1} + \cdots + h^{m-1}\} p(u_o,u_1) \\ &\le \frac{h^n}{1-h} p(u_o,u_1) \to 0. \end{aligned}$$

By Lemma 2.1, $\{u_n\}$ is a Cauchy sequence . Thus, $\{u_n\}$ converges to some $v_o \in X$. Fix $n \in \mathbb{N}$. Since $\{u_m\}$ converges to v_o and $p(u_n,.)$ is lower semicontinuous, we have

$$p(u_n,v_o) \le \liminf_{m\to\infty} p(u_n,u_m) \le \frac{h^n}{1-h} p(u_o,u_1).$$

Since $u_n \in T(u_{n-1})$ and $v_o \in X$, by hypothesis there is $w_n \in T(v_o)$ such that

$$p(u_n,w_n) \le hp(u_{n-1},v_o) \le \frac{h^n}{1-h} p(u_o,u_1).$$

By Lemma 2.1, $\{w_n\}$ converges to v_o. Since $T(v_o)$ is closed, we have $v_o \in T(v_o)$. For such v_o, there exists $v_1 \in T(v_o)$ such that

$$p(v_o,v_1) \le hp(v_o,v_o).$$

Thus, we also have a sequence $\{v_n\}$ in X such that $v_{n+1} \in T(v_n)$ and

$$p(v_o,v_{n+1}) \le hp(v_o,v_n), \quad \text{for every } n \in \mathbb{N}.$$

Thus, we have

$$p(v_o, v_n) \le hp(v_o, v_{n-1}) \le \cdots \le h^n p(v_o, v_o) \to 0 \text{ as } n \to \infty.$$

By Lemma 2.1, $\{v_n\}$ is a Cauchy sequence. Then, $\{v_n\}$ converges to a point $x_o \in X$. Since $p(v_o, .)$ is lower semicontinuous, we have

$$p(v_o, x_o) \le \liminf_{n\to\infty} p(v_o, v_n) \le 0,$$

and hence, $p(v_o, x_o) = 0$. Thus, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} p(u_n, x_o) &\le p(u_n, v_o) + p(v_o, x_o) \\ &\le \frac{h^n}{1-h} p(u_o, u_1) \to 0 \; as \; n \to \infty. \end{aligned}$$

So, by Lemma 2.1, we obtain $v_o = x_o$, and hence, $p(v_o, v_o) = 0$. □

The following result is a generalization of the Banach contraction principle.

Corollary 2.4. *Let (X, d) be a complete metric space and p be a w-distance on X. Then, each contraction mapping f on X with respect to p has a unique fixed point $x_0 \in X$ and $p(x_0, x_0) = 0$.*

Proof. From Theorem 2.28, there exists $x_o \in X$ with $f(x_o) = x_o$ and $p(x_o, x_o) = 0$. If $y_o = f(y_o)$, then we have

$$p(x_o, y_o) = p(f(x_o), f(y_o)) \le hp(x_o, y_o).$$

Since $h \in (0,1)$, we have $p(x_o, y_o) = 0$. So, by $p(x_o, x_o) = 0$ and by Lemma 2.1, we have $x_o = y_o$. □

In [89], Suzuki established the following multivalued version of the BCP with respect to τ-distance, which is a generalization of Theorem 2.28.

Theorem 2.29. *Let (X, d) be a complete metric space and let p be a τ-distance on X. Let $T : X \to Cl(X)$ be a mapping such that for a fixed constant $h \in [0,1)$ and for any $x, y \in X$, $u \in T(x)$, there is $v \in T(y)$ satisfying*

$$p(u, v) \le h\, p(x, y).$$

Then, T has a fixed point.

Many other fixed point results with respect to τ-distance and other generalized distances have appeared in the literature, see, for example, [13, 25, 54–58, 89] and the references therein.

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