

Chapter 3

Modal Interval Extensions

3.1 Introduction

The problem discussed in this chapter is that of obtaining a class of interval functions $F : I^*(\mathbb{R}^k) \rightarrow I^*(\mathbb{R})$, consistently referring to the continuous functions f from \mathbb{R}^k to \mathbb{R} .

In classical interval analysis, an interval extension of a \mathbb{R}^k to \mathbb{R} continuous function $z = f(x_1, \dots, x_k)$ is the *interval united extension* R_f of f . Given an interval argument $X = (X_1, \dots, X_k) \in I(\mathbb{R}^k)$, it is defined as the range of f -values on X

$$\begin{aligned} R_f(X_1, \dots, X_k) &= \{f(x_1, \dots, x_k) \mid x_1 \in X_1, \dots, x_k \in X_k\} \\ &= [\min\{f(x_1, \dots, x_k) \mid x_1 \in X_1, \dots, x_k \in X_k\}, \\ &\quad \max\{f(x_1, \dots, x_k) \mid x_1 \in X_1, \dots, x_k \in X_k\}], \end{aligned}$$

which can be considered as a “semantic extension” of f , since it admits the logical interpretations

$$(\forall x_1 \in X_1) \cdots (\forall x_k \in X_k) (\exists z \in R_f(X_1, \dots, X_k)) z = f(x_1, \dots, x_k)$$

and

$$(\forall z \in R_f(X_1, \dots, X_k)) (\exists x_1 \in X_1) \cdots (\exists x_k \in X_k) z = f(x_1, \dots, x_k).$$

Since the domain of values of a continuous function is generally not easily computable, an interval syntactic extension $fR(X_1, \dots, X_k)$ is defined by replacing the real operators of the real functions $f(x_1, \dots, x_k)$ on \mathbb{R} by the homonymous operators defined on the system $(I(\mathbb{R}), I(\mathbb{D}))$, that is, replacing

1. their numerical arguments x_1, \dots, x_k by the interval arguments X_1, \dots, X_k , and
2. their real arithmetic operators ω by the corresponding interval operations \mathcal{O} which, in the common case of the truncated computations of any actual arithmetic, must be the outwards directed \mathcal{O}_R because of the inclusion

$$X\mathcal{O}Y \subseteq X\mathcal{O}_RY = \text{Out}(X\mathcal{O}Y).$$

The crucial relation between both extensions is

$$R_f(X_1, \dots, X_k) \subseteq fR(X_1, \dots, X_k),$$

under the condition that the function $fR(X_1, \dots, X_k)$ is well defined, i.e., that it does not imply division by an unspecified interval containing the value zero. Therefore a syntactic extension $fR(X_1, \dots, X_k)$ is computable from the bounds of the intervals X_1, \dots, X_k , and usually represents an overestimation of $R_f(X_1, \dots, X_k)$.

Example 3.1.1 The united extension of the continuous real function

$$f(x) = \frac{x}{1+x}$$

to the interval $[2, 4]$ is $R_f([2, 4]) = [2/3, 4/5]$, the range of f in this interval. The syntactic extension for the same interval is

$$fR([2, 4]) = \frac{[2, 4]}{1 + [2, 4]} = [2/5, 4/3],$$

and, in fact, $R_f([2, 4]) = [2/3, 4/5] \subseteq [2/5, 4/3] = fR([2, 4])$.

Syntactic interval functions have the property, fundamental to the whole field of Interval Analysis, of being “inclusive”, that is, for $A_1 \subseteq B_1, \dots, A_k \subseteq B_k$, the relation

$$fR(A_1, \dots, A_k) \subseteq fR(B_1, \dots, B_k)$$

holds.

A basic critical fact is that the interval syntactic extension fR of f satisfies only one kind of interval predicate compatible with outer rounding:

$$(\forall x_1 \in X_1) \cdots (\forall x_k \in X_k) (\exists z \in \text{Out}(fR(X_1, \dots, X_k))) z = f(x_1, \dots, x_k).$$

In the context of modal intervals, it may be expected, as a starting point, that as soon as the \mathbb{R} -predicate $P(x)$ results in the modal interval predicate $Q(x, X)P(x)$, the relation $z = f(x_1, \dots, x_k)$ must become some kind of interval relation $Z = F(X_1, \dots, X_k)$ guaranteeing some sort of $(k + 1)$ -dimensional interval predicate of the form

$$Q_1(x_1, X_1) \dots Q_k(x_k, X_k) Q_z(z, Z) z = f(x_1, \dots, x_k),$$

where an ordering problem obviously arises since the quantifying prefixes are not generally commutable.

3.1.1 Poor Computational Extension

To find a more general approach, we scale down the problem of digital computation to its bare essentials and start by considering the most elementary sort of computational functions able to get actual information about the ideal connections established by the continuous real functions $f : \mathbb{R}^k \rightarrow \mathbb{R}$. To prevent the restrictions of any extension by set of values due to the limited character of system $(I(\mathbb{R}), +, *)$, illustrated in Chap. 1, we will first introduce the definition of a “poor” computational extension of a continuous real function.

Definition 3.1.1 (Poor computational extension) The function $F : \mathbb{R}^k \rightarrow I(\mathbb{R})$ is a *poor computational extension* of a continuous real function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ if the existence of $F(\mathbf{a})'$ implies that $f(\mathbf{a}) \in F(\mathbf{a})'$.

These simplest partial computational functions are defined on a subset of \mathbb{R} and have, wherever defined, the two values $\text{Sup}(F(\mathbf{a})')$ and $\text{Inf}(F(\mathbf{a})')$, upper and lower bounds of the analytically defined value $f(\mathbf{a})$, the value of which the exact determination is, as a general matter of fact, out of reach for digital processing.

The usefulness of this definition is to induce a more general one for extensions of the kind $F : I^*(\mathbb{R}^k) \rightarrow I^*(\mathbb{R})$, as it results from the lemma that follows.

Lemma 3.1.1 (Semantic formulation of a poor computational extension) Let $F : \mathbb{R}^k \rightarrow I(\mathbb{R})$ be a poor computational extension of f , and let $f : \mathbb{R}^k \rightarrow \mathbb{R}$ be a continuous function. Supposing that $F(\mathbf{a})' \in I(\mathbb{R})$ exists, the condition $f(\mathbf{a}) \in F(\mathbf{a})'$ is equivalent to

$$(\forall X' \in I(\mathbb{R}^k)) ((\mathbf{x} \in X') \in \text{Pred}^*([a, \mathbf{a}]) \Rightarrow (z \in f(X')) \in \text{Pred}^*(\text{Prop}(F(\mathbf{a}))))),$$

where $f(X')$ is the united extension or domain of values of f on X' .

Proof This logical formula is equivalent to

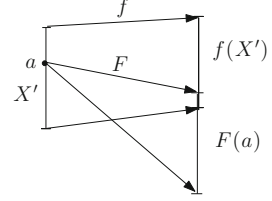
$$(\forall X' \in I(\mathbb{R}^k)) (\mathbf{a} \in X' \Rightarrow F(\mathbf{a})' \cap f(X') \neq \emptyset),$$

which is equivalent to $f(\mathbf{a}) \in F(\mathbf{a})'$ because:

(1) particularizing X' to $[a, \mathbf{a}]'$, becomes

$$(\mathbf{a} \in [a, \mathbf{a}]' \Rightarrow F(\mathbf{a})' \cap f([a, \mathbf{a}]') \neq \emptyset) \Rightarrow f(\mathbf{a}) \in F(\mathbf{a})';$$

Fig. 3.1 Poor extension diagram



(2) for the reverse implication, if $f(a) \in F(a)'$,

$$(\forall X' \in I(\mathbb{R}^k)) (a \in X' \Rightarrow f(a) \in f(X') \Rightarrow F(a)' \cap f(X') \neq \emptyset).$$

■

Figure 3.1 illustrates this proof.

Example 3.1.2 For the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x/3$, a poor computational extension can be $F : \{1\} \rightarrow I(\mathbb{R})$ given by $F(1)' = [0.33, 0.34]'$. In this case, $a = 1$ and $f(1) = 1/3$ satisfy $f(1) \in F(1)'$. This relation is equivalent to

$$(\forall X' \in I(\mathbb{R}))(1 \in X' \Rightarrow [0.33, 0.34]' \cap f(X') \neq \emptyset),$$

which can be written in the form

$$(\forall X' \in I(\mathbb{R}))(1 \in X' \Rightarrow (\exists z \in [0.33, 0.34]') z \in f(X'))$$

or

$$(\forall X' \in I(\mathbb{R}))((x \in X') \in \text{Pred}^*([1, 1]) \Rightarrow (z \in f(X')) \in \text{Pred}^*([0.33, 0.34]')).$$

Now, the equivalent definition for poor computational extensions, made available by this lemma, can be extended to define logically the “modal interval extensions” of continuous functions by formally substituting the element $[a, a]$ by a general modal interval $A \in I^*(\mathbb{R}^k)$, overcoming the rigidities of the theory of functions of the ordinary interval analysis which are induced by the set-theoretical domain-of-values approach.

3.1.2 Modal Interval Extension

In the logical formulation of the poor computational extension of a continuous function, let us replace the argument $[a, a]$ and its modal image $\text{Prop}(F(a)')$ by the more general argument and image, A and $F(A)$.

Definition 3.1.2 (Modal interval extension) If $f : \mathbb{R}^k \rightarrow \mathbb{R}$ is a real continuous function, then $F : I^*(\mathbb{R}^k) \rightarrow I^*(\mathbb{R})$ is its *modal interval extension*, if, wherever $F(A)$ exists,

$$(\forall X' \in I(\mathbb{R}^k)) ((x \in X') \in \text{Pred}^*(A) \Rightarrow (z \in f(X')) \in \text{Pred}^*(F(A))).$$

The logical form of the condition is acceptable. It is thus necessary only to emphasize the properties that make these functions interesting for a computation by intervals. The first indication of the nature of this definition is that it does not give a univocal value for each $F(A)$ which exists, but it gives only a lower limit for the modal inclusion. Indeed, by Lemma 2.2.13, the condition that $F(A)$ is a modal extension of f can be written in the following equivalent form:

$$(\forall X' \in I(\mathbb{R}^k)) (\text{Impr}(X') \subseteq A \Rightarrow \text{Impr}(f(X')) \subseteq F(A)).$$

There remains the task of uncovering the properties which characterize analytically and semantically these formally constructed “modal interval extensions”.

3.2 Semantic Functions

We will define the two “semantic” interval functions which play a grounding role in the theory because they are in close relation with the modal interval extensions of continuous functions.

Definition 3.2.1 (*-semantic extension) If f is an \mathbb{R}^k to \mathbb{R} continuous function and if $\mathbf{x} = (x_p, x_i)$ is the component-splitting corresponding to $X = (X_p, X_i) \in I^*(\mathbb{R}^k)$,

$$\begin{aligned} f^*(X) &= \bigvee_{x_p \in X'_p} \bigwedge_{x_i \in X'_i} [f(x_p, x_i), f(x_p, x_i)] \\ &= [\min_{x_p \in X'_p} \max_{x_i \in X'_i} f(x_p, x_i), \max_{x_p \in X'_p} \min_{x_i \in X'_i} f(x_p, x_i)]. \end{aligned}$$

and is called the *-semantic extension of f .

Definition 3.2.2 (-semantic extension)** With the same hypotheses as the previous definition,

$$\begin{aligned} f^{**}(X) &= \bigwedge_{x_i \in X'_i} \bigvee_{x_p \in X'_p} [f(x_p, x_i), f(x_p, x_i)] = \\ &= [\max_{x_i \in X'_i} \min_{x_p \in X'_p} f(x_p, x_i), \min_{x_i \in X'_i} \max_{x_p \in X'_p} f(x_p, x_i)]. \end{aligned}$$

and is called the **-semantic extension of f .

Remark 3.2.1 If all the X -components are proper intervals, i.e., $X_i = \emptyset$ allowing for the abuse of language, then

$$f^*(X) = f^{**}(X) = [\min\{f(x_1, \dots, x_k) \mid x_1 \in X'_1, \dots, x_k \in X'_k\}, \\ \max\{f(x_1, \dots, x_k) \mid x_1 \in X'_1, \dots, x_k \in X'_k\}],$$

which corresponds to the interval united extension R_f of the classical interval analysis, and $\text{Mod}(f^*(X)) = \exists$.

If all the X -components are improper intervals, i.e., $X_p = \emptyset$ allowing for the abuse of language, then, one has instead

$$f^*(X) = f^{**}(X) = [\max\{f(x_1, \dots, x_k) \mid x_1 \in X'_1, \dots, x_k \in X'_k\}, \\ \min\{f(x_1, \dots, x_k) \mid x_1 \in X'_1, \dots, x_k \in X'_k\}],$$

with $\text{Set}(f^*(X)) = R_f$ and $\text{Mod}(f^*(X)) = \forall$.

Example 3.2.1 For the continuous real continuous $f(x_1, x_2) = x_1^2 + x_2^2$, the computation of the $*$ -semantic and the $**$ -semantic functions for $X = ([-1, 1], [1, -1])$ yields the following results:

$$\begin{aligned} f^*([-1, 1], [1, -1]) &= \bigvee_{x_1 \in [-1, 1]'} \bigwedge_{x_2 \in [-1, 1]'} [x_1^2 + x_2^2, x_1^2 + x_2^2] \\ &= \bigvee_{x_1 \in [-1, 1]'} [x_1^2 + 1, x_1^2] = [1, 1]; \\ f^{**}([-1, 1], [1, -1]) &= \bigwedge_{x_2 \in [-1, 1]'} \bigvee_{x_1 \in [-1, 1]'} [x_1^2 + x_2^2, x_1^2 + x_2^2] \\ &= \bigwedge_{x_2 \in [-1, 1]'} [x_2^2, 1 + x_2^2] = [1, 1]. \end{aligned}$$

For the continuous real continuous function $g(x_1, x_2) = (x_1 + x_2)^2$, the corresponding $*$ -semantic and $**$ -semantic functions for $X = ([-1, 1], [1, -1])$ don't have coincident values:

$$\begin{aligned} g^*([-1, 1], [1, -1]) &= \bigvee_{x_1 \in [-1, 1]'} \bigwedge_{x_2 \in [-1, 1]'} [(x_1 + x_2)^2, (x_1 + x_2)^2] \\ &= \bigvee_{x_1 \in [-1, 1]'} [\text{if } x_1 < 0 \text{ then } (x_1 - 1)^2 \text{ else } (x_1 + 1)^2, 0] \\ &= [1, 0]; \\ g^{**}([-1, 1], [1, -1]) &= \bigwedge_{x_2 \in [-1, 1]'} \bigvee_{x_1 \in [-1, 1]'} [(x_1 + x_2)^2, (x_1 + x_2)^2] \\ &= \bigwedge_{x_2 \in [-1, 1]'} [0, \text{if } x_2 < 0 \text{ then } (x_2 - 1)^2 \text{ else } (x_2 + 1)^2] \\ &= [0, 1]. \end{aligned}$$

In general, both semantic extensions are out of reach of any direct computation except for some very simple continuous real functions, as the previous example may suggest, as unary operator (\exp, \ln, \dots) and the arithmetic operators of which semantic computations, properties and implementation are in Chap. 5.

The case of equality between both extensions characterizes the following important concept.

Definition 3.2.3 (JM-commutativity) A continuous real function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ is JM-commutable for $A \in I^*(\mathbb{R}^k)$ if $f^*(A) = f^{**}(A)$.

3.2.1 Properties of the Semantic Extensions

The semantic extensions are not independent: there exists a relation of duality.

Theorem 3.2.1 (Duality of the semantic functions) If f is an \mathbb{R}^k to \mathbb{R} continuous function and $X \in I^*(\mathbb{R}^k)$,

$$\text{Dual}(f^*(X)) = f^{**}(\text{Dual}(X)).$$

Proof From the definitions of f^* and f^{**} and Lemma 2.2.10

$$\begin{aligned} \text{Dual}(f^*(X)) &= \text{Dual}\left(\bigvee_{x_p \in X'_p} \bigwedge_{x_i \in X'_i} [f(x), f(x)]\right) \\ &= \bigwedge_{x_p \in X'_p} \bigvee_{x_i \in X'_i} [f(x), f(x)] = f^{**}(\text{Dual}(X)). \end{aligned}$$

■

The following result yields the basic relation of inclusion between the semantic extensions.

Theorem 3.2.2 (Min–max) If f is an \mathbb{R}^k to \mathbb{R} continuous function, and (X'_1, X'_2) is any component splitting of $X' \in I(\mathbb{R}^k)$, then

$$(\forall (x_1, x_2) \in (X'_1, X'_2)) \max_{x_1 \in X'_1} \min_{x_2 \in X'_2} f(x_1, x_2) \leq \min_{x_2 \in X'_2} \max_{x_1 \in X'_1} f(x_1, x_2)$$

and

$$\max_{x_1 \in X'_1} \min_{x_2 \in X'_2} f(x_1, x_2) \leq f(x_{1m}, x_{2M}) \leq \min_{x_2 \in X'_2} \max_{x_1 \in X'_1} f(x_1, x_2),$$

where x_{1m} is a point on which the function $\min_{x_2 \in X'_2} f(x_1, x_2)$ reaches its maximum and x_{2M} is a point on which the function $\max_{x_1 \in X'_1} f(x_1, x_2)$ reaches its minimum.

Proof The first inequality is true since

$$\begin{aligned} (\forall \mathbf{x}_1 \in X'_1) \min_{\mathbf{x}_2 \in X'_2} f(\mathbf{x}_1, \mathbf{x}_2) &\leq f(\mathbf{x}_1, \mathbf{x}_2) \\ (\forall \mathbf{x}_2 \in X'_2) \max_{\mathbf{x}_1 \in X'_1} f(\mathbf{x}_1, \mathbf{x}_2) &\geq f(\mathbf{x}_1, \mathbf{x}_2) \end{aligned}$$

For the second inequality, defining

$$\begin{aligned} f_m(\mathbf{x}'_1) &= \min_{\mathbf{x}_2 \in X'_2} f(\mathbf{x}'_1, \mathbf{x}_2), \\ f_M(\mathbf{x}'_2) &= \max_{\mathbf{x}_1 \in X'_1} f(\mathbf{x}_1, \mathbf{x}'_2) \end{aligned}$$

it follows that

$$(\forall \mathbf{x}'_1 \in X'_1) (\forall \mathbf{x}'_2 \in X'_2) (f_m(\mathbf{x}'_1) \leq f(\mathbf{x}'_1, \mathbf{x}'_2) \leq f_M(\mathbf{x}'_2)).$$

■

Remark 3.2.2 Since all the values $f_m(\mathbf{x}'_1)$ are less than or equal to all the values of $f_M(\mathbf{x}'_2)$, and the functions f_m and f_M are continuous, the sets $F'_m = \{f_m(\mathbf{x}'_1) \mid \mathbf{x}'_1 \in X'_1\}$ and $F'_M = \{f_M(\mathbf{x}'_2) \mid \mathbf{x}'_2 \in X'_2\}$ are intervals such that $\text{Sup}(F'_m) \leq \text{Inf}(F'_M)$, as is partially stated by this theorem.

Next, firstly, the inclusion relation between $f^*(X)$ and $f^{**}(X)$ will be shown:

Theorem 3.2.3 (Inclusion of f^* in f^{})** *If f is an \mathbb{R}^k to \mathbb{R} continuous real function and $X \in I^*(\mathbb{R}^k)$, then*

$$f^*(X) \subseteq f^{**}(X).$$

Proof

$$\begin{aligned} f^*(X) &= \bigvee_{\mathbf{x}_p \in X'_p} \bigwedge_{\mathbf{x}_i \in X'_i} [f(\mathbf{x}_p, \mathbf{x}_i), f(\mathbf{x}_p, \mathbf{x}_i)] \\ &= [\min_{\mathbf{x}_p \in X'_p} \max_{\mathbf{x}_i \in X'_i} f(\mathbf{x}_p, \mathbf{x}_i), \max_{\mathbf{x}_p \in X'_p} \min_{\mathbf{x}_i \in X'_i} f(\mathbf{x}_p, \mathbf{x}_i)] \\ &\subseteq [\max_{\mathbf{x}_i \in X'_i} \min_{\mathbf{x}_p \in X'_p} f(\mathbf{x}_p, \mathbf{x}_i), \min_{\mathbf{x}_i \in X'_i} \max_{\mathbf{x}_p \in X'_p} f(\mathbf{x}_p, \mathbf{x}_i)] \\ &= \bigwedge_{\mathbf{x}_i \in X'_i} \bigvee_{\mathbf{x}_p \in X'_p} [f(\mathbf{x}_p, \mathbf{x}_i), f(\mathbf{x}_p, \mathbf{x}_i)] \\ &= f^{**}(X). \end{aligned}$$

■

Secondly, there follows the \subseteq -monotonicity of f^* and f^{**} .

Lemma 3.2.1 For $X \in I^*(\mathbb{R})$ and any \mathbb{R} to \mathbb{R} continuous functions F_1, F_2

$$F_1(x) \subseteq F_2(x) \Rightarrow \bigcup_{(x,X)} F_1(x) \subseteq \bigcup_{(x,X)} F_2(x).$$

Proof In agreement with the Definition 2.2.17 of the meet-join operator,

a) If X is a proper interval,

$$\begin{aligned} \bigcup_{(x,X)} F_1(x) &= \bigvee_{x \in X'} F_1(x) \\ &= [\min_{x \in X'} \text{Inf}(F_1(x)), \max_{x \in X'} \text{Sup}(F_1(x))] \\ &\subseteq [\min_{x \in X'} \text{Inf}(F_2(x)), \max_{x \in X'} \text{Sup}(F_2(x))] \\ &= \bigvee_{x \in X'} F_2(x) \\ &= \bigcup_{(x,X)} F_2(x). \end{aligned}$$

b) A dual proof is valid if X is an improper interval.

Lemma 3.2.2 For $X_1, X_2 \in I^*(\mathbb{R})$ and $F : \mathbb{R} \rightarrow I^*(\mathbb{R})$,

$$X_1 \subseteq X_2 \Rightarrow \bigcup_{(x,X_1)} F(x) \subseteq \bigcup_{(x,X_2)} F(x).$$

Proof

a) If X_1 is a proper interval,

$$\bigcup_{(x,X_1)} F(x) = \bigvee_{x \in X'_1} F(x) = [\min_{x \in X'_1} \text{Inf}(F(x)), \max_{x \in X'_1} \text{Sup}(F(x))],$$

a1) if X_2 is a proper interval, then $X'_1 \subseteq X'_2$ and therefore

$$\begin{aligned} &[\min_{x \in X'_1} \text{Inf}(F(x)), \max_{x \in X'_1} \text{Sup}(F(x))] \\ &\subseteq [\min_{x \in X'_2} \text{Inf}(F(x)), \max_{x \in X'_2} \text{Sup}(F(x))] \\ &= \bigvee_{x \in X'_2} F(x) \\ &= \bigcup_{(x,X_2)} F(x), \end{aligned}$$

a2) if X_2 is an improper interval, then $X'_1 = X'_2 = \{a\}$ and $\bigcup_{(x,[a,a])}$ reduces to the identity operator.

b) The proof is completed by a similar reasoning if X_1 is an improper interval, and by a two step process through a point $x_0 \in X'_1 \cap X'_2$ for the case of X_1 improper and X_2 proper. ■

Lemma 3.2.3 For $X_1, X_2 \in I^*(\mathbb{R})$ and $F_1, F_2 : \mathbb{R} \rightarrow I^*(\mathbb{R})$,

$$(X_1 \subseteq X_2, F_1(x) \subseteq F_2(x)) \Rightarrow \bigcup_{(x,X_1)} F_1(x) \subseteq \bigcup_{(x,X_2)} F_2(x).$$

Proof From Lemmata 3.2.1 and 3.2.2,

$$\bigcup_{(x,X_1)} F_1(x) \subseteq \bigcup_{(x,X_2)} F_1(x) \subseteq \bigcup_{(x,X_2)} F_2(x).$$

■

Theorem 3.2.4 (Inclusivity of the semantic extensions) If $X, Y \in I^*(\mathbb{R}^k)$ and f from \mathbb{R}^k to \mathbb{R} is continuous,

$$X \subseteq Y \Rightarrow (f^*(X) \subseteq f^*(Y), f^{**}(X) \subseteq f^{**}(Y)).$$

Proof From the previous lemma,

$$\begin{aligned} \left. \begin{matrix} f^*(X) \\ f^{**}(X) \end{matrix} \right\} &= \bigcup_{(x_1,X_1)} \cdots \bigcup_{(x_k,X_k)} [f(x_1, \dots, x_k), f(x_1, \dots, x_k)] \\ &\subseteq \bigcup_{(x_1,Y_1)} \cdots \bigcup_{(x_k,Y_k)} [f(x_1, \dots, x_k), f(x_1, \dots, x_k)] = \left\{ \begin{matrix} f^*(Y) \\ f^{**}(Y) \end{matrix} \right\} \end{aligned}$$

■

3.2.2 Characterization of JM-Commutativity

Next the case $f^*(X) = f^{**}(X)$ will be characterized, when X is not uni-modal. The main role in this characterization is played by the saddle-points of the function f .

Definition 3.2.4 (Saddle-points set) Let $(X'_1, X'_2) = X'$ be a component splitting of $X' \in I(\mathbb{R}^k)$, and f be a continuous function from \mathbb{R}^k to \mathbb{R} . The set of saddle points of f in X' is

$$\begin{aligned} \text{SDP}(f, X'_1, X'_2) &= \{(x_{1m}, x_{2M}) \mid (\forall x_1 \in X'_1)(\forall x_2 \in X'_2) (f(x_{1m}, x_2) \\ &\leq f(x_{1m}, x_{2M}) \leq f(x_1, x_{2M}))\}. \end{aligned}$$

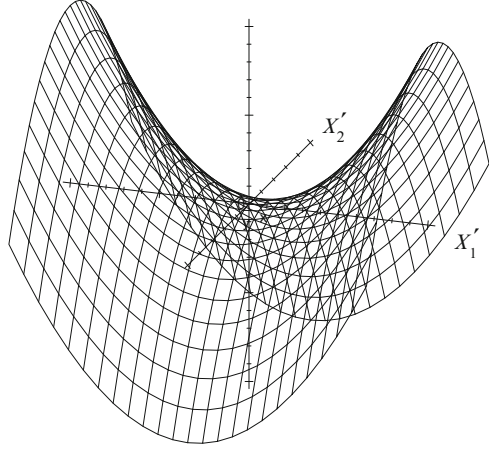
Fig. 3.2 Saddle-point

Figure 3.2 illustrates this definition.

Definition 3.2.5 (Saddle value). Let $(X'_1, X'_2) = X'$ be a component splitting of $X' \in I(\mathbb{R}^k)$, and f be a continuous function from \mathbb{R}^k to \mathbb{R} . The set of *saddle values* of f in X' is

$$\text{SDV}(f, X'_1, X'_2) = (\mathbf{x}_{1m}, \mathbf{x}_{2M})$$

if $(\text{SDP}(f, X'_1, X'_2) \neq \emptyset \text{ and } (\mathbf{x}_{1m}, \mathbf{x}_{2M}) \in \text{SDP}(f, X'_1, X'_2))$. Otherwise, it is undefined.

A well known property of saddle points is in the next lemma.

Lemma 3.2.4 *If $(\mathbf{x}_{1m}, \mathbf{x}_{2M})$ and $(\mathbf{x}'_{1m}, \mathbf{x}'_{2M})$ are two saddle points of f in (X'_1, X'_2) , then $(\mathbf{x}_{1m}, \mathbf{x}'_{2M})$ and $(\mathbf{x}'_{1m}, \mathbf{x}_{2M})$ are also saddle points. Moreover,*

$$f(\mathbf{x}_{1m}, \mathbf{x}_{2M}) = f(\mathbf{x}_{1m}, \mathbf{x}'_{2M}) = f(\mathbf{x}'_{1m}, \mathbf{x}_{2M}) = f(\mathbf{x}'_{1m}, \mathbf{x}'_{2M})$$

Proof For any $\mathbf{x}_1 \in X'_1$ and $\mathbf{x}_2 \in X'_2$ the inequalities

$$f(\mathbf{x}_{1m}, \mathbf{x}_2) \leq f(\mathbf{x}_{1m}, \mathbf{x}_{2M}) \leq f(\mathbf{x}_1, \mathbf{x}_{2M})$$

$$f(\mathbf{x}'_{1m}, \mathbf{x}_2) \leq f(\mathbf{x}'_{1m}, \mathbf{x}'_{2M}) \leq f(\mathbf{x}_1, \mathbf{x}'_{2M})$$

are true. So, particularizing the first one to \mathbf{x}'_{1m} and \mathbf{x}'_{2M} and the second one to \mathbf{x}_{1m} and \mathbf{x}_{2M} ,

$$f(\mathbf{x}'_{1m}, \mathbf{x}_{2M}) \leq f(\mathbf{x}'_{1m}, \mathbf{x}'_{2M}) \leq f(\mathbf{x}_{1m}, \mathbf{x}'_{2M}) \leq f(\mathbf{x}_{1m}, \mathbf{x}_{2M}) \leq f(\mathbf{x}'_{1m}, \mathbf{x}_{2M})$$

which implies the result. Moreover

$$f(\mathbf{x}'_{1m}, \mathbf{x}_{2M}) = f(\mathbf{x}_{1m}, \mathbf{x}_{2M}) \leq f(\mathbf{x}_1, \mathbf{x}_{2M})$$

and

$$f(\mathbf{x}'_{1m}, \mathbf{x}_2) \leq f(\mathbf{x}'_{1m}, \mathbf{x}'_{2M}) = f(\mathbf{x}'_{1m}, \mathbf{x}_{2M})$$

imply

$$f(\mathbf{x}'_{1m}, \mathbf{x}_2) \leq f(\mathbf{x}'_{1m}, \mathbf{x}_{2M}) \leq f(\mathbf{x}_1, \mathbf{x}_{2M}).$$

Therefore $(\mathbf{x}'_{1m}, \mathbf{x}_{2M})$ is a saddle point. Similarly for $(\mathbf{x}_{1m}, \mathbf{x}'_{2M})$. ■

In accordance with this result, the set of saddle values of f in (X'_1, X'_2) is either empty or contains a unique point.

Lemma 3.2.5 *In the context of the previous definition, if there exists a saddle point $(\mathbf{x}_{1m}, \mathbf{x}_{2M})$ of f in X' ,*

$$\begin{aligned} \text{SDV}(f, X'_1, X'_2) &= f(\mathbf{x}_{1m}, \mathbf{x}_{2M}) \\ &= \min_{\mathbf{x}_1 \in X'_1} \max_{\mathbf{x}_2 \in X'_2} f(\mathbf{x}_1, \mathbf{x}_2) = \max_{\mathbf{x}_2 \in X'_2} \min_{\mathbf{x}_1 \in X'_1} f(\mathbf{x}_1, \mathbf{x}_2). \end{aligned}$$

Proof From

$$\begin{aligned} \min_{\mathbf{x}_1 \in X'_1} \max_{\mathbf{x}_2 \in X'_2} f(\mathbf{x}_1, \mathbf{x}_2) &\leq \max_{\mathbf{x}_2 \in X'_2} f(\mathbf{x}_{1m}, \mathbf{x}_2) \leq f(\mathbf{x}_{1m}, \mathbf{x}_{2M}) \\ &\leq \min_{\mathbf{x}_1 \in X'_1} f(\mathbf{x}_1, \mathbf{x}_{2M}) \leq \max_{\mathbf{x}_2 \in X'_2} \min_{\mathbf{x}_1 \in X'_1} f(\mathbf{x}_1, \mathbf{x}_2) \end{aligned}$$

and Theorem 3.2.2 which closes the \leq -chain. ■

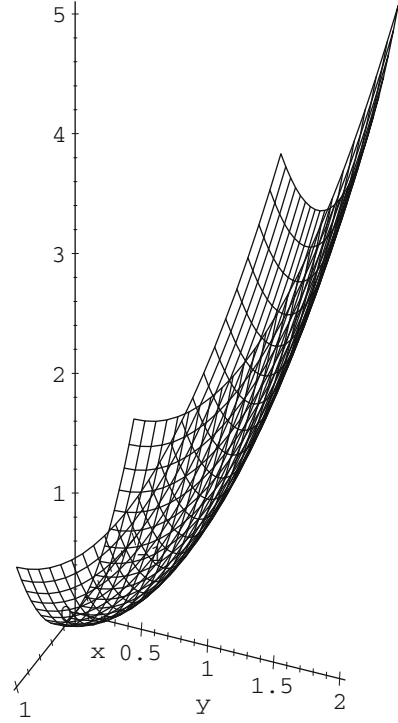
Theorem 3.2.5 (JM-commutativity) *For a given $X \in I^*(\mathbb{R}^k)$, the joint validity of $\text{SDP}(f, X'_p, X'_i) \neq \emptyset$, $\text{SDP}(f, X'_i, X'_p) \neq \emptyset$ is equivalent to $f^*(X) = f^{**}(X)$; in this case,*

$$f^*(X) = f^{**}(X) = [\text{SDV}(f, X'_p, X'_i), \text{SDV}(f, X'_i, X'_p)].$$

Proof As

$$\begin{aligned} \text{SDV}(f, X'_p, X'_i) &= \min_{\mathbf{x}_p \in X'_p} \max_{\mathbf{x}_i \in X'_i} f(\mathbf{x}_p, \mathbf{x}_i) = \max_{\mathbf{x}_i \in X'_i} \min_{\mathbf{x}_p \in X'_p} f(\mathbf{x}_p, \mathbf{x}_i), \\ \text{SDV}(f, X'_i, X'_p) &= \min_{\mathbf{x}_i \in X'_i} \max_{\mathbf{x}_p \in X'_p} f(\mathbf{x}_p, \mathbf{x}_i) = \max_{\mathbf{x}_p \in X'_p} \min_{\mathbf{x}_i \in X'_i} f(\mathbf{x}_p, \mathbf{x}_i), \end{aligned}$$

Fig. 3.3 Function
 $f(x_1, x_2) = x_1^2 + x_2^2$ in
 $X' = ([-1, 1]', [0, 2]')$



then

$$\begin{aligned}
 f^*(X) &= \bigvee_{x_p \in X'_p} \bigwedge_{x_i \in X'_i} [f(x_p, x_i), f(x_p, x_i)] \\
 &= [\min_{x_p \in X'_p} \max_{x_i \in X'_i} f(x_p, x_i), \max_{x_p \in X'_p} \min_{x_i \in X'_i} f(x_p, x_i)] \\
 &= [\max_{x_i \in X'_i} \min_{x_p \in X'_p} f(x_p, x_i), \min_{x_i \in X'_i} \max_{x_p \in X'_p} f(x_p, x_i)] \\
 &= \bigwedge_{x_i \in X'_i} \bigvee_{x_p \in X'_p} [f(x_p, x_i), f(x_p, x_i)] = f^{**}(X).
 \end{aligned}$$

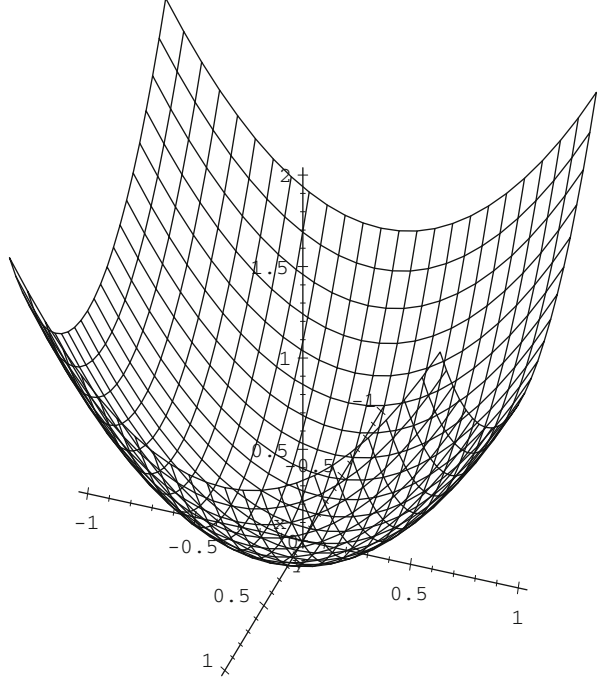
■

Remark 3.2.3 The *JM*-commutativity of a function f implies the applicability of one of the two semantic theorems, the direct or its dual. Which one will depend only on the truncation's sense, outer or inner, of the computation of f^* .

Example 3.2.2 For the continuous real function $f(x_1, x_2) = x_1^2 + x_2^2$ the $*$ -semantic and $**$ -semantic extensions for $X = ([-1, 1], [2, 0])$ (see Fig. 3.3) are

Fig. 3.4 Function

$f(x_1, x_2) = x_1^2 + x_2^2$ in
 $X' = ([-1, 1]', [-1, 1]')$



$$\begin{aligned}
 f^*([-1, 1], [2, 0]) &= \bigvee_{x_1 \in [-1, 1]'} \bigwedge_{x_2 \in [0, 2]'} [x_1^2 + x_2^2, x_1^2 + x_2^2] \\
 &= \bigvee_{x_1 \in [-1, 1]'} [x_1^2 + 4, x_1^2] = [4, 1], \\
 f^{**}([-1, 1], [2, 0]) &= \bigwedge_{x_2 \in [0, 2]'} \bigvee_{x_1 \in [-1, 1]'} [x_1^2 + x_2^2, x_1^2 + x_2^2] \\
 &= \bigwedge_{x_2 \in [0, 2]'} [x_2^2, x_2^2 + 1] = [4, 1]
 \end{aligned}$$

and

$$\begin{aligned}
 \text{SDP}(f, X'_p, X'_i) &= \text{SDP}(f, [-1, 1]', [0, 2]') = \{(0, 2)\} \\
 \text{SDV}(f, X'_p, X'_i) &= \text{SDV}(f, [-1, 1]', [0, 2]') = 4 \\
 \text{SDP}(f, X'_i, X'_p) &= \text{SDP}(f, [0, 2]', [-1, 1]') = \{(1, 0), (-1, 0)\} \\
 \text{SDV}(f, X'_i, X'_p) &= \text{SDV}(f, [0, 2]', [-1, 1]') = 1.
 \end{aligned}$$

For the same function $f(x_1, x_2) = x_1^2 + x_2^2$, the *-semantic and **-semantic extensions for $X = ([-1, 1], [1, -1])$ (see Fig. 3.4) are

$$\begin{aligned}
f^*([-1, 1], [1, -1]) &= \bigvee_{x_1 \in [-1, 1]'} \bigwedge_{x_2 \in [-1, 1]'} [x_1^2 + x_2^2, x_1^2 + x_2^2] \\
&= \bigvee_{x_1 \in [-1, 1]'} [x_1^2 + 1, x_1^2] = [1, 1], \\
f^{**}([-1, 1], [1, -1]) &= \bigwedge_{x_2 \in [-1, 1]'} \bigvee_{x_1 \in [-1, 1]'} [x_1^2 + x_2^2, x_1^2 + x_2^2] \\
&= \bigwedge_{x_2 \in [-1, 1]'} [x_2^2, x_2^2 + 1] = [1, 1]
\end{aligned}$$

and

$$\begin{aligned}
\text{SDP}(f, X'_p, X'_i) &= \text{SDP}(f, [-1, 1]', [-1, 1]') = \{(0, 1), (0, -1)\} \\
\text{SDV}(f, X'_p, X'_i) &= \text{SDV}(f, [-1, 1]', [-1, 1]') = 1 \\
\text{SDP}(f, X'_i, X'_p) &= \text{SDP}(f, [-1, 1]', [-1, 1]') = \{(-1, 0), (1, 0)\} \\
\text{SDV}(f, X'_i, X'_p) &= \text{SDV}(f, [-1, 1]', [-1, 1]') = 1.
\end{aligned}$$

3.3 Semantic Theorems

The values of the extensions f^* or f^{**} may not yield, without further thought, much clear meaning about the values of the real f on its domain. Two key theorems reverse this misimpression, uncovering completely the meaning of the interval results f^* and f^{**} and characterizing them as the key referents for the semantic interval extensions previously defined in logical terms.

Theorem 3.3.1 (*-semantic theorem) *Given a continuous real function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ and a modal vector $\mathbf{A} \in I^*(\mathbb{R}^k)$, whenever $F(\mathbf{A}) \in I^*(\mathbb{R})$ exists, we have that the following are equivalent propositions:*

- a) $f^*(\mathbf{A}) \subseteq F(\mathbf{A})$,
- b) $(\forall X' \in I(\mathbb{R}^k)) ((x \in X') \in \text{Pred}^*(\mathbf{A}) \Rightarrow (z \in f(X')) \in \text{Pred}^*(F(\mathbf{A})))$,
- c) $(\forall \mathbf{a}_p \in \mathbf{A}'_p) Q(z, F(\mathbf{A})) (\exists \mathbf{a}_i \in \mathbf{A}'_i) z = f(\mathbf{a}_p, \mathbf{a}_i)$

Proof If A_1, \dots, A_p are the proper components of \mathbf{A} and A_{p+1}, \dots, A_k the improper ones, then

$$\begin{aligned}
\text{Impr}(f(\mathbf{a}_p, \mathbf{A}'_i)) &= \text{Dual}(f^*(\mathbf{a}_p, \mathbf{A}'_i)) \\
&= \text{Dual}\left(\bigvee_{a_{p+1} \in A'_{p+1}} \dots \bigvee_{a_k \in A'_k} [f(a_1, \dots, a_p, a_{p+1}, \dots, a_k), \right. \\
&\quad \left. f(a_1, \dots, a_p, a_{p+1}, \dots, a_k)]\right)
\end{aligned}$$

$$\begin{aligned}
&= \bigwedge_{a_{p+1} \in A'_{p+1}} \dots \bigwedge_{a_k \in A'_k} [f(a_1, \dots, a_p, a_{p+1}, \dots, a_k), \\
&\quad f(a_1, \dots, a_p, a_{p+1}, \dots, a_k)] \\
&= f^*(a_p, A_i).
\end{aligned}$$

Therefore, (a) implies (b):

$$\begin{aligned}
&(\forall X' \in I(\mathbb{R}^k) (x \in X') \in \text{Pred}^*(A)) \\
&\Leftrightarrow \text{Impr}(X') \subseteq A \\
&\Leftrightarrow (\exists a_p \in A'_p) (a_p \in X'_1, X'_2 \supseteq A'_i) \\
&\quad // ((X_1, X_2) \text{ is the components' splitting corresponding to } (A_p, A_i). \\
&\Rightarrow (\exists a_p \in A'_p) f(X'_1, X'_2) \supseteq f(a_p, A'_i) \\
&\quad // f(X'_1, X'_2) \text{ or } f(X') \text{ designates the united extension of } f \text{ on } X'. \\
&\Leftrightarrow (\exists a_p \in A'_p) \text{Impr}(f(X')) \subseteq f^*(a_p, A_i) \\
&\Rightarrow \text{Impr}(f(X')) \subseteq f^*(A_p, A_i) \\
&\Rightarrow \text{Impr}(f(X')) \subseteq F(A) \\
&\quad // \text{see the hypothesis a).} \\
&\Leftrightarrow (z \in f(X')) \in \text{Pred}^*(F(A)).
\end{aligned}$$

(b) implies (a): Let a_p be any point of A'_p and X' the interval (a_p, A'_i) ,

$$\begin{aligned}
&(\forall a_p \in A'_p) (x \in (a_p, A'_i) \in \text{Pred}^*(A)) \\
&\Rightarrow (\forall a_p \in A'_p) (z \in f(a_p, A'_i)) \in \text{Pred}^*(F(A)) \\
&\quad // \text{Particularization of the hypothesis b).} \\
&\Leftrightarrow (\forall a_p \in A'_p) \text{Impr}(f(a_p, A'_i)) \subseteq F(A) \\
&\Leftrightarrow (\forall a_p \in A'_p) f^*(a_p, A_i) \subseteq F(A) \\
&\Leftrightarrow f^*(A_p, A_i) \subseteq F(A).
\end{aligned}$$

(a) is equivalent to (c):

$$\begin{aligned}
&f^*(A) \subseteq F(A) \\
&\Leftrightarrow (\forall a_p \in A'_p) f^*(a_p, A_i) \subseteq F(A) \\
&\Leftrightarrow (\forall a_p \in A'_p) (z \in f(a_p, A'_i)) \in \text{Pred}^*(F(A))
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow (\forall \mathbf{a}_p \in \mathbf{A}'_p) \mathbf{Q}(z, F(\mathbf{A})) z \in f(\mathbf{a}_p, \mathbf{A}'_i) \\
&\Leftrightarrow (\forall \mathbf{a}_p \in \mathbf{A}'_p) \mathbf{Q}(z, F(\mathbf{A})) (\exists \mathbf{a}_i \in \mathbf{A}'_i) z = f(\mathbf{a}_p, \mathbf{a}_i).
\end{aligned}$$

■

Remark 3.3.1 The *-semantic theorem allows interpreting universal intervals as “regulating or feedback ranges”, and existential intervals as “fluctuation or autonomous ranges” for the system consisting of the interval data and result $(\mathbf{A}_p, \mathbf{A}_i, F(\mathbf{A}))$, and the analytical connection $z = f(\mathbf{a}_p, \mathbf{a}_i)$.

Example 3.3.1 For the continuous real function $f(x, y) = x + y$, from the definition of f^* ,

$$f^*([x_1, x_2], [y_1, y_2]) = \bigcup_{(x, [x_1, x_2])} \bigcup_{(y, [y_1, y_2])} [x + y, x + y] = [x_1 + y_1, x_2 + y_2].$$

as will be proved in Sect. 5.3.1. For $X = [1, 2]$ and $Y = [2, 3]$ and since the result is $Z = [3, 5]$, we may write $[1, 2] + [2, 3] = [3, 5]$, with the meaning

$$(\forall x \in [1, 2]') (\forall y \in [2, 3]') (\exists z \in [3, 5]') x + y = z.$$

Similarly, for $X = [1, 2]$ and $Y = [4, 1]$ the result is $Z = [1, 2] + [4, 1] = [5, 3]$ which means, in this case,

$$(\forall x \in [1, 2]') (\forall z \in [3, 5]') (\exists y \in [1, 4]') x + y = z.$$

And so on, for $X = [2, 1]$ and $Y = [1, 4]$ the result is $Z = [2, 1] + [1, 4] = [3, 5]$, which means

$$(\forall y \in [1, 4]') (\exists x \in [1, 2]') (\exists z \in [3, 5]') x + y = z;$$

for $X = [2, 1]$ and $Y = [3, 2]$ the result is $Z = [2, 1] + [3, 2] = [5, 3]$ with the interpretation

$$(\forall z \in [3, 5]') (\exists x \in [1, 2]') (\exists y \in [2, 3]') x + y = z.$$

Moreover, these interval statements (or interpretations of the modal functional relation $f^*(X, Y) = Z$) are robust to “modal outer rounding” of the result, as is shown for example in the replacement of the Z -value $[3, 5]$ by $[2.9, 5.1] \supseteq [3, 5]$, or of $[5, 3]$ by $[4.9, 3.1] \supseteq [5, 3]$, the latter being equivalent to a set-theoretical inner rounding of $[3, 5]'$.

Example 3.3.2 Let us apply the result about the function $f(x, y) = x + y$ to a naturalistic context. Suppose we have two cable reels of lengths $a = 10$ and $b = 20$ units. When connected, they can cover an overall length $c = 30$. This most elementary situation can be expressed for all that computationally matters by the algebraic expression $a + b = c$.

Consider the parallel but more realistic interval-situation where the first reel of cable has a length a known only to lie in a range bounded by the interval $A' = [10, 20]'$, i.e., $a \in [10, 20]'$; about the second reel we know that $b \in B' = [10, 25]'$. Let us consider the connection between both reels and let us apply the *-Semantic Theorem 3.3.1 for f^* restricted to the function of addition $f(a, b) = a + b$.

Case 1: $[10, 20] + [25, 10] = [35, 30]$ means

$$(\forall a \in [10, 20]') (\forall c \in [30, 35]') (\exists b \in [10, 25]') c = a + b$$

that is, a determined length of a wider regulating interval $[25, 10]$ can be selected to get some, in principle, unknown but determinable length c lying within the improper interval $C = [35, 30]$, in spite of the value a belonging to the proper operand $A = [10, 20]$ being understood as coming out of some general random selection process.

Case 2: $[10, 20] + [10, 10] = [20, 30]$,

$$[10, 20] + [17, 17] = [27, 37],$$

$$[10, 20] + [25, 25] = [35, 45],$$

the variable b taking fixed values 10, 17 or 25 in the interval-set $B' = [10, 25]'$, the indeterminacy of $A = [10, 20]$ is carried to the interval C by the relation $c = a + b$ so that the value of c will range randomly and in parallel with a on one of the intervals $[20, 30]$, $[27, 37]$ or $[35, 45]$. The quantified statement (for example for the first equality) is, if b is bounded to the only value of the point interval $[10, 10]$,

$$(\forall a \in [10, 20]') (\exists c \in [20, 30]') c = a + 10.$$

Case 3: $[10, 20] + [10, 25] = [20, 45]$ means

$$(\forall a \in [10, 20]') (\forall b \in [10, 25]') (\exists c \in [20, 45]') c = a + b,$$

so that c will show the joint full indeterminacy coming from a and b .

Case 4: $[20, 10] + [25, 10] = [45, 20]$ will be interpreted by

$$(\forall c \in [20, 45]') (\exists a \in [10, 20]') (\exists b \in [10, 25]') c = a + b.$$

Case 5: $[10, 20] + [20, 15] = [30, 35]$ means

$$(\forall a \in [10, 20]') (\exists c \in [30, 35]') (\exists b \in [15, 20]') c = a + b$$

that is, with the same autonomous interval $A = [10, 20]$ and a narrower regulating interval $B = [20, 15]$, a determined length of $b \in B' = [15, 20]'$ should be selected (a regulation operation) just to get some length c lying within the domain of the proper interval $C = [30, 35]'$.

A dual feedback semantics for proper and improper modal intervals is established by the following Dual Semantic theorem.

Theorem 3.3.2 (-semantic theorem)** *Given a continuous real functions $f : \mathbb{R}^k \rightarrow \mathbb{R}$ and a modal vector $A \in I^*(\mathbb{R}^k)$, whenever $F(A) \in I^*(\mathbb{R})$ exists, we have that the following are equivalent propositions:*

- a) $f^{**}(A) \supseteq F(A)$,
- b) $(\forall X' \in I(\mathbb{R}^k)) ((x \notin X') \in \text{Copred}^*(A) \Rightarrow (z \notin f(X')) \in \text{Copred}^*(F(A)))$,
- c) $(\forall a_i \in A'_i) Q(z, \text{Dual}(F(A))) (\exists a_p \in A'_p) z = f(a_p, a_i)$.

Proof From the definitions of $f^*(X)$ and $f^{**}(X)$ we obtain

$$\text{Dual}(f^*(X)) = f^{**}(\text{Dual}(X)).$$

Applying Theorem 3.3.1 to $f^*(\text{Dual}(A)) \subseteq \text{Dual}(F(A))$, Theorem 3.3.2 follows. ■

Example 3.3.3 For the function $f(x, y) = xy$ and $X = [-1, 2]$, $Y = [5, 3]$ the values of f^* and f^{**} are $f^*([-1, 2], [5, 3]) = f^{**}([-1, 2], [5, 3]) = [-3, 6]$ (see Chap. 5). Then, in accordance with both semantic theorems,

$$\begin{aligned} (\forall x \in [-1, 2]') (\exists z \in [-3, 6]') (\exists y \in [3, 5]') z = xy, \\ (\forall y \in [3, 5]') (\forall z \in [-3, 6]') (\exists x \in [-1, 2]') z = xy. \end{aligned}$$

Remark 3.3.2 The Semantic Theorems show that:

- The semantic decision to apply the *-semantic theorem or the **-semantic theorem is made when one of the modal roundings, outer or inner, is selected.
- The functions $f^*(X)$ and $f^{**}(X)$ are semantically optimal for each semantic theorem.
- The effective computation of a modal extension $F(A)$ is not indicated by these two theorems which give only modal bounds, $f^*(X)$ or $f^{**}(X)$ according to the chosen rounding, to any modal extension $F(A)$.

Example 3.3.4 The solution of the equation $[3, 7] * X = [4, 6]$ is (see Chap. 5)

$$X = [4, 6]/\text{Dual}[3, 7] = [4, 6]/[7, 3] = [4/3, 6/7].$$

As an inner rounding of X is $[1.334, 0.857]$, then

$$[3, 7] * [1.334, 0.857] \subseteq [4, 6]$$

and the *-semantic theorem gives a meaning to this result

$$(\forall a \in [3, 7]') (\exists b \in [4, 6]') (\exists x \in [1.334, 0.857]') ax = b.$$

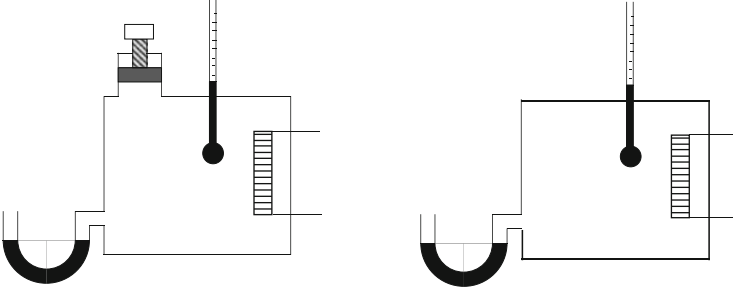


Fig. 3.5 Gas containers

The interpretation of the proper and improper intervals provided by the Semantic Theorems opens a wide field of technical applications for the theory of modal intervals, as the following suggestive example illustrates.

Example 3.3.5 Let us consider the volume v of a gas container, of which the temperature t takes values inside certain intervals. Assuming the validity of the equation

$$v = kt/p,$$

where k is the ideal gas constant, let us suppose the following intervals of variation

$$k \in K' = [0.00366, 0.00367]', \quad t \in T' = [263, 283]', \quad p \in P' = [0.99, 1.01]'$$

The problem is to determine the volume, keeping the pressure p within certain pre-established bounds, that is

$$(\forall k \in [0.00366, 0.00367]') (\forall t \in [263, 283]') Q(v, V) (\exists p \in [0.99, 1.01]') v = kt/p.$$

This semantic is equivalent to the interval inclusion

$$v^*(K, T, P) \subseteq V$$

with K and T proper intervals and P an improper one. Computing v^* , the result is

$$v^*([0.00366, 0.00367], [263, 283], [1.01, 0.99]) = [0.97 \dots, 1.02 \dots] \subseteq [0.97, 1.03]$$

which means that for every value of k and t there exists a volume v between 0.97 and 1.03, depending on k and t , which makes the pressure within the desired limits. The container is to be built with a feedback valve to allow its volume to be regulated within the computed bounds, to keep the stated conditions, as the left graph of Fig. 3.5 illustrates.

Allowing the pressure to vary within the domain $P' = [0.9, 1.1]'$, the resulting interval for the volume is

$$v^*([0.00366, 0.00367], [263, 283], [1.1, 0.9]) = [1.06 \dots, 0.94 \dots] \subseteq [1.06, 0.95].$$

In accordance with the $*$ -semantic theorem, this result means that for every k and t and every volume v between 0.95 and 1.06, the pressure falls within the limits. The container can be built without any feedback valve because, for any volume between the bounds, the pressure is inside the stated conditions, as the right graph of Fig. 3.5 illustrates.

3.4 Syntactic Functions

The two applications of the meet-join operators to a continuous function f from \mathbb{R}^k to \mathbb{R} , define the two semantic extensions f^* and f^{**} . From now on only real continuous functions with syntactic tree will be considered, so the existence of a syntactic tree for any function f is assumed and not explicitly repeated.

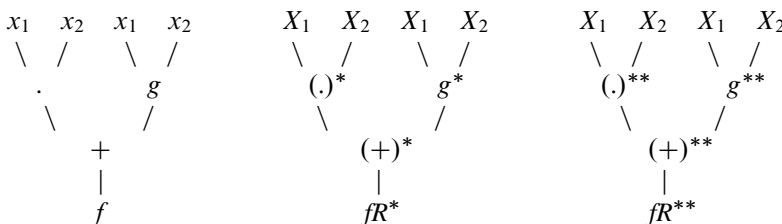
Looking at a syntactic tree of the continuous real function f , where the nodes are the operators, the leaves are the variables, and the branches define the domain of each operator, f can also be operationally extended to a syntactical function fR from $I^*(\mathbb{R}^k)$ to $I^*(\mathbb{R})$, by using the computational program implicitly defined by the syntactic tree of the expression defining the function.

3.4.1 Syntactic Extensions

Definition 3.4.1 (Modal syntactic $*$ -extension) The function fR^* from $I^*(\mathbb{R}^k)$ to $I^*(\mathbb{R})$, called the *Modal syntactic $*$ -extension* of f , is defined by the computational program indicated by a syntactic tree of the real function f from \mathbb{R}^k to \mathbb{R} , when the real operators are transformed into their $*$ -semantic extensions.

Definition 3.4.2 (Modal syntactic $$ -extension)** The function fR^{**} , called the *Modal syntactic $**$ -extension* of f , is defined similarly to fR^* , but with the operators transformed into their $**$ -semantic extensions.

Example 3.4.1 For the continuous real function $f(x_1, x_2) = x_1x_2 + g(x_1, x_2)$, with the operator $g(x_1, x_2) = (x_1 + x_2)^2$, syntactic trees of f , fR^* and fR^{**} are



If $X_1 = [-1, 1]$, $X_2 = [1, -1]$, fR^* and fR^{**} are computed as follows. For the x_1x_2 operator:

$$\begin{aligned} *{-extension} : \quad & \bigvee_{x_1 \in [-1, 1]'} \bigwedge_{x_2 \in [-1, 1]'} [x_1x_2, x_1x_2] = [0, 0], \\ **{-extension} : \quad & \bigwedge_{x_2 \in [-1, 1]'} \bigvee_{x_1 \in [-1, 1]'} [x_1x_2, x_1x_2] = [0, 0]. \end{aligned}$$

For the $g(x_1, x_2)$ operator:

$$\begin{aligned} *{-extension} : \quad & \bigvee_{x_1 \in [-1, 1]'} \bigwedge_{x_2 \in [-1, 1]'} [(x_1 + x_2)^2, (x_1 + x_2)^2] = [1, 0], \\ **{-extension} : \quad & \bigwedge_{x_2 \in [-1, 1]'} \bigvee_{x_1 \in [-1, 1]'} [(x_1 + x_2)^2, (x_1 + x_2)^2] = [0, 1]. \end{aligned}$$

Therefore,

$$\begin{aligned} fR^*([-1, 1], [1, -1]) &= \bigvee_{y_1 \in [0, 0]'} \bigwedge_{y_2 \in [0, 1]'} [y_1 + y_2, y_1 + y_2] = [1, 0], \\ fR^{**}([-1, 1], [1, -1]) &= \bigvee_{y_1 \in [0, 0]'} \bigwedge_{y_2 \in [0, 1]'} [y_1 + y_2, y_1 + y_2] = [0, 1]. \end{aligned}$$

Lemma 3.4.1 (Duality relation)

$$Dual(fR^*(X)) = fR^{**}(Dual(X)).$$

Proof If Ψ is the computational program indicated by a syntactic tree of f and w_i are the operators, then

$$Dual(fR^*(X)) = Dual(\Psi(w_i^*, X)) = \Psi(w_i^{**}, Dual(X)) = fR^{**}(Dual(X)).$$

■

Definition 3.4.3 (Modal syntactic operator) A modal syntactic operator is any continuous function f from \mathbb{R}^k to \mathbb{R} that is *JM*-commutable.

Definition 3.4.4 (Modal syntactic function) A modal syntactic function fR is a function defined similarly to fR^* or fR^{**} , but with all of its operators being *JM*-commutable, that is, modal syntactic operators.

This definition will extend considerably the framework of the four rational operators of real analysis $\{+, -, *, /\}$, since the constructive aspect which supports the four rational operators loses its interest within the numerical context where, obviously, all the operators are calculated with controlled deviations up to a certain degree.

A modal syntactic function will be, consequently, any function with the form of a continuous real function in which all its operators are modal syntactic operators, and where the functional correspondence *Arguments* \rightarrow *Values* is obtained by the computational program indicated by the syntactic tree of the function.

Later results will indicate the considerable repertoire of modal syntactic functions: $\text{Abs}(x)$, $\text{power}(x, n)$, $\log_a(x)$, $\text{root}(x, n)$ are modal syntactic operators, continuous and unary, and consequently possible nodes of the syntactic tree of a modal syntactic function.

At this stage, it is necessary to fix some notations to make easier the discussion of the problems presented by modal syntactic functions:

1. For a function f the symbols f^* and f^{**} indicate the semantic functions $f^* : I^*(\mathbb{R}^k) \rightarrow I^*(\mathbb{R})$ and $f^{**} : I^*(\mathbb{R}^k) \rightarrow I^*(\mathbb{R})$ defined by the correspondences $X \rightarrow f^*(X)$ and $X \rightarrow f^{**}(X)$, which do not depend on any syntactic tree of f .
2. $fR(x)$ and $fR(X)$ indicate the functions $fR : \mathbb{R}^k \rightarrow \mathbb{R}$ and $fR : I^*(\mathbb{R}^k) \rightarrow I^*(\mathbb{R})$ established by the computational program indicated by the syntactic tree with which these functions are indicated, where $fR(X)$ exists when the operators of the syntactic tree of f are modal syntactic and the computation of $fR(\text{Prop}(X))$ does not include any division by intervals containing zero.
3. Contrary to the equality $f(x) = fR(x)$ on \mathbb{R} , not only do the equalities between $f^*(X)$, $f^{**}(X)$ and $fR(X)$ not hold in general, but the forms of functions which are equivalent on \mathbb{R} , say f_1 and f_2 , in the sense $f_1 R(x) = f_2 R(x)$, do not necessarily maintain this same equality on $I^*(\mathbb{R})$.

Example 3.4.2 The expressions

$$f_1(x) = \frac{1}{1-x} + \frac{1}{1+x}$$

$$f_2(x) = \frac{2}{1-x^2}$$

define the same continuous real function, for $x > 1$. Nevertheless, their syntactic extensions to the interval $X = [2, 3]$ are

$$f_1 R([2, 3]) = \frac{1}{1-[2, 3]} + \frac{1}{1+[2, 3]} = [-3/4, -1/6]$$

$$f_2 R([2, 3]) = \frac{2}{1-[2, 3]^2} = [-2/3, -1/4],$$

which are different.

Theorem 3.4.1 (Inclusivity of the modal syntactic functions) *The modal syntactic extensions fR^* and fR^{**} (fR if it is the case) of a continuous real function f from \mathbb{R}^k to \mathbb{R} , are inclusion-isotonic.*

Proof If $X \subseteq Y$, Ψ is a syntactic tree of f and w_i are its operators, for any w_i the implication $X \subseteq Y \Rightarrow w_i(X) \subseteq w_i(Y)$ holds, and therefore

$$fR^*(X) = \Psi(w_i^*, X) \subseteq \Psi(w_i^*, Y) = fR^*(Y).$$

The same reasoning holds for fR^{**} . ■

3.4.2 Modal Syntactic Operators

The definition of modal syntactic operator extends the list of the real operators. Now, let us identify the most important classes of modal syntactic operators which will be the best interval operators for the syntactic tree of a modal syntactic extension.

Theorem 3.4.2 (One-variable operators) *Every one-variable continuous function is JM-commutable, and therefore a modal syntactic operator.*

Proof There is no commutation problem between the meet and join operations. ■

Remark 3.4.1 The interesting operators are the monotonic operators or other easily programable ones like $\text{abs}(x)$, $\text{power}(x, n)$, $\log(x)$ or $\text{root}(x, n)$ described in Chap. 5.

For the JM-commutativity of operators with two or more variables, the following definitions play an important role.

Definition 3.4.5 (Uniform monotonicity) A k -variable continuous function $f(x, y)$ is x -uniformly monotonic on a domain $(X', Y') \subseteq (\mathbb{R}, \mathbb{R}^{k-1})$, if it is monotonic for x on X' , and it is unary or keeps the same sense of monotonicity for all the values y on Y' .

Definition 3.4.6 (Partial monotonicity) A k -variable continuous function $f(x, y)$ is x -partially monotonic on $(X', Y') \subseteq (\mathbb{R}, \mathbb{R}^{k-1})$ if it may increase with x for some y -values, and may decrease with x for the rest of the y -values on the domain Y' .

Example 3.4.3 The functions xy and x/y are partially monotonic. Uniformly monotonic functions, which are monotonic increasing or monotonic decreasing for each component, include, for example $x + y$, $x - y$, $\min(x, y)$ $\max(x, y)$.

Theorem 3.4.3 (Two-variable operators) *Every two-variable continuous function $f(x, y)$ which is (x, y) -partially monotonic on a domain (X', Y') , is JM-commutable for the corresponding interval arguments (X, Y) .*

Proof If X and Y share the same modality, $f(x, y)$ is bounded by its values in the vertex of the domain (X', Y') . Otherwise, the possible cases, depending on the sign of the X, Y -bounds, are characterized by the behaviour of $f(x, y)$ on the borders of the two-dimensional interval domain, where the existence of two saddle-points is, case-by-case, easily assured by means of the continuity of f . These points are in the set of vertices of (X', Y') , or in some point of this domain. ■

Figure 3.6 illustrates a case of this reasoning, showing an interval domain, arrows indicating the sense of monotonicity of some two-variables operator, and the corresponding saddle point, which coincide with the origin for these senses of monotonicity.

Remark 3.4.2 This is the case of the operators $x + y$, $x - y$, $x * y$ or x/y , described in detail in Chap. 5, and the interesting operators $\max(x, y)$ or $\min(x, y)$.

Fig. 3.6 Saddle point for a two-variables operator

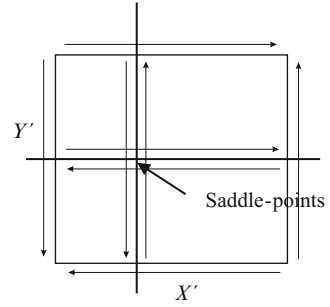
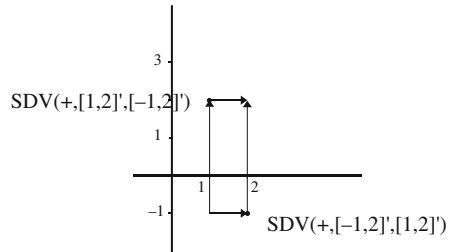


Fig. 3.7 Saddle points for addition



Example 3.4.4 For the addition of a proper interval and an improper one,

$$\begin{aligned}
 [1, 2] + [2, -1] &= [\text{SDV}(+, [1, 2]', [-1, 2,]'), \text{SDV}(+, [-1, 2]', [1, 2]')] \\
 &= [\min_{x \in [1, 2]'} \max_{y \in [-1, 2]'} (x + y), \min_{y \in [-1, 2]'} \max_{x \in [1, 2]'} (x + y)] = [3, 1]
 \end{aligned}$$

In Fig. 3.7 the sense of monotonicity of the sum in this interval and the saddle points are represented, where the arrows indicate the sense of monotonicity in the rectangular domain.

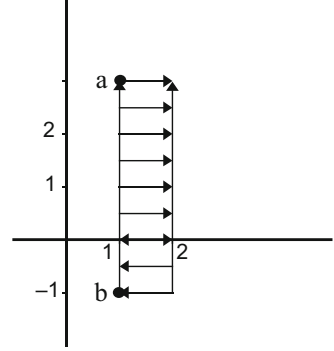
If both intervals are proper,

$$\begin{aligned}
 [1, 2] + [-1, 3] &= [\text{SDV}(+, ([1, 2]', [-1, 3,]')), \text{SDV}(+, \emptyset, ([-1, 3]', [1, 2]'))] \\
 &= [\min_{x \in [1, 2]'} \min_{y \in [-1, 3]'} x + y, \max_{y \in [1, 2]'} \max_{x \in [-1, 3]'} x + y] \\
 &= [0, 5]
 \end{aligned}$$

If both intervals are improper,

$$\begin{aligned}
 [1, -1] + [1, -2] &= [\text{SDV}(+, \emptyset, ([-1, 1]', [-2, 1]')), \text{SDV}(+, ([-1, 1]', [-2, 1]'), \emptyset)] \\
 &= [\max_{x \in [-1, 1]'} \max_{y \in [-2, 1]'} x + y, \min_{y \in [-1, 1]'} \min_{x \in [-2, 1]'} x + y] \\
 &= [2, -3]
 \end{aligned}$$

Fig. 3.8 Saddle points for the product function



For the product, the main problem appears when the domain intersects the axis. For $[1, 2] * [3, -1]$,

$$\begin{aligned}
 [1, 2] * [3, -1] &= [\text{SDV}(\cdot, [1, 2], [3, -1]), \text{SDV}(\cdot, [3, -1], [1, 2])] \\
 &= [\min_{x \in [1, 2]'} \max_{y \in [-1, 3]'} xy, \min_{y \in [-1, 3]'} \max_{x \in [1, 2]'} xy] \\
 &= [3, -1]
 \end{aligned}$$

Figure 3.8 depicts the sense of monotonicity of the product in this interval and the saddle points are represented.

Remark 3.4.3 Partial monotonicity does not guarantee the *JM*-commutativity for more than two variables, as is seen in the case of the function $f(x, y, z) = x(y + z)$.

Theorem 3.4.4 (Uniform monotonicity) *Every uniformly monotonic continuous function $f(x, y)$, with $(x, y) \in \mathbb{R}^k$, x -monotonic increasing and y -monotonic decreasing on (X', Y') , is *JM*-commutable for (X, Y) and*

$$\begin{aligned}
 fR(X, Y) &= f^*(X, Y) = f^{**}(X, Y) \\
 &= [f(\text{Inf}(X), \text{Sup}(Y)), f(\text{Sup}(X), \text{Inf}(Y))],
 \end{aligned}$$

where

$$\begin{aligned}
 \text{Inf}(X) &= (\text{Inf}(X_1), \dots, \text{Inf}(X_m)) \\
 \text{Sup}(X) &= (\text{Sup}(X_1), \dots, \text{Sup}(X_m)),
 \end{aligned}$$

and so on for Y .

Proof As f is uniformly monotonic,

$$X - \text{increasing} : \begin{cases} X_j \text{ proper} \Rightarrow \text{the minimum of } f \text{ is in } \text{Inf}(X_j) \\ X_j \text{ improper} \Rightarrow \text{the maximum of } f \text{ is in } \text{Inf}(X_j) \end{cases}$$

$$Y - \text{decreasing} : \begin{cases} Y_j \text{ proper} \Rightarrow \text{the minimum of } f \text{ is in } \text{Sup}(Y_j) \\ Y_j \text{ improper} \Rightarrow \text{the maximum of } f \text{ is in } \text{Sup}(Y_j). \end{cases}$$

Therefore

$$\begin{aligned} \min_{(x_p, y_p) \in (X_p, Y_p)'} \max_{(x_i, y_i) \in (X_i, Y_i)'} f(x_p, x_i, y_p, y_i) &= f(\text{Inf}(X), \text{Sup}(Y)) \\ &= \max_{(x_i, y_i) \in (X_i, Y_i)'} \min_{(x_p, y_p) \in (X_p, Y_p)'} \\ &\quad f(x_p, x_i, y_p, y_i) \end{aligned}$$

and, analogously,

$$\begin{aligned} \min_{(x_i, y_i) \in (X_i, Y_i)'} \max_{(x_p, y_p) \in (X_p, Y_p)'} f(x_p, x_i, y_p, y_i) &= f(\text{Sup}(X), \text{Inf}(Y)) \\ &= \max_{(x_p, y_p) \in (X_p, Y_p)'} \min_{(x_i, y_i) \in (X_i, Y_i)'} \\ &\quad f(x_p, x_i, y_p, y_i). \end{aligned}$$

So, $f^*(X, Y) = f^{**}(X, Y)$ and

$$f^*(X, Y) = [f(\text{Inf}(X), \text{Sup}(Y)), f(\text{Sup}(X), \text{Inf}(Y))].$$

■

Example 3.4.5 The function $f(x, y, z) = (x - y)/(z + y)$ in the domain $X = [0, 2]$, $Y = [4, 3]$, $Z = [2, 1]$ is x -monotonic increasing, y -monotonic decreasing, and z -monotonic increasing, as can be shown from the constancy of the signs of $(x - y)$ and of $(z + y)$ on the particular interval domain involved in this example. Then

$$\begin{aligned} fR(X, Y, Z) &= f^{**}(X, Y, Z) = f^*(X, Y, Z) \\ &= [f(\text{Inf}(X), \text{Sup}(Y), \text{Inf}(Z)), f(\text{Sup}(X), \text{Inf}(Y), \text{Sup}(Z))] \\ &= [(0 - 3)/(2 + 3), (2 - 4)/(1 + 4)] \\ &= [-3/5, -2/5], \end{aligned}$$

Example 3.4.6 A specially interesting example of continuous uniformly monotonic increasing operator is the function “limited identity”, that is,

$$\text{LID} : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$(t, x, y) \longrightarrow \text{LID}(t, x, y) = \begin{cases} \min(x, y) & \text{if } t \leq \min(x, y) \\ t & \text{if } \min(x, y) \leq t \leq \max(x, y) \\ \max(x, y) & \text{if } \max(x, y) \leq t \end{cases}$$

This operator is *JM*-commutable on its domain because it is a uniformly monotonic continuous function. Therefore

$$\begin{aligned} \text{LID}^{**}(T, X, Y) &= \text{LID}^*(T, X, Y) \\ &= [\text{LID}(\text{Inf}(T), \text{Inf}(X), \text{Inf}(Y)), \text{LID}(\text{Sup}(T), \text{Sup}(X), \text{Sup}(Y))] \end{aligned}$$

If T is the interval $[-\infty, +\infty]$, then

$$\begin{aligned} \text{LID}^{**}([-\infty, +\infty], X, Y) &= \text{LID}^*([-\infty, +\infty], X, Y) \\ &= [\min(\text{Inf}(X), \text{Inf}(Y)), \max(\text{Sup}(X), \text{Sup}(Y))] \\ &= X \vee Y \end{aligned}$$

If T is the interval $[+\infty, -\infty]$, then

$$\begin{aligned} \text{LID}^{**}([+\infty, -\infty], X, Y) &= \text{LID}^*([+\infty, -\infty], X, Y) \\ &= [\max(\text{Inf}(X), \text{Inf}(Y)), \min(\text{Sup}(X), \text{Sup}(Y))] \\ &= X \wedge Y \end{aligned}$$

After admitting these T -arguments we see that LID incorporates, among the modal syntactic operators, the \subseteq -lattice operators “meet” and “join”.

Actually, the enlargement of the set of modal syntactic operators from the classic one $(+, -, *, /)$, is quite important for applications, for example for control and approximation problems, mainly given the limitations imposed by the easy loss of information originating in multi-incidence and not-optimal syntactic trees of real expressions. In the computational context of $I^*(\mathbb{R})$, the classical rational operators $(+, -, *, /)$ have obviously no particular privilege over other programmable ones and holding the essential properties of continuity and *JM*-commutativity, since all of them are to be computed through the use of a suitably approximated arithmetic. Coming back to the “meet” and “join” operators, their use will only require an additional remark: the application of the semantic theorems will demand the consideration of the implicit t -variables they introduce.

Example 3.4.7 The operator

$$\text{CLIP}(t, x, y) = t - \text{LID}(t, x, y)$$

is uniformly t -monotonic increasing and (x, y) -monotonic decreasing and its program is

$$\text{CLIP}(t, x, y) = \begin{cases} t - \min(x, y) & \text{if } t \leq \min(x, y) \\ 0 & \text{if } \min(x, y) \leq t \leq \max(x, y) \\ t - \max(x, y) & \text{if } \max(x, y) \leq t \end{cases}$$

following a similar reasoning.

Theorem 3.4.5 (Partial uniform monotonicity) *If $f(x, y)$ is a x -uniform monotonic continuous function in the domain (X', Y') and $X = (U, V)$, $Y = (Y_p, Y_i)$ is the component-splitting of X and Y into their proper and improper components, then*

$$f^*(X, Y) = (f^*(\text{Inf}(U), \text{Inf}(V), Y) \vee f^*(\text{Sup}(U), \text{Inf}(V), Y)) \wedge (f^*(\text{Inf}(U), \text{Sup}(V), Y) \vee f^*(\text{Sup}(U), \text{Sup}(V), Y)). \quad (3.1)$$

Proof Taking into account Definition 3.2.1 of $*$ -semantic function, the associativity of the meet and join operators, and the x -monotonicity of f :

1) If $V = \emptyset$, $X = U$ is uni-modal proper and

$$\begin{aligned} f^*(X, Y) &= \bigvee_{u \in U'(y, Y)} \bigwedge f(u, y) = \bigvee_{u \in U'} f^*(u, Y) \\ &= f^*(\text{Inf}(U), Y) \vee f^*(\text{Sup}(U), Y). \end{aligned}$$

2) If $U = \emptyset$, $X = V$ is uni-modal improper, as

$$f^*(X, Y) = [\min_{y_p \in Y'_p} \max_{y_i \in Y'_i} \max_{v \in V'} f(v, y_p, y_i), \max_{y_p \in Y'_p} \min_{y_i \in Y'_i} \min_{v \in V'} f(v, y_p, y_i)],$$

if f is v -monotonic increasing

$$f^*(X, Y) = [\min_{y_p \in Y'_p} \max_{y_i \in Y'_i} f(\text{Inf}(V), y_p, y_i), \max_{y_p \in Y'_p} \min_{y_i \in Y'_i} f(\text{Sup}(V), y_p, y_i)],$$

and if f is v -monotonic decreasing

$$f^*(X, Y) = [\min_{y_p \in Y'_p} \max_{y_i \in Y'_i} f(\text{Sup}(V), y_p, y_i), \max_{y_p \in Y'_p} \min_{y_i \in Y'_i} f(\text{Inf}(V), y_p, y_i)].$$

On the other hand, as

$$\begin{aligned} &f^*(\text{Inf}(V), Y) \wedge f^*(\text{Sup}(V), Y) \\ &= [\min_{y_p \in Y'_p} \max_{y_i \in Y'_i} f(\text{Inf}(V), y_p, y_i), \max_{y_p \in Y'_p} \min_{y_i \in Y'_i} f(\text{Inf}(V), y_p, y_i)] \\ &\quad \wedge \\ &[\min_{y_p \in Y'_p} \max_{y_i \in Y'_i} f(\text{Sup}(V), y_p, y_i), \max_{y_p \in Y'_p} \min_{y_i \in Y'_i} f(\text{Sup}(V), y_p, y_i)], \end{aligned}$$

if f is ν -monotonic increasing

$$\begin{aligned}
 & f^*(\text{Inf}(V), Y) \wedge f^*(\text{Sup}(V), Y) \\
 &= [\min_{y_p \in Y'_p} \max_{y_i \in Y'_i} f(\text{Inf}(V), y_p, y_i), \max_{y_p \in Y'_p} \min_{y_i \in Y'_i} f(\text{Sup}(V), y_p, y_i)] \\
 &= f^*(X, Y),
 \end{aligned}$$

and if f is ν -monotonic decreasing

$$\begin{aligned}
 & f^*(\text{Inf}(V), Y) \wedge f^*(\text{Sup}(V), Y) \\
 &= [\min_{y_p \in Y'_p} \max_{y_i \in Y'_i} f(\text{Sup}(V), y_p, y_i), \max_{y_p \in Y'_p} \min_{y_i \in Y'_i} f(\text{Inf}(V), y_p, y_i)] \\
 &= f^*(X, Y).
 \end{aligned}$$

3) If $X = (U, V)$ is multi-modal, applying successively (2) and (1), we obtain (3.1). ■

Example 3.4.8 The function $f(x, y, z) = xy + 1/(x + y) + z - z^2$ in the domain $X = [0, 1]$, $Y = [2, 1]$, $Z = [4, 2]$ is x -monotonic increasing and z -monotonic decreasing. Its $*$ -semantic extension

$$f^*(X, Y, Z) = \bigvee_{x \in [0, 1]'} \bigwedge_{y \in [1, 2]'} \bigwedge_{z \in [2, 4]'} [xy + 1/(x + y) + z - z^2, xy + 1/(x + y) + z - z^2]$$

is not easily computable. But following Theorem 3.4.5, as

$$\begin{aligned}
 f^*(\text{Inf}(X), Y, \text{Inf}(Z)) &= f^*([0, 0], [2, 1], [4, 4]) \\
 &= \bigwedge_{y \in [1, 2]'} [1/y - 12, 1/y - 12] = [-11, -11.5] \\
 f^*(\text{Sup}(X), Y, \text{Inf}(Z)) &= f^*([1, 1], [2, 1], [4, 4]) \\
 &= \bigwedge_{y \in [1, 2]'} [y + 1/(1 + y) - 12, y + 1/(1 + y) - 12] \\
 &= [-9.66 \dots, -10.5] \\
 f^*(\text{Inf}(X), Y, \text{Sup}(Z)) &= f^*([0, 0], [2, 1], [2, 2]) \\
 &= \bigwedge_{y \in [1, 2]'} [1/y - 2, 1/y - 2] = [-1, -1.5] \\
 f^*(\text{Sup}(X), Y, \text{Sup}(Z)) &= f^*([1, 1], [2, 1], [2, 2]) \\
 &= \bigwedge_{y \in [1, 2]'} [y + 1/(1 + y) - 2, y + 1/(1 + y) - 2] \\
 &= [0.33 \dots, -0.5].
 \end{aligned}$$

then

$$f^*(X, Y, Z) = \left\{ \begin{array}{c} ([-11, -11.5] \vee [-9.66 \dots, -10.5]) \\ \wedge \\ ([-1, -1.5] \vee [0.33 \dots, -0.5]) \end{array} \right\} = [-1, -10.5].$$

Theorem 3.4.6 (k -Uniform monotonicity) *Every continuous function $f(x, y)$, with $x \in \mathbb{R}$ and $y = (u, v) \in \mathbb{R}^{k-1}$, which is uniformly monotonic for the y -arguments on a domain (X', Y') , u -monotonic increasing and v -monotonic decreasing, is JM -commutable for the corresponding interval arguments (X, Y) .*

Proof The continuity of $f(x, y_0)$ on X' , for any $y_0 \in \mathbb{R}^{k-1}$, implies the existence on X' of the x -minimum and x -maximum of $f(x, y_0)$. This allows showing the existence of the saddle-points, which means its JM -commutativity.

Let us denote by (x, u, v) the split of (x, y) , and by (x, u_m, v_M) the coordinates where $f(x, u, v)$ reaches a u -minimum and a v -maximum irrespective of the value for x . If

$$f(x_m, u_m, v_M) = \min_{x \in X'} f(x, u_m, v_M),$$

then $(x_m, u_m, v_M) \in \text{SDP}(f, (X', U'), V')$ because

$$\begin{aligned} &(\forall x \in X')(\forall u \in U')(\forall v \in V') \\ &(f(x_m, u_m, v) \leq f(x_m, u_m, v_M) \leq f(x, u_m, v_M) \leq f(x, u, v_M)) \end{aligned}$$

Let (x, u_M, v_m) be the coordinates where $f(x, u, v)$ reaches a u -maximum and a v -minimum irrespective of the value for x . If

$$f(x_M, u_M, v_m) = \max_{x \in X'} f(x, u_M, v_m),$$

then $(x_M, u_M, v_m) \in \text{SDP}(f, V', (X', U'))$ because

$$\begin{aligned} &(\forall x \in X')(\forall u \in U')(\forall v \in V') \\ &(f(x, u, v_m) \leq f(x, u_M, v_m) \leq f(x_M, u_M, v_m) \leq f(x_M, u_M, v)). \end{aligned}$$

Therefore, following the proof of the previous theorem,

$$\begin{aligned} f^{**}(X, Y) &= f^*(X, Y) \\ &= \underline{\text{if } X \text{ proper then}} \\ &\quad [\min_{x \in X'} f(x, \text{Inf}(U), \text{Sup}(V)), \max_{x \in X'} f(x, \text{Sup}(U), \text{Inf}(V))] \\ &\quad \underline{\text{if } X \text{ improper then}} \\ &\quad [\max_{x \in X'} f(x, \text{Inf}(U), \text{Sup}(V)), \min_{x \in X'} f(x, \text{Sup}(U), \text{Inf}(V))]. \end{aligned}$$

■

Example 3.4.9 The function $f(x, y, z) = (x - y)/(z + y)$ in the domains of $X = [0, 3]$, $Y = [2, 4]$, $Z = [3, 1]$ is x -monotonic increasing, y -monotonic decreasing and it is not monotonic in z . Then

$$\begin{aligned}
 f^*(X, Y, Z) &= f^{**}(X, Y, Z) = fR(X, Y, Z) \\
 &= [\max_{z \in [1, 3]'} f(\text{Inf}(X), \text{Sup}(Y), z), \min_{z \in [1, 3]'} f(\text{Sup}(X), \text{Inf}(Y), z)] \\
 &= [\max_{z \in [1, 3]'} ((0 - 4)/(z + 4)), \min_{z \in [1, 3]'} ((3 - 2)/(z + 2))] \\
 &= [-4/7, 1/5].
 \end{aligned}$$

Remark 3.4.4 The theorem is not essentially modified when x has more than one component if the condition of uni-modality is imposed on X .

Remark 3.4.5 Actually, those functions of the type $f(x, y)$, which are uniformly monotonic for $y \in \mathbb{R}^k$ and JM -commutable for $x \in \mathbb{R}^m$ on any vertex of the y -prism defined by the y -interval-arguments, can be admitted to the repertoire of modal syntactic operators. In fact,

$$f(x_m, y_M, u_m, v_M) = \min_{x \in X'} \max_{y \in Y'} f(x, y, u_m, v_M)$$

is a saddle-value of the (x, y) -function $f(x, y, u_m, v_M)$ because for every $x \in X'$, $y \in Y'$, $u \in U'$ and $v \in V'$

$$f(x_m, y, u_m, v) \leq f(x_m, y_M, u_m, v_M) \leq f(x, y_M, u_m, v_M) \leq f(x, y_M, u, v_M).$$

Remark 3.4.6 Anyway, the operators of the syntactic trees should be as simple as possible for actual practice, in spite of constituting a larger family than the classical ones for real functions.

3.4.3 Modal Syntactic Computations with Rounding

A modal syntactic computation with outer or inner rounding is defined by the syntax of $fR(X)$ where the interval value of every component and the exact value of every operator are replaced by their modal inner or outer rounding.

Definition 3.4.7 (Outer-rounding computation of $fR^*(X)$) The outer-rounding computation $\text{Out}(fR^*(X))$ is the function defined by the computational program of $fR^*(X)$, in which the value of every X -component is replaced by its modal outer rounding $\text{Out}(X_i) \supseteq X_i$, and also the exact value of every operator $\omega^*(X_i, \dots)$ is replaced by its computed actual outer-rounding $\text{Out}(\omega^*(X_i, \dots)) \supseteq \omega^*(X_i, \dots)$.

Definition 3.4.8 (Inner-rounding computation of $fR^{}(X)$)** The inner-rounding computation $\text{Inn}(fR^{**}(X))$ is the function defined by the program of $fR^{**}(X)$, in

which every X -component X_i is replaced by $\text{Inn}(X_i) \subseteq X_i$, and every exact value $\omega^{**}(X_i, \dots)$ by $\text{Inn}(\omega^{**}(X_i, \dots)) \subseteq \omega^{**}(X_i, \dots)$.

Remark 3.4.7 In a hypothetically ideal “real” arithmetic, the Out and Inn operators would reduce to the identity operator. If the real-arithmetics’ rounding supporting Out and Inn is supposed to be \leq -monotonic increasing and the elements of the corresponding digital scale are applied to themselves, it is usual to speak of an optimal rounding.

Lemma 3.4.2 (Duality relation)

$$\text{Dual}(\text{Out}(fR^*(X))) = \text{Inn}(fR^{**}(\text{Dual}(X))).$$

Proof

$$\text{Dual}(\text{Out}(fR^*(X))) = \text{Inn}(\text{Dual}(fR^*(X))) = \text{Inn}(fR^{**}(\text{Dual}(X))).$$

■

Theorem 3.4.7 (Inclusivity of the modal syntactic extensions) *The rounded modal syntactic extensions $\text{Out}(fR^*(X))$ and $\text{Inn}(fR^{**}(X))$ (or, $\text{Out}(fR(X))$ and $\text{Inn}(fR(X))$, if such be the case) of a continuous real function f from \mathbb{R}^k to \mathbb{R} , are inclusion-monotonic increasing, if the supporting interval rounding of the arguments and of the operators are also inclusion-monotonic increasing.*

Proof This property holds for the modal syntactic extensions $fR^*(X)$ and $fR^{**}(X)$ (or $fR(X)$ if such be the case). For computations with rounding the result may be obtained by considering the different roundings as ordinary inclusion-monotonic increasing operators interposed into the syntactic tree of fR . ■

Theorem 3.4.8 (Dual computing process) *If $fR(X)$ is a modal syntactic function, then*

$$\text{Inn}(fR(X)) = \text{Dual}(\text{Out}(fR(\text{Dual}(X)))).$$

Proof From Lemma 3.4.1. ■

Remark 3.4.8 Computations with modal intervals do not need a double arithmetic, with inner and outer rounding. This theorem allows the implementation of only the outer rounding interval arithmetic. Note the application of the Out operator to $\text{Dual}(X)$ in the second term: Dual is not a modal syntactic operator and the information about X implied by this expression will be $\text{Inn}(X)$.

3.5 Concluding Remarks

From the operational point of view, the most outstanding characteristic of the system of modal intervals $I^*(\mathbb{R})$ is the following: in a similar way that real numbers are associated in pairs having the same absolute value but opposite signs, the modal

intervals are associated in pairs too, each member corresponding to the same closed interval of the real line but having each one of the opposite selection modalities, existential or universal.

From the regularity point of view, $I^*(\mathbb{R})$ is a decisive improvement over $I(\mathbb{R})$ since $I^*(\mathbb{R})$ is not only the structural completion of $I(\mathbb{R})$, but also solves the referential character of interval computations to the isomorphic ones on the real line.

The system $I^*(\mathbb{R})$ provides a lot of properties immediately and consistently related with an informational approach to numeric data, as they arise from the procedures of measurement and digital computing. These properties are not obtained by using an additional over-imposed model, as in the models supported by probability, but are built on the inherent logic of the practical possibilities of the use of numbers. In fact, $I^*(\mathbb{R})$ is not a “model” for the numeric information, but the indispensable logical and operational frame for any geometrical model using numeric information. From the viewpoint of the technical constitution of the system $I^*(\mathbb{R})$, the main points are the following:

1. Association of each interval $A \in I^*(\mathbb{R})$ to the set $\text{Pred}(A)$ of the predicates P on the real line that A accepts, that is, those by which the modally-quantified statement $Q(x, A)P(x)$ is made true. This step brings out the particular set-theoretical character of the inclusion of modal intervals, and supplies the important theorem about the mutual transfers of information between the “exact” result of a naturally analytical relation and the outer and inner-rounding of its interval computation.
2. The logical re-formulation of the “poor interval extensions” of continuous real functions, allowing of defining the “modal interval extensions” of these functions, with their all-important $*$ and $**$ -semantic theorems, supports the application to modal intervals of the appealing intuition tied to the notions of “regulating” and “autonomous” ranges, and indicates the dependence between semantics and interval rounding. Starting from “poor interval extensions” selects also as meaningful only two of the different interval extensions of continuous functions that could be built if only the lattice completion of $I(\mathbb{R})$ was considered, a decision which would lead to an, in principle, different extension for each ordering of the meet and join operators.
3. The theory of interval modal syntactic functions clarifies the somewhat complicated relationship between the syntactic structure of the functions and their semantics, defined by their corresponding $*$ - and $**$ -semantic extensions. This is the key which solves the critical question of the dependence between computational process and the meaning of the computed results.

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