# Chapter 4
# Euclidean Geometry

The next step in our study of geometry, using the methods of linear algebra, is to introduce the notions of *distance* and *angle*. This will be achieved by adding a *scalar product* to a real affine space. We obtain what is called *a Euclidean space*.

We provide various examples and applications, study the metric properties of triangles, the orthogonal projections and orthogonal symmetries. We pay special attention to the *isometries*: the affine transformations which respect angles and distances, but also to the *similarities*, those which only respect angles.

## 4.1 Metric Geometry

*Geometry*, as the name indicates, is *the art of measuring the Earth*. The emphasis here is on the act of *measuring*—the physical nature of what we are measuring, if it has a physical nature, is besides the point. The question is thus: can we measure and compare lengths, angles, surfaces, volumes, in an affine space? For example in an affine plane, can we speak of a *square*: a figure with four sides of equal lengths and four "right" angles? Can we define the *perimeter* or the *surface* of such a figure?

In Definition 3.1.1 we have introduced the notion of a *segment* in a real affine space. But what about the *length* of such a segment? Of course in an affine space $(E, V)$ over a field $K$, when for $A, B, C, D \in E$ and $k \in K$

$$\overrightarrow{AB} = k\overrightarrow{CD},$$

we are tempted to say that $\overrightarrow{AB}$ *is* $k$ *times as long as* $\overrightarrow{CD}$. This is essentially what we have done in Definition 2.10.3. In this spirit, a *length* should be an element $k \in K$, an element that we probably want to be *positive* in the real case.

However, the argument above does not take us very far. What about the case where the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are not proportional?

In any case, if we want lengths to be positive numbers, we should once more restrict our attention to "ordered fields". We have already observed at the end of

F. Borceux, *An Algebraic Approach to Geometry*, DOI 10.1007/978-3-319-01733-4_4,

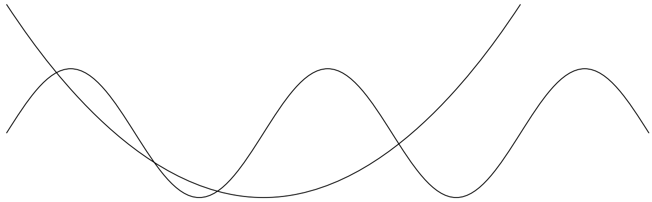

**Fig. 4.1**

Sect. 3.1 that the field $\mathbb{Q}$ of rational numbers does not seem to be adequate to generalize “classical geometry”. Let us give an additional reason. If a sensible metric geometry can be developed, a square whose side has length 1 should have a diagonal with length $\sqrt{2}$, which is no longer a rational number!

Finally, we seem to end up again with the single case: $K = \mathbb{R}$! Once more this conclusion is too severe, but we shall not insist on the possible generalizations.

If we fix $K = \mathbb{R}$, can we confidently state that we now have sound notions of length and angle? Take for example the vector space $\mathcal{C}(\mathbb{R}, \mathbb{R})$ of real continuous functions, regarded as a real affine space. What is the distance between a parabola and the “$\sin x$” function (see Fig. 4.1), viewed as points of the affine space? What is the angle between these two functions as vectors? The answer is not at all clear.

After all, even in the ordinary real plane, if you are French (and work with centimeters) or British (and work with inches), the measures that you will give for the same segment will be different real numbers.

So a real affine space does not carry an intrinsic notion of “measure” of lengths and angles. A way of measuring lengths and angles is an extra structure that you must put on your real affine space, even if in some cases—among other possibilities—some “canonical choice” may exist. The way to introduce such a “measure process” has been investigated in detail in Sect. 1.5: this is the notion of a *scalar product*.

## 4.2 Defining Lengths and Angles

With Proposition 1.5.2 in mind, let us recall a standard definition from every linear algebra course:

**Definition 4.2.1** A *scalar product* on a real vector space $V$ is a mapping

$$V \times V \longrightarrow \mathbb{R}, \qquad (x, y) \mapsto (x|y)$$

satisfying the following axioms

$$(\alpha x + \beta y|z) = \alpha(x|z) + \beta(y|z)$$
$$(x|\alpha y + \beta z) = \alpha(x|y) + \beta(x|z)$$

$$(x|y) = (y|x)$$

$$(x|x) \geq 0$$

$$(x|x) = 0 \Rightarrow x = 0,$$

for all vectors $x$, $y$, $z$ in $V$ and all scalars $\alpha$, $\beta$ in $\mathbb{R}$.

Of course this definition is redundant: the second axiom follows at once from the first and the third axioms. A scalar product is thus a *bilinear form* (first two axioms) which is *symmetric* (third axiom; see Definition G.1.1), *positive* (fourth axiom; see Definition G.3.3) and *definite* (fifth axiom; see Definition G.2.3). As a consequence, the mapping

$$V \longrightarrow \mathbb{R}, \qquad x \mapsto (x|x)$$

is a *positive definite quadratic form* (see Definitions G.1.1, G.3.3 and G.2.3).

We have the following matrix expression of a scalar product:

**Proposition 4.2.2** *Let $V$ be a real vector space with basis $e_1, \ldots, e_n$. In terms of coordinates in this basis, the scalar product of two vectors is given by*

$$(\overrightarrow{x} \,|\, \overrightarrow{y}) = \overrightarrow{x}^t A \overrightarrow{y}$$

*where the matrix $A$ is given by*

$$A = (e_i|e_i)_{1 \leq i,j \leq n}.$$

*Proof* Simply expand the scalar product

$$(x_1 e_1 + \cdots + x_n e_n | y_1 e_1 + \cdots + y_n e_n)$$

by bilinearity. □

**Definition 4.2.3** A *Euclidean space* consists of:

1. a real affine space $(E, V)$;
2. a scalar product on $V$.

With Proposition 1.5.3 in mind we make the following definition:

**Definition 4.2.4** Let $(E, V)$ be a Euclidean space.

1. The *norm* of a vector $v \in V$ is the positive real number

$$\|v\| = \sqrt{(v|v)}.$$

2. The *distance* between two points $A, B \in E$ is the norm of the vector joining them

$$d(A, B) = \|\overrightarrow{AB}\|.$$

Proposition 1.5.3 also suggests how to define an angle $\theta$ or at least, the cosine of such an angle. But a cosine should be a number between $-1$ and $+1$. So it is necessary to first prove:

**Proposition 4.2.5** (Schwarz inequality) *In a Euclidean space* $(E, V)$, *given two vectors* $x$, $y$ *in* $V$,

$$-\|x\| \cdot \|y\| \leq (x|y) \leq \|x\| \cdot \|y\|.$$

*Proof* Let $k$ be a real parameter. By bilinearity and symmetry of the scalar product

$$\|x + ky\|^2 = (x + ky|x + ky) = \|y\|^2 k^2 + 2(x|y)k + \|x\|^2.$$

This is a polynomial of degree 2 in $k$ which, as a norm, is always positive. Therefore

$$(x|y)^2 - \|x\|^2 \cdot \|y\|^2 \leq 0.$$ □

**Definition 4.2.6** Let $(E, V)$ be a Euclidean space.

1. The *angle* $\measuredangle(x, y)$ between two non-zero vectors $x$, $y$ is the unique real number $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{(x|y)}{\|x\| \cdot \|y\|}.$$

2. Let $(A, B, C)$ be a triangle in $E$ (Definition 2.10.1). The *angle* $\measuredangle(BAC)$ is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Observe further that

$$\measuredangle(x, y) = \frac{\pi}{2} \quad \Longleftrightarrow \quad \cos\measuredangle(x, y) = 0 \quad \Longleftrightarrow \quad (x|y) = 0.$$

Therefore we define:

**Definition 4.2.7** Let $(E, V)$ be a Euclidean space. Then:

1. two vectors $x$, $y$ are *orthogonal* or *perpendicular* when $(x|y) = 0$;
2. by a *right angle* we mean an angle $\frac{\pi}{2}$;
3. a *right triangle* is a triangle having a right angle.

We shall use the symbol $v \perp w$ to indicate the perpendicularity of two vectors $v$ and $w$.

## 4.3 Metric Properties of Euclidean Spaces

First we make the link with *normed vector spaces*:

**Proposition 4.3.1** *By a* norm *on a real vector space $V$ we mean a mapping*

$$\| - \| \colon V \longrightarrow \mathbb{R}$$

*such that, for all $v, w \in V$ and $r \in \mathbb{R}$:*

1. $\|v\| \geq 0$;
2. $\|v\| = 0 \implies v = 0$;
3. $\|rv\| = |r| \cdot \|v\|$;
4. $\|v + w\| \leq \|v\| + \|w\|$.

*When $(E, V)$ is a Euclidean space, the norm defined in Definition 4.2.4 satisfies these properties.*

*Proof* The first three statements follow at once from the definitions of a scalar product and a norm. For the last one, simply observe that by the Schwarz inequality (see 4.2.5)

$$\begin{aligned}(v + w|v + w) &= \|v\|^2 + 2(v|w) + \|w\|^2 \\ &\leq \|v\|^2 + 2\|v\| \cdot \|w\| + \|w\|^2 \\ &= \big(\|v\| + \|w\|\big)^2.\end{aligned}$$

□

Next, we exhibit the link with *metric spaces*.

**Proposition 4.3.2** *By a* metric space *we mean a set $E$ provided with a mapping*

$$d \colon E \times E \longrightarrow \mathbb{R}$$

*satisfying the following properties, for all $A, B, C \in E$:*

1. $d(A, B) \geq 0$;
2. $d(A, B) = 0 \implies A = B$;
3. $d(A, B) = d(B, A)$;
4. $d(A, B) + d(B, C) \geq d(A, C)$ (Minkowski inequality).

*Given a Euclidean space, the notion of distance as defined in Definition 4.2.4 provides $E$ with the structure of a metric space.*

*Proof* This follows at once from Proposition 4.3.1, keeping in mind that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. □

Notice that the Minkowski inequality can be rephrased as (Fig. 4.2):

*Given a triangle in a Euclidean space, the length of a side is shorter than the sum of the lengths of the other two sides.*

The case of angles is also worth some attention: the notion of angle is again "symmetric" while making a zero angle means as expected "being oriented in the same direction".

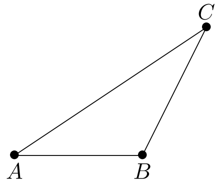

**Fig. 4.2**

**Proposition 4.3.3** *Let* $(E, V)$ *be a Euclidean space. Given distinct points* $A, B, C \in E$:

1. $\measuredangle(BAC) = \measuredangle(CAB)$;
2. $\measuredangle(BAC) = 0 \iff \overrightarrow{AC} = r\overrightarrow{AB}, r \geq 0$;
3. $\measuredangle(BAC) = \pi \iff \overrightarrow{AC} = r\overrightarrow{AB}, r \leq 0$.

*Proof* The first assertion holds by symmetry of the scalar product.

Next if $\overrightarrow{AC} = r\overrightarrow{AB}$, with $r \geq 0$, then by Proposition 4.3.1

$$(\overrightarrow{AB}|\overrightarrow{AC}) = r(\overrightarrow{AB}|\overrightarrow{AB}) = r\|\overrightarrow{AB}\|^2 = \|\overrightarrow{AB}\| \cdot \|\overrightarrow{AC}\|.$$

This immediately implies $\cos\measuredangle(BAC) = 1$, thus $\measuredangle(BAC) = 0$.

Notice further that we have

$$\|\overrightarrow{AC}\| = r\|\overrightarrow{AB}\| \quad \text{thus } r = \frac{\|\overrightarrow{AC}\|}{\|\overrightarrow{AB}\|}.$$

Conversely, $\measuredangle(BAC) = 0$ means that the cosine of this angle equals 1, that is

$$\frac{(\overrightarrow{AB}|\overrightarrow{AC})}{\|\overrightarrow{AB}\| \cdot \|\overrightarrow{AC}\|} = 1.$$

Keeping in mind the above observation concerning the only possible value of $r$, we put

$$C' = A + \frac{\|\overrightarrow{AC}\|}{\|\overrightarrow{AB}\|}\overrightarrow{AB}$$

and the thesis simply becomes $C = C'$. But

$$\overrightarrow{CC'} = \overrightarrow{CA} + \overrightarrow{AC'} = -\overrightarrow{AC} + \frac{\|\overrightarrow{AC}\|}{\|\overrightarrow{AB}\|}\overrightarrow{AB}.$$

We therefore obtain, since $(\overrightarrow{AB}|\overrightarrow{AC}) = \|\overrightarrow{AB}\| \cdot \|\overrightarrow{AC}\|$,

$$\|\overrightarrow{CC'}\|^2 = \|\overrightarrow{AC}\|^2 - 2\frac{\|\overrightarrow{AC}\|}{\|\overrightarrow{AB}\|}\|\overrightarrow{AC}\| \cdot \|\overrightarrow{AB}\| + \frac{\|\overrightarrow{AC}\|^2}{\|\overrightarrow{AB}\|^2}\|\overrightarrow{AB}\|^2 = 0$$

as expected.

The same proof, up to a change of sign, works for the last assertion. An alternative proof considers the point

$$C' = A + \overrightarrow{CA}$$

that is, $\overrightarrow{AC'} = -\overrightarrow{AC}$ and observes that

$$\measuredangle(BAC) = \pi \quad \Longleftrightarrow \quad \measuredangle\big(BAC'\big) = 0. \qquad \square$$

Coming back to the Minkowski inequality, we then obtain:

**Corollary 4.3.4** *Let $(E, V)$ be a Euclidean space. Given three points $A$, $B$, $C$ in $E$, the following conditions are equivalent*:

1. $d(A, B) + d(B, C) = d(A, C)$;
2. $\overrightarrow{AB} = r\overrightarrow{AC}$, $0 \leq r \leq 1$.

*Proof* If $\overrightarrow{AB} = r\overrightarrow{AC}$, $0 \leq r \leq 1$, then by Proposition 4.3.1, $d(A, B) = rd(A, C)$. On the other hand

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = (1 - r)\overrightarrow{AC}, \quad 0 \leq 1 - r \leq 1$$

and thus $d(B, C) = (1 - r)d(A, C)$. This forces at once the conclusion.

Conversely if the "Minkowski equality" holds, we have

$$d(A, C)^2 = d(A, B)^2 + 2d(A, B)d(B, C) + d(B, C)^2.$$

On the other hand since $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$, computing $(\overrightarrow{AC}|\overrightarrow{AC})$ yields

$$d(A, C)^2 = d(A, B)^2 + 2(\overrightarrow{AB}|\overrightarrow{BC}) + d(B, C)^2.$$

Comparing these two equalities, we obtain

$$(\overrightarrow{AB}|\overrightarrow{BC}) = \|\overrightarrow{AB}\| \cdot \|\overrightarrow{BC}\|.$$

By Proposition 4.3.3, this implies

$$\overrightarrow{AB} = k\overrightarrow{BC}, \quad k \geq 0.$$

This yields further

$$\overrightarrow{AB} = k(\overrightarrow{BA} + \overrightarrow{AC}) = k\overrightarrow{AC} - k\overrightarrow{AB}.$$

Finally

$$\overrightarrow{AB} = \frac{k}{1 + k}\overrightarrow{AC}, \quad 0 \leq \frac{k}{1 + k} \leq 1. \qquad \square$$

Notice that Corollary 4.3.4 can be rephrased as (see Definition 3.1.1):

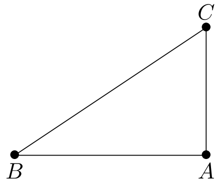


**Fig. 4.3**

*In a Euclidean space $(E, V)$, a point $B$ belongs to the segment $[A, C]$ if and only if $d(A, B) + d(B, C) = d(A, C)$.*

Thus "being on a same line" can be characterized by a metric equality.

Another important metric property of Euclidean spaces is:

**Theorem 4.3.5** (Pythagoras' theorem) *Consider a triangle $(A, B, C)$ in a Euclidean space $(E, V)$ (see Fig. 4.3). The following conditions are equivalent:*

1. $\measuredangle(BAC)$ *is a right angle*;
2. $d(A, B)^2 + d(A, C)^2 = d(B, C)^2$.

*Proof* Simply observe that

$$(\overrightarrow{BC}|\overrightarrow{BC}) = (\overrightarrow{BA} + \overrightarrow{AC}|\overrightarrow{BA} + \overrightarrow{AC}) = \|\overrightarrow{BA}\|^2 + 2(\overrightarrow{BA}|\overrightarrow{AC}) + \|\overrightarrow{AC}\|^2.$$

Thus the equality in condition 2 holds precisely when $(\overrightarrow{BA}|\overrightarrow{AC}) = 0$. □

## 4.4 Rectangles, Diamonds and Squares

A Euclidean space of dimension 2 is a model of Euclidean plane geometry, in the sense of *Hilbert* (see Example 8.5.3 in [7], *Trilogy I*). Thus all properties of plane Euclidean geometry considered in [7], *Trilogy I*, are valid. Let us give here a direct proof of some of these properties which will be useful in subsequent sections.

**Proposition 4.4.1** *In a right triangle (see Fig. 4.3) of a Euclidean space*

$$\cos \measuredangle(ABC) = \frac{d(B, A)}{d(B, C)}.$$

*Proof* Since $\overrightarrow{AB}$ is orthogonal to $\overrightarrow{AC}$

$$\cos \measuredangle(ABC) = \frac{(\overrightarrow{BA}|\overrightarrow{BC})}{\|\overrightarrow{BA}\| \cdot \|\overrightarrow{BC}\|} = \frac{(\overrightarrow{BA}|\overrightarrow{BA} + \overrightarrow{AC})}{\|\overrightarrow{BA}\| \cdot \|\overrightarrow{BC}\|}$$
$$= \frac{(\overrightarrow{BA}|\overrightarrow{BA})}{\|\overrightarrow{BA}\| \cdot \|\overrightarrow{BC}\|} = \frac{\|\overrightarrow{BA}\|}{\|\overrightarrow{BC}\|}.$$
□

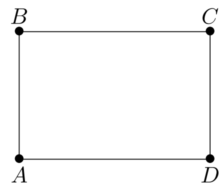


**Fig. 4.4**

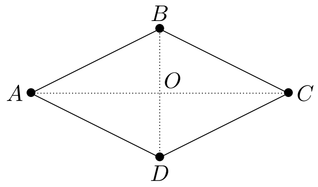


**Fig. 4.5**

**Definition 4.4.2** In a Euclidean space, a parallelogram is called:

1. a *rectangle*, when its four angles are right angles;
2. a *diamond*, when its four sides have the same length;
3. a *square*, when it is both a rectangle and a diamond.

Let us now observe some useful characterizations of these notions.

**Proposition 4.4.3** *In a Euclidean space, a parallelogram is a rectangle if and only if it admits a right angle.*

*Proof* Consider the parallelogram of Fig. 4.4 and assume that the angle $DAB$ is right. This means that the direction of $\overrightarrow{AB}$, which is also that of $\overrightarrow{DC}$, is orthogonal to the direction of $\overrightarrow{AD}$, which is also that of $\overrightarrow{BC}$ (see Definition 2.11.1). $\square$

**Proposition 4.4.4** *In a Euclidean space, a parallelogram is a diamond if and only if its diagonals are orthogonal.*

*Proof* Consider the parallelogram $ABCD$ of Fig. 4.5. Write $O$ for the intersection point of the two diagonals, which is thus such that $\overrightarrow{CO} = -\overrightarrow{AO}$ (see Proposition 2.11.5). This immediately implies

$$(\overrightarrow{AO}|\overrightarrow{OB}) = -(\overrightarrow{CO}|\overrightarrow{OB}).$$

On the other hand

$$\|\overrightarrow{AB}\|^2 = (\overrightarrow{AO} + \overrightarrow{OB}|\overrightarrow{AO} + \overrightarrow{OB}) = \|\overrightarrow{AO}\|^2 + 2(\overrightarrow{AO}|\overrightarrow{OB}) + \|\overrightarrow{OB}\|^2$$

and analogously

$$\|\overrightarrow{CB}\|^2 = (\overrightarrow{CO} + \overrightarrow{OB}|\overrightarrow{CO} + \overrightarrow{OB}) = \|\overrightarrow{CO}\|^2 + 2(\overrightarrow{CO}|\overrightarrow{OB}) + \|\overrightarrow{OB}\|^2.$$

If the diagonals are orthogonal, the scalar products disappear from these last expressions and since $d(A, O) = d(C, O)$, we obtain $d(A, B) = d(C, B)$. Since moreover $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{CB} = \overrightarrow{DA}$, all four sides have the same length.

If the parallelogram is a diamond, we have

$$\|\overrightarrow{AB}\| = \|\overrightarrow{CB}\|$$

from which the expressions above yield

$$(\overrightarrow{AO}|\overrightarrow{OB}) = (\overrightarrow{CO}|\overrightarrow{OB}).$$

Since we know already that

$$(\overrightarrow{AO}|\overrightarrow{OB}) = -(\overrightarrow{CO}|\overrightarrow{OB}),$$

we conclude that these quantities are zero and so the diagonals are orthogonal. □

**Proposition 4.4.5** *In a Euclidean space, a parallelogram is a square if and only if it its angles are right and its diagonals are perpendicular.*

*Proof* This follows by Definition 4.4.2 and Proposition 4.4.4. □

## 4.5 Examples of Euclidean Spaces

*Example 4.5.1* Given a basis $(O; e_1, \ldots, e_n)$ of a finite dimensional real affine space $(E, V)$, the formula

$$(\overrightarrow{x}\,|\,\overrightarrow{y}) = \sum_{i=1}^{n} x_i y_i$$

defines a scalar product on $V$.

*Proof* This is straightforward. Observe that in particular

$$(e_i|e_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

In the special case of the vector space $\mathbb{R}^n$ viewed as an affine space (see Example 2.2.1) and its "canonical basis", we recapture the "canonical scalar product" of Proposition 1.5.2:

$$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad (\overrightarrow{x}, \overrightarrow{y}) \mapsto \sum_{i=1}^{n} x_i y_i .$$

Unless otherwise specified, when we refer to the Euclidean space $\mathbb{R}^n$, we shall always mean $\mathbb{R}^n$ provided with this canonical scalar product. Sometimes, for the sake of precision, the notation $E^n(\mathbb{R})$ is used to indicate this particular Euclidean space $\mathbb{R}^n$.

Notice further that given strictly positive numbers $\omega_i$, the formula

$$(\overrightarrow{x} \mid \overrightarrow{y}) = \sum_{i=1}^{n} \omega_i x_i y_i$$

still defines a scalar product on $V$. □

*Example 4.5.2* Consider a symmetric $n \times n$ real matrix whose eigenvalues are all strictly positive. The mapping

$$\varphi \colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad (x, y) \mapsto x^t A y$$

is a scalar product on $\mathbb{R}^n$.

*Proof* Trivially, $\varphi$ is bilinear. It is also symmetric, by symmetry of $A$: indeed

$$x^t A y = \left(x^t A y\right)^t = y^t A^t x = y^t A x;$$

the first equality holds because $x^t A y$ is a $1 \times 1$-matrix.

Consider $\mathbb{R}^n$ with its canonical scalar product of Example 4.5.1. By Theorem G.4.1, we can choose another orthonormal basis of $\mathbb{R}^n$ with respect to which the matrix of $\varphi$ is diagonal. As observed in the proof of G.4.1, the diagonal elements will be the eigenvalues of $A$. With respect to this new basis $\varphi$ takes the form

$$\varphi(x, y) = \sum_{i=1}^{n} \lambda_i x_i y_i$$

with $\lambda_i > 0$ for all indices $i$. As in Example 4.5.1, we conclude that $\varphi$ is positive and definite. □

*Example 4.5.3* Consider a closed interval $[a, b]$ of the real line and the vector space $\mathcal{C}([a, b], \mathbb{R})$ of continuous functions $f \colon [a, b] \longrightarrow \mathbb{R}$. The formula

$$(f|g) = \int_a^b f(x)g(x)\,dx$$

defines a scalar product on $\mathcal{C}([a,b],\mathbb{R})$.

*Proof* Only the last axiom requires a comment. If $f(x_0) \neq 0$ for some $x_0 \in [a,b]$ then by continuity $|f(x)| > \frac{1}{2}|f(x_0)|$ on a neighborhood of $x_0$ in $[a,b]$. Writing $\varepsilon > 0$ for the length of an interval on which this is the case we conclude, since $f^2(x)$ is positive for all $x \in [a,b]$, that

$$\int_a^b f^2(x)dx \geq \frac{1}{4} f^2(x_0)\varepsilon > 0.$$

One should observe that in this example the distance between two functions is given by

$$d(f,g) = \sqrt{\int_a^b \big(f(x) - g(x)\big)^2\,dx}$$

which is not the same as "the area separating the two graphs"

$$\int_a^b \big|f(x) - g(x)\big|\,dx.$$

It should be mentioned here that "the area separating the two graphs" nevertheless yields a good notion of distance in the sense of metric spaces: but this notion of distance is not inherited from a scalar product.

Let us also mention that if

$$\omega\colon [a,b] \longrightarrow \mathbb{R}, \quad \forall x \in [a,b]\ \omega(x) > 0$$

is a strictly positive continuous function, the argument above can at once be adapted to prove that

$$(f|g) = \int_a^b \omega(x)f(x)g(x)dx$$

is another scalar product on $\mathcal{C}([a,b],\mathbb{R})$. At the end of Sect. 4.9, we shall remark on the benefits of introducing such a *weight*. □

*Example 4.5.4* Consider the vector space $\mathbb{R}_{(n)}[X]$ of polynomials with real coefficients and degree at most $n$. Consider further $n+1$ distinct real numbers

$$a_0 < a_1 < \cdots < a_{n-1} < a_n.$$

The formula

$$(p|q) = \sum_{i=0}^{n} p(a_i)q(a_i)$$

defines a scalar product on $\mathbb{R}_{(n)}[X]$.

*Proof* Again only the last axiom requires a comment. If

$$(p|p) = \sum_{i=0}^{n} p(a_i)^2 = 0$$

then $p(a_i) = 0$ for each index $i$. This proves that the polynomial $p(X)$ of degree at most $n$ admits $n+1$ distinct roots: it is therefore the zero polynomial.

Of course choosing arbitrary strictly positive real numbers $\omega_i > 0$, the formula

$$(p|q) = \sum_{i=1}^{n} \omega_i p(a_i)q(a_i)$$

still defines a scalar product on $\mathbb{R}_{(n)}[X]$. □

*Example 4.5.5* Every affine subspace of a Euclidean space is itself a Euclidean space with respect to the induced scalar product.

*Proof* Trivially, given a scalar product on a real vector space $V$, its restriction to any vector subspace $W \subseteq V$ remains a scalar product.

Going back to Example 2.2.2 and considering the canonical scalar product on $\mathbb{R}^n$ (see Example 4.5.1), we conclude that the affine space of solutions of a system $A\overrightarrow{x} = \overrightarrow{b}$ of linear equations can be provided with the structure of a Euclidean space.

Analogously, considering Examples 2.2.4 and 4.5.3, we conclude that the affine space of solutions, on an interval $[a, b]$, of a differential equation $ay'' + by' + cy = d$, can be provided with the structure of a Euclidean space. □

## 4.6 Orthonormal Bases

In a Euclidean space, using an affine basis closely related to the Euclidean structure allows drastic simplifications. As usual when we work with bases and coordinates, we reduce our attention to the finite dimensional case, even if various results hold (often with the same proofs) in arbitrary dimensions.

**Definition 4.6.1** By an *orthonormal basis* of a finite dimensional Euclidean space $(E, V)$ is meant an affine basis $(O; e_1, \dots, e_n)$ such that:

1. $\forall i \ \|e_i\| = 1$;
2. $\forall i \neq j \ e_i \perp e_j$.

Let us first list some advantages of working with orthonormal basis.

**Proposition 4.6.2** *Let $(O; e_1, \dots, e_n)$ be an orthonormal basis of a Euclidean space $(E, V)$. The coordinates of a vector $x \in V$ with respect to this basis are*

$$(x|e_i)_{1 \le i \le n}.$$

*Proof* If

$$x = x_1 e_1 + \cdots + x_n e_n$$

computing the scalar product with $e_i$ yields precisely $x_i$. □

**Proposition 4.6.3** *Let $(O; e_1, \dots, e_n)$ be an orthonormal basis of a Euclidean space $(E, V)$. Given two vectors $x, y \in V$, their scalar product is*

$$(x|y) = \sum_{i=1}^{n} x_i y_i.$$

*Proof* By Proposition 4.2.2, the matrix of the scalar product is the identity matrix because the basis is orthonormal. □

**Proposition 4.6.4** *Let $(O; e_1, \dots, e_n)$ and $(O'; e'_1, \dots, e'_n)$ be two orthonormal bases of a Euclidean space $(E, V)$. In the change of basis formula (see Proposition* 2.20.1)

$$\overrightarrow{x'} = M\overrightarrow{x} + \overrightarrow{b}$$

*the matrix $M$ is* orthogonal, *that is,* $M^{-1} = M^t$.

*Proof* The matrix $M$ is obtained by putting in columns the coordinates of the vectors $e_i$ with respect to the basis $(e'_1, \dots, e'_n)$. By Proposition 4.6.2,

$$M = (m_{ij})_{1 \le i,j \le n}, \quad m_{ij} = \big(e_j | e'_i\big).$$

Considering the inverse change of basis formula

$$\overrightarrow{x} = M^{-1}\overrightarrow{x'} + \overrightarrow{b'}$$

we obtain in the same way

$$M^{-1} = \big(m'_{ij}\big)_{1 \le i,j \le n}, \quad m'_{ij} = \big(e'_j | e_i\big).$$

The conclusion follows at once, by symmetry of the scalar product. □

This last result is certainly the most striking one, since computing an inverse matrix is generally rather hard work, especially when the dimension is high. However, all of these beautiful properties will only be made available to us if we can prove the existence of orthonormal basis. For that we observe first:

**Proposition 4.6.5** *Let $(E, V)$ be a Euclidean space. Given non-zero pairwise orthogonal vectors*

$$e_1, \ldots, e_n, \quad i \neq j \quad \Longrightarrow \quad e_i \perp e_j$$

*these vectors are necessarily linearly independent.*

*Proof* Suppose

$$x_1 e_1 + \cdots + x_n e_n = 0.$$

Computing the scalar product with $e_i$ yields

$$x_i (e_i | e_i) = 0$$

by perpendicularity of the vectors. But $(e_i|e_i) \neq 0$ because $e_i \neq 0$; therefore $x_i = 0$. □

**Theorem 4.6.6** (Gram-Schmidt process) *Let $(O; e_1, \ldots, e_n)$ be an arbitrary basis of a Euclidean space $(E, V)$. There exists an orthonormal basis $(O; v_1, \ldots, v_n)$ with the additional property that for every index $k$, the two subspaces*

$$\langle e_1, \ldots, e_k \rangle \quad \text{and} \quad \langle v_1, \ldots, v_k \rangle$$

*generated by the first $k$ vectors of each basis are equal.*

*Proof* We prove the result by induction on $n$. When $n = 1$, it suffices to put

$$v_1 = \frac{e_1}{\|e_1\|}.$$

Assuming the result up to the dimension $n - 1$, let us apply it to the vector subspace $\langle e_1, \ldots, e_{n-1} \rangle$ and its basis $e_1, \ldots, e_{n-1}$. We obtain an orthonormal basis $v_1, \ldots, v_{n-1}$ of this subspace, which satisfies the condition of the statement up to the index $n - 1$. Consider then

$$v_n' = e_n - (e_n|v_1)v_1 - \cdots - (e_n|v_{n-1})v_{n-1}.$$

We get at once, for $1 \leq i \leq n - 1$

$$\left(v_n'|v_i\right) = (e_n|v_i) - (e_n|v_i)(v_i|v_i) = (e_n|v_i) - (e_n|v_i) = 0.$$

Putting

$$v_n = \frac{v_n'}{\|v_n'\|}$$

thus yields a sequence $v_1, \dots, v_n$ pairwise orthogonal vectors of length 1. By Proposition 4.6.5, this is a basis of $V$. □

## 4.7 Polar Coordinates

In this section, we want to stress the fact that in Euclidean spaces, all the classical techniques mentioned in Chap. 1 (polar coordinates in the plane, polar or cylindrical coordinates in the three-dimensional space, and so on) now make perfect sense. We shall not dwell on these straightforward aspects. Just as an example, we focus on the case of polar coordinates in a Euclidean space.

Given a triangle $(A, B, C)$ in a Euclidean space, we have defined the angle $\measuredangle(BAC)$ (see Definition 4.2.6). Trivially, by symmetry of the scalar product

$$\measuredangle(BAC) = \measuredangle(CBA).$$

Of course we might be tempted to say instead that

$$\measuredangle(BAC) = -\measuredangle(CBA).$$

In order to be able to do this, we need to provide each angle with a sign. This is possible *only* in the special case of a Euclidean *plane*.

**Definition 4.7.1** When an orientation of a Euclidean plane has been fixed, the *relative angle* $\measuredangle(v, w)$ between two linearly independent vectors $v, w$ is the angle $\measuredangle(v, w)$ of Definition 4.2.6 provided with the sign $+$ when the basis $(v, w)$ has direct orientation and with the sign $-$ when this basis has inverse orientation. When two non-zero vectors $v$ and $w$ are linearly dependent, their relative angle is their ordinary angle as in Definition 4.2.6, that is, 0 or $\pi$ (see Proposition 4.3.3).

Of course choosing the opposite orientation of the plane interchanges the signs of all relative angles. Moreover, since an angle is defined via its cosine, the following convention certainly does not hurt:

**Convention 4.7.2** *Under the conditions of Definition 4.7.1, we shall freely identify a relative angle $\theta$ with any angle $\theta + 2k\pi$, for every integer $k \in \mathbb{Z}$.*

The reader is invited to explain why such definitions do not make sense in a three dimensional Euclidean space.

The existence of polar coordinates in every Euclidean plane is then attested by the following result:

**Proposition 4.7.3** *Let $(E, V)$ be a Euclidean plane provided with an orthonormal basis $(O; e_1, e_2)$ considered as having direct orientation. The coordinates of a point*

*$0 \neq P \in E$ are given by*

$$P = \|\overrightarrow{OP}\| \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \theta = \measuredangle(e_1, \overrightarrow{OP})$$

*where $\theta$ is the relative angle as in Definition 4.7.1.*

*Proof* Given real numbers $a$, $b$ such that $a^2 + b^2 = 1$, then $a^2 \leq 1$ thus $-1 \leq a \leq +1$. Therefore $a = \cos\tau$ for a unique $\tau \in [0, \pi]$, while $\sin\tau \geq 0$. But then

$$b^2 = 1 - a^2 = 1 - \cos^2\tau = \sin^2\tau$$

and thus $b = \pm\sin\tau$. If $b \geq 0$, we have at once

$$a = \cos\tau, \qquad b = \sin\tau.$$

If $b \leq 0$ we have

$$a = \cos\tau = \cos(-\tau), \qquad b = -\sin\tau = \sin(-\tau).$$

In both cases, we end up with

$$(a, b) = (\cos\sigma, \sin\sigma)$$

for a unique relative angle $\sigma$.

Now given $0 \neq P \in E$, we thus have a unique relative angle $\sigma$ such that

$$\overrightarrow{OP} = \|\overrightarrow{OP}\| \frac{\overrightarrow{OP}}{\|\overrightarrow{OP}\|} = \|\overrightarrow{OP}\| \begin{pmatrix} \cos\sigma \\ \sin\sigma \end{pmatrix}$$

since the vector $\frac{\overrightarrow{OP}}{\|\overrightarrow{OP}\|}$ has norm 1. It remains to show that $\sigma$ is also the relative angle $\theta$ between $e_1$ and $\overrightarrow{OP}$. But

$$\cos\theta = \frac{(e_1|\overrightarrow{OP})}{\|e_1\| \cdot \|\overrightarrow{OP}\|} = \cos\sigma.$$

The matrix having as columns the coordinates of $e_1$ and $\overrightarrow{OP}$ is

$$\begin{pmatrix} 1 & \|\overrightarrow{OP}\| \cos\sigma \\ 0 & \|\overrightarrow{OP}\| \sin\sigma \end{pmatrix};$$

its determinant is simply $\sin\sigma$. Thus by Definition 4.7.1, $\theta$ is positive or negative according to the sign of $\sin\sigma$, that is, it has the same sign as $\sigma$. Since $\sigma$ and $\theta$ have the same cosine and the same sign, they are equal. $\square$

Analogous arguments can be developed for the other systems of coordinates considered in Chap. 1. We are not really interested here in these considerations. The only reason for introducing the straightforward observations above is to emphasize the fact that we have now gathered all the necessary ingredients to make the link with "ordinary" geometrical notions: we need to have the affine structure, a notion of orientation and a notion of "measure" of angles and distances.

## 4.8 Orthogonal Projections

To avoid any ambiguity, let us make the following definition:

**Definition 4.8.1** Let $(E, V)$ be a Euclidean space and $(F_1, W_1)$, $(F_2, W_2)$ two affine subspaces. These subspaces are called *orthogonal* when every vector of $W_1$ is orthogonal to every vector of $W_2$.

Observe that this definition is more restrictive than the notion of perpendicularity "in real life". For example, you will probably say that a wall of your room is "perpendicular" to the floor. However this is not the situation described in Definition 4.8.1! Consider the line of intersection between the wall and the floor: a "vector" in this intersection is both on the wall and on the floor, but is not orthogonal to itself! This is a general fact:

**Lemma 4.8.2** *Let $(E, V)$ be a Euclidean space and $(F_1, W_1)$, $(F_2, W_2)$ two orthogonal affine subspaces. Then $W_1 \cap W_2 = \{0\}$ and thus $F_1 \cap F_2$ is either the empty set or is reduced to a singleton.*

*Proof* Indeed $w \in W_1 \cap W_2$ is such that $(w|w) = 0$, thus $w = 0$. The result follows by Propositions 2.3.4 and 2.1.2. □

Let us warn the reader: the following theorem is generally *not* valid for Euclidean spaces of infinite dimension.

**Theorem 4.8.3** *Let $(E, V)$ be a finite dimensional Euclidean space and $W$ a vector subspace of $V$. The set*

$$W^\perp = \{v \in V | \forall w \in W \ v \perp w\}$$

*is a vector subspace of $V$, orthogonal to $W$. The subspaces $W$ and $W^\perp$ are supplementary. Therefore $W^\perp$ is called the* orthogonal supplementary *of $W$.*

*Proof* The set $W^\perp$ is a vector subspace by bilinearity of the scalar product; it is trivially orthogonal to $W$. By Lemma 4.8.2 we know already that $W \cap W^\perp = \{0\}$; it remains to prove that $W + W^\perp = V$.

To prove this, consider a basis $e_1, \dots, e_k$ of $W$ and extend it to a basis $e_1, \dots, e_n$ of $V$. Apply the Gram–Schmidt construction (Theorem 4.6.6) to get an orthonormal basis $v_1, \dots, v_n$ such that in particular, $v_1, \dots, v_k$ is still a basis of $W$. The vector subspace $\langle v_{k+1}, \dots, v_n \rangle$ is contained in $W^\perp$ and is supplementary to $W$. Thus

$$W + W^\perp \supseteq W + \langle v_{k+1}, \dots, v_n \rangle = V$$

as expected.

Observe further (even if not needed for the proof) that since $\langle v_{k+1}, \dots, v_n \rangle$ and $W^\perp$ are two supplementary subspaces of $W$, they have the same dimension $n - \dim W$. Since $\langle v_{k+1}, \dots, v_n \rangle$ is contained in $W^\perp$, these two subspaces are necessarily equal. □

**Corollary 4.8.4** *Let $(E, V)$ be a finite dimensional Euclidean space with orthonormal basis $(O; e_1, \dots, e_n)$. Fix a point $A = (a_1, \dots, a_n) \in E$ and consider the vector line $W$ generated by a non-zero vector $w = (w_1, \dots, w_n)$ in $V$. The affine hyperplane $(F, W^\perp)$ containing $A$ and of direction $W^\perp$ (see Theorem 4.3.5) admits the equation*

$$\sum_{i=1}^{n} w_i (x_i - a_i) = 0.$$

*Proof* Since $A \in F$, the point $P = (x_1, \dots, x_n) \in E$ lies in $F$ when $\overrightarrow{AP}$ is perpendicular to $w$. The result follows by Proposition 4.6.3. □

**Definition 4.8.5** Let $(E, V)$ be a finite dimensional Euclidean space and $(F, W)$ an affine subspace. The projection on $W$, parallel to $W^\perp$ (see Definition 2.15.1) is called the *orthogonal projection on* $(F, W)$.

Here is a key property of orthogonal projections:

**Proposition 4.8.6** *Let $(E, V)$ be a finite dimensional Euclidean space and $(F, W)$ an affine subspace. Given a point $A \in E$ and its orthogonal projection $P \in F$, one has, for every other point $Q \in F$ (see Fig. 4.6)*

$$d(A, P) < d(A, Q), \qquad P \neq Q \in F.$$

*Proof* We have $\overrightarrow{AP} \in W^\perp$ and $\overrightarrow{QP} \in W$, thus the triangle $APQ$ is right angled. By Pythagoras' Theorem (see 4.3.5)

$$d(Q, P)^2 + d(A, P)^2 = d(A, Q)^2.$$

Since $Q \neq P$, $d(P, Q) \neq 0$ and it follows that $d(A, P) < d(A, Q)$. □

Proposition 4.8.6 can thus be rephrased in the following way:

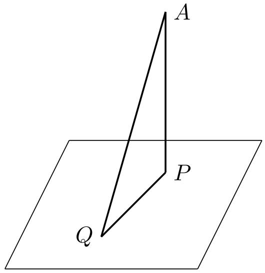


**Fig. 4.6**

*The orthogonal projection of a point $A$ on a given subspace is the best approximation of $A$ by a point of the subspace.*

Our next section will take full advantage of this observation.

It remains to establish an efficient formula for computing orthogonal projections.

**Proposition 4.8.7** *Let $(E, V)$ be a finite dimensional Euclidean space and $(F, W)$ an affine subspace. Given an orthonormal basis $(O; e_1, \dots, e_k)$ of the subspace $(F, W)$ and a point $A \in E$, the orthogonal projection $P$ of $A$ on $(F, W)$ is given by*

$$P = O + (\overrightarrow{OA}|e_1)e_1 + \cdots + (\overrightarrow{OA}|e_k)e_k.$$

*Proof* Extend $(e_1, \dots, e_k)$ to an orthonormal basis $(e_1, \dots, e_n)$ of $V$, as in the proof of Theorem 4.8.3. As observed at the end of that proof, $(e_{k+1}, \dots, e_n)$ is an orthonormal basis of $W^\perp$. By Proposition 4.6.2,

$$\overrightarrow{OA} = \sum_{i=1}^{k} (\overrightarrow{OA}|e_i) + \sum_{i=k+1}^{n} (\overrightarrow{OA}|e_i)$$

is thus the unique decomposition

$$\overrightarrow{OA} = w + w', \quad w \in W,\ w' \in W^\perp.$$

But $P = O + w$ (see the proof of Theorem 2.6.2): this yields the formula of the statement. □

## 4.9 Some Approximation Problems

Consider a subspace $(F, W)$ of a finite dimensional Euclidean space $(E, V)$. Given a point $A \in E$, what is *the best approximation of $A$ by a point of $F$*? This is the point $B \in F$ such that the distance $d(A, B)$ is as small as possible! Such a point exists and is unique: by Proposition 4.8.6, it is the orthogonal projection of $A$ on $(F, W)$.

*Example 4.9.1* (Overdetermined systems) How can one find the "best approximation" of a solution for a system $A\overrightarrow{x} = \overrightarrow{b}$ of $m$ equations with $n$ unknowns when $m$ is much bigger than $n$?

*Proof* This situation occurs when we want to determine the values of some physical quantities $X_1, \dots, X_n$ which are impossible to isolate experimentally, but we are able to measure experimentally the result of some linear combination of these quantities:

$$a_1X_1 + \cdots + a_nX_n.$$

Repeating the experiment with different values of the coefficients $a_i$, we obtain a system $A\overrightarrow{x} = \overrightarrow{b}$ of equations. We want to "statistically" correct the experimental imprecisions by performing a large number $m$ of experiments (i.e. of equations), a number $m$ which is much bigger than the number $n$ of quantities to measure. Due to imprecisions in the measurements, there is no hope that the system $A\overrightarrow{x} = \overrightarrow{b}$ will still have an "algebraic solution", but of course the problem has a "physical solution": the actual values of the quantities $X_1, \dots, X_n$.

Without any experimental error, $\overrightarrow{b}$ would be of the form $A\overrightarrow{x}$, that is, would be a linear combination of the columns of the matrix $A$. Consider the canonical scalar product on $\mathbb{R}^m$ (see Example 4.5.1) and the vector subspace $W \subseteq \mathbb{R}^m$ generated by the columns of $A$. It remains to replace $\overrightarrow{b}$ by its "best approximation by a vector $\overrightarrow{c} \in W$", that is, by its orthogonal projection on $W$. The system $A\overrightarrow{x} = \overrightarrow{c}$ now has a solution. □

*Example 4.9.2* (Approximation by the law of least squares) How can one find the polynomial $p(X)$ of degree $n$ whose values $p(a_i)$ are "as close as possible" from prescribed values $b_i$, when the number $m$ of indices $i$ is much bigger that the degree $n$ of $p(X)$?

*Proof* Assume that some physical law is expressed by a formula of degree 2, for example: *the resistance of the air is proportional to the square of the speed*. We want to determine the proportionality coefficient, under some specific conditions of pressure or shape.

More generally, the theory tells us that some physical quantity $Y$ can be expressed by a polynomial of degree $n$ in terms of the physical quantity $X$

$$Y = p(X) = k_nX^n + \cdots + k_1X + k_0.$$

We want to determine experimentally the coefficients of the polynomial $p(X)$. For this we perform a large number $m$ of experiments, for different values $X = a_i \in \mathbb{R}$, measuring the corresponding values $Y = b_i$. We are looking for the polynomial $p(X)$ of degree $n$ such that each $p(a_i)$ is as close as possible to $b_i$. Figure 4.7 presents an example with $n = 2$ and $m = 20$.

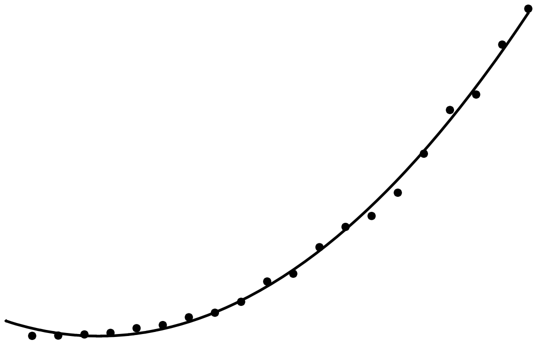

**Fig. 4.7**

First of all, observe that there is a polynomial $q(X)$ of degree $m-1$ such that $q(a_i)=b_i$ for each index $i$: this is simply

$$q(X)=\sum_{i=1}^{m} b_i\Bigg(\prod_{\substack{1\le j\le m\\ j\ne i}} \frac{X-a_j}{a_i-a_j}\Bigg).$$

Consider the Euclidean space $\mathbb{R}_{(m-1)}[X]$ of all polynomials of degree at most $m-1$, provided with the scalar product of Example 4.5.4

$$\big(\alpha(X)|\beta(X)\big)=\sum_{i=1}^{m}\alpha(a_i)\beta(a_i).$$

The orthogonal projection $p(X)$ of $q(X)$ on the subspace $\mathbb{R}_{(n)}[X]$ of polynomials of degree at most $n$ is the polynomial of degree $n$ such that the quantity

$$\sum_{i=1}^{m}\big(p(a_i)-q(a_i)\big)^2=\sum_{i=1}^{n}\big(p(a_i)-b_i\big)^2$$

is the smallest possible (Proposition 4.8.6).

This polynomial $p(X)$ is thus the solution to our problem according to *the law of least squares*: the sum of the squares of the "errors" has been made as small as possible. □

*Example 4.9.3* (Fourier approximation) How can one find a "best approximation" $g(X)$ of a periodic function $f(X)$ by a linear combination of sine and cosine functions?

*Proof* This time we need to realize a periodic electrical signal $y=f(x)$ with a prescribed shape. For example the signal in Fig. 4.8, which is the typical signal for the horizontal scanning of a screen.

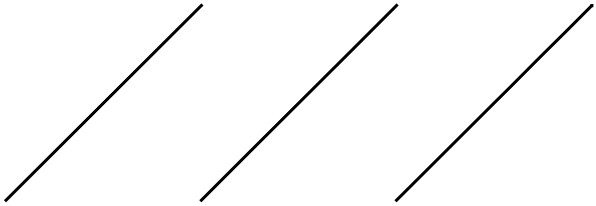

**Fig. 4.8**

The basic electrical signals that one can produce are continuous (a constant function) or alternating (a sine or cosine function, with an arbitrary frequency) electrical signals. We need to determine how to add such signals in order to get a result as close as possible to the prescribed periodic function.

Of course—up to a possible change of variable—there is no loss of generality in assuming that the period of $f(X)$ is equal to $2\pi$. Notice that each function $\sin kx$ or $\cos kx$, for $k \in \mathbb{N}$, itself admits $2\pi$ as a period, even if this is not the smallest possible period. But a linear combination of functions with period $2\pi$ remains a function with period $2\pi$. This proves that it suffices to compute the approximation on the interval $[-\pi, \pi]$: the approximation will automatically remain valid on the whole real line. A priori, $f(X)$ is not necessarily continuous, as in the example above. For simplicity, let us nevertheless assume that $f(X)$ is continuous on its period $]-\pi, \pi[$, with continuous extension to $[-\pi, \pi]$ (again, as in the example above).

We can now consider the Euclidean space $\mathcal{C}([-\pi, \pi], \mathbb{R})$ of Example 4.5.3. To switch back to a finite dimension, consider the Euclidean subspace $V$ generated by $f(X)$ and the functions

$$1,\ \sin X,\ \cos x,\ \sin 2X,\ \cos 2X,\ \ldots,\ \sin nX,\ \cos nX, \quad n \in \mathbb{N}.$$

Write $W$ for the subspace of $V$ generated by these last functions. The orthogonal projection $g(X)$ of $f(X)$ on $W$ thus yields the best approximation of $f(X)$ by a linear combination of $\sin kX$ and $\cos kX$ functions, for $k \leq n$.

It is interesting to observe that in this specific case, the orthogonal projection can be computed very easily. Indeed let us recall that

$$\int_{-\pi}^{\pi} \sin kx \cos lx \, dx = 0$$

$$\int_{-\pi}^{\pi} \sin kx \sin lx \, dx = \begin{cases} 0 & \text{if } k \neq l \\ \pi & \text{if } k = l \end{cases}$$

$$\int_{-\pi}^{\pi} \cos kx \cos lx \, dx = \begin{cases} 0 & \text{if } k \neq l \\ \pi & \text{if } k = l \end{cases}$$

$$\int_{-\pi}^{\pi} \sin kx \, dx = 0$$

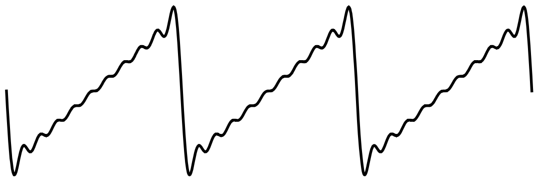

**Fig. 4.9**

$$\int_{-\pi}^{\pi} \cos kx \, dx = 0$$
$$\int_{-\pi}^{\pi} dx = 2\pi.$$

This can be rephrased by saying that the functions

$$\frac{1}{\sqrt{2\pi}}, \ \frac{1}{\sqrt{\pi}} \sin x, \ \frac{1}{\sqrt{\pi}} \cos x, \ \frac{1}{\sqrt{\pi}} \sin 2x, \ \frac{1}{\sqrt{\pi}} \cos 2x, \ \ldots$$

constitute an orthonormal sequence of functions in $\mathcal{C}([-\pi, \pi], \mathbb{R})$. Therefore, by Proposition 4.8.7, the function $g(X)$ above is simply

$$\begin{aligned} g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx & \\ + \frac{1}{\pi} \sum_{k=1}^{n} \sin kx \int_{-\pi}^{\pi} f(x) \sin kx \, dx & \\ + \frac{1}{\pi} \sum_{k=1}^{n} \cos kx \int_{-\pi}^{\pi} f(x) \cos kx \, dx. & \end{aligned}$$

This is a so-called *Fourier approximation* of $f(x)$.

For the "horizontal scanning" function above

$$f(x) = x, \quad -\pi \le x \le \pi$$

Figure 4.9 gives the Fourier approximation obtained when choosing $n = 10$.

Of course a Fourier approximation is always a continuous function, being a linear combination of continuous functions. Therefore the possible discontinuity $f(x)$ at $\pi$ can imply a lower quality of the Fourier approximation around this point. This is a typical case where one might want to use a *weight function* $\omega(x)$ as in Example 4.5.3: a strictly positive function whose values around $-\pi$ and $+\pi$ are slightly greater than at the middle of the interval. Doing this will improve the quality of the approximation at the extremities of the interval, but to the detriment of the quality of the approximation elsewhere! □

## 4.10 Isometries

As the name indicates, an *isometry* is "something" which leaves the "measures" unchanged.

**Definition 4.10.1** An *isometry*

$$(f, \overrightarrow{f}) \colon (E, V) \longrightarrow (F, W)$$

between Euclidean spaces is an affine transformation such that $\overrightarrow{f}$ respects the scalar product:

$$\bigl(\overrightarrow{f}(v) \big| \overrightarrow{f}(v')\bigr) = \bigl(v \big| v'\bigr).$$

Obviously:

**Proposition 4.10.2** *An isometry between two Euclidean spaces respects distances and angles and in particular, is injective.*

*Proof* Distances and angles are defined in terms of the scalar product (see Definitions 4.2.4 and 4.2.6). Moreover $A = B$ precisely when $d(A, B) = 0$ (see Proposition 4.3.2). □

**Proposition 4.10.3** *Let $(E, V)$ and $(F, W)$ be Euclidean spaces of respective finite dimensions $n$ and $m$. Consider an affine transformation*

$$(f, \overrightarrow{f}) \colon (E, V) \longrightarrow (F, W)$$

*and its matrix expression*

$$\overrightarrow{x} \mapsto A\overrightarrow{x} + \overrightarrow{b}$$

*with respect to orthonormal bases of $(E, V)$ and $(F, W)$. The following conditions are equivalent*:

1. *$(f, \overrightarrow{f})$ is an isometry*;
2. *the columns of $A$ constitute an orthonormal sequence of vectors in $\mathbb{R}^m$.*

*Proof* Let

$$(O; e_1, \ldots, e_n), \ \bigl(O'; e'_1, \ldots, e'_m\bigr)$$

be the two orthonormal bases. The columns of $A$ are the coordinates of the vectors $\overrightarrow{f}(e_i)$ in the second base. By Proposition 4.10.1 these vectors constitute an orthonormal sequence in $W$ and since the second base is orthonormal, their coordinates are orthonormal vectors in $\mathbb{R}^m$ (see Proposition 4.6.3).

Conversely, the assumption on $A$ can be rephrased as $A^t A = \mathsf{Id}_m$, where $\mathsf{Id}_n$ is the $n \times n$-identity matrix. Given two vectors of $V$ with coordinates $\overrightarrow{x}$, $\overrightarrow{y}$, we get once more by Proposition 4.6.3

$$\big(\overrightarrow{f}(\overrightarrow{x})|\overrightarrow{f}(\overrightarrow{y})\big) = (A\overrightarrow{x})^t(A\overrightarrow{y}) = \overrightarrow{x}^t A^t A\overrightarrow{y} = \overrightarrow{x}^t\overrightarrow{y} = (\overrightarrow{x}|\overrightarrow{y}). \qquad \square$$

Taking full advantage of Corollary 4.3.4, let us now give an interesting characterization of isometries.

**Theorem 4.10.4** *Let $(E, V)$ and $(F, W)$ be finite dimensional Euclidean spaces. There is a bijection between*:

1. *the isometries $(f, \overrightarrow{f}) : (E, V) \longrightarrow (F, W)$;*
2. *the mappings $f : E \longrightarrow F$ preserving distances.*

*Proof* By Proposition 4.10.2, it remains to show that a mapping $f : E \longrightarrow F$ preserving distances is the first component of a unique isometry $(f, \overrightarrow{f})$. The uniqueness is immediate since in an affine transformation $(f\,\overrightarrow{f})$, the linear mapping $\overrightarrow{f}$ is entirely determined by $f$ (see Proposition 2.12.4).

First observe that given points $A, B, C \in E$ and a scalar $r \in \mathbb{R}$:

$$\overrightarrow{AB} = r\overrightarrow{AC} \quad \Longrightarrow \quad \overrightarrow{f(A).f(B)} = r\,\overrightarrow{f(A).f(C)}.$$

When $0 \le r \le 1$, Corollary 4.3.4 reduces the condition $\overrightarrow{AB} = r\overrightarrow{AC}$ to

$$d(A, B) + d(B, C) = d(A, C).$$

Such a property is thus preserved by $f$. When $r \notin [0, 1]$, a permutation of the roles of $A, B, C$ reduces the problem to the first case. For example if $r > 1$, then

$$\overrightarrow{AC} = \frac{1}{r}\overrightarrow{AB}, \quad 0 \le \frac{1}{r} \le 1$$

and so on.

Thus we know already that $f$ transforms an affine line into an affine line. But by Pythagoras' Theorem 4.3.5, $f$ also transforms a right triangle into a right triangle. Thus $f$ respects the perpendicularity of two affine lines.

Fix now an orthonormal basis $(O; e_1, \ldots, e_n)$ of $(E, V)$. Write further $e_i = \overrightarrow{OA_i}$. The vectors $\overrightarrow{f(O)f(A_i)}$ then constitute an orthonormal sequence in $W$ and we can complete it to an orthonormal basis

$$\big(f(O); \overrightarrow{f(O)f(A_1)}, \ldots, \overrightarrow{f(O)f(A_n)}, e'_{n+1}, \ldots, e'_m\big)$$

of $(F, W)$.

Consider a point $P \in E$ and its $i$-th coordinate $x_i$ with respect to the orthonormal basis $(O; e_1, \ldots, e_n)$. The point $X_i$ such that

$$\overrightarrow{OX_i} = x_i e_i = x_i \overrightarrow{OA_i}$$

is thus the orthogonal projection of $P$ on the line $OA_i$, that is, the unique point of the line through $OA_i$ such that the triangle $OX_iP$ is right angled. But then $f(X_i)$ is the orthogonal projection of $f(P)$ on the line $f(O)f(A_i)$ and the $i$-th coordinate of $f(P)$ with respect to the orthonormal basis of $(F, W)$ is the scalar $x_i'$ such that

$$\overrightarrow{f(O)f(X_i)} = x_i' \overrightarrow{f(0)f(A_i)}.$$

Since we already know that $f$ preserves the proportionality of vectors with the same origin, we conclude that $x_i' = x_i$.

We can summarize our results by saying that, with respect to the two bases indicated, $f$ admits the following matrix description:

$$\overrightarrow{x} \mapsto M\overrightarrow{x}$$

where $M$ is an $m \times n$-matrix whose $n$ first lines are those of the $n \times n$-identity matrix and whose $m - n$ last lines are zero lines. Of course we define $\overrightarrow{f}$ to be the linear mapping $\overrightarrow{f} : V \longrightarrow W$ admitting the matrix $M$ with respect to the two orthonormal bases of $V$ and $W$ as above. The columns of $M$ are trivially orthonormal so that by Proposition 4.10.3, $(f, \overrightarrow{f})$ will be an isometry as soon as it is an affine transformation.

This last fact is obvious. Working in terms of coordinates in the orthonormal bases indicated, $f$ and $\overrightarrow{f}$ act simply by adding $m - n$ zero coordinates. Therefore axioms [AT1] and [AT2] are trivially satisfied. □

## 4.11 Classification of Isometries

In this section, we focus our attention on the isometries from a Euclidean space to itself. Proposition 4.10.3 can at once be rephrased as:

**Proposition 4.11.1** *Let $(E, V)$ be a finite dimensional vector space and*

$$(f, \overrightarrow{f}) \colon (E, V) \longrightarrow (E, V)$$

*be an affine transformation, with matrix expression*

$$\overrightarrow{x} \mapsto A\overrightarrow{x} + \overrightarrow{b}$$

*with respect to an orthonormal basis $(O; e_1, \ldots, e_n)$. The following conditions are equivalent*:

1. *$(f, \overrightarrow{f})$ is an isometry*;
2. *$A$ is an orthogonal matrix, i.e. $A^t = A^{-1}$.*

*In that case, one has* $\det A = \pm 1$ *and the isometry is an affine isomorphism.*

In order to determine all possible isometries on $(E, V)$, let us first review two well-known examples.

*Example 4.11.2* Every translation on a Euclidean space is an isometry.

*Proof* This follows by Proposition 2.14.2: of course the identity mapping preserves the scalar product. □

*Example 4.11.3* Let $(E, V)$ be a finite dimensional Euclidean space. Every orthogonal symmetry is an isometry.

*Proof* Consider an affine subspace $(F, W) \subseteq (E, V)$ and the orthogonal supplement $W^\perp$ of $W$ (see Theorem 4.8.3). Choose a point $O \in F$, an orthonormal basis $e_1, \dots, e_k$ of $W$ and an orthonormal basis $e_{k+1}, \dots, e_n$ of $W^\perp$. We thus obtain an orthonormal basis

$$(O; e_1, \dots, e_k, e_{k+1}, \dots, e_n)$$

of $(E, V)$. With respect to this basis the orthogonal symmetry with respect to $(F, W)$

$$(s, \overrightarrow{s}) \colon (E, V) \longrightarrow (E, V)$$

is such that

$$s(O) = O, \qquad \overrightarrow{s}(e_i) = \begin{cases} e_i & \text{if } 1 \le i \le k; \\ -e_i & \text{if } k+1 \le i \le n. \end{cases}$$

The matrix description of $(s, \overrightarrow{s})$ is thus

$$\overrightarrow{x} \mapsto A\overrightarrow{x}$$

where $A$ is a diagonal matrix with the first $k$ entries on the diagonal equal to $+1$ and the following entries equal to $-1$. The result follows by Proposition 4.11.1. □

Notice that the identity on $(E, V)$ is both the translation by the vector 0 and the orthogonal symmetry with respect to $(E, V)$. This observation helps us to better understand the following statement:

**Proposition 4.11.4** *Let $(E, V)$ be a finite dimensional Euclidean space. The isometries $(f, \overrightarrow{f})$ on $(E, V)$ are precisely the composites of*

- *a translation*;
- *an orthogonal symmetry*;

- *a direct isometry admitting a fixed point* (*see Definition* 2.13.1).

*Of course, each composite of such mappings is an isometry.*

*Proof* The last statement holds by Examples 4.11.2 and 4.11.3, since a composite of isometries is trivially an isometry.

Now let $(f, \overrightarrow{f})$ be an isometry on $(E, V)$. Fix a point $P \in E$ and consider the translation by the vector $\overrightarrow{f(P)P}$. We have

$$(t_{\overrightarrow{f(P)P}} \circ f)(P) = t_{\overrightarrow{f(P)P}}\big(f(P)\big) = P.$$

This proves that

$$g = t_{\overrightarrow{f(P)P}} \circ f$$

is an isometry admitting $P$ as a fixed point.

If $g$ is a direct isometry, we obtain

$$f = t_{Pf(P)} \circ \mathsf{id}_E \circ g$$

and $f$ is expressed as the composite of a translation, an orthogonal symmetry and a direct isometry with fixed point $P$.

If $g$ is an inverse symmetry and $V$ has dimension $n$, let $(F, W)$ be an affine subspace of dimension $n-1$ such that $P \in F$. Write $s$ for the orthogonal symmetry with respect to $(F, W)$. Since $P \in F$, we have $s(P) = P$. The proof of Example 4.11.3 tells us that in an ad-hoc orthonormal basis, the matrix of $s$ is diagonal with the first $n-1$ diagonal entries equal to $+1$ and the last one equal to $-1$. The determinant is thus equal to $-1$ and $s$ is an inverse isometry. But then $h = s \circ g$ is a direct isometry still admitting $P$ as a fixed point. Furthermore, since $s \circ s = \mathsf{id}_E$

$$f = t_{Pf(P)} \circ s \circ h$$

expresses $f$ as the composite of a translation, an orthogonal symmetry and a direct isometry with fixed point $P$. □

To describe all isometries of $(E, V)$, it thus remains to determine the form of the direct isometries with a fixed point.

## 4.12 Rotations

In this section we investigate the form of the direct isometries admitting a fixed point, in dimensions 0, 1, 2 and 3.

**Proposition 4.12.1** *On a Euclidean space of dimension* 0 *or* 1, *the only direct isometry with a fixed point is the identity.*

*Proof* In dimension 0 there is nothing to prove, since the only mapping from the singleton to itself is the identity.

In dimension 1, the only orthogonal matrix with positive determinant is the identity matrix (1); the result follows by Proposition 4.11.1. □

The notion of a *rotation of angle* $\theta$ *with center* $O$ in the usual plane makes perfect sense in a Euclidean plane: "all vectors $\overrightarrow{OP}$ turn around the center $O$ by the same angle $\theta$, in the same orientation". More precisely:

**Definition 4.12.2** Let $(E, V)$ be a Euclidean plane. Consider a relative angle $\theta \in ]-\pi, 0, \pi]$ (see Definition 4.7.1) and a point $O \in E$. A mapping $f\colon E \longrightarrow E$ is a *rotation of angle* $\theta$ *with center* $O$ when:

1. $f(O) = O$;
2. for all $P \neq O$ in $E$:
   (a) $d(O, P) = d(O, f(P))$;
   (b) $\measuredangle(POf(P)) = \theta$.

Let us clarify the situation concerning the two trivial cases $\theta = 0$ and $\theta = \pi$.

**Proposition 4.12.3** *Let* $(E, V)$ *be a Euclidean plane and* $O$ *a point of* $E$. *Then*:

1. *a rotation of angle* 0 *with center* $O$ *is the identity mapping on* $E$;
2. *a rotation of angle* $\pi$ *with center* $O$ *is the central symmetry with respect to* $O$.

*These two rotations are direct isometries.*

*Proof* The first two assertions follow immediately from Proposition 4.3.3. Of course the identity is a direct isometry. In an orthonormal basis $(O; e_1, e_2)$ the central symmetry admits as matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is an orthogonal matrix with determinant $+1$. The result follows by Proposition 4.11.1. □

Observe that a central symmetry is always an isometry (Example 4.11.3): but it is a direct isometry in even dimensions and an inverse isometry in odd dimensions, as the proof of 4.12.3 immediately suggests.

The key result is then:

**Theorem 4.12.4** *Let* $(E, V)$ *be a Euclidean plane and* $f\colon E \longrightarrow E$ *an arbitrary mapping. The following conditions are equivalent* (*see Theorem* 4.10.4):

1. $f$ *is a direct isometry with a fixed point*;
2. $f$ *is a rotation.*

*Proof* Let us work in an orthonormal basis $(0; e_1, e_2)$ considered as having direct orientation.

Suppose that $f$ is a rotation with center $O$ and relative angle $\theta$. If $P \neq O$, working in polar coordinates (see Proposition 4.7.3) the rotation $f$ is simply described by

$$P = \|\overrightarrow{OP}\| \begin{pmatrix} \cos\tau \\ \sin\tau \end{pmatrix} \mapsto f(P) = \|\overrightarrow{OP}\| \begin{pmatrix} \cos(\tau+\theta) \\ \sin(\tau+\theta) \end{pmatrix}.$$

But trivially

$$\begin{pmatrix} \cos(\tau+\theta) \\ \sin(\tau+\theta) \end{pmatrix} = \begin{pmatrix} \cos\tau\cos\theta - \sin\tau\sin\theta \\ \sin\tau\cos\theta + \cos\tau\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Therefore $f$ can be described by the matrix formula

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which is also trivially valid for the origin $O$, which is a fixed point. By Proposition 2.22.1, $f$ is thus an affine transformation. Since its matrix is trivially orthogonal with determinant $+1$, it is a direct isometry (see Proposition 4.10.3 and Definition 3.3.1). By assumption, it admits the fixed point $O$.

Conversely, let $(f, \overrightarrow{f})$ be a direct isometry with fixed point $O$. The matrix expression of the isometry is thus

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where the columns of the matrix are the coordinates of $\overrightarrow{f}(e_1)$ and $\overrightarrow{f}(e_2)$. The matrix is orthogonal (see Proposition 4.11.1) with determinant $+1$. Thus

$$a_1^2 + a_2^2 = 1, \qquad b_1^2 + b_2^2 = 1, \qquad a_1b_1 + a_2b_2 = 0, \qquad a_1b_2 - a_2b_1 = 1.$$

In particular (see the proof of Proposition 4.7.3)

$$a_1 = \cos\theta, \qquad a_2 = \sin\theta$$

for a unique relative angle $\theta$. The resolution of the system

$$b_1\cos\theta + b_2\sin\theta = 0$$
$$-b_1\sin\theta + b_2\cos\theta = 1$$

yields at once

$$b_1 = -\sin\theta, \qquad b_2 = \cos\theta.$$

Thus the matrix expression of $(f, \overrightarrow{f})$ with respect to the orthonormal basis $(O; e_1, e_2)$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This is precisely the form of a rotation of angle $\theta$, as observed in the first part of the proof. □

Let us now switch to dimension 3. The intuitive notion of *a rotation about an axis* can easily be formalized:

**Definition 4.12.5** Let $(E, V)$ be an affine space of dimension 3. Consider an affine line $(\ell, L) \subseteq (E, V)$ and a relative angle $\theta$. A mapping $f\colon E \longrightarrow E$ is a *rotation of axis $\ell$ and angle $\theta$* when:

1. $f(O) = O$ for all points $O \in \ell$;
2. for every point $O \in \ell$, $f$ restricts as a rotation of angle $\theta$ and center $O$ in the affine plane orthogonal to $(\ell, L)$ and containing $O$.

Again:

**Proposition 4.12.6** *Let $(E, V)$ be a Euclidean space of dimension* 3 *and $(\ell, L) \subseteq (E, V)$ an affine line. Then*:

- *a rotation of axis $\ell$ and angle* 0 *is the identity on $E$*;
- *a rotation of axis $\ell$ and angle $\pi$ is the orthogonal symmetry with respect to* $(\ell, L)$.

*These two rotations are direct isometries.*

*Proof* As for Proposition 4.12.3, via Proposition 4.3.3 and Example 4.11.3. The matrix of a rotation of angle $\pi$, with respect to an orthonormal basis $(0; e_1, e_2, e_3)$ now has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and thus has determinant $+1$. □

In dimension 3, the striking point about rotations is perhaps the "non-existence" of a rotation about a point. More precisely:

**Theorem 4.12.7** *Let $(E, V)$ be a Euclidean space of dimension* 3. *Consider an arbitrary mapping $f\colon E \longrightarrow E$. The following conditions are equivalent*:

1. *$f$ is a direct isometry with a fixed point*;
2. *$f$ is a rotation about an axis.*

Before giving the proof, let us recall that an orthogonal matrix $A$ can only have $+1$ and $-1$ as eigenvalues. Indeed

$$Av = \lambda v \quad \Longrightarrow \quad A^{-1}v = \frac{1}{\lambda}v \quad \Longrightarrow \quad A^t v = \frac{1}{\lambda}v.$$

Choosing $v$ of length 1 we get, since $v^t A v$ is a $(1 \times 1)$-matrix,

$$\lambda = \lambda v^t v = v^t A v = \left(v^t A v\right)^t = v^t A^t v = \frac{1}{\lambda}v^t v = \frac{1}{\lambda}.$$

Thus $\lambda^2 = 1$ and $\lambda = \pm 1$.

*Proof* Assume first that $f$ is a rotation of axis $(\ell, L)$ and relative angle $\theta$. Let us work in an orthonormal basis $(O; e_1, e_2, e_3)$ with $O \in \ell$ and $e_1 \in L$. We consider $(e_2, e_3)$ as having direct orientation in the subspace that these two vectors generate. The considerations in the proof of Theorem 4.12.4 indicate at once that $f$ can be described by the matrix formula

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

By Proposition 2.22.1, $f$ is thus an affine transformation. Since its matrix is trivially orthogonal with determinant $+1$, it is a direct isometry (see Proposition 4.10.3 and Definition 3.3.1).

Conversely, consider a direct isometry $(f, \overrightarrow{f})\colon (E, V) \longrightarrow (E, V)$ and a point $O \in E$ such that $f(O) = O$. Fix an orthonormal basis $(O; e'_1, e'_2, e'_3)$. The matrix expression of $f$ becomes

$$\overrightarrow{x} \mapsto A\overrightarrow{x}$$

with $A$ an orthogonal matrix with determinant $+1$ (see Proposition 4.11.1).

The characteristic polynomial of the matrix $A$ has the form

$$p(\lambda) = \det(A - \lambda \mathsf{Id}) = -\lambda^3 + \alpha\lambda^2 + \beta\lambda + \det A, \qquad \alpha, \beta \in \mathbb{R}.$$

We have thus

$$p(0) = \det A = +1, \qquad \lim_{\lambda \to \infty} p(\lambda) = -\infty.$$

By continuity of $p(\lambda)$, the *Intermediate Value Theorem* forces the existence of a positive root $\lambda$, that is, a positive eigenvalue of $A$. As we know, this eigenvalue must be $+1$.

Let $e_1$ be an eigenvector with eigenvalue 1; we choose it to be of length 1. Writing $L \subseteq V$ for the vector subspace generated by $e_1$, the line $(\ell, L)$ through $O$ (see Theorem 2.4.2) is then entirely composed of fixed points of $f$. Indeed,

$$P \in \ell \quad \Longrightarrow \quad P = O + ke_1, \quad k \in \mathbb{R}.$$

Therefore

$$f(P) = f(O) + k\overrightarrow{f}(e_1) = O + ke_1 = P.$$

We introduce an orthonormal basis $(O; e_1, e_2, e_3)$ of $(E, V)$, where $(O; e_2, e_3)$ is an orthonormal basis of the affine plane $(F, L^\perp)$ passing through $O$ and orthogonal to $(\ell, L)$ (see Proposition 4.10.2).

Since $(f, \overrightarrow{f})$ is an isometry, $\overrightarrow{f}$ respects the orthogonality. Therefore $(f, \overrightarrow{f})$ restricts as an isometry on $(F, L^\perp)$. Thus the matrix expression of $(f, \overrightarrow{f})$ with respect to the orthonormal basis $(O; e_1, e_2, e_3)$ has the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Write $B$ for this $3 \times 3$-matrix. Since $B$ is orthogonal with determinant $+1$, the same conclusion applies to the sub-matrix

$$B' = \begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix}$$

which is the matrix of the restriction of $(f\,\overrightarrow{f})$ to $(F, L^\perp)$. By Theorem 4.12.4, the restriction of $(f, \overrightarrow{f})$ to $(F, L^\perp)$ is a rotation with center $O$, for some relative angle $\theta$. Thus, as observed in the proof of Theorem 4.12.4,

$$B' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Finally

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

and as we have seen in the first part of the proof, this is the matrix of a rotation of angle $\theta$ about the axis $(\ell, L)$. □

## 4.13 Similarities

When representing geometrical objects, one often applies a *scaling factor*, just to get the picture at a reasonable size on the page. The scaling factor is somehow irrelevant: two pictures at two different scales "are the same, except for the size". We shall say that they are *similar*.

**Definition 4.13.1** Let $(E, V)$ be a Euclidean space and $k > 0$ a scalar. A *similarity of ratio* $k$ is an affine transformation $(f, \overrightarrow{f}): (E, V) \longrightarrow (E, V)$ such that:

1. $f$ respects all angles;
2. $f$ multiplies all distances by $k$.

We have at once:

**Proposition 4.13.2** *A similarity on a finite dimensional Euclidean space is an affine isomorphism.*

*Proof* With the notation of Definition 4.13.1, by Proposition 4.3.2

$$\begin{aligned} A \neq B \quad &\Longrightarrow \quad d(A, B) > 0 \\ &\Longrightarrow \quad d\big(f(A), f(B)\big) = kd(A, B) > 0 \\ &\Longrightarrow \quad f(A) \neq f(B). \end{aligned}$$

Thus $f$ is injective and the result follows by Proposition 2.13.3. □

*Example 4.13.3* Every isometry on a Euclidean space is a similarity.

*Proof* This follows by Proposition 4.10.2. □

*Example 4.13.4* Let $(E, V)$ be a finite dimensional Euclidean space. A homothety (see Definition 2.17.1) of ratio $k \neq 0$ is a similarity of ratio $k$.

*Proof* Consider a homothety with center $O$ and ratio $k$. With respect to an orthonormal basis $(O; e_1, \dots, e_n)$, the homothety admits the matrix expression

$$\overrightarrow{x} \mapsto k\overrightarrow{x}.$$

Thus all scalar products are multiplied by $k^2$, which forces at once the conclusion. □

A homothety is also called a *central similarity*. The two examples above are highly representative since:

**Proposition 4.13.5** *A similarity on a finite dimensional Euclidean space is the composite of an isometry and a homothety.*

*Proof* If the similarity $(f, \overrightarrow{f}): (E, V) \longrightarrow (E, V)$ has ratio $k$, fix an arbitrary point $O \in E$ and write the similarity as

$$(f, \overrightarrow{f}) = (h_k, \overrightarrow{h_k}) \circ (h_{\frac{1}{k}}, \overrightarrow{h_{\frac{1}{k}}}) \circ (f, \overrightarrow{f})$$

where $h_k$ and $h_{\frac{1}{k}}$ indicates respectively the homotheties with center $O$ and ratios $k$, $\frac{1}{k}$. By Example 4.13.4, $h_{\frac{1}{k}}$ multiplies all distances by $\frac{1}{k}$, thus $h_{\frac{1}{k}} \circ f$ respects distances and therefore is an isometry, by Theorem 4.10.4. □

**Proposition 4.13.6** *Let $(E, V)$ be a finite dimensional Euclidean space. A mapping $f: E \longrightarrow E$ which multiplies all distances by a fixed scalar $k > 0$ is necessarily a similarity.*

*Proof* The same argument as for Proposition 4.13.5 applies: one can write $f = h_k \circ h_{\frac{1}{k}} \circ f$ where $h_{\frac{1}{k}} \circ f$ respects distances and hence is an isometry by Theorem 4.10.4. By Examples 4.13.4 and 4.13.3, $f$ is a similarity, being a composite of two similarities. □

We obtain a nice characterization theorem for similarities, which can be regarded as an extension of Thales' (see 2.18.1).

**Theorem 4.13.7** *Consider a finite dimensional Euclidean space $(E, V)$ and an affine isomorphism $(f, \overrightarrow{f}): (E, V) \longrightarrow (E, V)$. The following conditions are equivalent*:

1. $(f, \overrightarrow{f})$ *is a similarity*;
2. $f$ *respects angles*;
3. $f$ *respects perpendicularity*;
4. $f$ *multiplies all distances by a fixed scalar* $k > 0$.

*Proof* $(1 \Rightarrow 2 \Rightarrow 3)$ are obvious and $(4 \Rightarrow 1)$ is Proposition 4.13.6. It thus suffices to prove $(3 \Rightarrow 4)$.

Consider an orthonormal basis $(O; e_1, \dots, e_n)$. For each pair $i \neq j$ of indices, the four points

$$O, \; X = O + e_i, \; Y = O + e_j, \; Z = O + e_i + e_j$$

are the four vertices of a parallelogram, since

$$\overrightarrow{OY} = e_j = \overrightarrow{XZ}.$$

This parallelogram is a square since $e_i$ is orthogonal to $e_j$ and both have length 1 (see Sect. 4.4). But being a square reduces to the perpendicularity of the sides and the diagonals (see Proposition 4.4.5). Since $f$ preserves the perpendicularity, it preserves squares and therefore, $e_i$ and $e_j$ are mapped by $\overrightarrow{f}$ to orthogonal vectors with the same length: let us say, length $k$. Of course $k \neq 0$ since $f$ is an isomorphism. We thus obtain a new orthonormal basis

$$\left( f(0); \frac{\overrightarrow{f}(e_1)}{k}, \dots, \frac{\overrightarrow{f}(e_n)}{k} \right).$$

The matrix expression of $f$ with respect to the original orthonormal basis and the new orthonormal basis is thus simply $k\mathsf{Id}$, where $\mathsf{Id}$ is the identity matrix. All distances are thus indeed multiplied by $k$. □

## 4.14 Euclidean Quadrics

In the Euclidean case, Theorem 2.24.2 can be improved in the expected way:

**Theorem 4.14.1** *Let $Q \subseteq E$ be a quadric in a finite dimensional Euclidean space $(E, V)$. There exists an orthonormal basis $(0; e_1, \dots, e_n)$ with respect to which the equation of the quadric takes one of the reduced forms*:

Type 1 $\sum_{i=1}^{n} a_i X_i^2 = 1$;
Type 2 $\sum_{i=1}^{n} a_i X_i^2 = 0$;
Type 3 $\sum_{i=1}^{n-1} a_i X_i^2 = X_n$.

*Proof* The proof is an easy adaptation of that of Theorem 2.24.2. We focus only on the necessary changes.

Applying Theorem G.4.1 instead of Corollary G.2.8 in the proof of the preliminary Lemma 2.24.1, we begin with an orthonormal basis $(P; \varepsilon_1, \dots, \varepsilon_n)$ with respect to which the equation of the quadric has the form

$$\sum_{i=1}^{n} \alpha_i Y_i^2 + \sum_{i=1}^{n} \beta_i Y_i + \gamma = 0.$$

The arguments in the proof of Theorem 2.24.2 apply as such to prove the existence of another origin $O$ so that with respect to the orthonormal basis $(O; \varepsilon_1, \dots, \varepsilon_n)$, the equation of the quadric now takes one of the three forms:

$$\sum_{i=1}^{n} \alpha_i Z_i^2 = 1$$

$$\sum_{i=1}^{n} \alpha_i Z_i^2 = 0$$

$$\sum_{i=1}^{m} \alpha_i Z_i^2 + \sum_{i=m+1}^{n} \beta_i Z_i = 0.$$

To conclude the proof, it suffices to find another orthonormal basis giving rise to a change of coordinates with the properties

$$X_i = Z_i \quad \text{for } 1 \le i \le m, \qquad X_n = -\frac{1}{k}\left(\sum_{i=m+1}^{n} \beta_i Z_i\right), \quad k \neq 0.$$

Multiplying the equation by $k$ will yield the expected result. Notice that if the change of coordinates matrix $M$ is orthogonal, the new basis will automatically be orthonormal. Indeed the vectors of the new basis, expressed in terms of the old orthonormal basis, will be the columns of the inverse change of coordinate matrix $M^{-1}$, that is, the lines of $M$ since $M^{-1} = M^t$.

We must therefore prove the existence of an orthogonal matrix of the form

$$\left(\begin{array}{ccc|ccc}
1 & & 0 & & & \\
 & \ddots & & & 0 & \\
0 & & 1 & & & \\
\hline
 & & & & & \\
\hline
0 & \cdots & 0 & -\frac{\beta_{m+1}}{k} & \cdots & -\frac{\beta_n}{k}
\end{array}\right).$$

Choosing

$$k = \left\|(0, \dots, 0, -\beta_{m+1}, \dots, -\beta_n)\right\| \neq 0,$$

the first $m$ lines of this matrix, together with the last line, constitute an orthonormal sequence of vectors in $\mathbb{R}^n$. It suffices to complete this sequence to an orthonormal basis of $\mathbb{R}^n$ to get an orthogonal matrix. □

Proposition 2.25.1 yields immediately, in the Euclidean context:

**Proposition 4.14.2** *Let $(O; e_1, \dots, e_n)$ be a given orthonormal basis in some Euclidean space $(E, V)$. Consider a quadric $Q \subseteq E$ which, with respect to this basis, has a reduced equation as in Theorem 4.14.1. Consider a vector subspace $W \subseteq V$ which:*

1. *in the case of an equation of type* 1 *or* 2, *is generated by some of the vectors* $e_1, \dots, e_n$;
2. *in the case of an equation of type* 3, *is generated by some of the vectors* $e_1, \dots, e_{n-1}$.

*Write $(F, W^\perp)$ for the affine subspace with direction $W^\perp$ passing through the origin $O$. The quadric $Q$ is stable under the orthogonal symmetry with respect to $(F, W^\perp)$.*

## 4.15 Problems

**4.15.1** In a Euclidean space, prove that the sum of the angles of an arbitrary triangle equals $\pi$.

**4.15.2** Let $(0; e_1, \dots, e_n)$ be a finite dimensional Euclidean space and $(F, W)$ a hyperplane with equation $\sum_{i=1}^n a_i X_i = b$. Find a formula giving the distance between a point $P$ and the subspace $F$.

**4.15.3** In a Euclidean plane, prove that a direct isometry is a rotation or a translation.

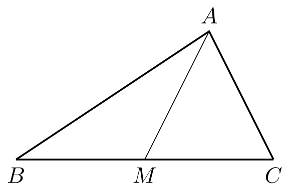


**Fig. 4.10**

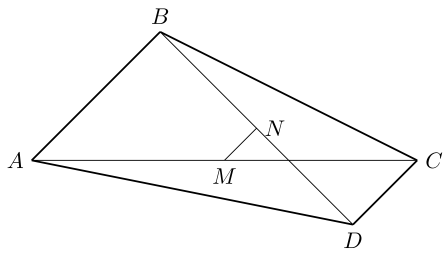


**Fig. 4.11**

**4.15.4** In a Euclidean plane $(E, V)$ consider a rotation $(r, \overrightarrow{r})$ by angle $0 < \theta < \pi$. Prove that for every vector $v \in V$, there exists a unique point $P \in E$ such that $\overrightarrow{Pr(P)} = v$.

**4.15.5** Consider a triangle $ABC$ in a Euclidean plane and the median $AM$ of the side $BC$ (see Fig. 4.10). Prove that

$$d(A, B)^2 + d(A, C)^2 = \frac{1}{2} d(B, C)^2 + 2d(A, M)^2.$$

**4.15.6** Consider a quadrilateral $ABCD$ in a Euclidean plane, together with the middle points $M$, $N$ of the two diagonals (see Fig. 4.11). Prove that

$$d(A, B)^2 + d(B, C)^2 + d(C, D)^2 + d(D, A)^2 = d(A, C)^2 + d(B, D)^2 + 4d(M, N)^2.$$

**4.15.7** Let $(O; e_1, \ldots, e_n)$ be an arbitrary basis of a Euclidean space. Prove that the basis is orthonormal if and only if the coordinates $\overrightarrow{x}$ of a point $P$ are such that $x_i = (\overrightarrow{OP}|e_i)$ for each index $i$.

**4.15.8** Consider two points $A \neq B$ in a finite dimensional Euclidean space $(E, V)$. Prove that the locus of points $P$ such that $d(P, A) = d(P, B)$ is a hyperplane. This hyperplane is called the *mediatrix hyperplane* of the segment $[AB]$.

**4.15.9** In a Euclidean plane $(E, V)$, prove that two distinct affine lines are parallel if and only if they are perpendicular to the same third affine line.

**4.15.10** In a finite dimensional Euclidean space, prove that every translation is the composite of two orthogonal symmetries with respect to parallel hyperplanes.

**4.15.11** Let $(E, V)$ be a finite dimensional Euclidean space. Every similarity on $(E, V)$ of ratio $k \neq 1$ has exactly one fixed point.

## 4.16 Exercises

**4.16.1** Let $(E, V, (-|-))$ be a Euclidean space.

1. If in Fig. 4.11, $ABCD$ is a parallelogram, prove that
$$2d(A, B)^2 + 2d(B, C)^2 = d(A, C)^2 + d(B, D)^2.$$
2. Infer the *median theorem* from this equality (see Fig. 4.10): *In a triangle $ABC$, if $M$ is the middle point of the side $BC$, then*
$$d(A, B)^2 + d(A, C)^2 = 2d(A, M)^2 + \frac{1}{2}d(B, C)^2.$$
3. Prove vectorially that a triangle $ABC$ "inscribed in a circle" (i.e. the three vertices of the triangle are points of a given circle) is a *right triangle* if and only if two of its vertices are on a diameter of this circle.

**4.16.2** Consider the Euclidean space $E^2(\mathbb{R})$, that is, $\mathbb{R}^2$ with its usual scalar product.

1. Consider the two bases
$$\mathcal{R}' = \left(P'; e_1', e_2'\right), \qquad P' = (1, -3), \qquad e_1' = (1, 2), \qquad e_2' = (2, -3),$$
$$\mathcal{R}'' = \left(P''; e_1'', e_2''\right), \qquad P'' = \left(17, \frac{3}{4}\right), \qquad e_1'' = \left(\frac{3}{5}, \frac{4}{5}\right),$$
$$e_2'' = \left(-\frac{4}{5}, \frac{3}{5}\right).$$
Give the matrix of the scalar product with respect to these two bases.
2. Let $A, B, C \in E^2(\mathbb{R})$ admit the coordinates
$$A = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$
with respect to the basis $\mathcal{R}'$. Calculate the scalar product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and the angle between these two vectors.

**4.16.3** Consider $\mathbb{R}^2$ as an affine space over itself.

1. Determine a scalar product such that:
   (a) $\|(1, 1)\| = 1$ and $\|(0, \frac{1}{2})\| = 1$;
   (b) the vectors $(1, 1)$ and $(0, \frac{1}{2})$ are orthogonal.
2. In the so-obtained Euclidean space, compute the angle between the two lines with equations $x = 1$ and $y = 17$.
3. In the same Euclidean structure, determine the circle of radius 1 centered at the origin.

**4.16.4** Consider the Euclidean space $\mathbb{R}_{(2)}[X]$ of real polynomials of degree at most 2, where the scalar product is defined by

$$\big(a_2X^2 + a_1X + a_0 | b_2X^2 + b_1X + b_0\big) = 2a_0b_0 + 2a_1b_1 + a_2b_2 + a_0b_1 + a_1b_0.$$

Consider the canonical basis $\mathcal{R}_c = (0; 1, X, X^2)$ of $\mathbb{R}_{(2)}[X]$ and the other basis $\mathcal{R}' = (P; e'_1, e'_2, e'_3)$ given by

$$P = 2X^2 + 2X + 2, \qquad e'_1 = X + 1, \qquad e'_2 = X^2 + X, \qquad e'_3 = X^2 + 1.$$

Consider next the point $A$, admitting the coordinates $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ with respect to the canonical basis $\mathcal{R}_c$, and the point $B$, admitting the coordinates $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ with respect to the basis $\mathcal{R}'$. Find the vectorial and Cartesian equations, with respect to the canonical basis $\mathcal{R}_c$, of the plane $\pi$ containing $A$ and perpendicular to $\overrightarrow{AB}$.

**4.16.5** Let $(E, V)$ be a real affine space of dimension 2. Fix $O \in E$ and two linearly independent vectors $e_1, e_2 \in V$.

1. Give the formulas of change of coordinates between the following two bases: $(0; e_1, e_2)$ and $(0 + e_2, -e_1, e_2 - e_1)$.
2. Is it possible to provide $V$ with a scalar product with respect to which both bases are orthonormal?

**4.16.6** In $E^3(\mathbb{R})$ consider a tetrahedron $ABCD$. Prove that, when the edges $AB$ and $CD$ are orthogonal, as well as the edges $AC$ and $BD$, then the edges $AD$ and $BC$ are necessarily also orthogonal.

**4.16.7** In a real affine space $(E, V)$ of dimension 2, consider a parallelogram $ABCD$ and denote by $O$ the intersection point of its diagonals. Consider further the two bases

$$\mathcal{R} = (A; \overrightarrow{AB}, \overrightarrow{AD}), \qquad \mathcal{S} = (0; \sqrt{2}\overrightarrow{OA}, \sqrt{2}\overrightarrow{OB}).$$

Show that for any scalar product on $V$, the basis $\mathcal{R}$ is orthonormal if and only if the basis $\mathcal{S}$ is orthonormal.

**4.16.8** Consider the vector space $\mathbb{R}_{(2)}[X]$ of real polynomials of degree at most 2, viewed as an affine space over itself. Provide this space with the scalar product

$$(P|Q) = P(-1)\cdot Q(-1) + P(0)\cdot Q(0) + P(1)\cdot Q(1).$$

1. Calculate the angle $\measuredangle(PQR)$ when

$$P = X+1, \qquad Q = -X+1, \qquad R = X^2 - X.$$

2. Show that

$$F = \left\{X^2 + aX + b \middle| a, b \in \mathbb{R}\right\}$$

   is an affine subspace of $\mathbb{R}_{(2)}[X]$ and compute the orthogonal projection of the zero polynomial on this subspace.

**4.16.9** Let $(f, \overrightarrow{f})$ be an affine transformation of $E^3(\mathbb{R})$. Suppose that $\overrightarrow{f}$ preserves the scalar product. Suppose further that $\overrightarrow{f}$ admits 1 as eigenvalue and that the corresponding subspace of eigenvectors of eigenvalue 1 is a vectorial plane $\pi_0$. Suppose finally that $f$ admits a fixed point $P$. Explain why $f$ is necessarily the orthogonal symmetry with respect to the plane $\pi$ of direction $\pi_0$ passing through $P$.

**4.16.10** Consider $E^3(\mathbb{R})$; we work in the canonical basis. Prove that the affine transformation defined by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is an isometry. Is this a symmetry? If *so*, determine the plane, the line or the center of symmetry. Is this a rotation? If *so*, determine the axis and the angle of rotation.

**4.16.11** Consider the affine transformation $f$ on $E^3(\mathbb{R})$ described, with respect to the canonical basis, by the formula

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}.$$

Prove that this is an isometry. Determine the type of isometry and its geometric elements.

**4.16.12** Consider the affine transformation $f$ on $E^3(\mathbb{R})$ described, in the canonical basis, by the formula

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ a \\ 0 \end{pmatrix}.$$

Prove that this is an isometry. Determine the type of this isometry according to the values of the parameter $a$. Determine the geometric elements of this isometry for the values $a = 1$ and $a = 2$.

**4.16.13** Let $(E, (-|-))$ be a Euclidean space and $O \in E$. Consider an isometry $(f, \overrightarrow{f})$ of this space to itself and put $v = \overrightarrow{Of(O)}$. Prove that $f$ admits a fixed point if and only if $v$ is perpendicular to every vector of $\mathsf{Fix}(\overrightarrow{f})$.

**4.16.14** In the Euclidean space $E^3(\mathbb{R})$ with its canonical scalar product, consider the basis $\mathcal{R}' = (O; e_1, e_2, e_3)$ with

$$O = (0, 0, 0), \qquad e_1 = (1, 0, 0), \qquad e_2 = (1, 1, 0), \qquad e_3 = (1, 1, 1).$$

Prove that the affine transformation defined in the basis $\mathcal{R}'$ by the formula

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

is an isometry. Determine its nature and its geometric elements.

**4.16.15** In $E^2(\mathbb{R})$, find the matrix representation, in the canonical basis, of an isometry $f$ mapping:

- the point (0, 1) to the point $(\frac{1}{\sqrt{3}}, 0)$;
- the point $(\frac{1}{\sqrt{3}}, 0)$ to the point $(-\frac{1}{\sqrt{3}}, 0)$;
- the point $(-\frac{1}{\sqrt{3}}, 0)$ to the point (0, 1).

Give the nature and the geometric elements of this isometry.

**4.16.16** Consider the vector space $\mathbb{R}_{(2)}[X]$ of real polynomials of degree at most 2 viewed as an affine space over itself. Provide it with the scalar product

$$\big(a_2X^2 + a_1X + a_0 \big| b_2X^2 + b_1X + b_0\big) = a_2b_2 + a_1b_1 + a_0b_0.$$

Determine the matrices $A \in \mathbb{R}^{3\times 3}$ and $B \in \mathbb{R}^{3\times 1}$ such that the affine transformation defined by the formula

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + B$$

in the canonical basis is the inverse isometry mapping

$$0 \text{ to } X^2, \qquad 1 \text{ to } 0, \qquad X \text{ to } X^2 + \frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}.$$

**4.16.17** Let $(E, V, (-|-))$ be a finite dimensional Euclidean space. Consider two isometries $(f, \overrightarrow{f})$ and $(g, \overrightarrow{g})$ of $E$ to itself. Suppose that both $f$ and $g$ admit fixed points and commute with each other, that is, $f \circ g = g \circ f$. Prove, via the following steps, that $f$ and $g$ have at least one common fixed point.

1. Prove first that $(\mathsf{Fix}(f), \mathsf{Fix}(\overrightarrow{f}))$ is an affine subspace.
2. Prove that this subspace is invariant under $(g, \overrightarrow{g})$, that is $(g, \overrightarrow{g})$ restricts to an affine isomorphism

$$(g, \overrightarrow{g}) \colon \big(\mathsf{Fix}(f), \mathsf{Fix}(\overrightarrow{f})\big) \longrightarrow \big(\mathsf{Fix}(f), \mathsf{Fix}(\overrightarrow{f})\big).$$

3. Let $\Omega$ be a fixed point of $g$ and $w$ the orthogonal projection of $\Omega$ on $\mathsf{Fix}(f)$. Prove that $w$ is a common fixed point of $f$ and $g$. (Hint: using the definition of an orthogonal projection, show that $g(w)$ is also the orthogonal projection of $\Omega$ on $\mathsf{Fix}(f)$.)

**4.16.18** In $E^3(\mathbb{R})$, consider the orthogonal symmetry with respect to the plane with equation $z + 2 = 0$ followed by the rotation by a half turn about the line with equations $x = 1,\ y = z$. Is this composite an isometry? If *so*, what type of isometry is it? Find its geometric elements.

**4.16.19** Consider the quadric $\mathcal{Q}$ of $\mathbb{R}^3$ whose equation with respect to the canonical basis is

$$3x^2 + 3y^2 - 2z^2 + 2xy + 6x + 2y + 4z + 3 = 0.$$

Find an orthonormal basis with respect to which this quadric admits a reduced equation. Infer the nature of this quadric.

**4.16.20** In $E^3(\mathbb{R})$ consider the quadric $\mathcal{Q}$ with equation

$$2x^2 + 4x - y^2 - 2yz - z^2 - 3z + 5 = 0$$

with respect to the canonical basis. Let $\mathcal{R} = (0; e_1, e_2, e_3)$ be an orthonormal basis with respect to which the equation of $\mathcal{Q}$ is reduced and let $\pi$ be the plane with equation $-x + y + z = 0$ in the canonical basis. What is the equation of the direction of $\pi$ with respect to the basis $(e_1, e_2, e_3)$?

**4.16.21** In $E^3(\mathbb{R})$ and its canonical basis, consider the point $A = (1, 0, 1)$ and the plane $\pi$ with equation $y = x - 1$. Let $\mathcal{Q}$ be the locus of those points whose distance to the point $A$ is equal to $\sqrt{2}$ times the distance to the plane $\pi$. Prove that $\mathcal{Q}$ is a quadric.

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