Preface

The reader is invited to immerse himself in a "love story" which has been unfolding for 35 centuries: the love story between mathematicians and geometry. In addition to accompanying the reader up to the present state of the art, the purpose of this *Trilogy* is precisely to tell this story. The *Geometric Trilogy* will introduce the reader to the multiple complementary aspects of geometry, first paying tribute to the historical work on which it is based, and then switching to a more contemporary treatment, making full use of modern logic, algebra and analysis. In this *Trilogy*, Geometry is definitely viewed as an autonomous discipline, never as a sub-product of algebra or analysis. The three volumes of the *Trilogy* have been written as three independent but complementary books, focusing respectively on the axiomatic, algebraic and differential approaches to geometry. They contain all the useful material for a wide range of possibly very different undergraduate geometry courses, depending on the choices made by the professor. They also provide the necessary geometrical background for researchers in other disciplines who need to master the geometric techniques.

It is a matter of fact that, for more than 2000 years, the Greek influence remained so strong that geometry was regarded as the only noble branch of mathematics. In [7], *Trilogy I*, we have described how Greek mathematicians handled the basic algebraic operations in purely geometrical terms. The reason was essentially that *geometric quantities are more general than numbers*, since at the time, only rational numbers were recognized as actual numbers. In particular, algebra was considered as a "lower level art"—if an "art" at all. Nevertheless, history provides evidence that some mathematicians sometimes thought "in algebraic terms"; but elegance required that the final solution of a problem always had to be expressed in purely geometrical terms. This attitude persisted up to the moment where some daring mathematicians succeeded in creating elegant and powerful algebraic methods which were able to compete with the classical synthetic geometric approach. Unexpectedly, it is to geometry that this new approach has been most profitable: a wide range of new problems, in front of which Greek geometry was simply helpless, could now be stated and solved. Let us recall that Greek geometry limited itself to the study of those problems which could be solved with ruler and compass constructions!

During the 17th century, Fermat and Descartes introduced the basic concepts of analytic geometry, allowing an efficient algebraic study of functions and curves. The successes of this new approach have been striking. However, as time went on, and the problems studied became more and more involved, the algebraic computations needed to solve the problems were themselves becoming so involved and heavy to handle that they had lost all traces of elegance. Clearly, the limits of this algebraic approach had more or less been reached.

But for those men believing in their art, a difficulty taking the form of a dead end is just the occasion to open new ways to unexpected horizons. This is what happened during the 19th century, with the birth of abstract algebra. The theory of groups, that of vector spaces, the development of matrix algebra and the abstract theory of polynomials have provided new efficient tools which, today, remain among the key ingredients in the development of an algebraic approach to geometry. Grothendieck's theory of schemes is probably the most important new stone that the 20th century offered to algebraic geometry, but this is rather clearly beyond the scope of this introductory text.

We devote the first chapter of this book to an historical survey of the birth of analytic geometry, in order to provide the useful intuitive support to the modern abstract approach, developed in the subsequent chapters.

The second chapter focuses on affine geometry over an arbitrary (always commutative) field: we study parallel subspaces, parallel projections, symmetries, quadrics and of course, the possible use of coordinates to transform a geometric problem into an algebraic one.

The three following chapters investigate the special cases where the base field is that of the real or complex numbers. In real affine spaces, there is a notion of "orientation" which in particular allows us to recapture the notion of a segment. The Euclidean spaces are the real affine spaces provided with a "scalar product", that is, a way of computing distances and angles. We pay special attention to various possible applications, such as approximations by the law of least squares and the Fourier approximations of a function. We also study the Hermitian case: the affine spaces, over the field of complex numbers, provided with an *ad hoc* "scalar (i.e. Hermitian) product".

Returning to the case of an arbitrary field, we next develop the theory of the corresponding projective spaces and generalize various results proved synthetically in [7], *Trilogy I*: the duality principle, the theory of the anharmonic ratio, the theorems of Desargues, Pappus, Pascal, and so on.

The last chapter of this book is a first approach to the theory of algebraic curves. We limit ourselves to the study of curves of an arbitrary degree in the complex projective plane. We focus on questions such as tangency, multiple points, the Bezout theorem, the rational curves, the cubics, and so on.

Each chapter ends with a section of "problems" and another section of "exercises". Problems are generally statements not treated in this book, but of theoretical interest, while exercises are more intended to allow the reader to practice the techniques and notions studied in the book.

Of course reading this book supposes some familiarity with the algebraic methods involved. Roughly speaking, we assume a reasonable familiarity with the content of a first course in linear algebra: vector spaces, bases, linear mappings, matrix calculus, and so on. We freely use these notions and results, sometimes with a very brief reminder for the more involved of them. We make two notable exceptions. First the theory of quadratic forms, whose diagonalization appears to be treated only in the real case in several standard textbooks on linear algebra. Since quadratic forms constitute the key tool for developing the theory of quadrics, we briefly present the results we need about them in an appendix. The second exception is that of dual vector spaces, often absent from a first course in linear algebra.

In the last chapter on algebraic curves, the fact that the field $\mathbb{C}$ of complex numbers is algebraically closed is of course essential, as is the theory of polynomials in several variables, including the theory of the resultant. These topics are certainly not part of a first course in algebra, even if the reader may get the (false) impression that many of the statements look very natural. We provide various appendices proving these results in elementary terms, accessible to undergraduate students. This is in particular the case for the proof that the field of complex numbers is algebraically closed and for the unique factorization in irreducible factors of a polynomial in several variables.

A selective bibliography for the topics discussed in this book is provided. Certain items, not otherwise mentioned in the book, have been included for further reading.

The author thanks the numerous collaborators who helped him, through the years, to improve the quality of his geometry courses and thus of this book. Among them he especially thanks *Pascal Dupont*, who also gave useful hints for drawing some of the illustrations, realized with *Mathematica* and *Tikz*.

http://www.springer.com/978-3-319-01732-7

An Algebraic Approach to Geometry
Geometric Trilogy II
Borceux, F.
2014, XVII, 430 p. 117 illus., Hardcover
ISBN: 978-3-319-01732-7