# Preface

The reader is invited to immerse himself in a “love story” which has been unfolding for 35 centuries: the love story between mathematicians and geometry. In addition to accompanying the reader up to the present state of the art, the purpose of this *Trilogy* is precisely to tell this story. The *Geometric Trilogy* will introduce the reader to the multiple complementary aspects of geometry, first paying tribute to the historical work on which it is based and then switching to a more contemporary treatment, making full use of modern logic, algebra and analysis. In this *Trilogy*, Geometry is definitely viewed as an autonomous discipline, never as a sub-product of algebra or analysis. The three volumes of the *Trilogy* have been written as three independent but complementary books, focusing respectively on the axiomatic, algebraic and differential approaches to geometry. They contain all the useful material for a wide range of possibly very different undergraduate geometry courses, depending on the choices made by the professor. They also provide the necessary geometrical background for researchers in other disciplines who need to master the geometric techniques.

In the 1630s Fermat and Descartes were already computing the tangents to some curves using arguments which today we would describe in terms of derivatives (see [4], *Trilogy II*). However, these arguments concerned algebraic curves, that is, curves whose equation is expressed by a polynomial, and the derivative of a polynomial is something that one can describe algebraically in terms of its coefficients and exponents, without having to handle limits. Some decades later, the development of differential calculus by Newton and Leibniz allowed these arguments to be formalized in terms of actual derivatives, for rather arbitrary curves. In the present book, we focus on this general setting of curves and surfaces described by functions which are no longer defined by polynomials, but are arbitrary functions having sufficiently well behaved properties with respect to differentiation.

We have deliberately chosen to restrict our attention to curves in the 2- and 3-dimensional real spaces and surfaces in the 3-dimensional real space. Although we occasionally give a hint on how to generalize several of our results to higher dimensions, our focus on lower dimensions provides the best possible intuition of the basic notions and techniques used today in advanced studies of differential geometry.

An important notion is the consideration of *parametric equations*, following an idea of *Euler* (see [4], *Trilogy II*). A closer look at such equations suggests that we should view a curve not as the set of points whose coordinates satisfy some equation(s), but as a continuous deformation of the real line in $\mathbb{R}^2$ or in $\mathbb{R}^3$, according to the case. When a parameter $t$ varies on the real line, the parametric equations describe successively all the points of the curve. Analogously, a surface can be seen as a continuous deformation of the real plane in the space $\mathbb{R}^3$. This is the notion of a *parametric representation*, which is the basic tool that we shall use in our study.

Our first chapter is essentially historical: its purpose is to explain where the ideas of differential geometry came from and why we choose this or that precise definition and not another possible one.

The formalized study of curves then begins with Chap. 2, where we restrict our attention to the simplest case: the plane curves. We pay special attention to basic notions like tangency, length and curvature, but we also prove very deep theorems, such as the *Hopf theorem* for simple closed curves. Working in the plane makes certainly things easier to grasp in a first approach. However, it is also a matter of fact that the study of plane curves offers many interesting aspects, such as envelopes, evolutes, involutes, which have beautiful applications. Many of these aspects do not generalize elegantly to higher dimensions.

Our Chap. 3 is a kind of parenthesis in our theory of differential geometry: we present a *museum* of some specimens of curves which have played an important historical role in the development of the theory.

Chapter 4 is then devoted to the study of curves in three dimensional space: the so-called *skew curves*. We focus our attention on the main aspects of the theory, namely, the study of the *curvature* and the *torsion* of skew curves and the famous *Frenet trihedron*.

Next we switch to surfaces in $\mathbb{R}^3$. In Chap. 5 we concentrate our attention on the *local properties* of surfaces, that is, properties "in a neighborhood of a given point of the surface", such as the tangent plane at that point or the various notions of curvature at that point: normal curvature, Gaussian curvature, and the information that we can get from these on the shape of the surface in a neighborhood of the point.

Chapter 6 then begins by repeating many of the arguments of Chap. 5, but using a different notation: the notation of Riemannian geometry. Our objective is to provide in this way a good intuitive approach to notions such as the *metric tensor*, the *Christoffel symbols*, the *Riemann tensor*, and so on. We provide evidence that these apparently very technical notions reduce, in the case of surfaces in $\mathbb{R}^3$, to very familiar notions studied in Chap. 5. We also devote special attention to the case of *geodesics* and establish the main properties (including the existence) of the systems of geodesic coordinates.

The last chapter of this book is devoted to some *global properties* of surfaces: properties for which one has to consider the full surface, not just what happens in a neighborhood of one of its points. We start with a basic study of the surfaces of revolution, the ruled and the developable surfaces and the surfaces with constant curvature. Next we switch to results and notions such as the *Gauss–Bonnet* theorem and the *Euler characteristic*, which represent some first bridges between the elementary theory of surfaces and more advanced topics.

Each chapter ends with a section of “problems” and another section of “exercises”. Problems are generally statements not treated in this book, but of theoretical interest, while exercises are more intended to allow the reader to practice the techniques and notions studied in the book.

Of course reading this book assumes some familiarity with the basic notions of linear algebra and differential calculus, but these can be found in all undergraduate courses on these topics. An appendix on general topology introduces the few ingredients of that theory which are needed to properly follow our approach to Riemannian geometry and the global theory of surfaces. A second appendix states with full precision (but without proofs this time) some theorems on the existence of solutions of differential equations and partial differential equations, which are required in some advanced geometrical results.

A selective bibliography for the topics discussed in this book is provided. Certain items, not otherwise mentioned in the book, have been included for further reading.

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