Classical and Stochastic Löwner–Kufarev Equations

Filippo Bracci, Manuel D. Contreras, Santiago Díaz-Madrigal, and Alexander Vasil'ev

Abstract In this paper we present a historical and scientific account of the development of the theory of the Löwner–Kufarev classical and stochastic equations spanning the 90-year period from the seminal paper by K. Löwner in 1923 to recent generalizations and stochastic versions and their relations to conformal field theory.

Keywords Brownian motion • Conformal mapping • Evolution family • Integrable system • Kufarev • Löwner • Pommerenke • Schramm • Subordination chain

Mathematics Subject Classification (2000). Primary 01A70 · 30C35; Secondary 17B68 · 70H06 · 81R10

Filippo Bracci (✉)
Dipartimento Di Matematica, Università di Roma "Tor Vergata", Via Della Ricerca Scientifica 1, 00133 Roma, Italy
e-mail: fbracci@mat.uniroma2.it

M.D. Contreras · S. Díaz-Madrigal
Departamento de Matemática Aplicada II, Escuela Técnica Superior de Ingeniería, Universidad de Sevilla, Camino de los Descubrimientos, s/n 41092 Sevilla, Spain
e-mail: contreras@us.es; madrigal@us.es

A. Vasil'ev
Department of Mathematics, University of Bergen, P.O. Box 7803, Bergen N-5020, Norway
e-mail: alexander.vasiliev@math.uib.no

A. Vasil'ev (ed.), *Harmonic and Complex Analysis and its Applications*, Trends in Mathematics, DOI 10.1007/978-3-319-01806-5_2,
© Springer International Publishing Switzerland 2014

1 Introduction

The Löwner theory went through several periods of its development. It was born in 1923 in the seminal paper by Löwner [165], its *formation* was completed in 1965 in the paper by Pommerenke [189] and was finally formulated thoroughly in his monograph [190] unifying Löwner's ideas of semigroups and evolution equations and Kufarev's contribution [144, 146] of the t-parameter differentiability (in the Carathéodory kernel) of a family of conformal maps of simply connected domains $\Omega(t)$ onto the canonical domain $\mathbb{D}$, the unit disk in particular, and of PDE for subordination Löwner chains of general type. It is worth mentioning that the 20-year period between papers [165] and [144] was illuminated by Goluzin's impact [90] on further applications of Löwner's parametric method to extremal problems for univalent functions, as well as works by Fekete and Szegö [75], Peschl [182], Robinson [215], Komatu [135], Bazilevich [27, 28] related to this period.

The next period became *applications* of the *parametric method* to solving concrete problems for conformal maps culminating at the de Branges proof [46] of the celebrating Bieberbach conjecture [37] in 1984. During approximately 16 years after this proof the interest to univalent functions had been somehow decaying. However, the *modern period* has been marked by burst of interest to stochastic (Schramm)–Löwner evolution (SLE) which has implied an elegant description of several 2D conformally invariant statistical physical systems at criticality by means of a solution to the Cauchy problem for the Löwner equation with a random driving term given by 1D Brownian motion, see [157, 230]. At the same time, several connections with mathematical physics were discovered recently, in particular, relations with a singular representation of the Virasoro algebra using Löwner martingales in [24, 81, 82, 130] and with a Hamiltonian formulation and a construction of the KP integrable hierarchies in [169–171].

2 Löwner

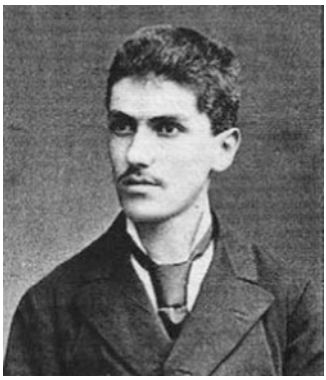
Karel Löwner was born on May 29, 1893 in Lány, Bohemia. He also used the German spelling Karl of his first name until his immigration to the U.S.A. in 1939 after Nazis occupied Prague, when he changed it to Charles Loewner as a new start in a new country. Although coming from a Jewish family and speaking Czech at home, all of his education was in German, following his father's wish. After finishing a German Gimnasium in Prague in 1912 he entered the Charles University of Prague and received his Ph.D. from this university in 1917 under the supervision of Georg Pick.

After four and a half years in German Technical University in Prague where the mathematical environment was not stimulating enough, Löwner took up a position at the University of Berlin (now Humboldt Universität Berlin) with great enthusiasm where he started to work surrounded by Schur, Brauer, Hopf, von Neumann, Szegö,



K. Löwner

and other famous names. Following a brief lectureship at Cologne in 1928 Löwner returned to the Charles University of Prague where he hold a chair in Mathematics until the occupation of Czechoslovakia in 1939. When the Nazis occupied Prague, he was put in jail. Luckily, after paying the "emigration tax" twice over he was allowed to leave the country with his family and moved, in 1939, to the U.S.A. After von Neumann arranged a position for him at the Louisville University, Löwner was able to start from the bottom of his career. Following further appointments at the Brown (1944) and Syracuse (1946) Universities, he moved in 1951 to his favorite place, California, where he worked at the Stanford University until his death. His former Ph.D. student in Prague, Lipman Bers, testifies [180] that Löwner was a man whom everybody liked, a man at peace with himself, a man who was incapable of malice. He was a great teacher, Lipman Bers, Adriano Garcia, Gerald Goodman, Carl Fitzgerald, Roger Horn, Ernst Lammel, Charles Titus were his students among 26 in total. Löwner died as a Professor Emeritus in Stanford on January 8, 1968, after a brief illness. For more about Löwner's biography, see [2, 180].



G. Pick

Löwner's work covers wide areas of complex analysis and differential geometry and displays his deep understanding of Lie theory and his passion for semigroup theory. Perhaps it is worth to mention here two of his discoveries. First was his introduction of infinitesimal methods for univalent mappings [165], where he defined basic notions of semigroups of conformal maps of $\mathbb{D}$ and their evolution families leading to the sharp estimate $|a_3| \leq 3$ in the Bieberbach conjecture $|a_n| \leq n$ in the class $\mathbf{S}$ of normalized univalent maps $f(z) = z + a_2 z^2 + \dots$ of the unit disk $\mathbb{D}$.

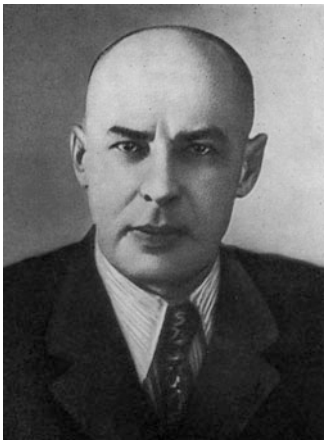
It is wellknown that the Bieberbach conjecture was finally proven by de Branges in 1984 [45, 46]. In his proof, de Branges introduced ideas related to Löwner's but he used only a distant relative of Löwner's equation in connection with his main apparatus, his own rather sophisticated theory of composition operators. However, elaborating on de Branges' ideas, FitzGerald and Pommerenke [76] discovered how to avoid altogether the composition operators and rewrote the proof in classical terms, applying the *bona fide* Löwner equation and providing in this way a direct and powerful application of Löwner's classical method.

The second paper [166], we want to emphasize on, is dedicated to properties of n-monotonic matrix functions, which turned to be of importance for electrical engineering and for quantum physics. Löwner was also interested in the problems of fluid mechanics, see, e.g., [167], while working at Brown University 1944–1946 on a war-related program. Starting with some unusual applications of the theory of univalent functions to the flow of incompressible fluids, he later applied his methods to the difficult problems of the compressible case. This led him naturally to the study of partial differential equations in which he obtained significant differential inequalities and theorems regarding general conservation laws.

3 Kufarev

The counterpart subordination chains and the Löwner–Kufarev PDE for them were introduced by Pavel Parfen'evich Kufarev (Tomsk, March 18, 1909–Tomsk, July 17, 1968). Kufarev entered the Tomsk State University in 1927, first, the Department of Chemistry, and then, the Department of Mathematics, which he successfully finished in 1931. After a year experience in industry in Leningrad he returned to Tomsk in 1932 and all his academic life was connected with this university, the first university in Siberia. Kufarev started to work at the Department of Mathematics led by professor Lev Aleksandrovich Vishnevskiĭ (1887–1937). At the same time, two prominent Western mathematicians came to Tomsk.

One was Stefan Bergman (1895–1977). Being of Jewish origin he was forced from his position in Berlin in 1933 due to the "Restoration of the civil service," an anti-Semitic Hitler's law. Bergman came to Tomsk in 1934 and was working there until 1936, then at Tbilisi in Georgia in 1937 and had to leave Soviet Union under Stalin's oppression towards foreign scientists. For 2 years he was working at the Institute Henri Poincaré in Paris, and then, left France for the USA in 1939 because of German invasion, he finally worked at Stanford from 1952 together with Löwner.



P. P. Kufarev



S. Bergman

The history of the other one is more tragic. Fritz Noether (1884–1941), the youngest son of Max Noether (1844–1921) and the younger brother of Emmy Noether (1882–1935), came to Tomsk in the same year as Bergman, because of the same reason, and remained there until 1937, when Vishnevskiĭ and "his group" were accused of espionage. Noether and Vishnevskiĭ were arrested by NKVD (later KGB) and Noether was transported to Orel concentration camp where he was jailed until 1941. Nazis approached Orel in 1941 and many prisoners were executed on September 10, 1941 (Stalin's order from September 8, 1941). Fritz Noether was among them following the official Soviet version of 1988, cf. [62].

Fitz's wife Regine returned to Germany from Soviet Union in 1935 under psychological depression and soon died. The same year his famous sister Emmy died in Pennsylvania after a feeble operation. His two sons were deported from Soviet Union. Fortunately, they were given refuge in Sweden as a first step. However, many years later Evgeniy Berkovich and Boris M. Schein published a correspondence [35], also [236], in which prof. Boris Shein referred to prof. Saveliĭ



F. Noether

Vladimirovich Falkovich (1911–1982, Saratov), who met Fritz Noether in Moscow metro in the late fall of 1941. Noether and Falkovich knew each other and Noether talked his story on the arrest and tortures in Tomsk NKVD. In particular, NKVD agents confiscated many of his things and books. Noether said that he finally was released from Orel Central and went to Lubyanka (NKVD/KGB headquarters) for some traveling documents to visit his family. Then the train stopped and their conversation was interrupted. Since he did not come to Tomsk (his son's Gottfried Noether testimony), he most probably was arrested again. This ruins the official Soviet version of Noether's death and indicates that the story is not completed yet.[1] One of the possible reasons is that several Jewish prisoners (first of all, originating from Poland, e.g., Henrik Erlich and Victor Alter; and from Germany) were released following Stalin's plans of creating an Anti-Hitler Jewish Committee, which further was not realized. Noether could be among them being rather known mathematician himself and having support from Einstein and Weyl.

Kufarev was also accused of espionage being a member of Vishnevskiĭ's "terrorist group" together with Boris A. Fuchs, but they were not arrested in contrast to Fritz Noether, see [38].

Bergman and Noether supported Kufarev's thesis defense in 1936. Then Kufarev was awarded the Doctor of Sciences degree (analogue of German habilitation) in 1943 and remained the unique mathematician in Siberia with this degree until 1957 when the *Akademgorodok* was founded and Mikhail A. Lavrentiev invited many first-class mathematicians to Novosibirsk. Kufarev became a full professor of the Tomsk State University in 1944, served as a dean of the Faculty of Mechanics and Mathematics 1952–1955, and remained a professor of the university until his death in 1968 after a hard illness.

[1] I have also some personal interest to this story because me, Falkovich and Shein worked in different periods at the same Saratov State University. *A. Vasil'ev*

The main interests of Kufarev were in the theory of univalent functions and applications to fluid mechanics, in particular, in Hele-Shaw flows, see an overview in [253]. His main results on the parametric method were published in two papers [144, 146], where he considered subordination chains of general type and wrote corresponding PDE for mapping functions, but he returned to this method all the time together with his students, combining variational and parametric methods, creating the parametric method for multiply connected domains, half-plane version of the parametric method, etc.

4 Pommerenke and Unification of the Parametric Method

Christian Pommerenke (born December 17, 1933 in Copenhagen, Professor Emeritus at the Technische Universität Berlin) became that person who unified Löwner and Kufarev's ideas and thoroughly combined analytic properties of the ordering of the images of univalent mappings of the unit disk with evolutionary aspects of semigroups of conformal maps. He seems to have been the first one to use the expression "Löwner chain" for describing the family of "increasing" univalent mappings in Löwner's theory.



Ch. Pommerenke

Recall that we denote the unit disk by $\mathbb{D} = \{\zeta : |\zeta| < 1\}$ and the class of normalized univalent maps $f : \mathbb{D} \to \mathbb{C}$, $f(z) = z + a_2 z^2 + \dots$ by $\mathbf{S}$. A t-parameter family $\Omega(t)$ of simply connected hyperbolic univalent domains forms a *Löwner subordination chain* in the complex plane $\mathbb{C}$, for $0 \le t < \tau$ (where τ may be ∞), if $\Omega(t) \subseteq \Omega(s)$, whenever $t < s$, and the family is continuous in the Carathéodory sense. We suppose that the origin is a point of $\Omega(0)$.

A Löwner subordination chain $\Omega(t)$ is described by a t-dependent family of conformal maps $z = f(\zeta, t)$ from $\mathbb{D}$ onto $\Omega(t)$, normalized by $f(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2 + \dots$, $a_1(t) > 0$, $\dot{a}_1(t) > 0$. Pommerenke [189, 190] described governing evolution equations in partial and ordinary derivatives, known now as the Löwner–Kufarev equations.

One can normalize the growth of evolution of a subordination chain by the conformal radius of $\Omega(t)$ with respect to the origin setting $a_1(t) = e^t$.

We say that the function p is from the Carathéodory class if it is analytic in $\mathbb{D}$, normalized as $p(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + \ldots, \quad \zeta \in \mathbb{D}$, and such that $\operatorname{Re} p(\zeta) > 0$ in $\mathbb{D}$. Given a Löwner subordination chain of domains $\Omega(t)$ defined for $t \in [0, \tau)$, there exists a function $p(\zeta, t)$, measurable in $t \in [0, \tau)$ for any fixed $z \in \mathbb{D}$, and from the Carathéodory class for almost all $t \in [0, \tau)$, such that the conformal mapping $f : \mathbb{D} \to \Omega(t)$ solves the equation

$$\frac{\partial f(\zeta, t)}{\partial t} = \zeta \frac{\partial f(\zeta, t)}{\partial \zeta} p(\zeta, t), \tag{1}$$

for $\zeta \in \mathbb{D}$ and for almost all $t \in [0, \tau)$. Equation (1) is called the Löwner–Kufarev equation due to two seminal papers: by Löwner [165] who considered the case when

$$p(\zeta, t) = \frac{e^{iu(t)} + \zeta}{e^{iu(t)} - \zeta}, \tag{2}$$

where $u(t)$ is a continuous function regarding $t \in [0, \tau)$, and by Kufarev [144] who proved differentiability of f in t for all ζ in the case of general p from the Carathéodory class.

Let us consider a reverse process. We are given an initial domain $\Omega(0) \equiv \Omega_0$ (and therefore, the initial mapping $f(\zeta, 0) \equiv f_0(\zeta)$), and an analytic function $p(\zeta, t)$ of positive real part normalized by $p(\zeta, t) = 1 + p_1\zeta + \ldots$. Let us solve (1) and ask ourselves, whether the solution $f(\zeta, t)$ defines a subordination chain of simply connected univalent domains $f(\mathbb{D}, t)$. The initial condition $f(\zeta, 0) = f_0(\zeta)$ is not given on the characteristics of the partial differential equation (1); hence, the solution exists and is unique but not necessarily univalent. Assuming s as a parameter along the characteristics we have

$$\frac{dt}{ds} = 1, \quad \frac{d\zeta}{ds} = -\zeta p(\zeta, t), \quad \frac{df}{ds} = 0,$$

with the initial conditions $t(0) = 0$, $\zeta(0) = z$, $f(\zeta, 0) = f_0(\zeta)$, where z is in $\mathbb{D}$. Obviously, we can assume $t = s$. Observe that the domain of ζ is the entire unit disk. However, the solutions to the second equation of the characteristic system range within the unit disk but do not fill it. Therefore, introducing another letter w (in order to distinguish the function $w(z, t)$ from the variable ζ) we arrive at the Cauchy problem for the Löwner–Kufarev equation in ordinary derivatives

$$\frac{dw}{dt} = -wp(w, t), \tag{3}$$

for a function $\zeta = w(z, t)$ with the initial condition $w(z, 0) = z$. Equation (3) is a nontrivial characteristic equation for (1). Unfortunately, this approach requires the extension of $f_0(w^{-1}(\zeta, t))$ into the whole $\mathbb{D}$ (here w^{-1} means the inverse function

in ζ) because the solution to (1) is the function $f(\zeta, t)$ given as $f_0(w^{-1}(\zeta, t))$, where $\zeta = w(z, s)$ is a solution of the initial value problem for the characteristic equation (3) that maps $\mathbb{D}$ into $\mathbb{D}$. Therefore, the solution of the initial value problem for (1) may be nonunivalent.

On the other hand, solutions to (3) are holomorphic univalent functions $w(z, t) = e^{-t}(z + a_2(t)z^2 + \dots)$ in the unit disk that map $\mathbb{D}$ into itself. Every function f from the class $\mathbf{S}$ can be represented by the limit

$$f(z) = \lim_{t\to\infty} e^t w(z, t), \tag{4}$$

where $w(z, t)$ is a solution to (3) with some function $p(z, t)$ of positive real part for almost all $t \geq 0$ (see [190, pp. 159–163]). Each function $p(z, t)$ generates a unique function from the class $\mathbf{S}$. The reciprocal statement is not true. In general, a function $f \in \mathbf{S}$ can be obtained using different functions $p(\cdot, t)$.

Now we are ready to formulate the condition of univalence of the solution to (1) in terms of the limiting function (4), which can be obtained by combination of known results of [190].

Theorem 1 ([190, 207]). *Given a function $p(\zeta, t)$ of positive real part normalized by $p(\zeta, t) = 1 + p_1\zeta + \dots$, the solution to (1) is unique, analytic, and univalent with respect to ζ for almost all $t \geq 0$, if and only if, the initial condition $f_0(\zeta)$ is taken in the form (4), where the function $w(\zeta, t)$ is the solution to (3) with the same driving function p.*

Concluding this section we remark that the Löwner and Löwner–Kufarev equations are described in several monographs [4, 14, 57, 72, 93, 108, 122, 190, 218].

5 Half-plane Version

In 1946, Kufarev [145, Introduction] first mentioned an evolution equation in the upper half-plane $\mathbb{H}$ analogous to the one introduced by Löwner in the unit disk and was first studied by Popova [196] in 1954. In 1968, Kufarev et al. [151] introduced a combination of Goluzin–Schiffer's variational and parametric methods for this equation for the class of univalent functions in the upper half-plane, which is known to be related to physical problems in hydrodynamics. They showed its application to the extremal problem of finding the range of $\{\operatorname{Re} e^{i\alpha} f(z), \operatorname{Im} f(z)\}$, $\operatorname{Im} z > 0$. Moreover, during the second half of the past century, the Soviet school intensively studied Kufarev's equations for $\mathbb{H}$. We ought to cite here at least contributions by Aleksandrov [14], Aleksandrov and Sobolev [15], Goryainov and Ba [101, 104]. However, this work was mostly unknown to many Western mathematicians, in particular, because some of it appeared in journals not easily accessible from the outside of the Soviet Union. In fact, some of Kufarev's papers were not even reviewed by Mathematical Reviews. Anyhow, we refer the reader to [7], which contains a complete bibliography of his papers.

In order to introduce Kufarev's equation properly, let us fix some notation. Let γ be a Jordan arc in the upper half-plane $\mathbb{H}$ with starting point $\gamma(0) = 0$. Then there exists a unique conformal map $g_t : \mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$ with the normalization

$$g_t(z) = z + \frac{c(t)}{z} + O\left(\frac{1}{z^2}\right), \quad z \sim \infty.$$

After a reparametrization of the curve γ, one can assume that $c(t) = 2t$. Under this normalization, one can show that g_t satisfies the following differential equation:

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi(t)}, \qquad g_0(z) = z. \tag{5}$$

The equation is valid up to a time $T_z \in (0, +\infty]$ that can be characterized as the first time t such that $g_t(z) \in \mathbb{R}$ and where h is a continuous real-valued function. Conversely, given a continuous function $h\colon [0, +\infty) \to \mathbb{R}$, one can consider the following initial value problem for each $z \in \mathbb{H}$:

$$\frac{dw}{dt} = \frac{2}{w - \xi(t)}, \qquad w(0) = z\,. \tag{6}$$

Let $t \mapsto w^z(t)$ denote the unique solution to this Cauchy problem and let $g_t(z) := w^z(t)$. Then g_t maps holomorphically a (not necessarily slit) subdomain of the upper half-plane $\mathbb{H}$ onto $\mathbb{H}$. Equation (6) is nowadays known as the *chordal Löwner differential equation* with the function h as the driving term. The name is due to the fact that the curve $\gamma[0,t]$ evolves in time as t tends to infinity into a sort of chord joining two boundary points. This kind of construction can be used to model evolutionary aspects of decreasing families of domains in the complex plane. Equation (3) with the function p given by (2) in this context is called the *radial Löwner equation*, because in the slit case, the tip of slit tends to the origin in the unit disk.

Quite often it is presented the half-plane version considering the inverse of the functions g_t (see, i.e., [151]). Namely, the conformal mappings $f_t = g_t^{-1}$ from $\mathbb{H}$ onto $\mathbb{H} \setminus \gamma\big([0,t)\big)$ satisfy the PDE

$$\frac{\partial f_t(z)}{\partial t} = -f_t'(z)\frac{2}{z - \xi(t)}. \tag{7}$$

We remark that using the Cayley transform $T(z) = \frac{z+1}{1-z}$ we obtain that the chordal Löwner equation in the unit disk takes the form

$$\frac{\partial h_t(z)}{\partial t} = -h_t'(z)(1-z)^2 p(z,t), \tag{8}$$

where $\operatorname{Re} p(z,t) \geq 0$ for all $t \geq 0$ and $z \in \mathbb{D}$. From a geometric point of view, the difference between this family of parametric functions and those described by the

Löwner–Kufarev equation (1) is clear: the ranges of the solutions of (8) decrease with t while in the former equation (1) increase. This duality "decreasing" versus "increasing" has recently been analyzed in [54] and, roughly speaking, we can say that the "decreasing" setting can be deduced from the "increasing" one.

The stochastic version of (6) with a random entry λ will be discussed in Sect. 13.

Analogues of the Löwner–Kufarev methods appeared also in the theory of planar quasiconformal maps but only few concrete problems were solved using them, see [233] and [87].



G. M. Goluzin

6 Applications to Extremal Problems: Optimal Control

After Löwner himself, one of the first who applied Löwner's method to extremal problems in the theory of univalent functions in 1936 was Gennadiĭ Mikhailovich Goluzin (1906–1952, St. Petersburg–Leningrad, Russia) [90, 91] and Ernst Peschl (1906, Passau–1986, Eitorf, Germany) [182]. Goluzin was a founder of Leningrad school in geometric function theory, a student of Vladimir I. Smirnov. He obtained in an elegant way several new and sharp estimates. The most important of them is the sharp estimate in the rotation theorem. Namely, if $f \in \mathbf{S}$, then

$$|\arg f'(z)| \le \begin{cases} 4 \arcsin |z| & \text{if } 0 < |z| \le \frac{1}{\sqrt{2}}, \\ \pi + \log \frac{|z|^2}{1-|z|^2} & \text{if } \frac{1}{\sqrt{2}} < |z| < 1. \end{cases}$$

Goluzin himself proved [91] sharpness only for the case $0 < |z| \le \frac{1}{\sqrt{2}}$, Bazilevich [27] completed the proof for $\frac{1}{\sqrt{2}} < |z| < 1$ later during the same year.



E. Peschl (right) with H. Weyl

Ernst Peschl was a student of Constantin Carathéodory and obtained his doctorate at Munich in 1931. He spent 2 years 1931–1933 in Jena working with Robert König, then moved to Münster, Bonn, Braunschweig and finally returned to the Rheinische Friedrich-Wilhelms University in Bonn where he worked until his retirement in 1974. In contrast to other heroes of our story Peschl remained in Germany under Nazi and even was a member of the Union of National Socialist Teachers 1936–1938 (he was thrown out of the Union of National Socialist Teachers since he had not paid his membership fees), however, it was only in order to continue his academic career. Being fluent in French, Peschl served as a military interpreter during the II-nd World War until 1943. Peschl [182] applied Löwner's method to prove the following statement. If $f \in \mathbf{S}$ and $f(z) = z + a_2 z^2 + \ldots$, then $2\Phi\left(\mathrm{Re}\,\frac{a_2}{2}\right) - 1 \leq \mathrm{Re}\,(a_3 - a_2^2) \leq 1$, where

$$\Phi(x) = \frac{x^2}{\varphi^2(x)}(2\varphi(x) - 1),$$

where $\varphi(x)$ is a unique solution to the equation $x + \varphi e^{1-\varphi} = 0$. The result is sharp. The paper is much deeper than only this inequality and became the first serious treatment of the problem of description of the coefficient body $(a_2, a_3, \ldots, a_n)$ in the class S, which was later treated in a nice monograph by Schaeffer and Spencer [226]. Let us mention that the above inequality was repeated by Goryainov [97] in 1980 who also presented all possible extremal functions.

Related result is a nice matter of the paper by Fekete and Szegö [75], see also [72, p. 104] who ingeniously applied Löwner's method in order to disprove a conjecture $|c_{2n+1}| \leq 1$ by Littlewood and Paley [164] on coefficients of odd univalent functions $\mathbf{S}^{(2)} \subset \mathbf{S}$ defined by the expansion $h(z) = z + c_3 z^3 + c_5 z^5 + \ldots$.

Theorem 2 ([75]). *Suppose that* $f \in \mathbf{S}$ *and* $0 < \alpha < 1$. *Then*

$$|a_3 - \alpha a_2^2| \le 1 + 2^{-2\alpha/(1-\alpha)}.$$

This bound is sharp for all $0 < \alpha < 1$.

The choice $\alpha = \frac{1}{4}$ and a simple recalculation of coefficients of functions from $\mathbf{S}^{(2)}$ versus $\mathbf{S}$

$$c_3 = \frac{1}{2}a_2, \quad c_5 = \frac{1}{2}\big(a_3 - \frac{1}{4}a_2\big),$$

lead to the corollary $|c_5| \le \frac{1}{2} + e^{-2/3} = 1.013\ldots$. Since the result is sharp, there exist odd univalent functions with coefficients bigger than 1. Using similar method Robertson [213] proved that $|c_3|^2 + |c_5|^2 \le 2$, which is a particular case of his conjecture $\sum_{k=1}^{n} |c_{2k-1}|^2 \le n$ for the class $\mathbf{S}^{(2)}$ with $c_1 = 1$.

Among other papers on coefficient estimates, let us distinguish the first proof of the Bieberbach conjecture for $n = 4$ by Garabedian and Schiffer [85] in 1955 where Löwner's method was also used. Walter Hayman wrote in his review on this paper that the method "not only carries Bieberbach's conjecture one step further from the point where Löwner placed it over 30 years ago, but also gives a hope, in principle at least, to prove the conjecture for the next one or two coefficients by an increase in labour, rather than an essentially new method." Finally, the complete proof of the Bieberbach conjecture by de Branges [45, 46] in 1984 used the parametric method. It is worth mentioning that the coefficient problem for the inverse functions is much simpler and was solved by Löwner in the same 1923 paper.

Theorem 3 ([165]). *Suppose that* $f \in \mathbf{S}$ *and that*

$$z = \phi(w) = f^{-1}(w) = w + b_2 w^2 + \ldots$$

is the inverse function. Then

$$|b_n| \le \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(n+1)!} 2^n,$$

with the equality for the function $f(z) = \frac{z}{(1+z)^2}$.

Let us just mention that most of the elementary estimates of functionals in the class $\mathbf{S}$, such as $|f(z)|$, $|\arg \frac{f(z)}{z}|$, $|f'(z)|$, can be obtained by Löwner's method, see, e.g., [14, 72, 122, 190]. In particular, the sharp estimate

$$\left|\arg \frac{z f'(z)}{f(z)}\right| \le \log \frac{1+|z|}{1-|z|}$$

implies that for every $0 < r \le \tanh(\pi/4)$ the disk $|z| < r$ is mapped by $f \in \mathbf{S}$ onto a starlike domain with respect to the origin. The constant $\tanh(\pi/4)$ is sharp and is

called the radius of starlikeness. Krzyż [143] used the Löwner method to obtain the radius of close-to-convexity 0.80... in the class $\mathbf{S}$.

In 1950–1952, Kufarev and his student Fales [148–150] solved a Lavrentiev problem on the weighted product of the conformal radii of two non-overlapping domains in $\mathbb{D}$ using his parametric method. Namely, let z_1 and z_2 be two points in $\mathbb{D}$ and let Γ be a Jordan curve splitting $\mathbb{D}$ into two domains B_1 and B_2 such that $z_1 \in B_1$ and $z_2 \in B_2$. Let f_1 and f_2 be two conformal maps of $\mathbb{D}$ onto B_1 and B_2, respectively, $f_k(0) = z_k$, $f_k'(0) > 0$, $k = 1, 2$. The problem is to find Γ which maximizes the functional $J(\Gamma; z_1, z_2) = |f_1'(0)|^\alpha |f_2'(0)|^\beta$, where $\alpha, \beta \geq 0$. For $\alpha = \beta$ the maximum is attained if Γ is the non-Euclidean line in $\mathbb{D}$ which bisects orthogonally the non-Euclidean segment which connects the two points z_1 and z_2. This was obtained by Lavrentiev [154] himself by different approach. For $\alpha \neq \beta$ the answer is much more complicated.

After Kufarev, the Tomsk school in geometric function theory was led by Igor Aleksandrovich Aleksandrov (born May 11, 1932, Novosibirsk, Russia) who completed his Ph.D. in the Tomsk State University in 1958 under supervision by Kufarev. Together with his students, he developed the parametric method combining it with the Goluzin–Schiffer variational method in 1960 and solved several important extremal problems. In 1963 he defended the Doctor of Sciences degree. This time specialists in univalent functions turned from estimation of functionals on the class $\mathbf{S}$ to evaluation of ranges of systems of functionals. One of the natural systems is $I(f) = \{\log \frac{f(z)}{z}, \log \frac{zf'(z)}{f(z)}\}$ for a chosen branch of log, such that $I(id) = (0, 0, 0, 0)$. When f runs over $\mathbf{S}$, the range $\mathfrak{B} = \{I(f) : f \in \mathbf{S}\}$ of $I(f)$ fills a closed bounded set in $\mathbb{R}^4$. The boundary of this set was completely described by Popov [194], Gutlyanskiĭ [115], Goryainov and Gutlyanskiĭ [96] (for the subclass $\mathbf{S}_M$ of bounded functions $|f(z)| \leq M$ from $\mathbf{S}$). Goryainov [98–100] moreover obtained the uniqueness of the extremal functions for the boundary of



I. A. Aleksandrov

$\mathfrak{B}$ and their form. Goryainov, based on Löwner–Kufarev equations and on results by Pommerenke [190] and Gutlyanskiĭ [115] on representation of functions from $\mathbf{S}$

by (4), created a method of determination of all boundary functions for this and other systems of functionals, see also [97]. It was time when a part of mathematicians led by prof. Georgiĭ Dmitrievich Suvorov moved from Tomsk to Donetsk (Ukraine) to a newly established (1966) Institute of Applied Mathematics of the Academy of Sciences of the Ukraine. Vladimir Ya. Gutlyanskiĭ was a student of Aleksandrov, Georgiĭ D. Suvorov of Kufarev, and Victor V. Goryainov of Gutlyanskiĭ. There were several related works on studying $\mathfrak{B}$, its projections, generalizations to other similar results for the classes of univalent functions by Chernetskiĭ, Astakhov, and other members of this active group. Let us mention earlier works by Aleksandrov and Kopanev [9] who determined the domain of variability $\log f'(z)$ for functions in $\mathbf{S}$ for each fixed z by Löwner–Kufarev method. Earlier Arthur Grad [226, pp. 263–291] obtained this domain by variational method. The same authors [8] found the range of the system of functionals $\{\log|\frac{f(z)}{z}|, \log|f'(z)|\}$ in the class $\mathbf{S}$.

That time several powerful methods for solution of extremal problems competed. Most known are the variational method of Schiffer and Goluzin [226, 227], area principle [159], method of extremal metrics [153]. The problem is hard first of all because the univalence condition for the class $\mathbf{S}$ makes it a nonlinear manifold and all variational methods must be very special. In this situation, a considerable progress was achieved when a general optimization principle [192] appeared in 1964, now known as the Pontryagin maximum principle (PMP). It turned out that the Löwner and Löwner–Kufarev equations being viewed as evolution equations give controllable systems of differential equations where the driving term or control function is provided by the Carathéodory function p. Perhaps it was Löwner himself who first noticed all advantages of combination of his parametric method with PMP. His last student Gerald S. Goodman [94, 95] first explicitly stated this combination in 1966. However, he did not return to this topic, and later (and independently) Aleksandrov and Popov [10] gave a real start to investigations in this direction. Further applications of this combination were performed in [11] finding the range of the system of functionals $\{\mathrm{Re}\,(e^{i\alpha}a_2), \mathrm{Re}\,(a_3 - a_2^2)\}$ in the class $\mathbf{S}$, $\alpha \in [0, \pi/2]$. We observe that during this period Vladimir Ivanovich Popov was a real driving force of this process, see, e.g., [195]. However, after some first success [12, 13] with already known problems or their modifications, some insuperable difficulties did not allow to continue with a significant progress although the ideas were rather clear. Real breakthrough happened in 1984 when Dmitri Prokhorov [199] started a series of papers in which he solved several new problems by applying PMP together with the Löwner–Kufarev equation (3). Let us describe a typical application of PMP in the Löwner–Kufarev theory. As an example we consider one of the most difficult problems in the theory of univalent functions, namely, description of the boundary of the coefficient body in the class $\mathbf{S}$. A beautiful book by Schaeffer and Spenser [226] can be our starting point. If V_n denotes the nth coefficient body, $V_n = V_n(f) = \{a_2, a_3, \dots, a_n\}$, $f = z + a_2z^2 + a_3z^3 + \dots$ in the class $\mathbf{S}$, then the first nontrivial body V_3 was completely described in [226]. The authors also give some qualitative description of general V_n. Several qualitative results on ∂V_n were obtained in the monograph [39]. In particular, Bobenko developed a method



D. V. Prokhorov

of second variation more general than that by Duren and Schiffer [71] and proved that ∂V_n is smooth except sets of smaller dimension.

Now let $f(z,t) = e^t w(z,t)$, where $w(z,t)$ is a solution to the Löwner equation (3) with the function p given by (2). We introduce the matrix $A(t)$ and the vector $a(t)$ as

$$A(t) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ a_1(t) & 0 & \dots & 0 & 0 \\ a_2(t) & a_1(t) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}(t) & a_{n-2}(t) & \dots & a_1(t) & 0 \end{pmatrix}, \quad a(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ \vdots \\ a_n(t) \end{pmatrix},$$

where $a_1(t) \equiv 1$ and $a_2(t), a_3(t), \dots$ are the coefficients of the function $f(z,t)$. Substituting $f(z,t)$ in (3) we obtain a controllable system

$$\frac{da(t)}{dt} = -2 \sum_{k=1}^{n-1} e^{-k(t+iu(t))} A^k(t) a(t), \tag{9}$$

with the initial condition $a^T(0) = (1,0,0,\dots,0)$. The coefficient body V_n is a reachable set for the controllable system (9).

The results concerning the structure and properties of V_n include

(i) V_n is homeomorphic to a $(2n-2)$-dimensional ball and its boundary ∂V_n is homeomorphic to a $(2n-3)$-dimensional sphere;
(ii) every point $x \in \partial V_n$ corresponds to exactly one function $f \in \mathbf{S}$ which will be called a *boundary function* for V_n;
(iii) with the exception for a set of smaller dimension, at every point $x \in \partial V_n$ there exists a normal vector satisfying the Lipschitz condition;

(iv) there exists a connected open set X_1 on ∂V_n, such that the boundary ∂V_n is an analytic hypersurface at every point of X_1. The points of ∂V_n corresponding to the functions that give the extremum to a linear functional belong to the closure of X_1.

It is worth noticing that all boundary functions have a similar structure. They map the unit disk $\mathbb{D}$ onto the complex plane $\mathbb{C}$ minus piecewise analytic Jordan arcs forming a tree with a root at infinity and having at most $n-1$ tips. This assertion underlines the importance of multi-slit maps in the coefficient problem for univalent functions.

Solutions $a(t)$ to (9) for different control function $u(t)$ (piecewise continuous, in general) represent all points of ∂V_n as $t \to \infty$. The trajectories $a(t)$, $0 \le t < \infty$, fill V_n so that every point of V_n belongs to a certain trajectory $a(t)$. The endpoints of these trajectories can be interior or else boundary points of V_n. This way, we set V_n as the closure of the reachable set for the control system (9).

According to property (ii) of V_n, every point $x \in \partial V_n$ is attained by exactly one trajectory $a(t)$ which is determined by a choice of the piecewise continuous control function $u(t)$. The function $f \in \mathbf{S}$ corresponding to x is a multi-slit map of $\mathbb{D}$. If the boundary tree of f has only one tip, then there is a unique continuous control function $u(t)$ in $t \in [0, \infty)$ that corresponds to f. Otherwise one obtains multi-slit maps for piecewise continuous control u.

In order to reach a boundary point $x \in \partial V_n$ corresponding to a one-slit map, the trajectory $a(t)$ has to obey extremal properties, i.e., to be an *optimal trajectory*. The continuous control function $u(t)$ must be optimal, and hence it satisfies a necessary condition of optimality. PMP is a powerful tool to be used that provides a joint interpretation of two classical necessary variational conditions: the Euler equations and the Weierstrass inequalities (see, e.g., [192]).

To realize the maximum principle we consider an *adjoint vector* (or momenta)

$$\psi(t) = \begin{pmatrix} \psi_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \psi_n(t) \end{pmatrix},$$

with the complex-valued coordinates $\psi_1, \ldots, \psi_n$, and the *Hamiltonian function*

$$H(a, \psi, u) = \mathrm{Re}\left[\left(-2 \sum_{k=1}^{n-1} e^{-k(t+iu(t))} A^k(t) a(t) \right)^T \bar{\psi} \right],$$

where $\bar{\psi}$ means the vector with complex conjugate coordinates. To come to the Hamiltonian formulation for the coefficient system we require that $\bar{\psi}$ satisfies the adjoint system of differential equations

$$\frac{d\bar{\psi}}{dt} = -\frac{\partial H}{\partial a}, \quad 0 \le t < \infty. \tag{10}$$

Taking into account (9) we rewrite (10) as

$$\frac{d\bar{\psi}}{dt} = 2\sum_{k=1}^{n-1} \left(e^{-k(t+iu(t))}(k+1)(A^T)^k\right)\bar{\psi}, \quad \psi(0) = \xi. \tag{11}$$

The maximum principle states that any optimal control function $u^*(t)$ possesses a maximizing property for the Hamiltonian function along the corresponding trajectory, i.e.,

$$\max_u H(a^*(t), \psi^*(t), u) = H(a^*(t), \bar{\psi}^*(t), u^*), \quad t \ge 0, \tag{12}$$

where a^* and ψ^* are solutions to the system (9), (11) with $u = u^*(t)$.

The maximum principle (12) yields that

$$\left.\frac{\partial H(a^*(t), \psi^*(t), u)}{\partial u}\right|_{u=u^*(t)} = 0. \tag{13}$$

Evidently, (9), (10), and (13) imply that

$$\frac{d\ H(a^*(t), \psi^*(t), u^*(t))}{dt} = 0, \tag{14}$$

for an optimal control function $u^*(t)$.

If the boundary function f gives a point at the boundary hypersurface ∂V_n, then its rotation is also a boundary function. The rotation operation $f(z) \to e^{-i\alpha} f(e^{i\alpha} z)$ gives a curve on ∂V_n and establishes a certain symmetry of the boundary hypersurface and a change of variables $u \to u + \alpha$, $\psi \to e^{i\alpha}\psi$. This allows us to normalize the adjoint vector as $\operatorname{Im} \psi_n = 0$ and $\psi_n = \pm 1$, which corresponds to the projection of V_n onto the hypersurface $\operatorname{Im} a_n = 0$ of dimension $2n - 3$. By abuse of notation, we continue to write V_n for this projection. Further study reduces to investigation of critical points of the equation $\frac{\partial}{\partial u} H(a(t), \psi(t), u) = 0$, and search and comparison of local extrema. In the case when two local maxima coincide, the Gamkrelidze theory [84] of sliding regimes comes into play. Then one considers the generalized Löwner equation in which the function p takes the form

$$p(z,t) = \sum_{k=1}^{n} \lambda_k \frac{e^{iu_k(t)} + z}{e^{iu_k(t)} - z}, \quad \sum_{k=1}^{n} \lambda_k = 1, \tag{15}$$

where $u_k(t)$ is a continuous functions regarding $t \in [0, \infty)$, and $\lambda_k \ge 0$ are constant.

Theorem 4 ([200, 202]). *Let $f \in \mathbf{S}$ give a nonsingular boundary point of the set V_n and let f map the unit disk onto the plane with piecewise-analytic slits having m finite tips. Then there exist m real-valued functions $u_1, \dots, u_m$ continuous on $[0, \infty)$ and positive numbers $\lambda_1, \dots, \lambda_m$, $\sum_{k=1}^{m} \lambda_k = 1$, such that a solution $w = w(z,t)$ to the Cauchy problem for the generalized Löwner differential equation (3,15) represents f according to the formula $f(z) = \lim_{t\to\infty} f(z,t)$. This representation is unique.*

The boundary ∂V_n is then parametrized by the initial conditions ξ in (11). If (11) allows to choose only one optimal control u^*, then the boundary point is given by a boundary function that maps the unit disk onto the plane with piecewise-analytic slit having a unique finite tips. If the Hamiltonian function has some m equal local maximums, then this case is called the sliding regime, and it is provided by $m-1$ equations with respect to ξ, which determine the equality of the values of the Hamiltonian function at m critical points. The constants λ_k represent additional controls of the problem.

Therefore, the sliding regime of the optimal control problem with m maximum points of the Hamiltonian function is realized when $\xi \in \mathfrak{M}_m$, where the manifold $\mathfrak{M}_m$ of dimension $2n-3-m$ is defined by $m-1$ equations for the values of the Hamiltonian function at m critical points for $t = 0$. It is shown in [202] that the sliding regime with m maximum points of function H at $t = 0$ preserves this property for $t > 0$. The number of maximum points may only decrease and only because of the joining of some of them at certain instants t.

Theorem 5 ([200, 202]). *The boundary hypersurface ∂V_n, $n \geq 2$, is a union of the sets $\Omega_1, \dots, \Omega_{n-1}$, every pair of which does not have mutual interior points. Each set Ω_m, $1 \geq m \geq n-1$, corresponds to the manifold $\mathfrak{M}_m$, $\mathfrak{M}_1 = \mathbb{R}^{2n-4}$, so that the parametric representation*

$$\Omega_m = \left\{ a(\infty, \xi, \lambda) \colon \xi \in \mathfrak{M}_m;\ \xi_n = \pm 1;\ \lambda_1, \dots, \lambda_m \geq 0;\ \sum_{k=1}^{m} \lambda_k = 1 \right\}$$

holds, where $a(\infty, \xi, \lambda)$ is the manifold coordinate of the system of bicharacteristics a, ψ for the Hamiltonian system (9), (11) with continuous branches of the optimal controls given by (13).

This theorem can be used in investigations of topologic, metric, smooth, and analytic properties of the boundary hypersurface ∂V_n. In particular, the qualitative results in [39, 226] can be obtained as corollaries. Equally, this method work for the subclass $\mathbf{S}^M$ of bounded functions $|f(z)| \leq M$ from $\mathbf{S}$. In this case, one stops at the moment $t = \log M$.

Let us here mention a slightly different approach proposed by Friedland and Schiffer [79, 80] in 1976–1977 to the problem of description of V_n. Instead of the ordinary Löwner differential equation (3), they considered the Löwner partial

differential equation (1) with the function p in the form (2). Their main conclusion states that if the initial condition for (1) is a boundary function for V_n, then the solutions to (1) form a one-parameter curves on ∂V_n (for special choice of u). The conclusions on the special character of the Koebe functions reflect the angular property of the corresponding points at the boundary hypersurface.

Besides these general results on the coefficient body several concrete problems and conjectures were solved by this method. In particular,several projections of V_3 were described in [250]. The range of the system of set of functionals $\{f(r_1), f(r_2)\}$, $0 < r_1 < r_2 < 1$, was given in [252] in the subclass $\mathbf{S}_R$ of functions from $\mathbf{S}$ with real coefficients. Prokhorov [201] described V_4 in the class $\mathbf{S}_R^M = \mathbf{S}_R \cap \mathbf{S}^M$ generalizing and improving Tammi's results [246, 247]. The range of the system of functionals $\{f(r), a_2, a_3\}$ was described in [251] in the class $\mathbf{S}_R^M$.

We would like to mention here some conjectures which were confirmed or disproved by the above described method. First let us introduce the so-called Pick function that plays in the class $\mathbf{S}^M$ a role analogous to the Koebe function $k_\theta(z) = \frac{z}{(1-e^{i\theta}z)^2}$ in the class $\mathbf{S}$. The Pick function $p_\theta(z)$ satisfies the equation

$$\frac{M^2 p_\theta}{(M - p_\theta)^2} = k_\theta(z), \quad M > 1, \quad z \in \mathbb{D}.$$

The coefficients of $p_0(z) = z + p_2(M)z^2 + \dots$ we denote by $p_n(M)$. The Jakubowski conjecture says that the estimate $|a_n| \le p_n(M)$ holds for even n in the class $\mathbf{S}^M$ for sufficiently large M. This conjecture was proved in [203]. As for odd coefficients of univalent functions, it is easy to verify that the necessary maximality conditions for the Pick functions fail for large M, see [203].

Bombieri [40] in 1967 posed the problem to find

$$\sigma_{mn} := \liminf_{f \to K, f \in S} \frac{n - \mathbb{R}a_n}{m - \mathbb{R}a_m}, \quad m, n \ge 2,$$

$f \to K$ locally uniformly in U. We call σ_{mn} the Bombieri numbers. He conjectured that $\sigma_{mn} = B_{mn}$, where

$$B_{mn} = \min_{\theta \in [0, 2\pi)} \frac{n \sin\theta - \sin(n\theta)}{m \sin\theta - \sin(m\theta)}.$$

and proved that $\sigma_{mn} \le B_{mn}$ for $m = 3$ and n odd. It is noteworthy that Bshouty and Hengartner [47] proved Bombieri's conjecture for functions from S having real coefficients in their Taylor's expansion. Continuing this contribution by Bshouty and Hengartner, the conjecture for the whole class S has been disproved by Greiner and Roth [112] for $n = 2, m = 3$, $f \in S$. Actually, they have got the sharp Bombieri number $\sigma_{32} = (e-1)/4e < 1/4 = B_{32}$.

It is easily seen that $\sigma_{43} = B_{43} = \sigma_{23} = B_{23} = 0$. Applying Löwner's parametric representation for univalent functions and the optimal control method we found [208] the exact Bombieri numbers $\sigma_{42}, \sigma_{24}, \sigma_{34}$ and their numerical

approximations $\sigma_{42} \approx 0.050057\ldots$, $\sigma_{24} \approx 0.969556\ldots$, and $\sigma_{34} \approx 0.791557\ldots$ (the Bombieri conjecture for these permutations of m, n suggests $B_{42} = 0.1$, $B_{24} = 1$, $B_{34} = 0.828427\ldots$). Of course, our method permits us to reprove the result of [112] about σ_{32}.

Our next target is the fourth coefficient a_4 of a function from $\mathbf{S}^M$. The sharp estimate $|a_2| \le 2(1-1/M) = p_2(M)$ in the class S^M is rather trivial and has been obtained by Pick [186] in 1917. The next coefficient a_3 was estimated independently by Schaeffer and Spencer [225] in 1945 and Tammi [244] in 1953. The Pick function does not give the maximum to $|a_3|$ and the estimate is much more difficult. Schiffer and Tammi [228] in 1965 found that $|a_4| \le p_4(M)$ for any $f \in S^M$ with $M > 300$. This result was repeated by Tammi [246, p. 210] in a weaker form ($M > 700$) and there it was conjectured that this constant could be decreased until 11. The case of function with real coefficients is simpler: the Pick function gives the maximum to $|a_4|$ for $M \ge 11$ and this constant is sharp (see [245], [247, p. 163]). By our suggested method we showed [208] that the Pick function locally maximizes $|a_4|$ on S^M if $M > M_0 = 22.9569\ldots$ and does not for $1 < M < M_0$. This disproves Tammi's conjecture.

Among other results obtained by PMP we mention here the Charzyński–Tammi conjecture on functions from $\mathbf{S}^M$ close to identity (or M close to 1). The suggested extremal function analogous to the Koebe and Pick functions is $n-1$-symmetric function

$$Q_{n-1}^M(z) = \left(p_0^{M^{n-1}}(z^{n-1})\right)^{1/(n-1)}.$$

The conjecture proved by Schiffer and Tammi [228] and Siewierski [237] states that the coefficient of the function $Q_{n-1}^M(z)$ gives the extremum to the coefficient a_n of a function from $\mathbf{S}^M$, namely

$$|a_n| \le \frac{2}{n-1}\left(1 - \frac{1}{M^{n-1}}\right), \quad n \ge 2.$$

Prokhorov [204] found an asymptotic estimate in the above problem in terms of $\log^2 M$ as M is close to 1. For $M \to \infty$, Prokhorov and Nikulin [206] obtained also asymptotic estimates in the coefficient problem for the class $\mathbf{S}^M$ with. In particular,

$$|a_n| \le n - \frac{n(n^2-1)}{3}\frac{1}{M} + o\left(\frac{1}{M}\right), \quad M \to \infty.$$

More about coefficient problems solved by PMP and Löwner–Kufarev theory, see [205].

7 One-Slit Maps

It is interesting that in spite of many known properties of the Löwner equations several geometric questions remained unclear until recently.

Let us return to the original Löwner equation (3) with the driving function p given by (2)

$$\frac{dw}{dt} = -w\frac{e^{iu(t)} + w}{e^{iu(t)} - w}, \quad w(z,0) \equiv z, \tag{16}$$

Solutions $w(z,t)$ to (16) map $\mathbb{D}$ onto $\Omega(t) \subset \mathbb{D}$. If $\Omega(t) = \mathbb{D} \setminus \gamma(t)$, where $\gamma(t)$ is a Jordan curve in $\mathbb{D}$ except one of its endpoints, then the driving term $u(t)$ is uniquely defined and we call the corresponding map w a *slit map*. However, from 1947 [147] it is known that solutions to (16) with continuous $u(t)$ may give non-slit maps, in particular, $\Omega(t)$ can be a family of hyperbolically convex digons in $\mathbb{D}$.

Marshall and Rohde [173] addressed the following question: *Under which condition on the driving term $u(t)$ the solution to (16) is a slit map?* Their result states that if $u(t)$ is Lip(1/2) (Hölder continuous with exponent 1/2), and if for a certain constant $C_{\mathbb{D}} > 0$, the norm $\|u\|_{1/2}$ is bounded $\|u\|_{1/2} < C_{\mathbb{D}}$, then the solution w is a slit map, and moreover, the Jordan arc $\gamma(t)$ is s quasislit (a quasiconformal image of an interval within a Stolz angle). As they also proved, a converse statement without the norm restriction holds. The absence of the norm restriction in the latter result is essential. On the one hand, Kufarev's example [147] contains $\|u\|_{1/2} = 3\sqrt{2}$, which means that $C_{\mathbb{D}} \le 3\sqrt{2}$. On the other hand, Kager et al. [129] constructed exact slit solutions to the half-plane version of the Löwner equation with arbitrary norms of the driving term.

Reminding the half-plane version of the Löwner equation let $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$, $\mathbb{R} = \partial\mathbb{H}$. The functions $h(z,t)$, normalized near infinity by $h(z,t) = z - 2t/z + b_{-2}(t)/z^2 + \ldots$, solving the equation

$$\frac{dh}{dt} = \frac{-2}{h - \lambda(t)}, \quad h(z,0) \equiv z, \tag{17}$$

where $\lambda(t)$ is a real-valued continuous driving term, map $\mathbb{H}$ onto a subdomain of $\mathbb{H}$. The difference in the sign between (17) and (6) is because in (6) the equation is for the inverse mapping. In some papers, e.g., [129, 162], the authors work with (16), (17) changing $(-)$ to $(+)$ in their right-hand sides, and with the mappings of slit domains onto $\mathbb{D}$ or $\mathbb{H}$. However, the results remain the same for both versions.

The question about the slit mappings and the behavior of the driving term $\lambda(t)$ in the case of the half-plane $\mathbb{H}$ was addressed by Lind [162]. The techniques used by Marshall and Rohde carry over to prove a similar result in the case of (17), see [173, p. 765]. Let us denote by $C_{\mathbb{H}}$ the corresponding bound for the norm $\|\lambda\|_{1/2}$. The main result by Lind is the sharp bound, namely $C_{\mathbb{H}} = 4$.

Marshall and Rohde [173] remarked that there exist many examples of driving terms $u(t)$ which are not Lip(1/2), but which generate slit solutions with simple arcs $\gamma(t)$. In particular, if $\gamma(t)$ is tangent to $\mathbb{T}$, then $u(t)$ is never Lip(1/2).

Our result [209] states that if $\gamma(t)$ is a circular arc tangent to $\mathbb{R}$, then the driving term $\lambda(t) \in$Lip(1/3). Besides, we prove that $C_{\mathbb{D}} = C_{\mathbb{H}} = 4$ and consider properties of singular solutions to the one-slit Löwner equation. Moreover, examples of non-slit solutions filling the whole spectrum $[4, \infty)$ were given in [127, 163].

The authors analyzed in [163] Löwner traces $\gamma(t)$ driven by $\lambda(t)$ asymptotic to $k\sqrt{1-t}$. They proved a form of stability of the self-intersection for such $\lambda(t)$. Being slightly rephrased it reads as follows.

Theorem 6 ([163]). *Let the driving term $\lambda\colon [0, 1] \to \mathbb{R}$ be sufficiently regular with the above asymptotic of $\lambda(t)$*

$$\lim_{t\to 1} \frac{|\lambda(1) - \lambda(t)|}{\sqrt{1-t}} = k > 4.$$

Then $\gamma(1-0)$ exists, is real, and γ intersects $\mathbb{R}$ at the same angle as the trace for $\lambda = k\sqrt{1-t}$.

Namely,

$$\lim_{t\to 1} \arg(\gamma(t) - \gamma(1)) = \pi \frac{1 - \sqrt{1 - 16/k^2}}{1 + \sqrt{1 - 16/k^2}}.$$

The method of proof of the above theorem also applies to the case $|k| < 4$. In this case the trace γ driven by λ is a Jordan arc, and γ is asymptotically similar to the logarithmic spiral at $\gamma(1) \in \mathbb{H}$.

Another our result [127] states that an analytic orthogonal slit requires the 1/2 Lipschitz vanishing norm, exactly as in Kadanoff's et al. examples [129] with a line-slit and a circular slit. In this case the conformal radius approaching the origin is of order Lip 1/2 (compare with Earle and Epstein [73]).

8 Univalence Criteria

Several important univalence criteria can be obtained by means of the Löwner–Kufarev differential equation. For example, a function $f(z) = z + \dots$ analytic in $\mathbb{D}$ is spirallike of type $\alpha \in (-\pi/2, \pi/2)$ (and therefore, univalent) if and only if

$$\operatorname{Re}\left(e^{i\alpha} z \frac{f'(z)}{f(z)}\right) > 0, \quad \text{in } \mathbb{D},$$

see [214, 242] and [190, p. 172]. Spirallikeness means that a function $f(z)$ is analytic, univalent, and if $w \in f(\mathbb{D})$, $\tau \geq 0$, then $we^{-e^{-i\alpha}\tau} \in f(\mathbb{D})$. If $\alpha = 0$, then we obtain the usual class of starlike functions S^*.

Next criterion is obtained by integration of the Löwner–Kufarev equation (3) with a special choice of the driving function p from the Carathéodory class. Let us choose

$$p(z,t) = \frac{1}{h(z) + th_0(z)}, \quad h_0(z) = i\beta + \alpha \frac{zg'(z)}{g(z)}, \quad g \in S^*.$$

Integrating (3) as a Bernoulli-type equation, and letting $t \to \infty$ leads to the limiting function

$$f(z) = \left(\frac{\alpha + i\beta}{1 + i\alpha} \int_0^z h(z) z^{i\beta - 1} g^{\alpha}(z)\, dz \right)^{1/(\alpha + i\beta)} = z + \dots,$$

$\alpha > 0$, $\beta \in \mathbb{R}$, which is the Bazilevich class $B_{\alpha,\beta}$ introduced in [29, 30]. Prokhorov [198] and Sheil-Small [235] proved that the class $B_{\alpha,\beta}$ is characterized by the condition

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} F(re^{i\theta})\, d\theta > -\pi, \qquad 0 < r < 1, \quad 0 < \theta_2 - \theta_1 < 2\pi,$$

$$F(z) = 1 + z\frac{f''(z)}{f'(z)} + (\alpha - 1 + i\beta) z \frac{f'(z)}{f(z)},$$

under the additional assumption that $f(z) f'(z)/z \neq 0$ on $\mathbb{T} = \partial\mathbb{D}$. Moreover, the boundary of $f(\mathbb{D})$ is accessible from outside by the curves $w = a(1 + bt)^{1/(\alpha + i\beta)}$.

In 1972, Becker [31] assumed that the driving function p in (3) satisfied the inequality

$$\left| \frac{p(z,t) - 1}{p(z,t) + 1} \right| \leq k < 1, \quad z \in \mathbb{D}, \quad 0 \leq t < \infty.$$

Then the solutions in the form (4) have k-quasiconformal extension to the Riemann sphere $\hat{\mathbb{C}}$. This allowed him to show that the inequality

$$(1 - |z|^2) \left| z \frac{f''(z)}{f'(z)} \right| \leq k < 1$$

in $\mathbb{D}$ for an analytic function $f(z) = z + \dots$ implies that f is univalent and has a k-quasiconformal extension to $\hat{\mathbb{C}}$. This improves a result of Duren et al. [68]. Later in 1984 Becker and Pommerenke [32] established the criteria

$$(1-|z|^2)|zf''(z)/f'(z)| \le 1,\ f'(0) \ne 0\ \ (z \in \mathbb{D})$$
$$2\,\mathrm{Re}\, z|f''(z)/f'(z)| \le 1\ \ (z \in \mathbb{H})$$
$$(|z|^2-1)|zf''(z)/f'(z)| \le 1\ \ (z \in \mathbb{D}^-),$$

where $\mathbb{H}$ is the right half-plane and $\mathbb{D}^-$ is the exterior of $\mathbb{D}$. In all inequalities the constant 1 is the best possible. The first criterium implies that the boundary of $f(\mathbb{D})$ is Jordan whereas the second and the third do not necessary imply this. Various univalence conditions were obtained later, see [5] for more complete list of them.

9 Semigroups

Looking at the classical radial Löwner–Kufarev equation (3) and the classical chordal Löwner–Kufarev equation (6), one notices that there is a similitude between the two. Indeed, we can write both equations in the form

$$\frac{dz(t)}{dt} = G(z,t),$$

with

$$G(z,t) = (\tau - z)(1-\overline{\tau}z)p(z,t),$$

where $\tau = 0, 1$ and $\mathrm{Re}\, p(z,t) \ge 0$ for all $z \in \mathbb{D}$ and $t \ge 0$.

The reason for the previous formula is not at all by chance, but it reflects a very important feature of "Herglotz vector fields" (see Sect. 10 for the definition). In order to give a rough idea of what we are aiming, consider the case $\tau = 0$ (the radial case). Fix $t_0 \in [0, +\infty)$. Consider the holomorphic vector field $G(z) := G(z, t_0)$. Let $g(z) := |z|^2$. Then,

$$dg_z(G(z)) = 2\mathrm{Re}\,\langle G(z), z\rangle = -|z|^2 \mathrm{Re}\, p(z,t_0) \le 0, \quad \forall z \in \mathbb{D}. \tag{18}$$

This Lyapunov-type inequality has a deep geometrical meaning. Indeed, (18) tells that G points toward the center of the level sets of g, which are concentric circles centered at 0. For each $z_0 \in \mathbb{D}$, consider then the Cauchy problem

$$\begin{cases} \frac{dw(t)}{dt} = G(w(t)), \\ w(0) = z_0 \end{cases} \tag{19}$$

and let $w^{z_0} : [0, \delta) \to \mathbb{D}$ be the maximal solution (such a solution can propagate also in the "past," but we just consider the "future" time). Since G points inward with respect to all circles centered at 0, the flow $t \mapsto w^{z_0}(t)$ cannot escape from the circle of radius $g(z_0)$. Therefore, the flow is defined for all future times, namely, $\delta = +\infty$. This holds for all $z_0 \in \mathbb{D}$.

Hence, the vector field G has the feature to be $\mathbb{R}^+$-semicomplete, that is, the maximal solution of the initial value problem (19) is defined in the interval $[0, +\infty)$. To understand how one can unify both radial and chordal Löwner theory, we dedicate this section to such vector fields and their flows.

A family of holomorphic self-maps of the unit disk (ϕ_t) is a *(one-parameter) semigroup (of holomorphic functions)* if $\phi\colon (\mathbb{R}^+, +) \to \mathsf{Hol}(\mathbb{D}, \mathbb{D})$ is a continuous homomorphism between the semigroup of nonnegative real numbers and the semigroup of holomorphic self-maps of the disk with respect to composition, endowed with the topology of uniform convergence on compact sets. In other words:

- $\phi_0 = \mathrm{id}_{\mathbb{D}}$;
- $\phi_{t+s} = \phi_s \circ \phi_t$ for all $s, t \geq 0$;
- ϕ_t converges to ϕ_{t_0} uniformly on compact sets as t goes to t_0.

It can be shown that if (ϕ_t) is a semigroup, then ϕ_t is univalent for all $t \geq 0$.

Semigroups of holomorphic maps are a classical subject of study, both as (local/global) flows of continuous dynamical systems and from the point of view of "fractional iteration," the problem of embedding the discrete set of iterates generated by a single self-map into a one-parameter family (a problem that is still open even in the disk). It is difficult to exactly date the birth of this notion but it seems that the first paper dealing with semigroups of holomorphic maps and their asymptotic behavior is due to Tricomi in 1917 [248]. Semigroups of holomorphic maps also appear in connection with the theory of Galton–Watson processes (branching processes) started in the 1940s by Kolmogorov and Dmitriev [118]. An extensive recent survey [106] gives a complete overview on details. Furthermore, they are an important tool in the theory of strongly continuous semigroups of operators between spaces of analytic functions (see, for example, [239]).

A very important contribution to the theory of semigroups of holomorphic self-maps of the unit disk is due to Berkson and Porta [36]. They proved that:

Theorem 7 ([36]). *A semigroup of holomorphic self-maps of the unit disk (ϕ_t) is in fact real-analytic in the variable t, and is the solution of the Cauchy problem*

$$\frac{\partial \phi_t(z)}{\partial t} = G(\phi_t(z)), \quad \phi_0(z) = z\,, \tag{20}$$

where the map G, the infinitesimal generator of the semigroup, has the form

$$G(z) = (z - \tau)(\overline{\tau} z - 1)\, p(z) \tag{21}$$

for some $\tau \in \overline{\mathbb{D}}$ and a holomorphic function $p\colon \mathbb{D} \to \mathbb{C}$ with $\operatorname{Re} p \geq 0$.

Conversely, any vector field of the form (21) is semicomplete and if, for $z \in \mathbb{D}$, we take w^z the solution of the initial value problem

$$\frac{dw}{dt} = G(w), \quad w(0) = z,$$

then $\phi_t(z) := w^z(t)$ *is a semigroup of analytic functions.*

Expression (21) of the infinitesimal generator is known as its *Berkson–Porta decomposition*. Other characterizations of vector fields which are infinitesimal generators of semigroups can be seen in [41] and references therein.

The dynamics of the semigroup (ϕ_t) are governed by the analytical properties of the infinitesimal generator G. For instance, all the functions of the semigroup have a common fixed point at τ (in the sense of non-tangential limit if τ belongs to the boundary of the unit disk) and asymptotically tends to τ, which can thus be considered a sink point of the dynamical system generated by G.

When $\tau = 0$, it is clear that (21) is a particular case of (3), because the infinitesimal generator G is of the form $-wp(w)$, where p belongs to the Carathéodory class. As a consequence, when the semigroup has a fixed point in the unit disk (which, up to a conjugation by an automorphism of the disk, amounts to taking $\tau = 0$), *once differentiability in* t *is proved* Berkson–Porta's theorem can be easily deduced from Löwner's theory. However, when the semigroup has no common fixed points in the interior of the unit disk, Berkson–Porta's result is really a new advance in the theory.

We have already remarked that semigroups give rise to evolution families; they also provide examples of Löwner chains. Indeed, Heins [123] and Siskasis [238] have independently proved that if (ϕ_t) is a semigroup of holomorphic self-maps of the unit disk, then there exists a (unique, when suitably normalized) holomorphic function $h\colon \mathbb{D} \to \mathbb{C}$, the *Königs function* of the semigroup, such that $h(\phi_t(z)) = m_t(h(z))$ for all $t \geq 0$, where m_t is an affine map (in other words, the semigroup is semiconjugated to a semigroup of affine maps). Then it is easy to see that the maps $f_t(z) := m_t^{-1}(h(z))$, for $t \geq 0$, form a Löwner chain (in the sense explained in the next section).

The theory of semigroups of holomorphic self-maps has been extensively studied and generalized: to Riemann surfaces (in particular, Heins [123] has shown that Riemann surfaces with non-Abelian fundamental group admit no nontrivial semigroup of holomorphic self-maps); to several complex variables; and to infinitely dimensional complex Banach spaces, by Baker, Cowen, Elin, Goryainov, Poggi-Corradini, Pommerenke, Reich, Shoikhet, Siskakis, Vesentini, and many others. We refer to [41] and the books [1] and [212] for references and more information on the subject.

10 General Theory: Herglotz Vector Fields, Evolution Families, and Löwner Chains

Although the chordal and radial versions of the Löwner theory share common ideas and structure, on their own they can only be regarded as parallel but independent theories. The approach of [70, 101–105] does show that there can be

(and actually are) much more independent variants of Löwner evolution bearing similar structures, but does not solve the problem of constructing a unified theory covering all the cases.

Recently a new unifying approach has been suggested by Gumenyuk and the three first authors [20, 42, 43, 53]. In the previous section we saw that the vector fields which appear in radial and chordal Löwner equations have the property to be infinitesimal generators for all fixed times. We will exploit such a fact to define a general family of Herglotz vector fields. Thus, relying partially on the theory of one-parametric semigroups, which can be regarded as the autonomous version of Löwner theory, we can build a new general theory.

Definition 1 ([42]). Let $d \in [1, +\infty]$. A *weak holomorphic vector field of order d* in the unit disk $\mathbb{D}$ is a function $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ with the following properties:

WHVF1. for all $z \in \mathbb{D}$, the function $[0, +\infty) \ni t \mapsto G(z, t)$ is measurable,
WHVF2. for all $t \in [0, +\infty)$, the function $\mathbb{D} \ni z \mapsto G(z, t)$ is holomorphic,
WHVF3. for any compact set $K \subset \mathbb{D}$ and any $T > 0$ there exists a nonnegative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$|G(z, t)| \leq k_{K,T}(t)$$

for all $z \in K$ and for almost every $t \in [0, T]$.

We say that G is a *(generalized) Herglotz vector field* of order d if, in addition to conditions WHVF1–WHVF3 above, for almost every $t \in [0, +\infty)$ the holomorphic function $G(\cdot, t)$ is an infinitesimal generator of a one-parametric semigroup in $\mathsf{Hol}(\mathbb{D}, \mathbb{D})$.

Herglotz vector fields in the unit disk can be decomposed by means of Herglotz functions (and this the reason for the name). We begin with the following definition:

Definition 2. Let $d \in [1, +\infty]$. A *Herglotz function of order d* is a function $p : \mathbb{D} \times [0, +\infty) \mapsto \mathbb{C}$ with the following properties:

1. For all $z \in \mathbb{D}$, the function $[0, +\infty) \ni t \mapsto p(z, t) \in \mathbb{C}$ belongs to $L^d_{loc}([0, +\infty), \mathbb{C})$;
2. For all $t \in [0, +\infty)$, the function $\mathbb{D} \ni z \mapsto p(z, t) \in \mathbb{C}$ is holomorphic;
3. For all $z \in \mathbb{D}$ and for all $t \in [0, +\infty)$, we have $\operatorname{Re} p(z, t) \geq 0$.

Then we have the following result which, using the Berkson–Porta formula, gives a general form of the classical Herglotz vector fields:

Theorem 8 ([42]). *Let $\tau : [0, +\infty) \to \overline{\mathbb{D}}$ be a measurable function and let $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ be a Herglotz function of order $d \in [1, +\infty)$. Then the map $G_{\tau,p} : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ given by*

$$G_{\tau,p}(z, t) = (z - \tau(t))(\overline{\tau(t)}z - 1)\, p(z, t),$$

for all $z \in \mathbb{D}$ and for all $t \in [0, +\infty)$, is a Herglotz vector field of order d on the unit disk.

Conversely, if $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a Herglotz vector field of order $d \in [1, +\infty)$ on the unit disk, then there exist a measurable function $\tau : [0, +\infty) \to \overline{\mathbb{D}}$ and a Herglotz function $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ of order d such that $G(z, t) = G_{\tau,p}(z, t)$ for almost every $t \in [0, +\infty)$ and all $z \in \mathbb{D}$.

Moreover, if $\tilde{\tau} : [0, +\infty) \to \overline{\mathbb{D}}$ is another measurable function and $\tilde{p} : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is another Herglotz function of order d such that $G = G_{\tilde{\tau},\tilde{p}}$ for almost every $t \in [0, +\infty)$, then $p(z, t) = \tilde{p}(z, t)$ for almost every $t \in [0, +\infty)$ and all $z \in \mathbb{D}$ and $\tau(t) = \tilde{\tau}(t)$ for almost all $t \in [0, +\infty)$ such that $G(\cdot, t) \not\equiv 0$.

Remark 1. The generalized Löwner–Kufarev equation

$$\frac{dw}{dt} = G(w, t), \quad t \geq s, \quad w(s) = z, \tag{22}$$

resembles the radial Löwner–Kufarev ODE when $\tau \equiv 0$ and $p(0, t) \equiv 1$. Furthermore, with the help of the Cayley map between $\mathbb{D}$ and $\mathbb{H}$, the chordal Löwner equation appears to be the special case of (22) with $\tau \equiv 1$.

We also give a generalization of the concept of evolution families in *the whole semigroup* $\mathsf{Hol}(\mathbb{D}, \mathbb{D})$ as follows:

Definition 3. ([42]) A family $(\varphi_{s,t})_{t \geq s \geq 0}$ of holomorphic self-maps of the unit disk is an *evolution family of order d* with $d \in [1, +\infty]$ if

EF1. $\varphi_{s,s} = \mathrm{Id}_{\mathbb{D}}$ for all $s \geq 0$,
EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $0 \leq s \leq u \leq t < +\infty$,
EF3. for every $z \in \mathbb{D}$ and every $T > 0$ there exists a nonnegative function $k_{z,T} \in L^d([0, T], \mathbb{R})$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi) d\xi$$

whenever $0 \leq s \leq u \leq t \leq T$.

Condition EF3 is to guarantee that any evolution family can be obtained via solutions of an ODE which resembles both the radial and chordal Löwner–Kufarev equations. The vector fields that drive this generalized Löwner–Kufarev ODE are referred to as *Herglotz vector fields.*

Remark 2. Definition 3 does not require elements of an evolution family to be univalent. However, this condition is satisfied. Indeed, by Theorem 9, any evolution family $(\varphi_{s,t})$ can be obtained via solutions to the generalized Löwner–Kufarev ODE. Hence the univalence of $\varphi_{s,t}$'s follows from the uniqueness of solutions to this ODE. For an essentially different direct proof, see [43, Proposition 3].

Remark 3. Different notions of evolution families considered previously in the literature can be reduced to special cases of L^d-evolution families defined above.

The Schwarz lemma and distortion estimates imply that solutions of the classical Löwner–Kufarev equation (3) are evolution families of order ∞. Also, it can be proved that for all semigroups of analytic maps (ϕ_t), the biparametric family $(\varphi_{s,t}) := (\phi_{t-s})$ is also an evolution family of order ∞.

Equation (22) establishes a 1-to-1 correspondence between evolution families of order d and Herglotz vector fields of the same order. Namely, the following theorem takes place.

Theorem 9 ([42, Theorem 1.1]). *For any evolution family $(\varphi_{s,t})$ of order $d \in [1, +\infty]$ there exists an (essentially) unique Herglotz vector field $G(z,t)$ of order d such that for every $z \in \mathbb{D}$ and every $s \geq 0$ the function $[s, +\infty) \ni t \mapsto w_{z,s}(t) := \varphi_{s,t}(z)$ solves the initial value problem (22).*

Conversely, given any Herglotz vector field $G(z,t)$ of order $d \in [1, +\infty]$, for every $z \in \mathbb{D}$ and every $s \geq 0$ there exists a unique solution $[s, +\infty) \ni t \mapsto w_{z,s}(t)$ to the initial value problem (22). The formula $\varphi_{s,t}(z) := w_{z,s}(t)$ for all $s \geq 0$, all $t \geq s$, and all $z \in \mathbb{D}$, defines an evolution family $(\varphi_{s,t})$ of order d.

Here by *essential uniqueness* we mean that two Herglotz vector fields $G_1(z,t)$ and $G_2(z,t)$ corresponding to the same evolution family must coincide for a.e. $t \geq 0$.

The strong relationship between semigroups and evolution families on the one side and Herglotz vector fields and infinitesimal generators on the other side is very much reflected by the so-called product formula in convex domains of Reich and Shoikhet [212]. Such a formula can be rephrased as follows: let $G(z,t)$ be a Herglotz vector field. For almost all $t \geq 0$, the holomorphic vector field $\mathbb{D} \ni z \mapsto G(z,t)$ is an infinitesimal generator. Let (ϕ_r^t) be the associated semigroups of holomorphic self-maps of $\mathbb{D}$. Let $(\varphi_{s,t})$ be the evolution family associated with $G(z,t)$. Then, uniformly on compacta of $\mathbb{D}$ it holds

$$\phi_t^r = \lim_{m\to\infty} \varphi_{t,t+\frac{r}{m}}^{\circ m} = \lim_{m\to\infty} \underbrace{(\varphi_{t,t+\frac{r}{m}} \circ \ldots \circ \varphi_{t,t+\frac{r}{m}})}_{m}.$$

Using such a formula for the case of the unit disk $\mathbb{D}$, in [44] it has been proved the following result which gives a description of semigroups-type evolution families:

Theorem 10. *Let $G(z,t)$ be a Herglotz vector field of order d in $\mathbb{D}$ and let $(\varphi_{s,t})$ be its associated evolution family. The following are equivalent:*

1. *there exists a function $g \in L^d_{loc}([0, +\infty), \mathbb{C})$ and an infinitesimal generator H such that $G(z,t) = g(t)H(z)$ for all $z \in \mathbb{D}$ and almost all $t \geq 0$,*
2. *$\varphi_{s,t} \circ \varphi_{u,v} = \varphi_{u,v} \circ \varphi_{s,t}$ for all $0 \leq s \leq t$ and $0 \leq u \leq v$.*

In order to end up the picture started with the classical Löwner theory, we should put in the frame also the Löwner chains. The general notion of a Löwner chain has been given[2] in [53].

Definition 4 ([53]). A family $(f_t)_{t\geq 0}$ of holomorphic maps of $\mathbb{D}$ is called a *Löwner chain of order d* with $d \in [1, +\infty]$ if it satisfies the following conditions:

LC1. each function $f_t : \mathbb{D} \to \mathbb{C}$ is univalent,
LC2. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s < t < +\infty$,
LC3. for any compact set $K \subset \mathbb{D}$ and any $T > 0$ there exists a nonnegative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi) d\xi$$

whenever $z \in K$ and $0 \leq s \leq t \leq T$.

This definition of (generalized) Löwner chains matches the abstract notion of evolution family introduced in [42]. In particular the following statement holds.

Theorem 11 ([53, Theorem 1.3]). *For any Löwner chain (f_t) of order $d \in [1, +\infty]$, if we define*

$$\varphi_{s,t} := f_t^{-1} \circ f_s \quad \textit{whenever } 0 \leq s \leq t,$$

then $(\varphi_{s,t})$ is an evolution family of the same order d. Conversely, for any evolution family $(\varphi_{s,t})$ of order $d \in [1, +\infty]$, there exists a Löwner chain (f_t) of the same order d such that

$$f_t \circ \varphi_{s,t} = f_s \quad \textit{whenever } 0 \leq s \leq t.$$

In the situation of this theorem we say that the Löwner chain (f_t) and the evolution family $(\varphi_{s,t})$ are *associated with* each other. It was proved in [53] that given an evolution family $(\varphi_{s,t})$, an associated Löwner chain (f_t) is unique up to conformal maps of $\cup_{t\geq 0} f_t(\mathbb{D})$. Thus there are essentially one two different types of Löwner chains: those such that $\cup_{t\geq 0} f_t(\mathbb{D}) = \mathbb{C}$ and those such that $\cup_{t\geq 0} f_t(\mathbb{D})$ is a simply connected domain different from $\mathbb{C}$ (see [53] for a characterization in terms of the evolution family associated with).

Thus in the framework of the approach described above the essence of Löwner theory is represented by the essentially 1-to-1 correspondence among Löwner chains, evolution families, and Herglotz vector fields.

[2] See also [20] for an extension of this notion to complex manifolds and with a complete different approach even in the unit disk. The construction of a Löwner chain associated with a given evolution family proposed there differs essentially from the one we used in [53, Theorems 1.3 and 1.6].

Once the previous correspondences are established, given a Löwner chain (f_t) of order d, the general Löwner–Kufarev PDE

$$\frac{\partial f_t(z)}{\partial t} = -G(z,t)\frac{\partial f_t(z)}{\partial z}. \tag{23}$$

follows by differentiating the structural equation

$$\frac{dz}{dt} = G(z,t), \qquad z(0) = z. \tag{24}$$

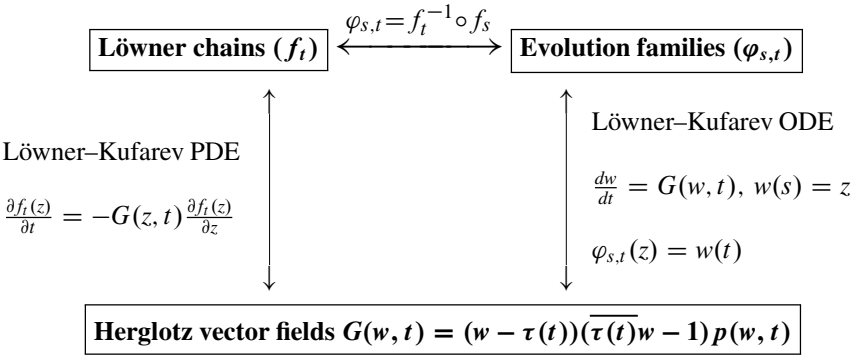
Conversely, given a Herglotz vector field $G(z,t)$ of order d, one can build the associated Löwner chain (of the same order d), solving (23) by means of the associated evolution family.

The Berkson–Porta decomposition of a Herglotz vector field $G(z,t)$ also gives information on the dynamics of the associated evolution family. For instance, when $\tau(t) \equiv \tau \in \mathbb{D}$, the point τ is a (common) fixed point of $(\varphi_{s,t})$ for all $0 \le s \le t < +\infty$. Moreover, it can be proved that, in such a case, there exists a unique locally absolutely continuous function $\lambda\colon [0,+\infty) \to \mathbb{C}$ with $\lambda' \in L^d_{loc}([0,+\infty),\mathbb{C})$, $\lambda(0) = 0$ and $\operatorname{Re}\lambda(t) \ge \operatorname{Re}\lambda(s) \ge 0$ for all $0 \le s \le t < +\infty$ such that for all $s \le t$

$$\varphi'_{s,t}(\tau) = \exp(\lambda(s) - \lambda(t)).$$

A similar characterization holds when $\tau(t) \equiv \tau \in \partial\mathbb{D}$ (see [41]).

Let us summarize this section. In the next scheme we show the main three notions of Löwner theory we are dealing with in this paper and the relationship between them:



where p is a Herglotz function.

11 Extensions to Multiply Connected Domains



Y. Komatu
(from Oberwolfach Photo Collection)

Yûsaku Komatu (1914–2004), in 1942 [135], was the first to generalize Löwner's parametric representation to univalent holomorphic functions defined in a circular annulus and with images in the exterior of a disc. Later, Goluzin [92] gave a much simpler way to establish Komatu's results. With the same techniques, Li [161] considered a slightly different case, when the image of the annulus is the complex plane with two slits (ending at infinity and at the origin, respectively). See also [116, 136, 158]. The monograph [14] contains a self-contained detailed account on the Parametric Representation in the multiply connected setting.

Another way of adapting Löwner's method to multiply connected domains was developed by Kufarev and Kuvaev [152]. They obtained a differential equation satisfied by automorphic functions realizing conformal covering mappings of the unit disc onto multiply connected domains with a gradually erased slit. This differential approach has also been followed by Tsai [249]. Roughly speaking, these results can be considered as a version for multiply connected domains of the slit-radial Löwner equation. In a similar way, Bauer and Friedrich have developed a slit-chordal theory for multiply connected domains. Moreover, they have even dealt with stochastic versions of both the radial and the chordal cases. In this framework the situation is more subtle than in the simply connected case, because moduli spaces enter the picture [25, 26].

Following the guide of the new and general approach to Löwner theory in the unit disk as described in Sect. 9, the second and the third author of this survey jointly with Pavel Gumenyuk have developed a global approach to Löwner theory for double connected domains which give a uniform framework to previous works of Komatu, Goluzin, Li en Pir, and Lebedev. More interestingly, this abstract theory shows some phenomena not considered before and also poses many new questions.

Besides the similarities between the general approach for simple and double connected domains, there are a number of significative differences both in concepts and in results. For instance, in order to develop an interesting substantial theory

for the double connected case, instead of a static reference domain D, one has to consider a family of expanding annuli $D_t = \mathbb{A}_{r(t)} := \{z : r(t) < |z| < 1\}$, where $r : [0, +\infty) \to [0, 1)$ is non-increasing and continuous. The first who noticed this fact, in a very special case, was already Komatu [135]. Such a family (D_t), it is usually called a canonical domain system of order d, whenever $-\pi/\log(r)$ belongs to $AC^d([0, +\infty), \mathbb{R})$.

This really non-optional dynamic context forces to modify the definitions of (again) the three basic elements of the general theory: semicomplete vector fields, evolution families, and Löwner chains. Nevertheless, there is still a (essentially) one-to-one correspondence between these three notions.

As in the unit disk, (weak) vector fields are introduced in this picture bearing in mind Carathéodory's theory of ODEs.

Definition 5. Let $d \in [1, +\infty]$. A function $G : \mathscr{D} \to \mathbb{C}$ is said to be a *weak holomorphic vector field* of order d in the domain

$$\mathscr{D} := \{(z, t) : t \geq 0, z \in D_t\},$$

if it satisfies the following three conditions:

WHVF1. For each $z \in \mathbb{C}$ the function $G(z, \cdot)$ is measurable in $E_z := \{t : (z, t) \in \mathscr{D}\}$.

WHVF2. For each $t \in E$ the function $G(\cdot, t)$ is holomorphic in D_t.

WHVF3. For each compact set $K \subset \mathscr{D}$ there exists a nonnegative function

$$k_K \in L^d\big(pr_{\mathbb{R}}(K), \mathbb{R}\big), \quad pr_{\mathbb{R}}(K) := \{t \in E : \exists z \in \mathbb{C} \quad (z, t) \in K\},$$

such that

$$|G(z, t)| \leq k_K(t), \quad \text{for all } (z, t) \in K.$$

Given a weak holomorphic vector field G in $\mathscr{D}$ and an initial condition $(z, s) \in \mathscr{D}$, it is possible to consider the initial value problem,

$$\dot{w} = G(w, t), \quad w(s) = z. \tag{25}$$

A solution to this problem is any continuous function $w : J \to \mathbb{C}$ such that $J \subset E$ is an interval, $s \in J$, $(w(t), t) \in \mathscr{D}$ for all $t \in J$ and

$$w(t) = z + \int_s^t G(w(\xi), \xi)\, d\xi, \quad t \in J. \tag{26}$$

When these kinds of problems have solutions well defined globally to the right for any initial condition, the vector field G is called semicomplete.

Putting together the main properties of the flows generated by semicomplete weak vector fields, we arrive to the concept of evolution families for the doubly connected setting.

Definition 6. A family $(\varphi_{s,t})_{0\le s\le t<+\infty}$ of holomorphic mappings $\varphi_{s,t} : D_s \to D_t$ is said to be an *evolution family of order d over* (D_t) (in short, an L^d*-evolution family*) if the following conditions are satisfied:

EF1. $\varphi_{s,s} = \mathsf{id}_{D_s}$,
EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $0 \le s \le u \le t < +\infty$,
EF3. for any closed interval $I := [S, T] \subset [0, +\infty)$ and any $z \in D_S$ there exists a nonnegative function $k_{z,I} \in L^d\big([S, T], \mathbb{R}\big)$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \le \int_u^t k_{z,I}(\xi)d\xi$$

whenever $S \le s \le u \le t \le T$.

As expected, there is a one-to-one correspondence between evolution families over canonical domain systems and semicomplete weak holomorphic vector fields, analogous to the correspondence between evolution families and Herglotz vector fields in the unit disk.

Theorem 12 ([55, Theorem 5.1]). *The following two assertions hold:*

(A) For any L^d-evolution family $(\varphi_{s,t})$ over the canonical domain system (D_t) there exists an essentially unique semicomplete weak holomorphic vector field $G : \mathscr{D} \to \mathbb{C}$ of order d and a null-set $N \subset [0, +\infty)$ such that for all $s \ge 0$ the following statements hold:

(i) the mapping $[s, +\infty) \ni t \mapsto \varphi_{s,t} \in \mathrm{Hol}(D_s, \mathbb{C})$ is differentiable for all $t \in [s, +\infty) \setminus N$;
(ii) $d\varphi_{s,t}/dt = G(\cdot, t) \circ \varphi_{s,t}$ for all $t \in [s, +\infty) \setminus N$.

(B) For any semicomplete weak holomorphic vector field $G : \mathscr{D} \to \mathbb{C}$ of order d the formula $\varphi_{s,t}(z) := w_s^*(z, t)$, $t \ge s \ge 0$, $z \in D_s$, where $w_s^*(z, \cdot)$ is the unique non-extendable solution to the initial value problem

$$\dot{w} = G(w, t), \quad w(s) = z, \tag{27}$$

defines an L^d-evolution family over the canonical domain system (D_t).

With the corresponding perspective, there is also a true version of the non-autonomous Berkson–Porta description of Herglotz vector fields. Now, the role played by functions associated with Scharwz kernel is fulfilled by a natural class of functions associated with the Villat kernel (see [55, Theorem 5.6]).

Löwner chains in the double connected setting are introduced in a similar way as it was done in the case of the unit disc.

Definition 7. A family $(f_t)_{t\geq 0}$ of holomorphic functions $f_t : D_t \to \mathbb{C}$ is called a *Loewner chain* of order d over (D_t) if it satisfies the following conditions:

LC1. each function $f_t : D_t \to \mathbb{C}$ is univalent,
LC2. $f_s(D_s) \subset f_t(D_t)$ whenever $0 \leq s < t < +\infty$,
LC3. for any compact interval $I := [S, T] \subset [0, +\infty)$ and any compact set $K \subset D_S$ there exists a nonnegative function $k_{K,I} \in L^d([S, T], \mathbb{R})$ such that

$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,I}(\xi)d\xi$$

for all $z \in K$ and all (s, t) such that $S \leq s \leq t \leq T$.

The following theorem shows that every Löwner chain generates an evolution family of the same order.

Theorem 13 ([56, Theorem 1.9]). *Let (f_t) be a Löwner chain of order d over a canonical domain system (D_t) of order d. If we define*

$$\varphi_{s,t} := f_t^{-1} \circ f_s, \qquad 0 \leq s \leq t < \infty, \tag{28}$$

then $(\varphi_{s,t})$ is an evolution family of order d over (D_t).

An interesting consequence of this result is that any Löwner chain over a canonical system of annuli satisfies a PDE driven by a semicomplete weak holomorphic vector field. Moreover, the concrete formulation of this PDE clearly resembles the celebrated Löwner–Kufarev PDE appearing in the simple connected case.

The corresponding converse of the above theorem is more subtle than the one shown in the simple connected setting. Indeed, for any evolution family $(\varphi_{s,t})$ there exists a (essentially unique) Löwner chain (f_t) of the same order such that (28) holds but the selection of this fundamental chain is affected by the different conformal types of the domains D_t as well as the behavior of their elements with the index $I(\gamma)$ (with respect to zero) of certain closed curves $\gamma \subset \mathbb{D}_t$.

Theorem 14 ([56, Theorem 1.10]). *Let $(\varphi_{s,t})$ be an evolution family of order $d \in [1, +\infty]$ over the canonical domain system $D_t := \mathbb{A}_{r(t)}$ with $r(t) > 0$ (a non-degenerate system). Let $r_\infty := \lim_{t\to+\infty} r(t)$. Then there exists a Löwner chain (f_t) of order d over (D_t) such that*

1. *$f_s = f_t \circ \varphi_{s,t}$ for all $0 \leq s \leq t < +\infty$, i.e. (f_t) is associated with $(\varphi_{s,t})$;*
2. *$I(f_t \circ \gamma) = I(\gamma)$ for any closed curve $\gamma \subset D_t$ and any $t \geq 0$;*
3. *If $0 < r_\infty < 1$, then $\cup_{t\in[0,+\infty)} f_t(D_t) = \mathbb{A}_{r_\infty}$;*
4. *If $r_\infty = 0$, then $\cup_{t\in[0,+\infty)} f_t(D_t)$ is either $\mathbb{D}^*$, $\mathbb{C} \setminus \overline{\mathbb{D}}$, or $\mathbb{C}^*$.*

If (g_t) is another Löwner chain over (D_t) associated with $(\varphi_{s,t})$, then there is a biholomorphism $F : \cup_{t\in[0,+\infty)} g_t(D_t) \to \cup_{t\in[0,+\infty)} f_t(D_t)$ such that $f_t = F \circ g_t$ for all $t \geq 0$.

In general, a Löwner chain associated with a given evolution family is not unique. We call a Löwner chain (f_t) to be *standard* if it satisfies conditions (2)–(4) from Theorem 14. It follows from this theorem that the standard Löwner chain (f_t) associated with a given evolution family is defined uniquely up to a rotation (and scaling if $\cup_{t\in[0,+\infty)} f_t(D_t) = \mathbb{C}^*$).

Furthermore, combining Theorems 13 and 14 one can easily conclude that for any Löwner chain (g_t) of order d over a canonical domain system (D_t) there exists a biholomorphism $F : \cup_{t\in[0,+\infty)} g_t(D_t) \to L[(g_t)]$, where $L[(g_t)]$ is either $\mathbb{D}^*$, $\mathbb{C}\setminus\overline{\mathbb{D}}$, $\mathbb{C}^*$, or $\mathbb{A}_\rho$ for some $\rho > 0$, and such that the formula $f_t = F \circ g_t$, $t \geq 0$, defines a standard Löwner chain of order d over the canonical domain system (D_t). Hence, the conformal type of any Löwner chain $\cup_{t\in[0,+\infty)} g_t(D_t)$ can be identified with $\mathbb{D}^*$, $\mathbb{C}\setminus\overline{\mathbb{D}}$, $\mathbb{C}^*$, or $\mathbb{A}_\rho$ for some $\rho > 0$. Moreover, it is natural (well defined) to say that the (conformal) type of an evolution family $(\varphi_{s,t})$ is the conformal type of any Löwner chain associated with it.

The following statement characterizes the conformal type of a Löwner chain via dynamical properties of two associated evolution families over (D_t). One of them is the usual one $(\varphi_{s,t})$ and the other one is defined as follows: for each $s \geq 0$ and $t \geq s$,

$$\tilde{\varphi}_{s,t}(z) := r(t)/\varphi_{s,t}(r(s)/z).$$

It is worth mentioning that at least one of the families $(\varphi_{0,t})$ and $(\tilde{\varphi}_{0,t})$ converges to 0 as $t \to +\infty$ provided $r_\infty = \lim_{t\to+\infty} r(t) = 0$.

Theorem 15 ([56, Theorem 1.13]). *Let $\big((D_t), (\varphi_{s,t})\big)$ be a non-degenerate evolution family and denote as before $r_\infty := \lim_{t\to+\infty} r(t)$. In the above notation, the following statements hold:*

(i) the conformal type of the evolution family $(\varphi_{s,t})$ is $\mathbb{A}_\rho$ for some $\rho > 0$ if and only if $r_\infty > 0$;
(ii) the conformal type of evolution family $(\varphi_{s,t})$ is $\mathbb{D}^*$ if and only if $r_\infty = 0$ and $\varphi_{0,t}$ does not converge to 0 as $t \to +\infty$;
(iii) the conformal type of evolution family $(\varphi_{s,t})$ is $\mathbb{C}\setminus\overline{\mathbb{D}}$ if and only if $r_\infty = 0$ and $\tilde{\varphi}_{0,t}$ does not converge to 0 as $t \to +\infty$;
(iv) the conformal type of evolution family $(\varphi_{s,t})$ is $\mathbb{C}^*$ if and only if $r_\infty = 0$ and both $\varphi_{0,t} \to 0$ and $\tilde{\varphi}_{0,t} \to 0$ as $t \to +\infty$.

12 Integrability

In this section we plan to reveal relations between contour dynamics tuned by the Löwner–Kufarev equations and the Liouville (infinite dimensional) integrability, which was exploited actively since establishment of the Korteweg–de Vries equation as an equation for spectral stability in the Sturm–Liouville problem, and

construction of integrable hierarchies in a series of papers by Gardner, Green, Kruskal, Miura, Zabusky, et al., see, e.g., [86, 262], and exact integrability results by Zakharov and Faddeev [263]. In this section we mostly follow our results in [170, 171].

Recently, it has become clear that one-parameter expanding evolution families of simply connected domains in the complex plane in some special models has been governed by infinite systems of evolution parameters, conservation laws. This phenomenon reveals a bridge between a nonlinear evolution of complex shapes emerged in physical problems, dissipative in most of the cases, and exactly solvable models. One of such processes is the Laplacian growth, in which the harmonic (Richardson's) moments are conserved under the evolution, see, e.g., [114, 174]. The infinite number of evolution parameters reflects the infinite number of degrees of freedom of the system and clearly suggests to apply field theory methods as a natural tool of study. The Virasoro algebra provides a structural background in most of field theories, and it is not surprising that it appears in soliton-like problems, e.g., KP, KdV, or Toda hierarchies, see [74, 88].

Another group of models, in which the evolution is governed by an infinite number of parameters, can be observed in controllable dynamical systems, where the infinite number of degrees of freedom follows from the infinite number of driving terms. Surprisingly, the same algebraic structural background appears again for this group. We develop this viewpoint here.

One of the general approaches to the homotopic evolution of shapes starting from a canonical shape, the unit disk in our case, is given by the Löwner–Kufarev theory. A shape evolution is described by a time-dependent conformal parametric map from the canonical domain onto the domain bounded by the shape at any fixed instant. In fact, these one-parameter conformal maps satisfy the Löwner–Kufarev differential equation (3), or an infinite dimensional controllable system, for which the infinite number of conservation laws is given by the *Virasoro generators* in their covariant form.

Recently, Friedrich and Werner [82], and independently Bauer and Bernard [24], found relations between SLE (stochastic or Schramm–Löwner evolution) and the highest weight representation of the Virasoro algebra. Moreover, Friedrich developed the Grassmannian approach to relate SLE to the highest weight representation of the Virasoro algebra in [81].

All the above results encourage us to conclude that the *Virasoro algebra* is a common algebraic structural basis for these and possibly other types of contour dynamics and we present the development in this direction here. At the same time, the infinite number of conservation laws suggests a relation with exactly solvable models.

The geometry underlying classical integrable systems is reflected in Sato's and Segal–Wilson's constructions of the infinite dimensional *Grassmannian* Gr. Based on the idea that the evolution of shapes in the plane is related to an evolution in a general universal space, the Segal–Wilson Grassmannian in our case, we provide an embedding of the Löwner–Kufarev evolution into a fiber bundle with the cotangent bundle over $\mathscr{F}_0$ as a base space, and with the smooth Grassmannian Gr_∞ as a

typical fiber. Here $\mathscr{F}_0 \subset \mathbf{S}$ denotes the space of all conformal embeddings f of the unit disk into $\mathbb{C}$ normalized by $f(z) = z\left(1 + \sum_{n=1}^{\infty} c_n z^n\right)$ smooth on the boundary S^1, and under the *smooth Grassmannian* Gr_∞ we understand a dense subspace Gr_∞ of Gr defined further on.

So our plan is as follows. We will

- Consider homotopy in the space of shapes starting from the unit disk: $\mathbb{D} \to \Omega(t)$ given by the Löwner–Kufarev equation;
- Give its Hamiltonian formulation;
- Find conservation laws;
- Explore their algebraic structure;
- Embed Löwner–Kufarev trajectories into a moduli space (Grassmannian);
- Construct τ-function, Baker–Akhiezer function, and finally KP hierarchy.

Finally we present a class of solutions to KP hierarchy, which are preserving their form along the Löwner–Kufarev trajectories.

Let us start with two useful lemmas [170, 171].

Lemma 1. *Let the function $w(z,t)$ be a solution to the Cauchy problem (3). If the driving function $p(\cdot,t)$, being from the Carathéodory class for almost all $t \geq 0$, is C^∞ smooth in the closure $\hat{\mathbb{D}}$ of the unit disk $\mathbb{D}$ and summable with respect to t, then the boundaries of the domains $\Omega(t) = w(\mathbb{D},t) \subset \mathbb{D}$ are smooth for all t and $w(\cdot,t)$ extended to S^1 is injective on S^1.*

Lemma 2. *With the above notations let $f(z) \in \mathscr{F}_0$. Then there exists a function $p(\cdot,t)$ from the Carathéodory class for almost all $t \geq 0$, and C^∞ smooth in $\hat{\mathbb{D}}$, such that $f(z) = \lim_{t\to\infty} f(z,t)$ is the final point of the Löwner–Kufarev trajectory with the driving term $p(z,t)$.*

12.1 Witt and Virasoro Algebras

The complex *Witt algebra* is the Lie algebra of holomorphic vector fields defined on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ acting by derivation over the ring of Laurent polynomials $\mathbb{C}[z, z^{-1}]$. It is spanned by the basis $L_n = z^{n+1}\frac{\partial}{\partial z}$, $n \in \mathbb{Z}$. The Lie bracket of two basis vector fields is given by the commutator $[L_n, L_m] = (m-n)L_{n+m}$. Its central extension is the complex *Virasoro algebra* $\mathfrak{vir}_{\mathbb{C}}$ with the central element c commuting with all L_n, $[L_n, c] = 0$, and with the Virasoro commutation relation

$$[L_n, L_m] = (m-n)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n,-m}, \quad n, m \in \mathbb{Z},$$

where $c \in \mathbb{C}$ is the central charge denoted by the same character. These algebras play important role in conformal field theory. In order to construct their representations one can use an analytic realization.

12.2 Segal–Wilson Grassmannian

Sato's (universial) Grassmannian appeared first in 1982 in [222] as an infinite dimensional generalization of the classical finite dimensional Grassmannian manifolds and they are described as "the topological closure of the inductive limit of" a finite dimensional Grassmanian as the dimensions of the ambient vector space and its subspaces tend to infinity.



M. Sato

It turned out to be a very important infinite dimensional manifold being related to the representation theory of loop groups, integrable hierarchies, micrological analysis, conformal and quantum field theories, the second quantization of fermions, and to many other topics [59,177,224,259]. In the Segal and Wilson approach [224] the infinite dimensional Grassmannian $\mathrm{Gr}(H)$ is taken over the separable Hilbert space H. The first systematic description of the infinite dimensional Grassmannian can be found in [197].

We present here a general definition of the infinite dimensional smooth Grassmannian $\mathrm{Gr}_\infty(H)$. As a separable Hilbert space we take the space $L^2(S^1)$ and consider its dense subspace $H = C^\infty_{\|\cdot\|_2}(S^1)$ of smooth complex-valued functions defined on the unit circle endowed with $L^2(S^1)$ inner product $\langle f, g\rangle = \frac{1}{2\pi}\int\limits_{S^1} f\bar{g}\, dw$, $f, g \in H$. The orthonormal basis of H is $\{z^k\}_{k\in\mathbb{Z}} = \{e^{ik\theta}\}_{k\in\mathbb{Z}}$, $e^{i\theta} \in S^1$.

Let us split all integers $\mathbb{Z}$ into two sets $\mathbb{Z}^+ = \{0, 1, 2, 3, \dots\}$ and $\mathbb{Z}^- = \{\dots, -3, -2, -1\}$, and let us define a polarization by

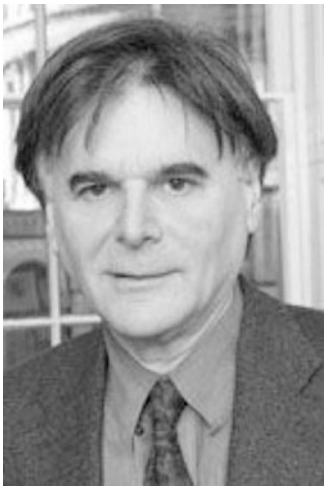
$$H_+ = \mathrm{span}_H\{z^k,\ k \in \mathbb{Z}^+\}, \qquad H_- = \mathrm{span}_H\{z^k,\ k \in \mathbb{Z}^-\}.$$

Here and further span is taken in the appropriate space indicated as a subscription. The Grassmanian is thought of as the set of closed linear subspaces W of H, which are commensurable with H_+ in the sense that they have finite codimension in both H_+ and W. This can be defined by means of the descriptions of the orthogonal projections of the subspace $W \subset H$ to H_+ and H_-.

Definition 8. The infinite dimensional smooth Grassmannian $\mathrm{Gr}_\infty(H)$ over the space H is the set of subspaces W of H, such that

1. the orthogonal projection $pr_+ \colon W \to H_+$ is a Fredholm operator,
2. the orthogonal projection $pr_- \colon W \to H_-$ is a compact operator.

The requirement that pr_+ is Fredholm means that the kernel and cokernel of pr_+ are finite dimensional. More information about Fredholm operators the reader can find in [64]. It was proved in [197], that $\mathrm{Gr}_\infty(H)$ is a dense submanifold in a Hilbert manifold modeled over the space $\mathscr{L}_2(H_+, H_-)$ of Hilbert–Schmidt operators from H_+ to H_-, that itself has the structure of a Hilbert space, see [210]. Any $W \in \mathrm{Gr}_\infty(H)$ can be thought of as a graph W_T of a Hilbert–Schmidt operator $T \colon W \to W^\perp$, and points of a neighborhood U_W of $W \in \mathrm{Gr}_\infty(H)$ are in one-to-one correspondence with operators from $\mathscr{L}_2(W, W^\perp)$.



G. Segal

Let us denote by $\mathfrak{S}$ the set of all collections $\mathbb{S} \subset \mathbb{Z}$ of integers such that $\mathbb{S} \setminus \mathbb{Z}^+$ and $\mathbb{Z}^+ \setminus \mathbb{S}$ are finite. Thus, any sequence $\mathbb{S}$ of integers is bounded from below and contains all positive numbers starting from some number. It is clear that the sets $H_{\mathbb{S}} = \mathrm{span}_H\{z^k,\ k \in \mathbb{S}\}$ are elements of the Grassmanian $\mathrm{Gr}_\infty(H)$ and they are usually called *special points*. The collection of neighborhoods $\{U_{\mathbb{S}}\}_{\mathbb{S}\in\mathfrak{S}}$,

$U_{\mathbb{S}} = \{W \mid \text{there is an orthogonal projection } \pi \colon W \to H_{\mathbb{S}} \text{ that is an isomorphism}\}$

forms an open cover of $\mathrm{Gr}_\infty(H)$. The virtual cardinality of $\mathbb{S}$ defines the *virtual dimension* (v.d.) of $H_\mathbb{S}$, namely:

$$\mathrm{virtcard}(\mathbb{S}) = \mathrm{virtdim}(H_\mathbb{S}) = \dim(\mathbb{N} \setminus \mathbb{S}) - \dim(\mathbb{S} \setminus \mathbb{N}) = \mathrm{ind}(pr_+). \tag{29}$$

The expression $\mathrm{ind}(pr_+) = \dim\ker(pr_+) - \dim\mathrm{coker}(pr_-)$ is called the index of the Fredholm operator pr_+. According to their virtual dimensions the points of $\mathrm{Gr}_\infty(H)$ belong to different components of connectivity. The Grassmannian is the disjoint union of connected components parametrized by their virtual dimensions.

12.3 Hamiltonian Formalism

Let the driving term $p(z,t)$ in the Löwner–Kufarev ODE (3) be from the Carathéodory class for almost all $t \geq 0$, C^∞-smooth in $\hat{\mathbb{D}}$, and summable with respect to t as in Lemma 1. Then the domains $\Omega(t) = f(\mathbb{D},t) = e^t w(\mathbb{D},t)$ have smooth boundaries $\partial\Omega(t)$ and the function f is injective on S^1, i.e.; $f \in \mathscr{F}_0$. So the Löwner–Kufarev equation can be extended to the closed unit disk $\hat{\mathbb{D}} = \mathbb{D} \cup S^1$.

Let us consider the sections ψ of $T^*\mathscr{F}_0 \otimes \mathbb{C}$, that are from the class $C^\infty_{\|\cdot\|_2}$ of smooth complex-valued functions $S^1 \to \mathbb{C}$ endowed with L^2 norm,

$$\psi(z) = \sum_{k\in\mathbb{Z}} \psi_k z^{k-1}, \quad |z| = 1.$$

We also introduce the space of observables on $T^*\mathscr{F}_0 \otimes \mathbb{C}$, given by integral functionals

$$\mathscr{R}(f,\bar\psi,t) = \frac{1}{2\pi}\int_{z\in S^1} r(f(z),\bar\psi(z),t)\frac{dz}{iz},$$

where the function $r(\xi,\eta,t)$ is smooth in variables ξ,η and measurable in t.

We define a special observable, the time-dependent pseudo-Hamiltonian $\mathscr{H}$, by

$$\mathscr{H}(f,\bar\psi,p,t) = \frac{1}{2\pi}\int_{z\in S^1} \bar z^2 f(z,t)(1 - p(e^{-t}f(z,t),t))\bar\psi(z,t)\frac{dz}{iz}, \tag{30}$$

with the driving function (control) $p(z,t)$ satisfying the above properties. The Poisson structure on the space of observables is given by the canonical brackets

$$\{\mathscr{R}_1,\mathscr{R}_2\} = 2\pi\int_{z\in S^1} z^2\left(\frac{\delta\mathscr{R}_1}{\delta f}\frac{\delta\mathscr{R}_2}{\delta\bar\psi} - \frac{\delta\mathscr{R}_1}{\delta\bar\psi}\frac{\delta\mathscr{R}_2}{\delta f}\right)\frac{dz}{iz},$$

where $\frac{\delta}{\delta f}$ and $\frac{\delta}{\delta\bar\psi}$ are the variational derivatives, $\frac{\delta}{\delta f}\mathscr{R} = \frac{1}{2\pi}\frac{\partial}{\partial f}r$, $\frac{\delta}{\delta\bar\psi}\mathscr{R} = \frac{1}{2\pi}\frac{\partial}{\partial\bar\psi}r$.

Representing the coefficients c_n and $\bar{\psi}_m$ of f and $\bar{\psi}$ as integral functionals

$$c_n = \frac{1}{2\pi}\int_{z\in S^1} \bar{z}^{n+1} f(z,t)\frac{dz}{iz}, \quad \bar{\psi}_m = \frac{1}{2\pi}\int_{z\in S^1} z^{m-1}\bar{\psi}(z,t)\frac{dz}{iz},$$

$n \in \mathbb{N}$, $m \in \mathbb{Z}$, we obtain $\{c_n, \bar{\psi}_m\} = \delta_{n,m}$, $\{c_n, c_k\} = 0$, and $\{\bar{\psi}_l, \bar{\psi}_m\} = 0$, where $n, k \in \mathbb{N}$, $l, m \in \mathbb{Z}$.

The infinite dimensional Hamiltonian system is written as

$$\frac{dc_k}{dt} = \{c_k, \mathscr{H}\}, \quad c_k(0) = 0, \tag{31}$$

$$\frac{d\bar{\psi}_k}{dt} = \{\bar{\psi}_k, \mathscr{H}\}, \quad \psi_k(0) = \xi_k, \tag{32}$$

where $k \in \mathbb{Z}$ and $c_0 = c_{-1} = c_{-2} = \cdots = 0$, or equivalently, multiplying by corresponding powers of z and summing up,

$$\frac{df(z,t)}{dt} = f(1 - p(e^{-t}f, t)) = 2\pi\frac{\delta\mathscr{H}}{\delta\bar{\psi}}z^2 = \{f, \mathscr{H}\}, \quad f(z,0) \equiv z, \tag{33}$$

$$\frac{d\bar{\psi}}{dt} = -(1 - p(e^{-t}f, t) - e^{-t}fp'(e^{-t}f, t))\bar{\psi} = -2\pi\frac{\delta\mathscr{H}}{\delta f}z^2 = \{\bar{\psi}, \mathscr{H}\}, \tag{34}$$

where $\psi(z,0) = \xi(z) = \sum_{k\in\mathbb{Z}} \xi_k z^{k-1}$ and $z \in S^1$. So the phase coordinates $(f, \bar{\psi})$ play the role of the canonical Hamiltonian pair and the coefficients ξ_k are free parameters. Observe that the Eq. (33) is the Löwner–Kufarev equation (3) for the function $f = e^t w$.

Let us set up the *generating function* $\mathscr{G}(z) = \sum_{k\in\mathbb{Z}} \mathscr{G}_k z^{k-1}$, such that

$$\bar{\mathscr{G}}(z) := f'(z,t)\bar{\psi}(z,t).$$

Consider the "nonpositive" $(\bar{\mathscr{G}}(z))_{\le 0}$ and "positive" $(\bar{\mathscr{G}}(z))_{>0}$ parts of the Laurent series for $\bar{\mathscr{G}}(z)$:

$$\begin{aligned}(\bar{\mathscr{G}}(z))_{\le 0} &= (\bar{\psi}_1 + 2c_1\bar{\psi}_2 + 3c_2\bar{\psi}_3 + \dots) + (\bar{\psi}_2 + 2c_1\bar{\psi}_3 + \dots)z^{-1} + \dots \\ &= \sum_{k=0}^{\infty} \bar{\mathscr{G}}_{k+1} z^{-k}.\end{aligned}$$

$$\begin{aligned}(\bar{\mathscr{G}}(z))_{>0} &= (\bar{\psi}_0 + 2c_1\bar{\psi}_1 + 3c_2\bar{\psi}_2 + \dots)z + (\bar{\psi}_{-1} + 2c_1\bar{\psi}_0 + 3c_2\bar{\psi}_1 \dots)z^2 + \dots \\ &= \sum_{k=1}^{\infty} \bar{\mathscr{G}}_{-k+1} z^{k}.\end{aligned}$$

Proposition 1. *Let the driving term $p(z,t)$ in the Löwner–Kufarev ODE be from the Carathéodory class for almost all $t \geq 0$, C^∞-smooth in $\hat{\mathbb{D}}$, and summable with respect to t. The functions $\mathscr{G}(z)$, $(\mathscr{G}(z))_{<0}$, $(\mathscr{G}(z))_{\geq 0}$, and all coefficients $\mathscr{G}_n$ are time-independent for all $z \in S^1$.*

Proof. It is sufficient to check the equality $\dot{\bar{\mathscr{G}}} = \{\bar{\mathscr{G}}, \mathscr{H}\} = 0$ for the function $\mathscr{G}$, and then, the same holds for the coefficients of the Laurent series for $\mathscr{G}$. □

Proposition 2. *The conjugates $\bar{\mathscr{G}}_k$, $k = 1, 2, \ldots$, to the coefficients of the generating function satisfy the Witt commutation relation $\{\bar{\mathscr{G}}_m, \bar{\mathscr{G}}_n\} = (n-m)\bar{\mathscr{G}}_{n+m}$ for $n, m \geq 1$, with respect to our Poisson structure.*

The isomorphism $\iota : \bar{\psi}_k \to \partial_k = \frac{\partial}{\partial c_k}$, $k > 0$, is a Lie algebra isomorphism

$$(T_f^{*(0,1)}\mathscr{F}_0, \{\ ,\ \}) \to (T_f^{(1,0)}\mathscr{F}_0, [\ ,\ ]).$$

It makes a correspondence between the conjugates $\bar{\mathscr{G}}_n$ of the coefficients $\mathscr{G}_n$ of $(\mathscr{G}(z))_{\geq 0}$ at the point $(f, \bar{\psi})$ and the Kirillov vectors $L_n[f] = \partial_n + \sum_{k=1}^{\infty}(k+1)c_k\partial_{n+k}$, $n \in \mathbb{N}$, see [133]. Both satisfy the Witt commutation relations

$$[L_n, L_m] = (m-n)L_{n+m}.$$

12.4 Curves in Grassmannian

Let us recall that the underlying space for the universal smooth Grassmannian $\mathrm{Gr}_\infty(H)$ is $H = C^\infty_{\|\cdot\|_2}(S^1)$ with the canonical L^2 inner product of functions defined on the unit circle. Its natural polarization

$$H_+ = \mathrm{span}_H\{1, z, z^2, z^3, \ldots\}, \qquad H_- = \mathrm{span}_H\{z^{-1}, z^{-2}, \ldots\},$$

was introduced before. The pseudo-Hamiltonian $\mathscr{H}(f, \bar{\psi}, t)$ is defined for an arbitrary $\psi \in L^2(S^1)$, but we consider only smooth solutions of the Hamiltonian system, therefore, $\psi \in H$. We identify this space with the dense subspace of $T_f^*\mathscr{F}_0 \otimes \mathbb{C}$, $f \in \mathscr{F}_0$. The generating function $\mathscr{G}$ defines a linear map $\bar{\mathscr{G}}$ from the dense subspace of $T_f^*\mathscr{F}_0 \otimes \mathbb{C}$ to H, which being written in a block matrix form becomes

$$\begin{pmatrix} \bar{\mathscr{G}}_{>0} \\ \bar{\mathscr{G}}_{\leq 0} \end{pmatrix} = \begin{pmatrix} C_{1,1} & C_{1,2} \\ 0 & C_{1,1} \end{pmatrix} \begin{pmatrix} \bar{\psi}_{>0} \\ \bar{\psi}_{\leq 0} \end{pmatrix}, \tag{35}$$

where

$$\begin{pmatrix} C_{1,1} & C_{1,2} \\ 0 & C_{1,1} \end{pmatrix} = \left(\begin{array}{ccccc|ccccc} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & 5c_4 & 6c_5 & 7c_6 & \cdots \\ \cdots & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & 5c_4 & 6c_5 & \cdots \\ \cdots & 0 & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & 5c_4 & \cdots \\ \hline \cdots & 0 & 0 & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 2c_1 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right).$$

Proposition 3. *The operator $C_{1,1}\colon H_+ \to H_+$ is invertible.*

The generating function also defines a map $\mathscr{G}\colon T^*\mathscr{F}_0 \otimes \mathbb{C} \to H$ by

$$T^*\mathscr{F}_0 \otimes \mathbb{C} \ni (f(z), \psi(z)) \mapsto \mathscr{G} = \bar{f}'(z)\psi(z) \in H.$$

Observe that any solution $\left(f(z,t), \bar{\psi}(z,t)\right)$ of the Hamiltonian system is mapped into a single point of the space H, since all $\mathscr{G}_k$, $k \in \mathbb{Z}$ are time-independent by Proposition 1.

Consider a bundle $\pi\colon \mathscr{B} \to T^*\mathscr{F}_0 \otimes \mathbb{C}$ with a typical fiber isomorphic to $\mathrm{Gr}_\infty(H)$. We are aimed at construction of a curve $\Gamma\colon [0,T] \to \mathscr{B}$ that is traced by the solutions to the Hamiltonian system, or in other words, by the Löwner–Kufarev evolution. The curve Γ will have the form

$$\Gamma(t) = \left(f(z,t), \psi(z,t), W_{T_n}(t)\right)$$

in the local trivialization. Here W_{T_n} is the graph of a finite rank operator $T_n\colon H_+ \to H_-$, such that W_{T_n} belongs to the connected component of U_{H_+} of virtual dimension 0. In other words, we build a hierarchy of finite rank operators $T_n\colon H_+ \to H_-$, $n \in \mathbb{Z}^+$, whose graphs in the neighborhood U_{H_+} of the point $H_+ \in \mathrm{Gr}_\infty(H)$ are

$$T_n((\mathscr{G}(z))_{>0}) = T_n(\mathscr{G}_1, \mathscr{G}_2, \dots, \mathscr{G}_k, \dots) = \begin{cases} G_0(\mathscr{G}_1, \mathscr{G}_2, \dots, \mathscr{G}_k, \dots) \\ G_{-1}(\mathscr{G}_1, \mathscr{G}_2, \dots, \mathscr{G}_k, \dots) \\ \dots \\ G_{-n+1}(\mathscr{G}_1, \mathscr{G}_2, \dots, \mathscr{G}_k, \dots), \end{cases}$$

with $G_0 z^{-1} + G_{-1} z^{-2} + \ldots + G_{-n+1} z^{-n} \in H_-$. Let us denote by $G_k = \mathscr{G}_k$, $k \in \mathbb{N}$. The elements $G_0, G_{-1}, G_{-2}, \ldots$ are constructed so that all $\{\bar{G}_k\}_{k=-n+1}^{\infty}$ satisfy the truncated Witt commutation relations

$$\{\bar{G}_k, \bar{G}_l\}_n = \begin{cases} (l-k)\bar{G}_{k+l}, & \text{for } k+l \geq -n+1, \\ 0, & \text{otherwise,} \end{cases}$$

and are related to Kirilov's vector fields [133] under the isomorphism ι. The projective limit as $n \leftarrow \infty$ recovers the whole Witt algebra and the Witt commutation relations. Then the operators T_n such that their conjugates are $\bar{T}_n = (\tilde{B}^{(n)} + C_{2,1}^{(n)}) \circ C_{1,1}^{-1}$, are operators from H_+ to H_- of finite rank and their graphs $W_{T_n} = (\mathrm{id} + T_n)(H_+)$ are elements of the component of virtual dimension 0 in $\mathrm{Gr}_\infty(H)$. We can construct a basis $\{e_0, e_1, e_2, \ldots\}$ in W_{T_n} as a set of Laurent polynomials defined by means of operators T_n and $\bar{C}_{1,1}$ as a mapping

$$\{\psi_1, \psi_2, \ldots\} \xrightarrow{\bar{C}_{1,1}} \{G_1, G_2, \ldots\} \xrightarrow{\mathrm{id}+T_n} \{G_{-n+1}, G_{-n+2}, \ldots, G_0, G_1, G_2, \ldots\},$$

of the canonical basis $\{1, 0, 0, \ldots\}, \{0, 1, 0, \ldots\}, \{0, 0, 1, \ldots\}, \ldots$

Let us formulate the result as the following main statement.

Proposition 4. *The operator T_n defines a graph $W_{T_n} = \mathrm{span}\{e_0, e_1, e_2, \ldots\}$ in the Grassmannian Gr_∞ of virtual dimension 0. Given any*

$$\psi = \sum_{k=0}^{\infty} \psi_{k+1} z^k \in H_+ \subset H,$$

the function

$$G(z) = \sum_{k=-n}^{\infty} G_{k+1} z^k = \sum_{k=0}^{\infty} \psi_{k+1} e_k,$$

is an element of W_{T_n}.

Proposition 5. *In the autonomous case of the Cauchy problem (3), when the function $p(z,t)$ does not depend on t, the pseudo-Hamiltonian $\mathscr{H}$ plays the role of time-dependent energy and $\mathscr{H}(t) = \bar{G}_0(t) + const$, where $\bar{G}_0\big|_{t=0} = 0$. The constant is defined as $\sum_{n=1}^{\infty} p_k \bar{\psi}_k(0)$.*

Remark 4. The Virasoro generator L_0 plays the role of the energy functional in CFT. In view of the isomorphism ι, the observable $\bar{G}_0 = \iota^{-1}(L_0)$ plays an analogous role.

Thus, we constructed a countable family of curves $\Gamma_n : [0, T] \to \mathscr{B}$ in the trivial bundle $\mathscr{B} = T^*\mathscr{F}_0 \otimes \mathbb{C} \times \mathrm{Gr}_\infty(H)$, such that the curve Γ_n admits the form

$\Gamma_n(t) = \Big(f(z,t), \psi(z,t), W_{T_n}(t)\Big)$, for $t \in [0,T]$ in the local trivialization. Here $\big(f(z,t), \bar{\psi}(z,t)\big)$ is the solution of the Hamiltonian system (31–32). Each operator $T_n(t)\colon H_+ \to H_-$ that maps $\mathscr{G}_{>0}$ to

$$\big(G_0(t), G_{-1}(t), \ldots, G_{-n+1}(t)\big)$$

defined for any $t \in [0,T]$, $n = 1,2,\ldots$, is of finite rank and its graph $W_{T_n}(t)$ is a point in $\mathrm{Gr}_\infty(H)$ for any t. The graphs W_{T_n} belong to the connected component of the virtual dimension 0 for every time $t \in [0,T]$ and for fixed n. The coordinates $(G_{-n+1}, \ldots, G_{-2}, G_{-1}, G_0, G_1, G_2, \ldots)$ of a point in the graph W_{T_n} considered as a function of ψ are isomorphic to the Kirilov vector fields

$$(L_{-n+1}, \ldots, L_{-2}, L_{-1}, L_0, L_1, L_1, L_2, \ldots)$$

under the isomorphism ι.

12.5 τ-Function

Remind that any function g holomorphic in the unit disc, nonvanishing on the boundary and normalized by $g(0) = 1$ defines the multiplication operator $g\varphi$, $\varphi(z) = \sum_{k\in\mathbb{Z}} \varphi_k z^k$, that can be written in the matrix form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \varphi_{\geq 0} \\ \varphi_{<0} \end{pmatrix}. \tag{36}$$

All these upper triangular matrices form a subgroup GL^+_{res} of the group of automorphisms GL_{res} of the Grassmannian $\mathrm{Gr}_\infty(H)$.

With any function g and any graph W_{T_n} constructed in the previous section (which is transverse to H_-) we can relate the τ-function $\tau_{W_{T_n}}(g)$ by the following formula

$$\tau_{W_{T_n}}(g) = \det(1 + a^{-1} b T_n),$$

where a, b are the blocks in the multiplication operator generated by g^{-1}. If we write the function g in the form $g(z) = \exp(\sum_{n=1}^\infty t_n z^n) = 1 + \sum_{k=1}^\infty S_k(\mathbf{t}) z^k$, where the coefficients $S_k(\mathbf{t})$ are the homogeneous elementary Schur polynomials, then the coefficients $\mathbf{t} = (t_1, t_2, \ldots)$ are called generalized times. For any fixed W_{T_n} we get an orbit in $\mathrm{Gr}_\infty(H)$ of curves Γ_n constructed in the previous section under the action of the elements of the subgroup GL^+_{res} defined by the function g. On the other hand, the τ-function defines a section in the determinant bundle over $\mathrm{Gr}_\infty(H)$ for any fixed $f \in \mathscr{F}_0$ at each point of the curve Γ_n.

12.6 Baker–Akhiezer Function, KP Flows, and KP Equation

Let us consider the component Gr^0 of the Grassmannian Gr_∞ of virtual dimension 0, and let g be a holomorphic function in $\mathbb{D}$ considered as an element of GL^+_{res} analogously to the previous section. Then g is an upper triangular matrix with 1s on the principal diagonal. Observe that $g(0) = 1$ and g does not vanish on S^1. Given a point $W \in \mathrm{Gr}^0$ let us define a subset $\Gamma^+ \subset GL^+_{\mathrm{res}}$ as

$$\Gamma^+ = \{g \in GL^+_{\mathrm{res}} : \ g^{-1}W \text{ is transverse to } \ H_-\}.$$

Then there exists [224] a unique function $\Psi_W[g](z)$ defined on S^1, such that for each $g \in \Gamma^+$, the function $\Psi_W[g]$ is in W, and it admits the form

$$\Psi_W[g](z) = g(z)\left(1 + \sum_{k=1}^{\infty} \omega_k(g, W)\frac{1}{z^k}\right).$$

The coefficients $\omega_k = \omega_k(g, W)$ depend both on $g \in \Gamma^+$ and on $W \in \mathrm{Gr}^0$, besides they are holomorphic on Γ^+ and extend to meromorphic functions on GL^+_{res}. The function $\Psi_W[g](z)$ is called the *Baker–Akhiezer function* of W.

Henry Frederick Baker (1866–1956) was British mathematician known for his contribution in algebraic geometry, PDE, and Lie theory. Naum Ilyich Akhiezer (1901–1980) was a Soviet mathematician known for his contributions in approximation theory and the theory of differential and integral operators, mathematical physics and history of mathematics. His brother Alexander was known theoretical physicist.

The Baker–Akhiezer function plays a crucial role in the definition of the KP (Kadomtsev–Petviashvili) hierarchy which we will define later. We are going to construct the Baker–Akhiezer function explicitly in our case.

Let $W = W_{T_n}$ be a point of Gr^0 defined in Proposition 4. Take a function $g(z) = 1 + a_1 z + a_2 z^2 + \cdots \in \Gamma^+$, and let us write the corresponding bi-infinite series for the Baker–Akhiezer function $\Psi_W[g](z)$ explicitly as

$$\begin{aligned}
\Psi_W[g](z) = \sum_{k\in\mathbb{Z}} \mathscr{W}_k z^k &= (1 + a_1 z + a_2 z^2 + \dots)\left(1 + \frac{\omega_1}{z} + \frac{\omega_2}{z^2} + \dots\right) \\
&= \dots + (a_2 + a_3\omega_1 + a_4\omega_2 + a_5\omega_3 + \dots)z^2 \\
&\quad + (a_1 + a_2\omega_1 + a_3\omega_2 + a_4\omega_3 + \dots)z \\
&\quad + (1 + a_1\omega_1 + a_2\omega_2 + a_3\omega_3 + \dots) \\
&\quad + (\omega_1 + a_1\omega_2 + a_2\omega_3 + \dots)\frac{1}{z} \\
&\quad + (\omega_2 + a_1\omega_3 + a_2\omega_4 + \dots)\frac{1}{z^2} + \dots \\
&\quad \dots + (\omega_k + a_1\omega_{k+1} + a_2\omega_{k+2} + \dots)\frac{1}{z^k} + \dots
\end{aligned}$$

The Baker–Akhiezer function for g and W_{T_n} must be of the form

$$\Psi_{W_{T_n}}[g](z) = g(z)\left(1 + \sum_{k=1}^{n} \omega_k(g)\frac{1}{z^k}\right) = \sum_{k=-n}^{\infty} \mathscr{W}_k z^k.$$

For a fixed $n \in \mathbb{N}$ we truncate this bi-infinite series by putting $\omega_k = 0$ for all $k > n$. In order to satisfy the definition of W_{T_n}, and determine the coefficients $\omega_1, \omega_2, \ldots, \omega_n$, we must check that there exists a vector $\{\psi_1, \psi_2, \ldots\}$, such that $\Psi_{W_{T_n}}[g](z) = \sum_{k=0}^{\infty} e_k \psi_{k+1}$. First we express ψ_k as linear functions $\psi_k = \psi_k(\omega_1, \omega_2, \ldots, \omega_n)$ by

$$(\psi_1, \psi_2, \psi_3, \ldots) = \bar{C}_{1,1}^{-1}\Big(\mathscr{W}_0(\omega_1, \omega_2, \ldots, \omega_n), \mathscr{W}_1(\omega_1, \omega_2, \ldots, \omega_n), \ldots\Big). \quad (37)$$

Using Wronski formula we can write

$$\begin{aligned}
\psi_1 &= \mathscr{W}_0 - 2\bar{c}_1\mathscr{W}_1 - (3\bar{c}_2 - 4\bar{c}_1^2)\mathscr{W}_2 - (4\bar{c}_3 - 12\bar{c}_2\bar{c}_1 + 8\bar{c}_1^3)\mathscr{W}_3 + \ldots,\\
\psi_2 &= \mathscr{W}_1 - 2\bar{c}_1\mathscr{W}_2 - (3\bar{c}_2 - 4\bar{c}_1^2)\mathscr{W}_3 - (4\bar{c}_3 - 12\bar{c}_2\bar{c}_1 + 8\bar{c}_1^3)\mathscr{W}_4 + \ldots,\\
\psi_3 &= \mathscr{W}_2 - 2\bar{c}_1\mathscr{W}_3 - (3\bar{c}_2 - 4\bar{c}_1^2)\mathscr{W}_4 - (4\bar{c}_3 - 12\bar{c}_2\bar{c}_1 + 8\bar{c}_1^3)\mathscr{W}_5 + \ldots,\\
&\ldots\ldots\ldots
\end{aligned}$$

Next we define $\omega_1, \omega_2, \ldots, \omega_n$ as functions of g and W_{T_n}, or in other words, as functions of $a_k, \bar{c}_k$ by solving linear equations

$$\begin{aligned}
\omega_1 &= \bar{c}_1\psi_1 + 2\bar{c}_2\psi_2 + \ldots k\bar{c}_k\psi_k + \ldots,\\
\omega_2 &= \sum_{k=1}^{\infty}\Big((k+2)\bar{c}_{k+1} - 2\bar{c}_1\bar{c}_k\Big)\psi_k,\\
&\ldots\ldots\ldots
\end{aligned}$$

where ψ_k are taken from (37). The solution exists and is unique because of the general fact of the existence of the Baker–Akhiezer function. It is quite difficult task in general, however, in the case $n = 1$, it is possible to write the solution explicitly in the matrix form. If

$$A = \begin{pmatrix} \ldots \\ 3\bar{c}_3 \\ 2\bar{c}_2 \\ \bar{c}_1 \end{pmatrix}^T \bar{C}_{1,1}^{-1} \begin{pmatrix} \ldots \\ a_3 \\ a_2 \\ a_1 \end{pmatrix}, \quad B = \begin{pmatrix} \ldots \\ 3\bar{c}_3 \\ 2\bar{c}_2 \\ \bar{c}_1 \end{pmatrix}^T \bar{C}_{1,1}^{-1} \begin{pmatrix} \ldots \\ a_2 \\ a_1 \\ 1 \end{pmatrix}.$$

then $\omega_1 = \frac{B}{1-A}$.

In order to apply further theory of integrable systems we need to change variables $a_n \to a_n(\mathbf{t})$, $n > 0$, $\mathbf{t} = \{t_1, t_2, \dots\}$ in the following way

$$a_n = a_n(t_1, \dots, t_n) = S_n(t_1, \dots, t_n),$$

where S_n is the n-th elementary homogeneous Schur polynomial

$$1 + \sum_{k=1}^{\infty} S_k(\mathbf{t}) z^k = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) = e^{\xi(\mathbf{t},z)}.$$

In particular,

$$S_1 = t_1, \quad S_2 = \frac{t_1^2}{2} + t_2, \quad S_3 = \frac{t_1^3}{6} + t_1 t_2 + t_3,$$
$$S_4 = \frac{t_1^4}{24} + \frac{t_2^2}{2} + \frac{t_1^2 t_2}{2} + t_1 t_3 + t_4.$$

Then the Baker–Akhiezer function corresponding to the graph W_{T_n} is written as

$$\Psi_{W_{T_n}}[g](z) = \sum_{k=-n}^{\infty} \mathscr{W}_k z^k = e^{\xi(\mathbf{t},z)} \left(1 + \sum_{k=1}^{n} \frac{\omega_k(\mathbf{t}, W_{T_n})}{z^k}\right),$$

and $\mathbf{t} = \{t_1, t_2, \dots\}$ is called the vector of generalized times. It is easy to see that

$$\partial_{t_k} a_m = 0, \quad \text{for all } m = 1, 2 \dots, k-1,$$

$\partial_{t_k} a_m = 1$ and

$$\partial_{t_k} a_m = a_{m-k}, \quad \text{for all } m > k.$$

In particular, $B = \partial_{t_1} A$. Let us denote $\partial := \partial_{t_1}$. Then in the case $n = 1$ we have

$$\omega_1 = \frac{\partial A}{1 - A}. \tag{38}$$

Now we consider the associative algebra of pseudo-differential operators $\mathscr{A} = \sum_{k=-\infty}^{n} a_k \partial^k$ over the space of smooth functions, where the derivation symbol ∂ satisfies the Leibniz rule and the integration symbol and its powers satisfy the algebraic rules $\partial^{-1}\partial = \partial\partial^{-1} = 1$ and $\partial^{-1} a$ is the operator $\partial^{-1} a = \sum_{k=0}^{\infty} (-1)^k (\partial^k a) \partial^{-k-1}$ (see, e.g., [63]). The action of the operator ∂^m, $m \in \mathbb{Z}$, is well defined over the function $e^{\xi(\mathbf{t},z)}$, where $\xi(\mathbf{t}, z) = \sum_{k=1}^{\infty} t_k z^k$, so that the function $e^{\xi(\mathbf{t},z)}$ is the eigenfunction of the operator ∂^m for any integer m, i.e., it satisfies the equation

$$\partial^m e^{\xi(\mathbf{t},z)} = z^m e^{\xi(\mathbf{t},z)}, \quad m \in \mathbb{Z}, \tag{39}$$

see, e.g., [22, 63]. As usual, we identify $\partial = \partial_{t_1}$, and $\partial^0 = 1$.

Let us introduce the dressing operator $\Lambda = \phi\partial\phi^{-1} = \partial + \sum_{k=1}^{\infty} \lambda_k \partial^{-k}$, where ϕ is a pseudo-differential operator $\phi = 1 + \sum_{k=1}^{\infty} w_k(\mathbf{t})\partial^{-k}$. The operator Λ is defined up to the multiplication on the right by a series $1 + \sum_{k=1}^{\infty} b_k \partial^{-k}$ with constant coefficients b_k. The m-th KP flow is defined by making use of the vector field

$$\partial_m \phi := -\Lambda_{<0}^m \phi, \quad \partial_m = \frac{\partial}{\partial t_m},$$

and the flows commute. In the Lax form the KP flows are written as

$$\partial_m \Lambda = [\Lambda_{\geq 0}^m, \Lambda]. \tag{40}$$

If $m = 1$, then $\partial\Lambda = [\partial, \Lambda] = \sum_{k=1}^{\infty}(\partial\lambda_k)\partial^{-k}$, which justifies the identification $\partial = \partial_{t_1}$.

Thus, the Baker–Akhiezer function $\Psi_{W_{T_n}}[g](z)$ admits the form

$$\Psi_{W_{T_n}}[g](z) = \phi \exp(\xi(\mathbf{t}, z)),$$

where ϕ is a pseudo-differential operator $\phi = 1 + \sum_{k=1}^{n} \omega_k(\mathbf{t}, W_{T_n})\partial^{-k}$. The function $\Psi_{W_{T_n}}[g](z)$ becomes the eigenfunction of the operator Λ^m, namely $\Lambda^m w = z^m w$, for $m \in \mathbb{Z}$. Besides, $\partial_m w = \Lambda_{>0}^m w$. In view of (39) we can write this function as previously,

$$\Psi_{W_{T_n}}[g](z) = \Big(1 + \sum_{k=1}^{n} \omega_k(\mathbf{t}, W_{T_n}) z^{-k}\Big) e^{\xi(\mathbf{t},z)}.$$

Proposition 6. *Let $n = 1$, and let the Baker–Akhiezer function be of the form*

$$\Psi_{W_{T_n}}[g](z) = e^{\xi(\mathbf{t},z)}\left(1 + \frac{\omega}{z}\right),$$

where $\omega = \omega_1$ is given by the formula (38). Then

$$\partial\omega = \frac{\partial^2 A}{1 - A} + \left(\frac{\partial A}{1 - A}\right)^2$$

is a solution to the KP equation with the Lax operator $L = \partial^2 - 2(\partial\omega)$.

Remark 5. Observe that the condition $\operatorname{Re} p > 0$ in $\mathbb{D}$ defines a Löwner–Kufarev cone of trajectories in the class $\mathscr{F}_0$ of univalent smooth functions which allowed us to construct Löwner–Kufarev trajectories in the neighborhood U_{H_+}, which is a cone in the Grassmannian Gr_∞. The form of solutions to the KP hierarchy is preserved along the Löwner–Kufarev trajectories. The solutions are parametrized by the initial conditions in the system (32).

Of course, one can express the Baker–Akhiezer function directly from the τ-function by the Sato formula

$$\Psi_{W_{T_n}}[g](z) = e^{\xi(\mathbf{t},z)} \frac{\tau_{W_{T_n}}(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, t_3 - \frac{1}{3z^3}, \dots)}{\tau_{W_{T_n}}(t_1, t_2, t_3 \dots)},$$

or applying the *vertex operator* V acting on the Fock space $\mathbb{C}[\mathbf{t}]$ of homogeneous polynomials

$$\Psi_{W_{T_n}}[g](z) = \frac{1}{\tau_{W_{T_n}}} V \tau_{W_{T_n}},$$

where

$$V = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} \frac{\partial}{\partial t_k} z^{-k}\right).$$

In the latter expression exp denotes the formal exponential series and z is another formal variable that commutes with all Heisenberg operators t_k and $\frac{\partial}{\partial t_k}$. Observe that the exponents in V do not commute and the product of exponentials is calculated by the Baker–Campbell–Hausdorff formula. The operator V is a vertex operator in which the coefficient V_k in the expansion of V is a well-defined linear operator on the space $\mathbb{C}[\mathbf{t}]$. The Lie algebra of operators spanned by $1, t_k, \frac{\partial}{\partial t_k}$, and V_k, is isomorphic to the affine Lie algebra $\hat{\mathfrak{sl}}(2)$. The vertex operator V plays a central role in the highest weight representation of affine Kac–Moody algebras [128, 176] and can be interpreted as the infinitesimal Bäcklund transformation for the Korteweg–de Vries equation [58].

The vertex operator V recovers the Virasoro algebra in the following sense. Taken in two close points $z+\lambda/2$ and $z-\lambda/2$ the operator product can be expanded in to the following formal Laurent–Fourier series

$$:V(z+\frac{\lambda}{2})V(z-\frac{\lambda}{2}) := \sum_{k\in\mathbb{Z}} W_k(z)\lambda^k,$$

where $: ab :$ stands for the bosonic normal ordering. Then $W_2(z) = T(z)$ is the stress–energy tensor which we expand again as

$$T(z) = \sum_{n\in\mathbb{Z}} L_n(\mathbf{t}) z^{n-2},$$

where the operators L_n are the Virasoro generators in the highest weight representation over $\mathbb{C}[\mathbf{t}]$. Observe that the generators L_n span the full Virasoro algebra with central extension and with the central charge 1. This can also be interpreted as a quantization of the shape evolution. We shall define a representation over the space

of conformal Fock space fields based on the Gaussian free field (GFF) in the next section.

Remark 6. Let us remark that the relation of Löwner equation to the integrable hierarchies of nonlinear PDE was noticed by Gibbons and Tsarev [89]. They observed that the chordal Löwner equation plays an essential role in the classification of reductions of Benney's equations. Later Takebe, Teo, and Zabrodin [243] showed that the chordal and radial Löwner PDE serve as consistency conditions for one-variable reductions of dispersionless KP and Toda hierarchies, respectively.

Remark 7. We also mention here relations between Löwner half-plane multi-slit equations and the estimates of spectral gaps of changing length for the periodic Zakharov–Shabat operators and for Hamiltonians in KdV and nonlinear Schrödinger equations elaborated in [137–139].

13 Stochastic Löwner Evolutions, Schramm, and Connections to CFT

13.1 SLE

This section we dedicate to the stochastic counterpart of the Löwner–Kufarev theory first recalling that one of the last (but definitely not least) contributions to this growing theory was the description by Oded Schramm in 1999–2000 [230] of the stochastic Löwner evolution (SLE), also known as the Schramm–Löwner evolution.



O. Schramm

The SLE is a conformally invariant stochastic process; more precisely, it is a family of random planar curves generated by solving Löwner's differential equation with

the Brownian motion as a driving term. This equation was studied and developed by Oded Schramm together with Greg Lawler and Wendelin Werner in a series of joint papers that led, among other things, to a proof of Mandelbrot's conjecture about the Hausdorff dimension of the Brownian frontier [157]. This achievement was one of the reasons Werner was awarded the Fields Medal in 2006. Sadly, Oded Schramm, born 10 December 1961 in Jerusalem, died in a tragic hiking accident on 01 September 2008 while climbing Guye Peak, north of Snoqualmie Pass in Washington.

The *(chordal) stochastic Löwner evolution* with parameter $k \geq 0$ (SLE_k) starting at a point $x \in \mathbb{R}$ is the random family of maps (g_t) obtained from the chordal Löwner equation (5) by letting $\xi(t) = \sqrt{k}B_t$, where B_t is a standard one-dimensional Brownian motion such that $\sqrt{k}B_0 = x$. Namely, let us consider the equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi(t)}, \qquad g_0(z) = z, \tag{41}$$

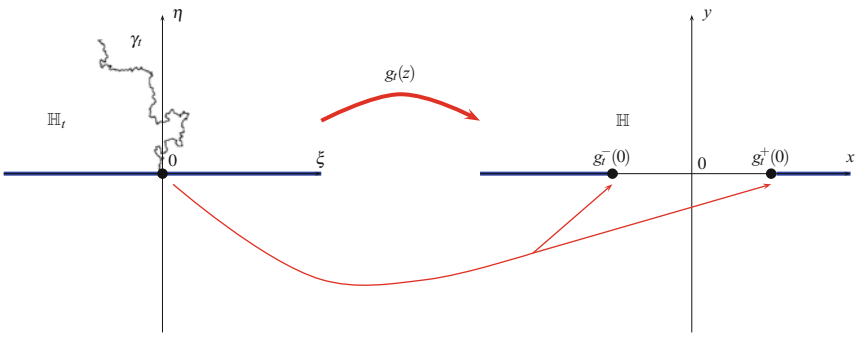
where $\xi(t) = \sqrt{k}B_t = \sqrt{k}B_t(\omega)$, $B_t(\omega)$ is the standard one-dimensional Brownian motion defined on the standard filtered probability space $(\Omega, \mathscr{G}, (\mathscr{G}_t), P)$ of Brownian motion with the sample space $\omega \in \Omega$, and $t \in [0, \infty)$, $B_0 = 0$. The solution to (41) exists as long as $g_t(z) - h(t)$ remains away from zero and we denote by T_z the first time such that $\lim_{t \to T_z - 0}(g_t(z) - h(t)) = 0$. The function g_t satisfies the hydrodynamic normalization at infinity $g_t(z) = z + \frac{2t}{z} + \dots$. Let $K_t = \{z \in \hat{\mathbb{H}} \colon T_z \leq t\}$ and let $\mathbb{H}_t$ be its complement $\mathbb{H} \setminus K_t = \{z \in \mathbb{H} \colon T_z > t\}$. The set K_t is called *SLE hull*. It is compact, $\mathbb{H}_t$ is a simply connected domain and g_t maps $\mathbb{H}_t$ onto $\mathbb{H}$. SLE hulls grow in time. The *trace* γ_t is defined as $\lim_{z \to \xi(t)} g_t^{-1}(z)$, where the limit is taken in $\mathbb{H}$. The unbounded component of $\mathbb{H} \setminus \gamma_t$ is $\mathbb{H}_t$. The Hausdorff dimension of the SLE_k trace is $\min(1 + \frac{8}{k}, 2)$, see [33]. Similarly, one can define the radial stochastic Löwner evolution. The terminology comes from the fact that the Löwner trace tip tends almost surely to a boundary point in the chordal case (∞ in the half-plane version) or to the origin in the disk version of the radial case.

Chordal SLE enjoys two important properties: scaling invariance and the Markovian property. Namely,

- $g_t(z)$ and $\frac{1}{\lambda} g_{\lambda^2 t}(\lambda z)$ are identically distributed;
- $h_t(z) = g_t(z) - \xi(t)$ possesses the Markov property. Furthermore, $h_s \circ h_t^{-1}$ is distributed as h_{s-t} for $s > t$.

The SLE_k depends on the choice of ω and it comes in several flavors depending on the type of Brownian motion exploited. For example, it might start at a fixed point or start at a uniformly distributed point, or might have a built in drift and so on. The parameter k controls the rate of diffusion of the Brownian motion and the behavior of the SLE_k critically depends on the value of k.

The SLE_2 corresponds to the loop-erased random walk and the uniform spanning tree. The $\mathrm{SLE}_{8/3}$ is conjectured to be the scaling limit of self-avoiding random walks. The SLE_3 is proved [52] to be the limit of interfaces for the Ising model (another Fields Medal 2010 awarded to Stanislav Smirnov), while the SLE_4 corresponds to the harmonic explorer and the GFF. For all $0 \le k \le 4$ SLE gives slit maps. The SLE_6 was used by Lawler, Schramm, and Werner in 2001 [157] to prove the conjecture of Mandelbrot (1982) that the boundary of planar Brownian motion has fractal dimension $4/3$. Moreover, Smirnov [240] proved the SLE_6 is the scaling limit of critical site percolation on the triangular lattice. This result follows from his celebrated proof of Cardy's formula. SLE_8 corresponds to the uniform spanning tree. For $4 < k < 8$ the curve intersects itself and every point is contained in a loop but the curve is not space-filling almost surely. For $k \ge 8$ the curve is almost sure space-filling. This phase change is due to the Bessel process interpretation of SLE, see (42).



An invariant approach to SLE starts with probability measures on non-self-crossing random curves in a domain Ω connecting two given points $a, b \in \partial\Omega$ and satisfying the properties of

- *Conformal invariance*. Consider a triple (Ω, a, b) and a conformal map ϕ. If γ is a trace $\mathrm{SLE}_k(\Omega, a, b)$, then $\phi(\gamma)$ is a trace $\mathrm{SLE}_k(\phi(\Omega), \phi(a), \phi(b))$.;
- *Domain Markov property*. Let $\{\mathscr{F}_t\}_{t\ge 0}$ be the filtration in $\mathscr{F}$ by $\{B_t\}_{t\ge 0}$ and let g_t be a Löwner flow generated by $\xi(t) = \sqrt{k}B_t$. Then the hulls $(g_t(K_{s+t} \cap \mathbb{H}_t) - \xi_t)_{s\ge 0}$ are also generated by SLE_k and independent of the sigma-algebra $\mathscr{F}$.

The expository paper [155] is perhaps the best option to start an exploration of this fascinating branch of mathematics. Nice papers [216, 217] give up-to-date exposition of developments of SLE so we do not intend to survey SLE in detail here. Rather, we are going to show relations with CFT and other related stochastic variants of Löwner (generalized) evolution.

Here let us also mention the work by Carleson and Makarov [49] studying growth processes motivated by diffusion-limited aggregation (DLA) via Löwner's equations.

In this section we review the connections between conformal field theory (CFT) and Schramm–Löwner evolution (SLE) following, e.g., [24, 82, 130]. Indeed, SLE, being, e.g., a continuous limit of CFT's archetypical Ising model at its critical point, gives an approach to CFT which emphasizes CFT's roots in statistical physics.

Equation (41) is deterministic with a random entry and we solve it for every fixed ω. The corresponding stochastic differential equation (SDE) in the Itô form for the function $h_t(z) = g_t(z) - \xi(t)$ is

$$dh_t(z) = \frac{2}{h_t(z)}dt - \sqrt{k}dB_t, \tag{42}$$

where $-h_t/\sqrt{k}$ represents a Bessel process (of order $(4+k)/k$). For any holomorphic function $M(z)$ we have the Itô formula

$$(dM)(h_t) = -d\xi_t \mathscr{L}_{-1} M(h_t) + dt(\frac{k}{2}\mathscr{L}_{-1}^2 - 2\mathscr{L}_{-2})M(h_t), \tag{43}$$

where $\mathscr{L}_n = -z^{n+1}\partial$. From the form of (42) one can see immediately that h_t is a (time-homogeneous) diffusion, i. e., a continuous strong Markov process. The infinitesimal generator of h_t is given by $A = (\frac{k}{2}\mathscr{L}_{-1}^2 - 2\mathscr{L}_{-2})$ and this operator appears here for the first time. This differential operator makes it possible to reformulate many probabilistic questions about h_t in the language of PDE theory. If we consider $h_t(z)$ with fixed z, then (42) for h_t describes the motion of particles in the time-dependent field v with $dv = -d\xi_t \mathscr{L}_{-1} + dtA$. For instance, if we denote by $u_t(z)$ the mean function of $h_t(z)$

$$u_t(z) = \mathsf{E}h_t(z),$$

then it follows from Kolmogorov's backward equation that u_t satisfies

$$\begin{cases} \frac{\partial u_t}{\partial t} = Au_t, \\ u_0(z) = z, \end{cases} \quad z \in \mathbb{H}.$$

The kernel of the operator A describes driftless observables with time-independent expectation known as local martingales or conservation (in mean) laws of the process.

13.2 SLE and CFT

A general picture of the connections between SLE and CFT can be viewed as follows. The axiomatic approach to CFT grew up from the Hilbert sixth problem [258], and the Euclidean axioms were suggested by Osterwalder and Schrader [181].

They are centered around group symmetry, relative to unitary representations of Lie groups in Hilbert space. They define first *correlators* (complex values amplitudes) dependent on n complex variables, and a Lie group of conformal transformations of the correlators under the Möbius group. This can be extended at the infinitesimal level of the Lie algebra to invariance under infinitesimal conformal transforms, and therefore, to the so-called Ward identities. The adjoint representation of this group is given with the help of the enveloping algebra of an algebra of special operators, that act on a Fock space of *fields*. The existence of fields and relation between correlators and fields are given by a *reconstruction theorem*, see, e.g., [229]. The boundary version of this approach BCFT, i.e., CFT on domains with boundary, was developed by Cardy [48]. SLE approach starts with a family of statistical fields generated by nonrandom central charge modification of the random fields defined initially by the *GFF* and the algebra of Fock space fields, and then, defines SLE correlators, which turn to be local martingales after coupling of modified GFF on SLE random domains. These correlators satisfy the axiomatic properties of BCFT in which the infinitesimal boundary distortion leads to Ward identities involving a special boundary changing operator of conformal dimension that depends on the amplitude $\sqrt{k}$ of the Brownian motion in SLE.



N. G. Makarov

Connections between SLE and CFT were considered for the first time by Bauer and Bernard [23]. General motivation was as follows. Belavin et al. [34] defined in 1984 a class of conformal theories "minimal models," which described some discrete models (Ising, Potts, etc.) at criticality. Central theme is *universality*, i.e., the properties of a system close to the critical point are independent of its microscopic realization. Universal classes are characterized by a special parameter, central charge. Schramm's approach is based on a special evolution of conformal maps describing possible candidates for the scaling limit of interface curves. How these approaches are related? BPZ conjectured that the behavior of the system at criticality should be described by critical exponents identified as highest weights of degenerate representations of infinite dimensional Lie algebras, Virasoro in our case.

In order to make a bridge to CFT, let us address the definition of *fields* under consideration, which will be in fact *Fock space fields* and nonrandom *martingale-observables* following Kang and Makarov exposition [130], see also [221] for physical encouragement. The underlying idea is to construct the representation Fock space based on the GFF. GFF is a particular case of the Lévy Brownian motion (1947) [160] as a space extension of the classical Brownian motion. Nelson (1973) [178] considered relations of Markov properties of generalized random fields and QFT, in particular, he proved that GFF possesses the Markov property. Albeverio and Høegh-Krohn [6] and Röckner [219, 220] proved the Markov property of measure-indexed GFF and revealed relations of GFF to potential theory. Difficulty in the approach described below comes from the fact that in the definition of the Markov property, the domain of reference is chosen to be random which requires a special coupling between random fields and domains, which was realized by Schramm and Sheffield [232] and Dubédat [69].

GFF $\Phi(z)$ is defined on a simply connected domain D with the Dirichlet boundary conditions, i.e., GFF is indexed by the Hilbert space $\mathscr{E}(D)$, the completion of test functions $f \in C_0^\infty(D)$ (with compact support in D) equipped with the norm

$$\|f\|^2 = \int_D \int_D f(\zeta)\bar{f}(z)G(z,\zeta)d\zeta dz,$$

where $G(z,\zeta)$ is the Green function of the domain D, so $\Phi\colon \mathscr{E}(D) \to L^2(\Omega)$. For example, if $\mathbb{H} = D$, then $G(z,\zeta) = \frac{1}{2\pi}\log\left|\frac{\zeta-\bar{z}}{\zeta-z}\right|$. One can think of GFF as a generalization of the Brownian motion to complex time; however, such analogy is very rough since for a fixed $z \in D$ the expression $\Phi(z)$ is not well defined as a random variable, e.g., the correlation is $\mathsf{E}(\Phi(z)\Phi(\zeta)) = G(z,\zeta)$, but the variance does not exist in a usual sense. Instead, in terms of distributions (Φ, f) over the space $\mathscr{E}(D)$ we have covariances

$$\mathrm{Cov}((\Phi, f_1), (\Phi, f_2)) = \frac{1}{2\pi}\int_D \int_D f_1(\zeta)\bar{f}_2(z)G(z,\zeta)d\zeta dz.$$

The distributional derivatives $J = \partial\Phi$ and $\bar{J} = \bar{\partial}\Phi$ are well defined as, e.g., $J(f) = -\Phi(\partial f)$, $J\colon \mathscr{E}(D) \to L^2(\Omega)$. They are again Gaussian distributional fields.

The tensor nth symmetric product of Hilbert spaces $\mathscr{H}(D)$ we denote by $\mathscr{H}^{\odot n}$, $\mathscr{H}^{\odot 0} = \mathbb{C}$ and the Fock symmetric space is defined as the direct sum $\bigoplus_{n=0}^\infty \mathscr{H}^{\odot n}$. Here the sign $\odot$ of the Wick product (defined below) denotes an isomorphism to the symmetric tensor algebra multiplication. Gian Carlo Wick (1909–1992) introduced originally the product $:\cdot:\equiv\odot$ in [256] in 1950, in order to provide useful information of infinite quantities in Quantum Field Theory. In physics, Wick product is related to the normal ordering of operators over a representation space, namely, in terms of annihilation and creation operators all the creation operators appear to the left of all annihilation operators. The Wick product $\odot$ of random

variables x_j is a random variable, commutative, and is defined formally as $:\ := 1$, $\frac{\partial}{\partial x_j}(x_1 \odot \cdots \odot x_n) = (x_1 \odot \cdots \odot x_{j-1} \odot x_{j+1} \odot \cdots \odot x_n)$, $\mathsf{E}(x_1 \odot \cdots \odot x_n) = 0$ for any $k \geq 1$. For example, $: x := \odot x = x - \mathsf{E}(x)$. However, we have to understand that $\Phi(z)$ and its derivatives are not random variables in the usual sense and must be think of as distributional random variables. Besides, $\odot\Phi = \Phi$. The mean of the field $\mathsf{E}\Phi$ is harmonic.

The *Fock space correlation functionals* are defined as span of basis correlation functionals 1 and $X(z_1) \odot \cdots \odot X(z_n)$, $z_1, \ldots, z_n \in D$, $X_k = \partial^\alpha \bar\partial^\beta \Phi$ as well as infinite combinations (exponentials) which will play a special role in the definition of the vertex operator. We have $\partial\bar\partial\Phi \approx 0$, i.e., $\mathsf{E}((\partial\bar\partial\Phi(z)Y(\zeta)) = 0$, for any functional $Y(\zeta)$, $\zeta \neq z$. In view of this notation $\Delta\Phi \approx 0$.

The *basis Fock space fields* are formal Wick products of the derivatives of the GFF 1, Φ, $\Phi \odot \Phi$, $\partial\Phi \odot \Phi$, etc. Again, since GFF and its derivatives are understood in distributional sense, the above Wick product is formal. A *Fock space field* F_D is a linear combination of basis Fock space fields over the ring of smooth functions in D. If $F_1, \ldots, F_n$ are Fock space fields and $z_1, \ldots, z_n$ are distinct points of D, then $F_1(z_1) \ldots F_n(z_n)$ is a correlation functional and $\mathsf{E}(F_1(z_1) \ldots F_n(z_n))$ is a *correlation function* of simply correlator. The product here is thought of as a tensor product defined by the Wick formula over Feynman diagrams.

Being formally defined all these objects can be recovered through approximation by well-defined objects (scaling limit of lattice GFF) and expectations and correlators can be calculated.

We continue specifying the Markov property of domains and fields. For a shrinking deterministic subordination $D(t) \subset D(s)$ for $t < s$, the Markov property of $\{D(t)\}_{t\geq 0}$ means that domains $D(t)\big|D(s)$ and D_{t-s} coincide in law, which simply means in the Löwner case that if $\varphi_t \colon \mathbb{H}_t \to \mathbb{H}$ preserves ∞, and such that the tip $\gamma(t) \to 0$, then $\varphi_s \circ g_t$ and g_{t-s} satisfy the same equation fixing s. In the random case we have that the domains $D(t)\big|D(s)$ and D_{t-s} coincide in law in the sense of distribution, or $\mathrm{Law}(D(s)\big|D(t)) = \mathrm{Law}(D_{s-t})$. For more thorough treatment of the Markov property of domains, see [156].

In order to formulate the Markov property for fields and domains, we coming at coupling of $\{D(t)\}_{t\geq 0}$ SLE and GFF to be defined on the same probability space $(\Omega, \mathscr{G}, (\mathscr{G}_t), P)$, see [69, 231].

Let us start with a toy example of a classical real-valued field F_D defined on a simply connected domain D. Similarly to the Markov property of g_t, we say that F possesses the Markov property if for a decreasing t-dependent family of domains D_t, $F_{D_t}\big|D_s(z) = F_{D_{s-t}}$, where $0 < s < t$, $D_0 = D$. If in (41) we put $k = 0$, then the family of domains $\mathbb{H}_t$ is the family $D_t = \mathbb{H} \setminus \gamma(t)$, $\gamma(t) = (0, 2i\sqrt{t}]$. As a trivial example, construct the field $F_{D_t}(z) = \arg \varphi_t'(z)$, where $\varphi_t = z^2 + 4t^2$. The function φ_t maps D_t onto $\mathbb{C} \setminus [0, \infty)$ and F_{D_t} satisfies the Markov property.

A collection $\mathscr{F} = \{F_k\}$ of random holomorphic Fock space fields defined on D_t is said to satisfy the Markov property, if the process

$$M_t(z_1, \ldots, z_n) := \mathsf{E}[F_{t1}(z_1) \ldots F_{tn}(z_n)\big|D_t],$$

is a local martingale. Here $F_t(z_j)$ is push-forward of $F(z_j)$ with respect to a conformal map $D \to D_t$. That is, let $\mathrm{Aut}(D, p, q)$ is the group of conformal automorphisms of D fixing $p, q \in \partial D$ ($0, \infty$ for $\mathbb{H}$ or $\gamma(t), \infty$ for $\mathbb{H}_t$). The first requirement is that F is invariant under $\mathrm{Aut}(D, p, q)$. The second is that if $\varphi_t \colon D_t \to D$ is a conformal map fixing q, then primary fields F are $(\lambda, \bar{\lambda})$-forms, i.e., $F_t(z) = (\varphi_t')^{\lambda}(\overline{\varphi_t'})^{\bar{\lambda}} F(\varphi_t(z))$. A Schwarzian form F satisfies $F_t(z) = (\varphi_t')^2 F(\varphi_t(z)) + \mu S_{\varphi_t}(z)$, etc.

The space of conformal Fock space fields is a graded commutative differential algebra over the ring of nonrandom conformal fields. Conformal invariance in D is assumed for nonrandom conformal fields, and for random conformal fields this means that all correlations are conformally invariant as nonrandom conformal fields. Observe that all definitions can be given for Riemann surfaces as in [130]. The nonrandom field $M(z_1, \dots, z_n) := M_0(z_1, \dots, z_n)$ is called an *SLE martingale-observable*.

Now we want to use some Fock space fields as *states* and others as *observables* acting on states by *operator product expansion* (OPE). OPE is defined being based on the expansion for GFF

$$\Phi(\zeta)\Phi(z) = \log \frac{1}{|\zeta - z|^2} + 2 \log R_D(z) + \Phi(z) \odot \Phi(z) + o(1), \quad \text{as } \zeta \to z,$$

where $R_D(z)$ is the conformal radius of D with respect to z. The product $\Phi(\zeta)\Phi(z)$ is defined as a tensor product and is given by the Wick formula, see, e.g., [183, Sect. 4.3]. A Fock space field $F(z)$ is called holomorphic if $\mathsf{E}(F(z)Y(\zeta))$ is a holomorphic function of z for any field $Y(\zeta)$, $z \neq \zeta$. The OPE of a holomorphic field is given as a Laurent series

$$F(\zeta)Y(z) = \sum_{n \in \mathbb{Z}} C_n(z)(\zeta - z)^n, \quad \text{as } \zeta \to z.$$

Obviously, OPE is neither commutative nor associative unless one of the fields is nonrandom. The coefficients C_n are also Fock space fields. We denote $F * Y = C_0$ and $F *_{(n)} Y = C_n$ for holomorphic Fock space fields. This product satisfies the Leibniz rule.

The complex Virasoro algebra was introduced in Sect. 12.1. Let us define a special field, Virasoro field of GFF, $T(z)$ by the equality $T = -\frac{1}{2} J * J$. In particular,

$$J(\zeta)J(z) = -\frac{1}{(\zeta - z)^2} - 2T(z) + o(1), \quad \text{as } \zeta \to z \text{ in } \mathbb{H},$$

where $J(z) = \partial\Phi(z)$.

Let us define a nonrandom modification $\Phi_{(b)}$ on a simply connected domain D of the GFF $\Phi_{(0)} = \Phi$ on D parametrized by a real constant b, $\Phi_{(b)} = \Phi_{(0)} - 2b \arg \varphi'$,

where $\varphi\colon D \to D$ fixing a point $\in \partial D$. Then $J_{(b)} = J_{(0)} + i\,b\,\frac{\varphi''}{\varphi'}$ is a pre-Schwarzian form and $T_{(b)} = T_{(0)} - b^2 S_\varphi$ is a Schwarzian form. The modified families of Fock space fields $\mathscr{F}_{(b)}$ have the central charge $c = 1-12b^2$. In order to simplify notations we omit subscript writing simply $T := T_{(b)}$, $J := J_{(b)}$, etc.

The field T becomes a stress–energy tensor in Quantum Field Theory and satisfies the equality

$$T(\zeta)T(z) = \frac{c/2}{(\zeta - z)^4} + \frac{2T(z)}{(\zeta - z)^2} + \frac{\partial T(z)}{\zeta - z} + o(1),$$

It is not a primary field and changes under conformal change of variables as $T(\zeta') = T(\zeta) - \frac{c}{12} S_{\zeta'}(\zeta)$, where $S_{\zeta'}(\zeta)$, as usual, is the Schwarzian derivative of $\zeta'(\zeta)$.

The Virasoro field has the expansion

$$T(\zeta)X(z) = \sum_{n\in\mathbb{Z}} \frac{L_n X(z)}{(\zeta - z)^{n+2}},$$

for any Fock space field $X(z)$. This way the Virasoro generators act on a field X: $L_n X = T *_{(-n-2)} X$ and can be viewed as operators over the space of Fock space fields. The result is again a Fock space field and one can define iteratively the field $L_{n_k} L_{n_{k-1}} \dots L_{n_1} X \equiv L_{n_k} L_{n_{k-1}} \dots L_{n_1} |X\rangle$, understanding $L_j := L_j *$ as operators acting on a "vector" X, where we adapt Dirac's notations "bra" and "ket" for vectors, operators and correlators. So we obtained a representation of the Virasoro algebra on Fock space fields

$$[L_n, L_m] = L_n * L_m - L_m * L_n = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n,-m}, \quad n, m \in \mathbb{Z}.$$

Let us recall that a field X is called *primary* of conformal weight $(\lambda, \bar\lambda)$ if $X(z) = X(\phi(z))(\phi')^\lambda(\bar\phi')^{\bar\lambda}$. The *Virasoro primary field* $|V\rangle$ of conformal dimension λ, or $((\lambda, \bar\lambda)$, is defined to satisfy $L_n|V\rangle = 0$, $n > 0$, $L_0|V\rangle = \lambda|V\rangle$ and $L_{-1}|V\rangle = \partial|V\rangle$. Analogously, the conjugate part of this definition is valid for the dimension $\bar\lambda$. Further on we omit the conjugate part because of its complete symmetry with the non-conjugate one. The *Virasoro–Verma module* $\mathscr{V}(\lambda, c)$ is constructed spanning and completing the basis vectors $L_{n_k} L_{n_{k-1}} \dots L_{n_1} |V\rangle$, where $n_k < n_{k-1} < \dots < n_1 < 0$, and $|V\rangle$ is taken to be the highest weight vector. We have a decomposition

$$\mathscr{V}(\lambda, c) = \bigoplus_{m=0}^{\infty} \mathscr{V}_m(\lambda, c),$$

where the level m space $\mathscr{V}_m(\lambda, c)$ is the eigenspace of L_0 with eigenvalue $\lambda + m$. A singular vector $|X\rangle$ by definition lies in some $\mathscr{V}_m(\lambda, c)$ and $L_n|X\rangle = 0$ for any $n > 0$. The Virasoro–Verma module is generically irreducible, having only

singular-generated submodules. The Virasoro primary field becomes the highest weight vector in the representation of the Virasoro algebra $\mathfrak{vir}_{\mathbb{C}}$.

Proposition 7. *Let $|V\rangle$ be a Virasoro primary field in $\mathscr{F}$ of conformal dimension λ with central charge c. Then the field*

$$[L_{-m} + \eta_1 L_{-1}[L_{-m+1} + \eta_2 L_{-1}[L_{-m+2} + \cdots + \eta_{m-2} L_{-1}[L_{-2} + \eta_{m-1} L_{-1}^2] \dots]|V\rangle$$

is a primary singular field of dimension $(\lambda + m)$, if and only if $\eta_1, \dots, \eta_{m-1}$, c, and λ satisfy a system of m linear equations.

For example, if $m = 2$, then the singular field is

$$[L_{-2} + \eta_1 L_{-1}^2]|V\rangle, \tag{44}$$

and

$$\begin{cases} 3 + 2\eta_1 + 4\eta_1\lambda = 0, \\ c + 8\lambda + 12\eta_1\lambda = 0, \end{cases}$$

if $m = 3$, then the singular field is $[L_{-3} + \eta_2 L_{-1} L_{-2} + \eta_1\eta_2 L_{-1}^3]|V\rangle$ and

$$\begin{cases} 2 + (\lambda + 2)\eta_2 = 0, \\ 1 + 4(\lambda + 1)\eta_1 = 0, \\ 5 + (4\lambda + \frac{c}{2} + 3)\eta_2 + 6(3\lambda + 1)\eta_1\eta_2 = 0. \end{cases}$$

Now we consider the "holomorphic part" of GFF $\phi(z) = \int^z J(z)$, where $J(z) = \partial\Phi$. Of course the definition requires more work because the field constructed this way is ramified at all points. However, in correlation with a Fock space functional, the monodromy group is well-defined and finitely generated. A vertex operator is a field $\mathscr{V}_\star^\alpha(z) = e^{*i\alpha\phi(z)}$. It is a primary field of conformal dimension $\lambda = \frac{\alpha^2}{2} + \alpha\sqrt{\frac{1-c}{12}}$. Considering an infinitesimal boundary distortion $w_\varepsilon(z) = z + \frac{\varepsilon}{z-\xi} + o(\varepsilon)$ is equivalent to the insertion operator $|X(z)\rangle \to T(\xi)|X(z)\rangle$. The Ward identity implies the BCFT Cardy equation

$$(\mathscr{L}_{-1}^2 - 2\alpha^2\mathscr{L}_{-2})\mathsf{E}\left[\mathscr{V}_\star^\alpha(\xi)|X(z)\rangle\right] = 0, \quad \text{for } \alpha\left(\alpha + \sqrt{\frac{1-c}{12}}\right) = 1. \tag{45}$$

Here the representation of the Virasoro algebra is

$$\mathscr{L}_n = -(z-\xi)^{n+1}\partial - \lambda(n+1)(z-\xi)^n.$$

Merging the form of the infinitesimal generator A in (43), Virasoro primary singular field (44) and the Cardy equation (45) we arrive at the SLE numerology $\eta = -2\alpha^2$, $\alpha = \frac{\sqrt{k}}{2}$,

$$c = \frac{(6-k)(3k-8)}{2k}, \quad \lambda = \frac{6-k}{2k},$$

with the unique free parameter k.

Given a Fock space field $X(z)$, its push-forward $X_t(z)$ generically does not possess the Markov property. However, summarizing above, we can formulate the following statement.

Proposition 8. *For SLE evolution we have*

- $\mathscr{V}^\alpha_\star$ *is a Virasoro primary field;*
- $V = \mathscr{V}^\alpha_{\star,t}(\xi_t)X_t(z)$ *possesses the Markov property,* $\xi_t = \sqrt{k}\,B_t$*;*
- $M(z) = \mathsf{E}\left[\mathscr{V}^\alpha_\star(0)X(z)\right]$ *is a one-point martingale-observable;*
- *The process* $M_t(z) = \mathsf{E}\left[\mathscr{V}^\alpha_{\star,t}(\xi_t)X_t(z)\middle|D_t\right]$ *is a one-point martingale;*
- $(\frac{k}{2}L^2_{-1} - 2L_{-2})|\mathscr{V}^\alpha_\star\rangle$ *is Virasoro primary singular field of level 2;*
- $(\frac{k}{2}\mathscr{L}^2_{-1} - 2\mathscr{L}_{-2})M(z) = 0$ *is the Cardy equation for the SLE martingale observables.*

As a simple example, consider the Fock space field $V = \mathscr{V}^\alpha_\star(0)T(z)$. Then $M(z) = \mathsf{E}\left[\mathscr{V}^\alpha_\star(0)T(z)\right]$ is a one-point martingale-observable and $M(z) = 1/z^2$. If $k = 8/3$, then $M_t(z) = (h_t'(z)/h_t(z))^2$ is a local martingale.

More general construction including multi-point observables requires more technical work related to a vertex field V and boundary condition changing operators, see [130, Sect. 8.4]. Examples of martingale observables were found, e.g., by Friedrich and Werner [82] and Schramm and Sheffield [231, 232]. A radial version of SLE and relations to conformal field theory one can find in [131].

Another construction of the stress–energy tensor of CFT comes as a local observable of the conformal loop ensemble (CLE) (see, e.g., [234]) loops for any central charge, see [65, 66]. More general construction is performed on a groupoid of conformal maps of a simply connected domain, a natural generalization of the finite dimensional conformal group. The underlying manifold structure is Fréchet. Similarly to moduli (Teichmüller) spaces, the elements of the cotangent bundle are analogues of quadratic differentials, see [67]. It is shown there that the stress–energy tensor of CFT is exactly such a quadratic differential.

13.3 Generalized Löwner–Kufarev Stochastic Evolution

Another attempt to construct random interfaces different from SLE has been launched by conformal welding in [21].

We considered a setup [126] in which the sample paths are represented by the trajectories of a point (e.g., the origin) in the unit disk $\mathbb{D}$ evolving randomly under

the generalized Löwner equation. The driving mechanism differs from SLE. In the SLE case the Denjoy–Wolff attracting point (∞ in the chordal case or a boundary point of the unit disk in the radial case) is fixed. In our case, the attracting point is the driving mechanism and the Denjoy–Wolff point is different from it. Relation with this model to CFT is the subject of a forthcoming paper. Let us consider the generalized Löwner evolution driven by a Brownian particle on the unit circle. In other words, we study the following initial value problem.

$$\begin{cases} \frac{d}{dt}\phi_t(z,\omega) = \frac{(\tau(t,\omega)-\phi_t(z,\omega))^2}{\tau(t,\omega)}\, p(\phi_t(z,\omega),t,\omega), \\ \phi_0(z,\omega) = z, \end{cases} \qquad t \geq 0,\ z \in \mathbb{D},\ \omega \in \Omega. \tag{46}$$



The function $p(z,t,\omega)$ is a Herglotz function for each fixed $\omega \in \Omega$. In order for $\phi_t(z,\omega)$ to be an Itô process adapted to the Brownian filtration, we require that the function $p(z,t,\omega)$ is adapted to the Brownian filtration for each $z \in \mathbb{D}$. Even though the driving mechanism in our case differs from that of SLE, the generated families of conformal maps still possess the important time-homogeneous Markov property.

For each fixed $\omega \in \Omega$, (46) similarly to SLE, may be considered as a deterministic generalized Löwner equation with the Berkson–Porta data $(\tau(\cdot,\omega), p(\cdot,\cdot,\omega))$. In particular, the solution $\phi_t(z,\omega)$ exists, is unique for each $t > 0$ and $\omega \in \Omega$, and moreover, is a family of holomorphic self-maps of the unit disk.

The equation in (46) is an example of a so-called *random differential equation* (see, for instance, [241]). Since for each fixed $\omega \in \Omega$ it may be regarded as an ordinary differential equation, the sample paths $t \mapsto \phi_t(z,\omega)$ have continuous first derivatives for almost all ω. See an example of a sample path of $\phi_t(0,\omega)$ for $p(z,t) = \frac{\tau(t)+z}{\tau(t)-z}$, $\tau(t) = e^{ikB_t}$, $k = 5$, $t \in [0,30]$ in the figure to the left.

In order to give an explicitly solvable example let $p(z,t,\omega) = \frac{\tau(t,\omega)}{\tau(t,\omega)-z} = \frac{e^{ikB_t(\omega)}}{e^{ikB_t(\omega)}-z}$. It makes (46) linear:

$$\frac{d}{dt}\phi_t(z,\omega) = e^{ikB_t(\omega)} - \phi_t(z,\omega),$$

and a well-known formula from the theory of ordinary differential equation yields

$$\phi_t(z,\omega) = e^{-t}\left(z + \int_0^t e^s e^{ikB_s(\omega)} ds\right).$$

Taking into account the fact that $\mathsf{E}e^{ikB_t(\omega)} = e^{-\frac{1}{2}tk^2}$, we can also write the expression for the mean function $\mathsf{E}\phi_t(z,\omega)$

$$\mathsf{E}\phi_t(z,\omega) = \begin{cases} e^{-t}(z+t), & k^2 = 2, \\ e^{-t}z + \frac{e^{-tk^2/2}-e^{-t}}{1-k^2/2}, & \text{otherwise.} \end{cases} \tag{47}$$

Thus, in this example all maps ϕ_t and $\mathsf{E}\phi_t$ are affine transformations (compositions of a scaling and a translation).

In general, solving the random differential equation (46) is much more complicated than solving its deterministic counterpart, mostly because of the fact that for almost all ω the function $t \mapsto \tau(t,\omega)$ is nowhere differentiable.

If we assume that the Herglotz function has the form $p(z,t,\omega) = \tilde{p}(z/\tau(t,\omega))$, then it turns out that the process $\phi_t(z,\omega)$ has an important invariance property, that were crucial in development of SLE.

Let $s > 0$ and introduce the notation

$$\tilde{\phi}_t(z) = \frac{\phi_{s+t}(z)}{\tau(s)}.$$

Then $\tilde{\phi}_t(z)$ is the solution to the initial-value problem

$$\begin{cases} \frac{d}{dt}\tilde{\phi}_t(z,\omega) = \frac{\left(\tilde{\tau}(t,\omega)-\tilde{\phi}_t(z,\omega)\right)^2}{\tilde{\tau}(t,\omega)}\,\tilde{p}\left(\tilde{\phi}_t(z,\omega)/\tilde{\tau}(t)\right), \\ \tilde{\phi}_0(z,\omega) = \phi_s(z,\omega)/\tilde{\tau}(s), \end{cases}$$

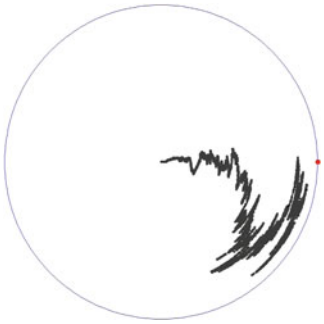
where $\tilde{\tau}(t) = \tau(s+t)/\tau(s) = e^{ik(B_{s+t}-B_s)}$ is again a Brownian motion on $\mathbb{T}$ (because $\tilde{B}_t = B_{s+t} - B_s$ is a standard Brownian motion). In other words, the conditional distribution of $\tilde{\phi}_t$ given ϕ_r, $r \in [0,s]$ is the same as the distribution of ϕ_t.

By the complex Itô formula, the process $\frac{1}{\tau(t,\omega)} = e^{-ikB_t}$ satisfies the equation

$$de^{-ikB_t} = -ike^{-ikB_t}dB_t - \frac{k^2}{2}e^{-ikB_t}dt.$$

Let us denote $\frac{\phi_t(z,\omega)}{\tau(t,\omega)}$ by $\Psi_t(z,\omega)$. Applying the integration by parts formula to Ψ_t, we arrive at the following initial value problem for the Itô SDE

$$\begin{cases} d\Psi_t = -ik\Psi_t dB_t + \left(-\frac{k^2}{2}\Psi_t + (\Psi_t - 1)^2 p(\Psi_t e^{ikB_t(\omega)}, t, \omega)\right)dt, \\ \Psi_0(z) = z. \end{cases} \tag{48}$$



A numerical solution to this equation for a specific choice for $p(z,t) \equiv i$, $k = 1$, and $t \in [0,2]$, is shown in the figure to the right.

Analyzing the process $\frac{\phi_t(z,\omega)}{\tau(t,\omega)}$ instead of the original process $\phi_t(z,\omega)$ is in many ways similar to one of the approaches used in SLE theory.

The image domains $\Psi_t(\mathbb{D},\omega)$ differ from $\phi_t(\mathbb{D},\omega)$ only by rotation. Due to the fact that $|\Psi_t(z,\omega)| = |\phi_t(z,\omega)|$, if we compare the processes $\phi_t(0,\omega)$ and $\Psi_t(0,\omega)$, we note that their first hit times of the circle $\mathbb{T}_r$ with radius $r < 1$ coincide, i. e.,

$$\inf\{t \geq 0, |\phi_t(0,\omega)| = r\} = \inf\{t \geq 0, |\Psi_t(0,\omega)| = r\}.$$

In other words, the answers to probabilistic questions about the expected time of hitting the circle $\mathbb{T}_r$, the probability of exit from the disk $\mathbb{D}_r = \{z : |z| < r\}$, etc. are the same for $\phi_t(0,\omega)$ and $\Psi_t(0,\omega)$.

If the Herglotz function has the form $p(z,t,\omega) = \tilde{p}(z/\tau(t,\omega))$, then (48) becomes

$$\begin{cases} d\Psi_t = -ik\Psi_t dB_t + \left(-\frac{k^2}{2}\Psi_t + (\Psi_t - 1)^2\tilde{p}(\Psi_t)\right)dt, \\ \Psi_0(z) = z, \end{cases} \tag{49}$$

and may be regarded as an equation of a two-dimensional time-homogeneous real diffusion written in complex form. This implies, in particular, that Ψ_t is a time-homogeneous strong Markov process. By construction, $\Psi_t(z)$ always stays in the unit disk.

Analogously to SLE we can consider random conformal Fock space fields defined on $\mathbb{D}$ changing correspondingly the definition using the Green function for $\mathbb{D}$ instead of $\mathbb{H}$. Coupling of (46) and the GFF in $\mathbb{D}$ we define the correlators $f_t(z_1,\ldots,z_n) = f(z_1,\ldots,z_n)\big|_{\mathbb{D}_t}$ as martingale-observables.

For a smooth function $f(z)$ defined in $\mathbb{D}$ we derive the Itô differential in the complex form

$$df(\Psi_t) = -ikdB_t(L_{-1} - \bar{L}_{-1})f(\Psi_t) + dtAf(\Psi_t),$$

where A is the infinitesimal generator of Ψ_t,

$$A = \left(-\frac{k^2}{2}z + (z-1)^2\tilde{p}(z)\right)\frac{\partial}{\partial z} - \frac{1}{2}k^2z^2\frac{\partial^2}{\partial z^2} + \left(-\frac{k^2}{2}\bar{z} + (\bar{z}-1)^2\overline{\tilde{p}(z)}\right)\frac{\partial}{\partial \bar{z}} - \frac{1}{2}k^2\bar{z}^2\frac{\partial^2}{\partial \bar{z}^2} + k^2|z|^2\frac{\partial^2}{\partial z\partial \bar{z}}.$$

In particular, if f is holomorphic, then

$$A = \left(-\frac{k^2}{2}z + (z-1)^2\tilde{p}(z)\right)\frac{\partial}{\partial z} - \frac{1}{2}k^2z^2\frac{\partial^2}{\partial z^2}. \tag{50}$$

If $f(z)$ is a martingale-observable, then $Af = 0$.

In [126] we proved the existence of a unique stationary point of Ψ_t in terms of the stochastic vector field

$$\frac{d}{dt}\Psi_t(z,\omega) = G_0(\Psi_t(z,\omega)),$$

where the Herglotz vector field $G_0(z,\omega)$ is given by

$$G_0(z,\omega) = -ikzW_t(\omega) - \frac{k^2}{2}z + (z-1)^2\tilde{p}(z).$$

Here, $W_t(\omega)$ denotes a generalized stochastic process known as *white noise*. Also nth moments were calculated and the boundary diffusion on the unit circle was considered, which corresponds, in particular, to North–South flow, see, e.g., [50].

14 Related Topics

14.1 Hele-Shaw Flows, Laplacian Growth

One of the most influential works in fluid dynamics at the end of the nineteenth century was a series of papers, see, e.g., [124] written by *Henry Selby Hele-Shaw* (1854–1941). There Hele-Shaw first described his famous cell that became a subject of deep investigation only more than 50 years later. A Hele-Shaw cell is a device for investigating two-dimensional flow (Hele-Shaw flow or Laplacian growth) of a viscous fluid in a narrow gap between two parallel plates.

This cell is the simplest system in which multi-dimensional convection is present. Probably the most important characteristic of flows in such a cell is that when the Reynolds number based on gap width is sufficiently small, the Navier–Stokes equations averaged over the gap reduce to a linear relation for the velocity similar to Darcy's law and then to a Laplace equation for the fluid pressure. Different driving mechanisms can be considered, such as surface tension or external forces



H. S. Hele-Shaw

(e.g., suction, injection). Through the similarity in the governing equations, Hele-Shaw flows are particularly useful for visualization of saturated flows in porous media, assuming they are slow enough to be governed by Darcy's law. Nowadays, the Hele-Shaw cell is used as a powerful tool in several fields of natural sciences and engineering, in particular, soft condensed matter physics, materials science, crystal growth and, of course, fluid mechanics. See more on historical and scientific account in [253].

The century-long development connecting the original Hele-Shaw experiments, the conformal mapping formulation of the Hele-Shaw flow by *Pelageya Yakovlevna Polubarinova-Kochina* (1899–1999) and *Lev Aleksandrovich Galin* (1912–1981) [83, 187, 188], and the modern treatment of the Hele-Shaw evolution based on integrable systems and on the general theory of plane contour motion, was marked by several important contributions by individuals and groups.

The main idea of Polubarinova-Kochina and Galin was to apply the Riemann mapping from an appropriate canonical domain (the unit disk $\mathbb{D}$ in most situations) onto the phase domain in order to parameterize the free boundary. The evolution equation for this map, named after its creators, allows to construct many explicit solutions and to apply methods of conformal analysis and geometric function theory to investigate Hele-Shaw flows. In particular, solutions to this equation in the case of advancing fluid give subordination chains of simply connected domains which have been studied by Löwner and Kufarev. The resulting equation for the family of functions $f(z,t) = a_1(t)z + a_2(t)z^2 + \dots$ from $\mathbb{D}$ onto domains occupied by viscous fluid is

$$\mathrm{Re}\,[\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}] = \frac{Q}{2\pi}, \quad |\zeta| = 1. \tag{51}$$



P. Ya. Polubarinova-Kochina

The corresponding equation in $\mathbb{D}$ is a first-order integro-PDE

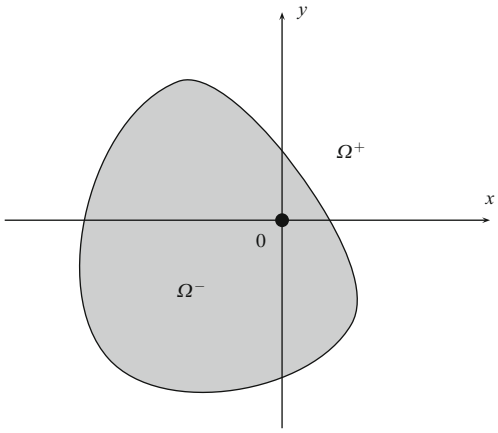
$$\dot{f}(\zeta,t) = \zeta f'(\zeta,t)\int_0^{2\pi} \frac{Q}{4\pi^2|f'(e^{i\theta},t)|^2}\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}d\theta, \quad |\zeta|<1.$$



L. A. Galin

Here Q is positive in the case of injection of negative in the case of suction. The Polubarinova–Galin and the Löwner–Kufarev equations, having some evident geometric connections, are of somewhat different nature. While the evolution of the Laplacian growth given by the Polubarinova–Galin equation is completely defined by the initial conditions, the Löwner–Kufarev evolution depends also on an arbitrary control function. The Polubarinova–Galin equation is essentially nonlinear and the corresponding subordination chains are of rather complicated nature. The treatment of classical Laplacian growth was given in the monograph [114].

The newest direction in the development of Hele-Shaw flow is related to Integrable Systems and Mathematical Physics, as a particular case of a contour dynamics. This story started in 2000 with a short note [174] by Mineev-Weinstein, Wiegmann, and Zabrodin, where the idea of the equivalence of 2D contour dynamics and the dispersionless limit of the integrable 2-D Toda hierarchy appeared for the first time. A list of complete references to corresponding works would be rather long so we only list the names of some key contributors: Wiegmann, Mineev-Weinstein, Zabrodin, Krichever, Kostov, Marshakov, Takebe, Teo et al., and some important references: [140–142, 172, 175, 243, 257]. Let us consider an "exterior" version of



- $M_{-k} = -\int_{\Omega^+} z^{-k} d\sigma_z$;
- $M_0 = |\Omega^-|$;
- $M_k = \int_{\Omega^-} z^k d\sigma_z$;
- $k \geq 1$,
- $t = M_0/\pi$, $t_k = M_k/\pi k$ generalized times.

the process when source/sink of viscous fluid is at ∞ and the bounded domain Ω^- is occupied by the inviscid one. Then the conformal map f of the exterior of the unit disk onto Ω^+

$$f(\zeta, t) = b(t)\zeta + b_0(t) + \sum_{k=1}^{\infty} \frac{b_j(t)}{\zeta^k}, \quad b(t) > 0;$$

satisfies the analogous boundary equation

$$\mathrm{Re}\,[\dot{f}(\zeta, t)\overline{\zeta f'(\zeta, t)}] = -Q, \tag{52}$$

Following the definition of Richardson's moments let us define interior and exterior moments as in the above figure. The integrals for $k = 1, 2$ are assumed to be properly regularized. Then the properties of the moments are

- $M_0(t) = M_0(0) - Qt$ is "physical time";
- M_k are conserved for $k \geq 1$;
- M_0 and $\{M_k\}_{k\geq 1}$ determine the domain Ω^+ locally (given $\partial\Omega^+$ is smooth);

– $\{M_k\}_{k<0}$ evolve in time in a quite complicated manner;
– M_0 and $\{M_k\}_{k\geq 1}$ can be viewed as local parameters on the space of "shapes";

Suppose $\Gamma = \partial\Omega^- = \partial\Omega^+$ is analytic and $\phi(x_1, x_2, t) = 0$ is an implicit representation of the free boundary $\Gamma(t)$. Substituting $x_1 = (z + \bar{z})/2$ and $x_2 = (z - \bar{z})/2i$ into this equation and solving it for $\bar{z}$ we obtain that $\Gamma = \{z : \ S(z,t) = \bar{z}\}$. is given by the *Schwarz function* $S(z,t)$ which is defined and analytic in a neighborhood of Γ. The Schwarz function can be constructed by means of the Cauchy integral

$$g(z) = -\frac{1}{2\pi i}\int\limits_{\partial\Omega}\frac{\bar{\zeta}d\zeta}{\zeta - z}.$$

Define the analytic functions, $g_e(z)$, in Ω^+ and $g_i(z)$, in Ω^-. On $\Gamma = \partial\Omega$ the jump condition is

$$g_e(z) - g_i(z) = \bar{z}, \quad z \in \partial\Omega.$$

Then the Schwarz function $S(z)$ is defined formally by $S(z) = g_e(z) - g_i(z)$. The Cauchy integral implies the Cauchy transform of Ω^+

$$g_e(z) = -\frac{1}{2\pi i}\int\limits_{\partial\Omega}\frac{\bar{\zeta}d\zeta}{\zeta - z} = -\frac{1}{\pi}\int\limits_{\Omega^+}\frac{d\sigma_\zeta}{\zeta - z},$$

with the Laurent expansion

$$g_e(z) = \sum_{k=0}^{\infty}\frac{M_k}{z^{k+1}}, \quad z \sim \infty.$$

Similarly for $g_i(z)$

$$g_i(z) = -\sum_{k=1}^{\infty} M_{-k} z^{k-1}, \quad z \sim 0;$$

So formally

$$S(z) = \sum_{k=-\infty}^{\infty}\frac{M_k}{z^{k+1}} = \sum_{k=1}^{\infty} M_{-k} z^{k-1} + \frac{M_0}{z} + \sum_{k=1}^{\infty}\frac{M_k}{z^{k+1}}.$$

Let us write the logarithmic energy as

$$\mathscr{F}(\Omega^+) = -\frac{1}{\pi^2}\int_{\mathbb{C}}\int_{\mathbb{C}}\log|z - \zeta| d\sigma_z d\sigma_\zeta,$$

where σ_z is a measure supported in Ω^+. It is the potential for the momens

$$\frac{1}{k}M_{-k}(\Omega) = \frac{\partial \mathscr{F}(\Omega)}{\partial M_k}.$$

So (see, e.g., [174]) the moments satisfy the 2-D Toda dispersionless lattice hierarchy

$$\frac{\partial M_{-k}}{\partial t_j} = \frac{\partial M_{-j}}{\partial t_k}, \qquad \frac{\partial M_{-k}}{\partial \bar{t}_j} = \frac{\partial \bar{M}_{-j}}{\partial t_k}.$$

The function $\exp(\mathscr{F}(\Omega^+)) =: \tau(\Omega^+) = \tau(\mathbf{t})$ is the τ-function, and $t = M_0/\pi, \ldots,$ $t_k = M_k/\pi k$ are generalized times.

The real-valued τ-function becomes the solution to the Hirota equation

$$S_{f^{-1}}(z) = \frac{6}{z^2}\sum_{k,n=1}^{\infty} \frac{1}{z^{n+k}} \frac{\partial^2 \log \tau}{\partial t_k \partial t_n},$$

where $z = f(\zeta)$ is the parametric map of the unit disk onto the exterior phase domain and $S_{f^{-1}}(z)$ denotes the Schwarzian derivative of the inverse to f.

$$\frac{M_{-k}}{\pi} = \frac{\partial \log \tau}{\partial t_k}, \qquad \frac{\bar{M}_{-k}}{\pi} = \frac{\partial \log \tau}{\partial \bar{t}_k}, \qquad k \geq 1.$$

The τ-function introduced by the "Kyoto School" as a central element in the description of soliton equation hierarchies.

If $\zeta = e^{i\theta}$, $M_0(t) = M_0(0) - Qt$, then the derivatives are

$$\frac{\partial f}{\partial \theta} = i\zeta \frac{\partial f}{\partial \zeta}, \quad \frac{\partial f}{\partial t} = Q\frac{\partial f}{\partial M_0}.$$

Let $f^*(\zeta) = \overline{f(1/\bar{\zeta})}$, and let

$$\{f, g\} = \zeta \frac{\partial f}{\partial \zeta}\frac{\partial g}{\partial M_0} - \zeta \frac{\partial g}{\partial \zeta}\frac{\partial f}{\partial M_0}.$$

In view of this the Polubarinova–Galin equation (52) can be rewritten as $\{f, f^*\} = 1$. This equation is known as the *string constraint*. The equation for the τ-function with a proper initial condition provided by the string equation solves the inverse moments problem for small deformations of a simply connected domain with analytic boundary. Indeed, define the Schwarzian derivative $S_F = \frac{F'''}{F''} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2$, $F = f^{-1}$. If we know the Schwarzian derivative $S(\zeta)$, we know

the conformal map $w = F(\zeta) = \eta_1/\eta_2$ normalized accordingly, where η_1 and η_2 are linearly independent solutions to the Fuchs equation

$$w'' + \frac{1}{2}S(\zeta)w = 0.$$

The connection extends to the Lax–Sato approach to the dispersionless 2-D Toda hierarchy. In this setting it is shown that a Laurent series for a univalent function that provides an invertible conformal map of the exterior of the unit circle to the exterior of the domain can be identified with the Lax function. The τ-function appears to be a generating function for the inverse map. The formalism allows one to associate a notion of τ-function to the analytic curves.

The conformal map

$$f(\zeta, t) = b(t)\zeta + b_0(t) + \sum_{k=1}^{\infty} \frac{b_j(t)}{\zeta^k}, \quad b(t) > 0$$

obeys the relations

$$\frac{\partial f}{\partial t_k} = \{H_k, f\}, \quad \frac{\partial f}{\partial \bar{t}_k} = \{\bar{H}_k, f\},$$

where

- $H_k = (f^k(\zeta))_+ + \frac{1}{2}(f^k(\zeta))_0$,
- $\bar{H}_k = (\bar{f}^k(1/\zeta))_- + \frac{1}{2}(\bar{f}^k(1/\zeta))_0$.

In the paper [142], an analog of this theory for multiply-connected domains is developed. The answers are formulated in terms of the so-called Schottky double of the plane with holes. The Laurent basis used in the simply connected case is replaced by the Krichever–Novikov basis. As a corollary, analogs of the 2-D Toda hierarchy depend on standard times plus a finite set of additional variables. The solution of the Dirichlet problem is written in terms of the τ-function of this hierarchy. The relation to some matrix problems is briefly discussed.

14.2 Fractal Growth

Benoît Mandelbrot (1924, Warsaw, Poland–2010, Cambridge, Massachusetts, United States) brought to the world's attention that many natural objects simply do not have a preconceived form determined by a characteristic scale. He [168] first saw a visualization of the set named after him, at IBM's Thomas J. Watson Research Center in upstate New York.

Many of the structures in space and processes reveal new features when magnified beyond their usual scale in a wide variety of natural and industrial



B. Mandelbrot

processes, such as crystal growth, vapor deposition, chemical dissolution, corrosion, erosion, fluid flow in porous media and biological growth a surface or an interface, biological processes. A fractal, a structure coined by Mandelbrot in 1975 ("fractal" from Latin "fractus"), is a rough or fragmented geometric shape that can be subdivided into parts, each of which is (at least approximately) a reduced-size copy of the whole. Fractals are generally self-similar, independent of scale, and have (by Mandelbrot's own definition) the Hausdorff dimension strictly greater than the topological dimension. There are many mathematical structures that are fractals, e.g., the Sierpinski triangle, the Koch snowflake, the Peano curve, the Mandelbrot set, and the Lorenz attractor. One of the ways to model a fractal is the process of fractal growth that can be either stochastic or deterministic. A nice overview of fractal growth phenomena is found in [254].

Many models of fractal growth patterns combine complex geometry with randomness. A typical and important model for pattern formation is *DLA* (see a survey in [119]). Considering colloidal particles undergoing Brownian motion in some fluid and letting them adhere irreversibly on contact with another one bring us to the basics of DLA. Fix a seed particle at the origin and start another one form infinity letting it perform a random walk. Ultimately, that second particle will either escape to infinity or contact the seed, to which it will stick irreversibly. Next another particle starts at infinity to walk randomly until it either sticks to the two-particle cluster or escapes to infinity. This process is repeated to an extent limited only by modeler's patience. The clusters generated by this process are highly branched and fractal, see figure to the left.

The DLA model was introduced in 1981 by Witten and Sander [260,261]. It was shown to have relation to dielectric breakdown [179], one-phase fluid flow in porous media [51], electro-chemical deposition [113], medical sciences [223], etc. A new conformal mapping language to study DLA was proposed by Hastings and Levitov [120, 121]. They showed that two-dimensional DLA can be grown by iterating stochastic conformal maps. Later this method was thoroughly handled in [60].



For a continuous random walk in 2-D the diffusion equation provides the law for the probability $u(z,t)$ that the walk reaches a point z at the time t,

$$\frac{\partial u}{\partial t} = \eta \Delta u,$$

where η is the diffusion coefficient. When the cluster growth rate per surface site is negligible compared to the diffusive relaxation time, the time dependence of the relaxation may be neglected (see, e.g., [261]). With a steady flux from infinity and the slow growth of the cluster the left-hand side derivative can be neglected and we have just the Laplacian equation for u. If $K(t)$ is the closed aggregate at the time t and $\Omega(t)$ is the connected part of the complement of $K(t)$ containing infinity, then the probability of the appearance of the random walker in $\mathbb{C} \setminus \Omega(t)$ is zero. Thus, the boundary condition $u(z,t)\big|_{\Gamma(t)} = 0$, $\Gamma(t) = \partial\Omega(t)$ is set. The only source of time dependence of u is the motion of $\Gamma(t)$. The problem resembles the classical Hele-Shaw problem, but the complex structure of $\Gamma(t)$ does not allow us to define the normal velocity in a good way although it is possible to do this in the discrete models.



Now let us construct a Riemann conformal map $f : \mathbb{D}^- \to \hat{\mathbb{C}}$, $\mathbb{D}^- = \{z\colon |z| > 1\}$, which is meromorphic in $\mathbb{D}^-$, $f(\zeta,t) = \alpha(t)\zeta + a_0(t) + \dfrac{a_1(t)}{\zeta} + \ldots$, $\alpha(t) > 0$, and maps $\mathbb{D}^-$ onto $\Omega(t)$. The boundary $\Gamma(t)$ need not even be a quasidisk, as considered earlier. While we are not able to construct a differential equation

analogous to the Polubarinova–Galin one on the unit circle, the retracting Löwner subordination chain still exists, and the function $f(\zeta, t)$ satisfies the equation

$$\dot{f}(\zeta, t) = \zeta f'(\zeta, t) p_f(\zeta, t), \quad \zeta \in \mathbb{D}^-, \tag{53}$$

where $p_f(\zeta, t) = p_0(t) + p_1(t)/\zeta + \dots$ is a Carathéodory function: $\operatorname{Re} p(z, t) > 0$ for all $\zeta \in \mathbb{D}^-$ and for almost all $t \in [0, \infty)$. A difference from the Hele-Shaw problem is that the DLA problem is well posed on each level of discreteness by construction. An analogue of DLA model was treated by means of Löwner chains by Carleson and Makarov in [49]. In this section we follow their ideas as well as those from [125].

Of course, the fractal growth phenomena can be seen without randomness. A simplest example of such growth is the Koch snowflake (Helge von Koch, 1870–1924).DLA-like fractal growth without randomness can be found, e.g., in [61].

Returning to the fractal growth we want to study a rather wide class of models with complex growing structure. We note that $\alpha(t) = \operatorname{cap} K(t) = \operatorname{cap} \Gamma(t)$. Let $M(0, 2\pi)$ be the class of positive measures γ on $[0, 2\pi]$. The control function $p_f(\zeta, t)$ in (53) can be represented by the Riesz–Herglotz formula

$$p_f(\zeta, t) = \int_0^{2\pi} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} d\gamma_t(\theta),$$

and $p_0(t) = \|\gamma_t\|$, where $\gamma_t(\theta) \in M(0, 2\pi)$ for almost all $t \geq 0$ and absolutely continuous in $t \geq 0$. Consequently, $\dot{\alpha}(t) = \alpha(t)\|\gamma_t\|$. There is a one-to-one correspondence between one-parameter (t) families of measures γ_t and Löwner chains $\Omega(t)$ (in our case of growing domains $\mathbb{C} \setminus \Omega(t)$ we have only surjective correspondence).

Example 1. Suppose we have an initial domain $\Omega(0)$. If the derivative of the measure γ_t with respect to the Lebesgue measure is the Dirac measure $d\gamma_t(\theta) \equiv \delta_{\theta_0}(\theta) d\theta$, then

$$p_f(\zeta, t) \equiv \frac{e^{i\theta_0} + \zeta}{e^{i\theta_0} - \zeta},$$

and $\Omega(t)$ is obtained by cutting $\Omega(0)$ along a geodesic arc. The preimage of the endpoint of this slit is exactly $e^{i\theta_0}$. In particular, if $\Omega(0)$ is a complement of a disk, then $\Omega(t)$ is $\Omega(0)$ minus a radial slit.

Example 2. Let $\Omega(0)$ be a domain bounded by an analytic curve $\Gamma(t)$. If the derivative of the measure γ_t with respect to the Lebesgue measure is

$$\frac{d\gamma_t(\theta)}{d\theta} = \frac{1}{2\pi |f'(e^{i\theta}, t)|^2},$$

then

$$p_f(\zeta,t)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|f'(e^{i\theta},t)|^2}\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}d\theta,$$

and letting ζ tend to the unit circle we obtain $\mathrm{Re}\,[\dot{f}\ \overline{\zeta f'}]=1$, which corresponds to the classical Hele-Shaw case, for which the solution exists locally in time.

In the classical Hele-Shaw process the boundary develops by fluid particles moving in the normal direction. In the discrete DLA models either lattice or with circular patterns the attaching are developed in the normal direction too. However, in the continuous limit it is usually impossible to speak of any normal direction because of the irregularity of $\Gamma(t)$.

In [49, Sect. 2.3] this difficulty was circumvented by evaluating the derivative of f occurring in γ_t in the above Löwner model slightly outside the boundary of the unit disk.

Let $\Omega(0)$ be any simply connected domain, $\infty\in\Omega(0)$, $0\notin\Omega(0)$. The derivative of the measure γ_t with respect to the Lebesgue measure is

$$\frac{d\gamma_t(\theta)}{d\theta}=\frac{1}{2\pi|f'((1+\varepsilon)e^{i\theta},t)|^2},$$

with sufficiently small positive ε. In this case the derivative is well defined.

It is worth to mention that the estimate

$$\frac{\partial\mathrm{cap}\,\Gamma(t)}{\partial t}=\dot{\alpha}(t)\lesssim\frac{1}{\varepsilon}$$

would be equivalent to the Brennan conjecture (see [191, Chap. 8]) which is still unproved. However, Theorem 2.1 [49] states that if

$$R(t)=\max_{\theta\in[0,2\pi)}|f((1+\varepsilon)e^{i\theta},t)|,$$

then

$$\limsup_{\Delta t\to 0}\frac{R(t+\Delta t)-R(t)}{\Delta t}\le\frac{C}{\varepsilon},$$

for some absolute constant C. Carleson and Makarov [49] were, with the above model, able to establish an estimate for the growth of the cluster or aggregate given as a lower bound for the time needed to multiply the capacity of the aggregate by a suitable constant. This is an analogue of the upper bound for the size of the cluster in two-dimensional stochastic DLA given by [132].

14.3 Extension to Several Complex Variables

Pfaltzgraff in 1974 was the first one who extended the basic Löwner theory to $\mathbb{C}^n$ with the aim of giving bounds and growth estimates to some classes of univalent mappings from the unit ball of $\mathbb{C}^n$. The theory was later developed by Poreda, Graham, Kohr, Kohr, Hamada, and others (see [111] and [43]).

Since then, a lot of work was devoted to successfully extend the theory to several complex variables, and finally, it has been accomplished. The main and dramatic difference between the one-dimensional case and the higher dimensional case is essentially due to the lack of a Riemann mapping theorem or, which is the same, to the existence of the so-called Fatou–Bieberbach phenomena, that is, the existence of proper open subsets of $\mathbb{C}^n$, $n \geq 2$, which are biholomorphic to $\mathbb{C}^n$. This in turn implies that there are no satisfactory growth estimates for univalent functions on the ball (nor in any other simply connected domain) of $\mathbb{C}^n$, $n \geq 2$.

The right way to proceed is then to look at the Löwner theory in higher dimension as a discrete complex dynamical system, in the sense of random iteration, and to consider abstract basins of attraction as the analogous of the Löwner chains. In order to state the most general results, we first give some definitions and comment on that.

Most estimates in the unit disc can be rephrased in terms of the Poincaré distance, which gives a more intrinsic point of view. In higher dimension one can replace the Poincaré distance with the Kobayashi distance. First, we recall the definition of Kobayashi distance (see [134] for details and properties). Let M be a complex manifold and let $z, w \in M$. A *chain of analytic discs* between z and w is a finite family of holomorphic mappings $f_j : \mathbb{D} \to M$, $j = 1, \ldots, m$ and points $t_j \in (0, 1)$ such that

$$f_1(0) = z,\ f_1(t_1) = f_2(0), \ldots, f_{m-1}(t_{m-1}) = f_m(0),\ f_m(t_m) = w.$$

We denote by $\mathscr{C}_{z,w}$ the set of all chains of analytic discs joining z to w. Let $L \in \mathscr{C}_{z,w}$. The *length of* L, denoted by $\ell(L)$, is given by

$$\ell(L) := \sum_{j=1}^{m} \omega(0, t_j) = \sum_{j=1}^{m} \frac{1}{2} \log \frac{1 + t_j}{1 - t_j}.$$

We define the *Kobayashi (pseudo)distance* $k_M(z, w)$ as follows:

$$k_M(z, w) := \inf_{L \in \mathscr{C}_{z,w}} \ell(L).$$

If M is connected, then $k_M(z, w) < +\infty$ for all $z, w \in M$. Moreover, by construction, it satisfies the triangular inequality. However, it might be that $k_M(z, w) = 0$ even if $z \neq w$ (a simple example is represented by $M = \mathbb{C}$, where $k_{\mathbb{C}} \equiv 0$). In the unit disc $\mathbb{D}$, $k_{\mathbb{D}}$ coincides with the Poincaré distance.

Definition 9. A complex manifold M is said to be *(Kobayashi) hyperbolic* if $k_M(z,w) > 0$ for all $z, w \in M$ such that $z \neq w$. Moreover, M is said *complete hyperbolic* if k_M is complete.

Important examples of complete hyperbolic manifolds are given by bounded convex domains in $\mathbb{C}^n$.

The main property of the Kobayashi distance is the following: let M, N be two complex manifolds and let $f : M \to N$ be holomorphic. Then for all $z, w \in M$ it holds

$$k_N(f(z), f(w)) \leq k_M(z,w).$$

It can be proved that if M is complete hyperbolic, then k_M is Lipschitz continuous (see [18]). If M is a bounded strongly convex domain in $\mathbb{C}^n$ with smooth boundary, Lempert (see, e.g., [134]) proved that the Kobayashi distance is of class C^∞ outside the diagonal. In any case, even if k_M is not smooth, one can consider the differential dk_M as the Dini-derivative of k_M, which coincides with the usual differential at almost every point in $M \times M$.

As it is clear from the one-dimensional general theory of Löwner's equations, evolution families and Herglotz vector fields are pretty much related to semigroups and infinitesimal generators. Kobayashi distance can be used to characterize infinitesimal generators of continuous semigroups of holomorphic self-maps of complete hyperbolic manifolds. The following characterization of infinitesimal generators is proved for strongly convex domains in [41], and in general in [18]:

Theorem 16. *Let M be a complete hyperbolic complex manifold and let $H : M \to TM$ be an holomorphic vector field on M. Then the following are equivalent.*

1. *H is an infinitesimal generator,*
2. *For all $z, w \in M$ with $z \neq w$ it holds*

$$(dk_M)_{(z,w)} \cdot (H(z), H(w)) \leq 0.$$

This apparently harmless characterization contains instead all the needed information to get good growth estimates. In particular, it is equivalent to the Berkson–Porta representation formula in the unit disc.

14.4 L^d-Herglotz Vector Fields and Evolution Families on Complete Hyperbolic Manifolds

Let M be a complex manifold, and denote by $\|\cdot\|$ a Hermitian metric on TM and by d_M the corresponding integrated distance.

Definition 10. Let M be a complex manifold. A *weak holomorphic vector field of order* $d \geq 1$ on M is a mapping $G : M \times \mathbb{R}^+ \to TM$ with the following properties:

(i) The mapping $G(z, \cdot)$ is measurable on $\mathbb{R}^+$ for all $z \in M$.
(ii) The mapping $G(\cdot, t)$ is a holomorphic vector field on M for all $t \in \mathbb{R}^+$.
(iii) For any compact set $K \subset M$ and all $T > 0$, there exists a function $C_{K,T} \in L^d([0, T], \mathbb{R}^+)$ such that

$$\|G(z, t)\| \leq C_{K,T}(t), \quad z \in K, \text{ a.e. } t \in [0, T].$$

A *Herglotz vector field of order* $d \geq 1$ is a weak holomorphic vector field $G(z, t)$ of order d with the property that $M \ni z \mapsto G(z, t)$ is an infinitesimal generator for almost all $t \in [0, +\infty)$.

If M is complete hyperbolic, due to the previous characterization of infinitesimal generators, a weak holomorphic vector field $G(z, t)$ of order d if a Herglotz vector field of order d if and only if

$$(dk_M)_{(z,w)} \cdot (G(z, t), G(w, t)) \leq 0, \quad z, w \in M, z \neq w, \text{ a.e. } t \geq 0. \tag{54}$$

This was proved in [41] for strongly convex domains, and in [18] for the general case.

One can also generalize the concept of evolution families:

Definition 11. Let M be a complex manifold. A family $(\varphi_{s,t})_{0 \leq s \leq t}$ of holomorphic self-mappings of M is an *evolution family of order* $d \geq 1$ (or L^d-evolution family) if it satisfies the *evolution property*

$$\varphi_{s,s} = \mathsf{id}, \quad \varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}, \quad 0 \leq s \leq u \leq t, \tag{55}$$

and if for any $T > 0$ and for any compact set $K \subset\subset M$ there exists a function $c_{T,K} \in L^d([0, T], \mathbb{R}^+)$ such that

$$d_M(\varphi_{s,t}(z), \varphi_{s,u}(z)) \leq \int_u^t c_{T,K}(\xi) d\xi, \quad z \in K, \ 0 \leq s \leq u \leq t \leq T. \tag{56}$$

It can be proved that all elements of an evolution family are univalent (cf. [20, Proposition 2.3]).

The classical Löwner and Kufarev–Löwner equations can now be completely generalized as follows:

Theorem 17. *Let M be a complete hyperbolic complex manifold. Then for any Herglotz vector field G of order $d \in [1, +\infty]$ there exists a unique L^d-evolution family $(\varphi_{s,t})$ over M such that for all $z \in M$*

$$\frac{\partial \varphi_{s,t}}{\partial t}(z) = G(\varphi_{s,t}(z), t) \quad \text{a.e. } t \in [s, +\infty). \tag{57}$$

Conversely for any L^d-evolution family $(\varphi_{s,t})$ over M there exists a Herglotz vector field G of order d such that (57) is satisfied. Moreover, if H is another weak holomorphic vector field which satisfies (57), then $G(z,t) = H(z,t)$ for all $z \in M$ and almost every $t \in \mathbb{R}^+$.

Equation (57) is the bridge between the L^d-Herglotz vector fields and L^d-evolution families. In [42] the result has been proved for any complete hyperbolic complex manifold M with Kobayashi distance of class C^1 outside the diagonal, but the construction given there only allowed to start with evolution families of order $d = +\infty$. Next, in [117] the case of L^d-evolution families has been proved for the case $M = \mathbb{B}^n$ the unit ball in $\mathbb{C}^n$. Finally, in [18], Theorem 17 was proved in full generality.

The previous equation, especially in the case of the unit ball of $\mathbb{C}^n$ and for the case $d = +\infty$, with evolution families fixing the origin and having some particular first jets at the origin has been studied by many authors, we cite here Pfaltzgraff [184, 185], Poreda [193], Graham et al. [107], Graham et al. [109] (see also [110]).

Using the so-called product formula, proved in convex domains by Reich and Shoikhet [211] (see also [212]), and later generalized on complete hyperbolic manifold in [18] we get a strong relation between the semigroups generated at a fixed time by a Herglotz vector field and the associated evolution family. Let $G(z,t)$ be a Herglotz vector field on a complete hyperbolic complex manifold M. For almost all $t \geq 0$, the holomorphic vector field $M \ni z \mapsto G(z,t)$ is an infinitesimal generator. Let (ϕ_r^t) be the associated semigroups of holomorphic self-maps of M. Let $(\varphi_{s,t})$ be the evolution family associated with $G(z,t)$. Then, uniformly on compacta of M it holds

$$\phi_t^r = \lim_{m\to\infty} \varphi_{t,t+\frac{r}{m}}^{\circ m} = \lim_{m\to\infty} \underbrace{(\varphi_{t,t+\frac{r}{m}} \circ \ldots \circ \varphi_{t,t+\frac{r}{m}})}_{m}.$$

14.5 Löwner Chains on Complete Hyperbolic Manifolds

Although one could easily guess how to extend the notion of Herglotz vector fields and evolution families to several complex variables, the concept of Löwner chains is not so easy to extend. For instance, starting from a Herglotz vector field on the unit ball of $\mathbb{C}^n$, one would be tempted to define in a natural way Löwner chains with range in $\mathbb{C}^n$. However, sticking with such a definition, it is rather hard to get a complete solution to the Löwner PDE. In fact, in case $D = \mathbb{B}^n$ the unit ball, much effort has been done to show that, given an evolution family $(\varphi_{s,t})$ on $\mathbb{B}^n$ such that $\varphi_{s,t}(0) = 0$ and $d(\varphi_{s,t})_0$ has a special form, then there exists an associated Löwner chain. We cite here the contributions of Pfaltzgraff [184, 185], Poreda [193], Graham et al. [107], Graham et al. [109], Arosio [16], Voda [255]. In the last two mentioned papers, resonances phenomena among the eigenvalues of $d(\varphi_{s,t})_0$ are taken into account.

The reason for these difficulties is due to the fact that, although apparently natural, the definition of Löwner chains as a more or less regular family of univalent mappings from the unit ball to $\mathbb{C}^n$ is not the right one. And the reason why this is meaningful in one dimension is just because of the Riemann mapping theorem, as we will explain.

Indeed, as shown before, there is essentially no difference in considering evolution families or Herglotz vector fields in the unit ball of $\mathbb{C}^n$ or on complete hyperbolic manifolds, since the right estimates to produce the Löwner equation are provided just by the completeness of the Kobayashi distance and its contractiveness properties.

The right point of view is to consider evolution families as random iteration families, and thus, the "Löwner chains" are just the charts of the abstract basins of attraction of such a dynamical system. To be more precise, let us recall the theory developed in [20]. Interesting and surprisingly enough, regularity conditions—which were basic in the classical theory for assuming the classical limiting process to converge—do not play any role.

Definition 12. Let M be a complex manifold. An *algebraic evolution family* is a family $(\varphi_{s,t})_{0\le s\le t}$ of univalent self-mappings of M satisfying the evolution property (55).

A L^d-evolution family is an algebraic evolution family because all elements of a L^d-evolution family are injective as we said before.

Definition 13. Let M, N be two complex manifolds of the same dimension. A family $(f_t)_{t\ge 0}$ of holomorphic mappings $f_t : M \to N$ is a *subordination chain* if for each $0 \le s \le t$ there exists a holomorphic mapping $v_{s,t} : M \to M$ such that $f_s = f_t \circ v_{s,t}$. A subordination chain (f_t) and an algebraic evolution family $(\varphi_{s,t})$ are *associated* if

$$f_s = f_t \circ \varphi_{s,t}, \quad 0 \le s \le t.$$

An *algebraic Löwner chain* is a subordination chain such that each mapping $f_t : M \to N$ is univalent. The *range* of an algebraic Löwner chain is defined as

$$\mathsf{rg}\,(f_t) := \bigcup_{t\ge 0} f_t(M).$$

Note that an algebraic Löwner chain (f_t) has the property that

$$f_s(M) \subset f_t(M), \quad 0 \le s \le t.$$

We have the following result which relates algebraic evolution families to algebraic Löwner chains, whose proof is essentially based on abstract categorial analysis:

Theorem 18 ([20]). *Let M be a complex manifold. Then any algebraic evolution family $(\varphi_{s,t})$ on M admits an associated algebraic Löwner chain $(f_t : M \to N)$.*

Moreover if $(g_t\colon M \to Q)$ *is a subordination chain associated with* $(\varphi_{s,t})$ *then there exist a holomorphic mapping* $\Lambda\colon \mathsf{rg}\,(f_t) \to Q$ *such that*

$$g_t = \Lambda \circ f_t, \quad \forall t \geq 0.$$

The mapping Λ *is univalent if and only if* (g_t) *is an algebraic Löwner chain, and in that case* $\mathsf{rg}\,(g_t) = \Lambda(\mathsf{rg}\,(f_t))$.

The previous theorem shows that the range $\mathsf{rg}\,(f_t)$ of an algebraic Löwner chain (f_t) is uniquely defined up to biholomorphisms. In particular, given an algebraic evolution family $(\varphi_{s,t})$ one can define its *Löwner range* $\mathsf{Lr}(\varphi_{s,t})$ as the biholomorphism class of the range of any associated algebraic Löwner chain.

In particular, if $M = \mathbb{D}$ the unit disc, then the Löwner range of any evolution family on $\mathbb{D}$ is a simply connected non compact Riemann surface, thus, by the uniformization theorem, the Löwner range is either the unit disc $\mathbb{D}$ or $\mathbb{C}$. Therefore, in the one-dimensional case, one can harmlessly stay with the classical definition of Löwner chains as a family of univalent mappings with image in $\mathbb{C}$.

One can also impose L^d-regularity as follows:

Definition 14. Let $d \in [1, +\infty]$. Let M, N be two complex manifolds of the same dimension. Let d_N be the distance induced by a Hermitian metric on N. An algebraic Löwner chain $(f_t\colon M \to N)$ is a *L^d-Löwner chain* (for $d \in [1, +\infty]$) if for any compact set $K \subset\subset M$ and any $T > 0$ there exists a $k_{K,T} \in L^d([0, T], \mathbb{R}^+)$ such that

$$d_N(f_s(z), f_t(z)) \leq \int_s^t k_{K,T}(\xi)d\xi \tag{58}$$

for all $z \in K$ and for all $0 \leq s \leq t \leq T$.

The L^d-regularity passes from evolution family to Löwner chains:

Theorem 19 ([20]). *Let* M *be a complete hyperbolic manifold with a given Hermitian metric and* $d \in [1, +\infty]$. *Let* $(\varphi_{s,t})$ *be an algebraic evolution family on* M *and let* $(f_t\colon M \to N)$ *be an associated algebraic Löwner chain. Then* $(\varphi_{s,t})$ *is a* L^d*-evolution family on* M *if and only if* (f_t) *is a* L^d*-Löwner chain.*

Once the general Löwner equation is established and Löwner chains have been well defined, even the Löwner–Kufarev PDE can be generalized:

Theorem 20 ([20]). *Let* M *be a complete hyperbolic complex manifold, and let* N *be a complex manifold of the same dimension. Let* $G : M \times \mathbb{R}^+ \to TM$ *be a Herglotz vector field of order* $d \in [1, +\infty]$ *associated with the* L^d*-evolution family* $(\varphi_{s,t})$. *Then a family of univalent mappings* $(f_t\colon M \to N)$ *is an* L^d*-Löwner chain associated with* $(\varphi_{s,t})$ *if and only if it is locally absolutely continuous on* $\mathbb{R}^+$ *locally uniformly with respect to* $z \in M$ *and solves the Löwner–Kufarev PDE*

$$\frac{\partial f_s}{\partial s}(z) = -(df_s)_z G(z, s), \quad \text{a.e. } s \geq 0, z \in M.$$

14.6 The Löwner Range and the General Löwner PDE in $\mathbb{C}^n$

As we saw before, given a L^d-evolution family (or just an algebraic evolution family) on a complex manifold, it is well defined the Löwner range $\mathsf{Lr}(\varphi_{s,t})$ as the class of biholomorphism of the range of any associated Löwner chain.

In practice, it is interesting to understand the Löwner range of an evolution family on a given manifold. For instance, one may ask whether, starting from an evolution family on the ball, the Löwner range is always biholomorphic to an open subset of $\mathbb{C}^n$. This problem turns out to be related to the so-called *Bedford's conjecture*. Such a conjecture states that given a complex manifold M, an automorphism $f : M \to M$ and a f-invariant compact subset $K \subset M$ on which the action of f is hyperbolic, then the stable manifold of K is biholomorphic to $\mathbb{C}^m$ for some $m \leq \dim M$. The equivalent formulation which resembles the problem of finding the Löwner range of an evolution family in the unit ball is in [78], see also [17] where such a relation is well explained.

In [16, Sect. 9.4] it is shown that there exists an algebraic evolution family $(\varphi_{s,t})$ on $\mathbb{B}^3$ which does not admit any associated algebraic Löwner chain with range in $\mathbb{C}^3$. Such an evolution family is, however, not L^d for any $d \in [1, +\infty]$.

In the recent paper [19] it has been proved the following result:

Theorem 21. *Let $D \subset \mathbb{C}^n$ be a complete hyperbolic starlike domain (for instance, the unit ball). Let $(\varphi_{s,t})$ be an L^d-evolution family, $d \in [1, +\infty]$. Then the Löwner range $\mathsf{Lr}(\varphi_{s,t})$ is biholomorphic to a Runge and Stein open domain in $\mathbb{C}^n$.*

The proof, which starts from the existence of a Löwner chain with abstract range, is based on the study of manifolds which are union of balls, using a result by Docquier and Grauert to show that the regularity hypothesis guarantees Rungeness and then one can use approximation results of Andersén and Lempert in order to construct a suitable embedding.

As a corollary of the previous consideration, we have a general solution to Löwner PDE in higher dimension, which is the full analogue of the one-dimensional situation:

Theorem 22 ([19]). *Let $D \subset \mathbb{C}^N$ be a complete hyperbolic starlike domain. Let $G : D \times \mathbb{R}^+ \to \mathbb{C}^N$ be a Herglotz vector field of order $d \in [1, +\infty]$. Then there exists a family of univalent mappings $(f_t \colon D \to \mathbb{C}^N)$ of order d which solves the Löwner PDE*

$$\frac{\partial f_t}{\partial t}(z) = -df_t(z)G(z,t), \quad a.a.\ t \geq 0, \forall z \in D. \tag{59}$$

Moreover, $R := \cup_{t\geq 0} f_t(D)$ is a Runge and Stein domain in $\mathbb{C}^N$ and any other solution to (59) is of the form $(\Phi \circ f_t)$ for a suitable holomorphic map $\Phi : R \to \mathbb{C}^N$.

In general, one can infer some property of the Löwner range from the dynamics of the evolution family. In order to state the result, let us recall what the Kobayashi pseudometric is:

Definition 15. Let M be a complex manifold. The *Kobayashi pseudometric* $\kappa_M : TM \to \mathbb{R}^+$ is defined by

$$\kappa_M(z; v) := \inf\{r > 0 : \exists g : \mathbb{D} \to M \text{ holomorphic } : g(0) = z, g'(0) = \frac{1}{r}v\}.$$

The Kobayashi pseudometric has the remarkable property of being contracted by holomorphic maps, and its integrated distance is exactly the Kobayashi pseudodistance. We refer the reader to [134] for details.

Definition 16. Let $(\varphi_{s,t})$ be an algebraic evolution family on a complex manifold M. For $v \in T_z M$ and $s \geq 0$ we define

$$\beta_z^s(v) := \lim_{t\to\infty} \kappa_M(\varphi_{s,t}(z); (d\varphi_{s,t})_z(v)). \tag{60}$$

Since the Kobayashi pseudometric is contracted by holomorphic mappings the limit in (60) is well defined.

The function β is the bridge between the dynamics of an algebraic evolution family $(\varphi_{s,t})$ and the geometry of its Löwner range. Indeed, in [20] it is proved that if N is a representative of the Löwner range of $(\varphi_{s,t})$ and $(f_t : M \to N)$ is an associated algebraic Löwner chain, then for all $z \in M$ and $v \in T_z M$ it follows

$$f_s^* \kappa_N(z; v) = \beta_z^s(v).$$

In the unit disc case, if $(\varphi_{s,t})$ is an algebraic evolution family, the previous formula allows to determine the Löwner range: by the Riemann mapping theorem the Löwner range is either $\mathbb{C}$ or $\mathbb{D}$. The first being non-hyperbolic, if $\beta_z^s(v) = 0$ for some $s > 0, z \in \mathbb{D}$ (v can be taken to be 1), then the Löwner range is $\mathbb{C}$, otherwise it is $\mathbb{D}$.

Such a result can be generalized to a complex manifold M. Let $\mathsf{aut}(M)$ denote the group of holomorphic automorphisms of a complex manifold M. Using a result by Fornæss and Sibony [77], in [20] it is shown that the previous formula implies

Theorem 23. *Let M be a complete hyperbolic complex manifold and assume that $M/\mathsf{aut}(M)$ is compact. Let $(\varphi_{s,t})$ be an algebraic evolution family on M. Then*

1. *If there exists $z \in M$, $s \geq 0$ such that $\beta_z^s(v) \neq 0$ for all $v \in T_z M$ with $v \neq 0$, then $\mathsf{Lr}(\varphi_{s,t})$ is biholomorphic to M.*
2. *If there exists $z \in M$, $s \geq 0$ such that $\dim_{\mathbb{C}}\{v \in T_z M : \beta_z^s(v) = 0\} = 1$, then $\mathsf{Lr}(\varphi_{s,t})$ is a fiber bundle with fiber $\mathbb{C}$ over a closed complex submanifold of M.*

15 Acknowledgements

All the authors have been supported by the ESF Networking Programme HCAA. Filippo Bracci has been supported by ERC grant "HEVO—Holomorphic Evolution Equations," no. 277691; Manuel D. Contreras and Santiago Díaz-Madrigal have been partially supported by the Ministerio de Economía y Competitividad and the European Union (FEDER), projects MTM2009-14694-C02-02 and MTM2012-37436-C02-01 and by La Consejería de Educación y Ciencia de la Junta de Andalucía; Alexander Vasil'ev has been supported by the grants of the Norwegian Research Council #204726/V30 and #213440/BG.

References

1. M. Abate, *Iteration Theory of Holomorphic Maps on Taut Manifolds* (Mediterranean Press, Rende, Cosenza, 1989)
2. M. Abate, F. Bracci, M.D. Contreras, S. Díaz-Madrigal, The evolution of Loewner's differential equations. Newslett. Eur. Math. Soc. **78**, 31–38 (2010)
3. D. Aharonov, S. Reich, D. Shoikhet, Flow invariance conditions for holomorphic mappings in Banach spaces. Math. Proc. R. Ir. Acad. **99A**(1), 93–104 (1999)
4. L.V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory* (AMS Chelsea Publishing, Providence, RI, 2010) (Reprint of the 1973 original)
5. L.A. Aksent'ev, P.L. Shabalin, Sufficient conditions for univalence and quasiconformal extendibility of analytic functions, in *Handbook of Complex Analysis: Geometric Function Theory*, vol. 1 (North-Holland, Amsterdam, 2002), pp. 169–206
6. S. Albeverio, R. Høegh-Krohn, Uniqueness and the global markov property for euclidean fields. The case of trigonometric interactions. Commun. Math. Phys. **68**, 95–128 (1979)
7. I.A. Aleksandrov, V.V. Chernikov, Pavel Parfen'evich Kufarev. Obituary (in Russian), Uspehi Mat. Nauk **24**, 181–184 (1969)
8. I.A. Aleksandrov, S.A. Kopanev, On mutual growth of the modulus of a univalent function and of the modulus of its derivative. Sibirsk. Mat. Zh. **7**, 23–30 (1966)
9. I.A. Aleksandrov, S.A. Kopanev, The range of values of the derivative on the class of holomorphic univalent functions. Ukrain. Mat. Zh. **22**, 660–664 (1970)
10. I.A. Aleksandrov, V.I. Popov, Optimal controls and univalent functions. Ann. Univ. M. Curie-Sklodowska, Sec. A **22–24**, 13–20 (1968–1970)
11. I.A. Aleksandrov, B.Ja. Krjučkov, V.I. Popov, The first coefficients of bounded holomorphic univalent functions. Dopovīdī Akad. Nauk Ukraïn. RSR Ser. A **1**, 3–5 (1973)
12. I.A. Aleksandrov, G.A. Popova, Extremal properties of univalent holomorphic functions with real coefficients. Sibirsk. Mat. Ž. **14**, 915–926 (1973)
13. I.A. Aleksandrov, G.G. Zavozin, S.A. Kopanev, Optimal controls in coefficient problems for univalent functions. Differencial'nye Uravnenija **12**(4), 599–611 (1976)
14. I.A. Aleksandrov, *Parametric Continuations in the Theory of Univalent Functions* (Nauka, Moscow, 1976)
15. S.T. Aleksandrov, V.V. Sobolev, Extremal problems in some classes of functions, univalent in the half plane, having a finite angular residue at infinity. Siberian Math. J. **27**, 145–154 (1986). Translation from Sibirsk. Mat. Zh. **27**, 3–13 (1986)
16. L. Arosio, Resonances in Loewner equations. Adv. Math. **227**, 1413–1435 (2011)
17. L. Arosio, Basins of attraction in Loewner equations. Ann. Acad. Sci. Fenn. Math. **37**, 563–570 (2012)

18. L. Arosio, F. Bracci, Infinitesimal generators and the Loewner equation on complete hyperbolic manifolds. Anal. Math. Phys. **1**(4), 337–350 (2011)
19. L. Arosio, F. Bracci, E. Fornæss Wold, *Solving the Loewner PDE in complete hyperbolic starlike domains of* $\mathbb{C}^N$, Adv. Math. **242**, 209–216 (2013)
20. L. Arosio, F. Bracci, H. Hamada, G. Kohr, An abstract approach to Löwner chains. J. Anal. Math. **119**(1), 89–114 (2013)
21. K. Astala, P. Jones, A. Kupiainen, E. Saksman, Random curves by conformal welding. Compt. Rend. Math. **348**(5–6), 257–262 (2010)
22. O. Babelon, D. Bernard, M. Talon, *Introduction to Classical Integrable Systems* (Cambridge University Press, Cambridge, 2003)
23. M. Bauer, D. Bernard, SLE_k growth processes and conformal field theories. Phys. Lett. B **543**(1–2), 135–138 (2002)
24. M. Bauer, D. Bernard, Conformal field theories of stochastic Loewner evolutions. Comm. Math. Phys. **239**(3), 493–521 (2003)
25. R.O. Bauer, R.M. Friedrich, On radial stochastic Loewner evolution in multiply connected domains. J. Funct. Anal. **237**, 565–588 (2006)
26. R.O. Bauer, R.M. Friedrich, On chordal and bilateral SLE in multiply connected domains. Math. Z. **258**, 241–265 (2008)
27. J. Basilewitsch, Sur les théorèmes de Koebe-Bieberbach. Rec. Math. [Mat. Sbornik] N.S. **1(43)**(3), 283–291 (1936)
28. J. Basilewitsch, D'une propriété extrémale de la fonction $F(z) = \frac{z}{(1-z)^2}$. Rec. Math. [Mat. Sbornik] N.S. **12(54)**(3), 315–319 (1943)
29. I.E. Bazilevič, On a case of integrability in quadratures of the Loewner-Kufarev equation. Mat. Sb. N.S. **37**(79), 471–476 (1955)
30. I.E. Bazilevič, Generalization of an integral formula for a subclass of univalent functions. Mat. Sb. N.S. **64 (106)**, 628–630 (1964)
31. J. Becker, Löwnersche, Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen. J. Reine Angew. Math. **255**, 23–43 (1972)
32. J. Becker, Ch. Pommerenke, Schlichtheitskriterien und Jordangebiete. J. Reine Angew. Math. **354**, 74–94 (1984)
33. V. Beffara, The dimension of the SLE curves. Ann. Probab. **36**(4), 1421–1452 (2008)
34. A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory. Nuclear Phys. B **241**(2), 333–380 (1984)
35. E. Berkovich, B. Shein, *Odyssey of Fritz Noether. Epilogue*, Notes in Jewish History (2009), no. 11(114), July
36. E. Berkson, H. Porta, Semigroups of holomorphic functions and composition operators. Mich. Math. J. **25**, 101–115 (1978)
37. L. Bieberbach, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Ein- heitskreises vermitteln. S.-B. Preuss. Akad. Wiss. 940–955 (1916)
38. Bill of indictment on L.A. Vishnevskiĭ. February 15, 1938. Novosibirsk
39. K.I. Bobenko, *On the theory of extremal problems for univalent functions of class* S. Trudy Mat. Inst. Steklov. vol. 101 (Izdat. "Nauka", Moscow, 1972)
40. E. Bombieri, On the local maximum property of the Koebe function. Inv. Math. **4**, 26–67 (1967)
41. F. Bracci, M. D. Contreras, S. Díaz-Madrigal, Pluripotential theory, semigroups and boundary behaviour of infinitesimal generators in strongly convex domains. J. Eur. Math. Soc. **12**, 23–53 (2010)
42. F. Bracci, M.D. Contreras, S. Díaz-Madrigal, Evolution families and the Löwner equation I: the unit disk. J. Reine Angew. Math. **672**, 1–37 (2012)
43. F. Bracci, M.D. Contreras, and S. Díaz-Madrigal, Evolution Families and the Löwner Equation II: complex hyperbolic manifolds. Math. Ann. **344**, 947–962 (2009)
44. F. Bracci, M.D. Contreras, S. Díaz-Madrigal, Semigroups versus evolution families in the Löwner theory. J. Anal. Math. **115**(1), 273–292 (2011)

45. L. de Branges, A proof of the Bieberbach conjecture. Preprint E-5-84, Leningrad Branch of the Steklov Institute of Mathematics, LOMI, Leningrad, 1984
46. L. de Branges, A proof of the Bieberbach conjecture. Acta Math. **154**(1–2), 137–152 (1985)
47. D. Bshouty, W. Hengartner, A variation of the Koebe mapping in a dense subset of S. Can. J. Math. **39**(7), 54–73 (1987)
48. J. Cardy, Boundary conformal field theory, in *Encyclopedia of Mathematical Physics*, ed. by J.-P. Francoise, G. Naber, S. Tsun Tsou (Elsevier, Amsterdam, 2006), pp. 333–340
49. L. Carleson, N. Makarov, Aggregation in the plane and Loewner's equation. Comm. Math. Phys. **216**, 583–607 (2001)
50. A. Carverhill, Flows of stochastic dynamical systems: ergodic theory. Stochastics **14**(4), 273–317 (1985)
51. D.Y.C. Chan, B.D. Hughes, L. Paterson, Ch. Sirakoff, Simulating flow in porous media. Phys. Rev. A **38**(8), 4106–4120 (1988)
52. D. Chelkak, S. Smirnov, Universality in the 2D Ising model and conformal invariance of fermionic observables. Invent. Math. **189**(3), 515–580 (2012)
53. M.D. Contreras, S. Díaz-Madrigal, P. Gumenyuk, Löwner chains in the unit disk. Rev. Mat. Iberoam. **26**(3), 975–1012 (2010)
54. M.D. Contreras, S. Díaz-Madrigal, P. Gumenyuk, Local duality in Löwner equations. J. Nonlinear Convex Anal. (to appear). Available in http://arxiv.org/pdf/1202.2334.pdf
55. M.D. Contreras, S. Díaz-Madrigal, P. Gumenyuk, Loewner Theory in annulus I: evolution families and differential equations. Trans. Am. Math. Soc. **365**, 2505–2543 (2013)
56. M.D. Contreras, S. Díaz-Madrigal, P. Gumenyuk, Loewner Theory in annulus II: Loewner chains. Anal. Math. Phys. **1**, 351–385 (2011)
57. J. Conway, *Functions of One Complex Variable II* (Springer, New York, 1995)
58. E. Date, M. Kashiwara, T. Miwa, Vertex operators and functions: transformation groups for soliton equations II. Proc. Jpn. Acad. Ser. A Math. Sci. **57**, 387–392 (1981)
59. E. Date, M. Kashiwara, M. Jimbo, T. Miwa, Transformation groups for soliton equations. *Nonlinear Integrable Systems—Classical Theory and Quantum Theory (Kyoto, 1981)* (World Sci. Publishing, Singapore, 1983), pp. 39–119
60. B. Davidovitch, H.G.E. Hentschel, Z. Olami, I. Procaccia, L.M. Sander, E. Somfai, Diffusion limited aggregation and iterated conformal maps. Phys. Rev. E **59**(2), 1368–1378 (1999)
61. B. Davidovitch, M.J. Feigenbaum, H.G.E. Hentschel, I. Procaccia, Conformal dynamics of fractal growth patterns without randomness. Phys. Rev. E **62**(2), 1706–1715 (2000)
62. A. Dick, *Emmy Noether. 1882–1935* (Birkhäuser, Basel, 1970)
63. L. A. Dickey, *Soliton Equations and Hamiltonian Systems*, 2nd edn. Advanced Series in Mathematical Physics, vol. 26 (World Scientific, Singapore, 2003)
64. R.G. Douglas, *Banach Algebra Techniques in Operator Theory*, 2nd edn. Graduate Texts in Mathematics, vol. 179. (Springer, New York, 1998)
65. B. Doyon, Conformal loop ensembles and the stress–energy tensor: I. Fundamental notions of CLE. arXiv:0903.0372 (2009)
66. B. Doyon, Conformal loop ensembles and the stress–energy tensor: II. Construction of the stress–energy tensor. arXiv:0908.1511 (2009)
67. B. Doyon, Calculus on manifolds of conformal maps and CFT. J. Phys. A Math. Theor. **45**(31), 315202 (2012)
68. P.L. Duren, H.S. Shapiro, A.L. Shields, Singular measures and domains not of Smirnov type. Duke Math. J. **33**, 247–254 (1966)
69. J. Dubédat, SLE and the free field: partition functions and couplings. J. Am. Math. Soc. **22**(4), 995–1054 (2009)
70. D.A. Dubovikov, An analog of the Loëwner equation for mappings of strip. Izv. Vyssh. Uchebn. Zaved. Mat. **51**(8), 77–80 2007 (Russian); translation in Russian Math. (Iz. VUZ) **51**(8), 74–77 (2007)
71. P.L. Duren, M. Schiffer, The theory of the second variation in extremum problems for univalent functions. J. Anal. Math. **10**, 193–252 (1962/1963)
72. P.L. Duren, *Univalent Functions* (Springer, New York, 1983)

73. C.J. Earle, A.L. Epstein, Quasiconformal variation of slit domains. Proc. Am. Math. Soc. **129**(11), 3363–3372 (2001)
74. L.D. Faddeev, *Discretized Virasoro Algebra*. Contemporary Mathematics, vol. 391 (American Mathematical Society, Providence, RI, 2005), pp. 59–67
75. M. Fekete, G. Szegö, Eine Bemerkung über ungerade schlichte Funktionen. J. Lond. Math. Soc. **8**(2), 85–89 (1933)
76. C.H. FitzGerald, Ch. Pommerenke, The de Branges theorem on univalent functions. Trans. Am. Math. Soc. **290**, 683–690 (1985)
77. J.E. Fornæss, N. Sibony, Increasing sequences of complex manifolds. Math. Ann. 255, 351–360 (1981)
78. J.E. Fornæss, B. Stensønes, Stable manifolds of holomorphic hyperbolic maps. Int. J. Math. **15**(8), 749–758 (2004)
79. S. Friedland, M. Schiffer, Global results in control theory with applications to univalent functions. Bull. Am. Math. Soc. **82**, 913–915 (1976)
80. S. Friedland, M. Schiffer, On coefficient regions of univalent functions. J. Anal. Math. **31**, 125–168 (1977)
81. R. Friedrich, The global geometry of stochastic Loewner evolutions. *Probabilistic Approach to Geometry*. Advanced Studies in Pure Mathematics, vol. 57 (Mathematical Society of Japan, Tokyo, 2010), pp. 79–117
82. R. Friedrich, W. Werner, Conformal restriction, highest-weight representations and SLE. Comm. Math. Phys. **243**(1), 105–122 (2003)
83. L.A. Galin, Unsteady filtration with a free surface. Dokl. Akad. Nauk USSR **47**, 246–249 (1945). (in Russian)
84. R.V. Gamkrelidze, *Foundations of Optimal Control Theory* (Tbilisi University, Tbilisi, 1977)
85. P.R. Garabedian, M.A. Schiffer, A proof of the Bieberbach conjecture for the fourth coefficient. J. Rational Mech. Anal. **4**, 427–465 (1955)
86. C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Korteweg-deVries equation and generalization. VI. Methods for exact solution. Comm. Pure Appl. Math. **27**, 97–133 (1974)
87. F.W. Gehring, E. Reich, Area distortion under quasiconformal mappings. Ann. Acad. Sci. Fenn. Ser. A I **388**, 1–14 (1966)
88. J.-L. Gervais, Infinite family of polynomial functions of the Virasoro generators with vanishing Poisson brackets. Phys. Lett. B **160**(4–5), 277–278 (1985)
89. l. Gibbons, S.P. Tsarev, Conformal maps and reductions of the Benney equations. Phys. Lett. A **258**(4–6), 263–271 (1999)
90. G.M. Goluzin, Über die Verzerrungssätze der schlichten konformen Abbildungen. Rec. Math. [Mat. Sbornik] N.S. **1(34)**(1), 127–135 (1936)
91. G.M. Goluzin, Sur les théorèmes de rotation dans la théorie des fonctions univalentes. Rec. Math. [Mat. Sbornik] N.S. **1(43)**(1), 293–296 (1936)
92. G.M. Goluzin, On the parametric representation of functions univalent in a ring. Mat. Sbornik N.S. **29(71)**, 469–476 (1951)
93. G.M. Goluzin, *Geometrical Theory of Functions of a Complex Variable*, 2nd edn. (Nauka, Moscow, 1966)
94. G. S. Goodman, Pontryagins Maximal Principle as a variational method in conformal mapping. Abstracts of Brief Scientific Communications, sec. 13 (ICM Moscow, 1966), pp. 7–8
95. G.S. Goodman, Univalent functions and optimal control, Ph.D. Thesis, Stanford University, 1968
96. V.V. Goryainov, V.Ya. Gutlyanskiĭ, *Extremal Problems in the Class* S_M, Mathematics collection (Russian) (Izdat. "Naukova Dumka", Kiev, 1976), pp. 242–246
97. V.V. Gorjainov, On a parametric method of the theory of univalent functions. Math. Notes **27**(3–4), 275–279 (1980)
98. V.V. Gorjainov, Boundary functions of a system of functionals composed of values of a univalent function and its derivative. Soviet Math. (Iz. VUZ) **26**(7), 91–94 (1982)

99. V.V. Gorjainov, *Extremals in Estimates of Functionals Depending on Values of a Univalent Function and Its Derivative*. Theory of Mappings and Approximation of Functions ("Naukova Dumka", Kiev, 1983), pp. 38–49
100. V.V. Gorjainov, The Parametric Method and Extremal Conformal Mappings. Soviet Math. Dokl. **28**(1), 205–208 (1983)
101. V.V. Goryainov, Semigroups of conformal mappings. Mat. Sb. (N.S.) **129(171)**(4), 451–472 (1986) (Russian); translation in Math. USSR Sbornik **57**, 463–483 (1987)
102. V.V. Goryajnov, Evolution families of analytic functions and time-inhomogeneous Markov branching processes. Dokl. Akad. Nauk **347**(6) 729–731 (1996); translation in Dokl. Math. **53**(2), 256–258 (1996)
103. V.V. Goryainov, *The Löwner – Kufarev equation and extremal problems for conformal mappings with fixed points on the boundary* (in Russian). Talk at the International Workshop on Complex Analysis and its Applications, Steklov Mathematical Institute, Moscow, 26 December 2011
104. V.V. Goryainov, I. Ba, Semigroups of conformal mappings of the upper half-plane into itself with hydrodynamic normalization at infinity. Ukrainian Math. J. **44**, 1209–1217 (1992)
105. V.V. Goryainov, O.S. Kudryavtseva, One-parameter semigroups of analytic functions, fixed points and the Koenigs function. Mat. Sb. **202**(7), 43–74 (2011) (Russian); translation in Sbornik: Mathematics **202**(7), 971–1000 (2011)
106. V.V. Goryainov, Semigroups of analytic functions in analysis and applications. Uspekhi Mat. Nauk **67**(6), 5–52 (2012)
107. I. Graham, H. Hamada, G. Kohr, Parametric representation of univalent mappings in several complex variables. Can. J. Math. **54**, 324–351 (2002)
108. I. Graham, G. Kohr, *Geometric Function Theory in One and Higher Dimensions* (Marcel Dekker, New York, 2003)
109. I. Graham, H. Hamada, G. Kohr, M. Kohr, Asymptotically spirallike mappings in several complex variables. J. Anal. Math. **105** 267–302 (2008)
110. I. Graham, G. Kohr, J.A. Pfaltzgraff, The general solution of the Loewner differential equation on the unit ball in $\mathbb{C}^n$. Contemp. Math. (AMS) **382**, 191–203 (2005)
111. I. Graham, G. Kohr, *Geometric Function Theory in One and Higher Dimensions* (Marcel Dekker, New York, 2003)
112. R. Greiner, O. Roth, On support points of univalent functions and a disproof of a conjecture of Bombieri. Proc. Am. Math. Soc. **129**(12), 3657–3664 (2001)
113. D. Grier, E. Ben-Jacob, R. Clarke, L.M. Sander, Morphology and Microstructure in Electrochemical Deposition of Zinc. Phys. Rev. Lett. **56**, 1264–1267 (1986)
114. B. Gustafsson, A. Vasil'ev, *Conformal and Potential Analysis in Hele-Shaw Cells* (Birkhäuser, Basel, 2006)
115. V.Ya. Gutlyanskiĭ, Parametric representation of univalent functions. Dokl. Akad. Nauk SSSR **194**, 750–753 (1970)
116. V.Ya. Gutljanskǐ, *Parametric Representations and Extremal Problems in the Theory of Univalent Functions* (in Russian), Dissertation Dokt. Fiz.-Mat. Nauk, Mat. Inst. Akad. Nauk Ukrain. SSR, Kiev, 1972
117. H. Hamada, G. Kohr, J.R. Muir, Extension of L^d-Loewner chains to higher dimensions. J. Anal. Math. 120(1), 357–392 (2013)
118. T.E. Harris, *The Theory of Branching Processes*. Corrected reprint of the 1963 original, Dover Phoenix Editions (Dover Publications, Mineola, NY, 2002)
119. T.C. Halsey, Diffusion-Limited Aggregation: A Model for Pattern Formation. Phys. Today **53**(11), 36 (2000)
120. M.B. Hastings, Renormalization theory of stochastic growth. Phys. Rev. E **55**(1), 135–152 (1987)
121. M.B. Hastings, L.S. Levitov, Laplacian growth as one-dimensional turbulence. Phys. D **116**(1–2), 244–252 (1998)
122. W.K. Hayman, *Multivalent Functions* (Cambridge University Press, Cambridge, 1958)

123. M.H. Heins, Semigroups of holomorphic maps of a Riemann surface into itself which are homomorphs of the set of positive reals considered additively, in *E.B. Christoffel—The Influence of His Work on Mathematics and the Physical Sciences* (Bikhäuser, Basel, 1981), pp. 314–331
124. H.S. Hele-Shaw, The flow of water. Nature **58**, 33–36 (1898)
125. Yu.E. Hohlov, S.D. Howison, C. Huntingford, J.R. Ockendon, A.A. Lacey, A model for non-smooth free boundaries in Hele-Shaw flows. Q. J. Mech. Appl. Math. **47**, 107–128 (1994)
126. G. Ivanov, A. Vasil'ev, Lï£¡wner evolution driven by a stochastic boundary point. Anal. Math. Phys. **1**(4), 387–412 (2011)
127. G. Ivanov, D. Prokhorov, A. Vasil'ev, Non-slit and singular solutions to the Loewner equation. Bull. Sci. Math. **136**(3), 328–341 (2012)
128. V.G. Kac, Simple irreducible graded Lie algebras of finite growth. Math. USSR Izv. **2**, 1271–1311 (1968); Izv. Akad. Nauk USSR Ser. Mat. **32**, 1923–1967 (1968)
129. W. Kager, B. Nienhuis, L.P. Kadanoff, Exact solutions for Löwner evolutions. J. Stat. Phys. **115**(3–4), 805–822 (2004)
130. N.-G. Kang, N.G. Makarov, Gaussian free field and conformal field theory. arXiv: 1101.1024 [math.PR], 2011
131. N.-G. Kang, N.G. Makarov, Radial SLE martingale-observables. arXiv:1209.1615 [math.PR], 2012
132. H. Kesten, Hitting probabilities of random walks on $\mathbb{Z}^d$. Stoch. Proc. Appl. **25**, 165–184 (1987)
133. A.A. Kirillov, D.V. Yuriev, Representations of the Virasoro algebra by the orbit method. J. Geom. Phys. **5**(3), 351–363 (1988)
134. S. Kobayashi, *Hyperbolic Complex Spaces* (Springer, Berlin Heidelberg, 1998)
135. Y. Komatu, Über eine Verschärfung des Löwnerschen Hilfssatzes. Proc. Imp. Acad. Tokyo **18**, 354–359 (1942)
136. Y. Komatu, On conformal slit mapping of multiply-connected domains. Proc. Jpn. Acad. **26**(7), 26–31 (1950)
137. E.L. Korotyaev, P. Kargaev, Inverse problems generated by conformal mappings on complex plane with parallel slits. 2009, unpublished preprint.
138. E.L. Korotyaev, P. Kargaev, Estimates for periodic Zakharov–Shabat operators. J. Differ. Equat. **249**, 76–93 (2010)
139. E.L. Korotyaev, S. Kuksin, KdV Hamiltonian as function of actions. arXiv 1110.4475, 2011
140. I.K. Kostov, I. Krichever, M. Mineev-Weinstein, P.B. Wiegmann, A. Zabrodin, The τ-function for analytic curves. *Random Matrix Models and Their Applications*. Mathematical Sciences Research Institute Publications, vol. 40 (Cambridge University Press, Cambridge, 2001), pp. 285–299
141. I. Krichever, M. Mineev-Weinstein, P. Wiegmann, A. Zabrodin, Laplacian growth and Whitham equations of soliton theory. Phys. D **198**(1–2), 1–28 (2004)
142. I. Krichever, A. Marshakov, A. Zabrodin, Integrable structure of the Dirichlet boundary problem in multiply-connected domains. Comm. Math. Phys. **259**(1), 1–44 (2005)
143. J. Krzyż, The radius of close-to-convexity within the family of univalent functions. Bull. Acad. Polon. Sci. **10**, 201–204 (1962)
144. P.P. Kufarev, On one-parameter families of analytic functions. Rec. Math. [Mat. Sbornik] N.S. **13(55)**, 87–118 (1943)
145. P.P. Kufarev, On integrals of a simplest differential equation with movable polar singularity in the right-hand side. Tomsk. Gos. Univ. Uchen. Zap. **1**, 35–48 (1946)
146. P.P. Kufarev, Theorem on solutions to some differential equation. Tomsk. Gos. Univ. Učenye Zapiski **5**, 20–21 (1947)
147. P.P. Kufarev, A remark on integrals of Löwner's equation. Dokl. Akad. Nauk SSSR (N.S.) **57**, 655–656 (1947)
148. P.P. Kufarev, On conformal mapping of complementary regions. Dokl. Akad. Nauk SSSR (N.S.) **73**(5), 881–884 (1950)

149. P.P. Kufarev, A.E. Fales, On an extremal problem for complementary regions. Dokl. Akad. Nauk SSSR (N.S.) **81**(6), 995–998 (1950)
150. P.P. Kufarev, A.E. Fales, On an extremal problem for complementary regions. Tomsk. Gos. Univ. Ucen. Zap. **17**, 25–35 (1947)
151. P.P. Kufarev, V.V. Sobolev, L.V. Sporysheva, A certain method of investigation of extremal problems for functions that are univalent in the half-plane. Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **200**, 142–164 (1968)
152. M.P. Kuvaev, P.P. Kufarev, An equation of Löwner's type for multiply connected regions. Tomsk. Gos. Univ. Uv. c. Zap. Mat. Meh. **25**, 19–34 (1955)
153. G.V. Kuz'mina, Moduli of families of curves and quadratic differentials. Trudy Mat. Inst. Steklov. **139**, 241 (1980)
154. M.A. Lavrent'ev, To the theory of conformal maps. Trav. Inst. Math. Stekloff **5**, 159–245 (1934)
155. G.F. Lawler, An introduction to the stochastic Loewner evolution, in *Random Walks and Geometry*, ed. by V. Kaimonovich (de Gruyter, Berlin , 2004), pp. 261–293
156. G.F. Lawler, *Conformally Invariant Processes in the Plane*, Mathematical Surveys and Monographs, vol. 114 (American Mathematical Society, Providence, RI, 2005)
157. G. Lawler, O. Schramm, W. Werner, Values of Brownian Intersection Exponents. I. Half-plane Exponents. Acta Math. **187**(2), 237–273 (2001); Values of Brownian intersection exponents. II. Plane exponents. Acta Math. **187**(2), 275–308 (2001); Values of Brownian intersection exponents. III. Two-sided exponents. Ann. Inst. H. Poincaré. Probab. Statist. **38**(1), 109–123 (2002)
158. N.A. Lebedev, On parametric representation of functions regular and univalent in a ring (in Russian). Dokl. Akad. Nauk SSSR (N.S.) **103**, 767–768 (1955)
159. N.A. Lebedev, *The Area Principle in the Theory of Univalent Functions* (Izdat. "Nauka", Moscow, 1975)
160. P. Lévy, *Processus Stochastiques et Mouvement Brownian* (Gauthier-Villars, Paris, 1947)
161. E.P. Li, On the theory of univalent functions on a circular ring. Dokl. Akad. Nauk SSSR (N.S.) **92**, 475–477 (1953)
162. J. Lind, A sharp condition for the Löwner equation to generate slits. Ann. Acad. Sci. Fenn. Math. **30**(1), 143–158 (2005)
163. J. Lind, D. Marshall, S. Rohde, Collisions and spirals of Loewner traces. Duke Math. J. **154**(3), 527–573 (2010)
164. J.E. Littlewood, R.E.A.C. Paley, A proof that an odd schlicht function has bounded coefficients. J. Lond. Math. Soc. **7**, 167–169 (1932)
165. K. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. Math. Ann. **89**, 103–121 (1923)
166. K. Löwner, Über monotone Matrixfunktionen. Math. Z. **38**, 177–216 (1934)
167. Ch. Loewner, Conservation laws in compressible fluid flow and associated mapping. J. Ration. Mech. Anal. **2**, 537–561 (1953)
168. B. Mandelbrot, Fractal aspects of the iteration of $z \to \lambda z(1-z)$ for complex λ and z, in *Non-Linear Dynamics*, ed. by Robert H.G. Helleman. Annals of the New York Academy of Sciences, vol. 357 (New York Academy of Sciences, New York, 1979), pp. 249–259
169. I. Markina, D. Prokhorov, A. Vasil'ev, Sub-Riemannian geometry of the coefficients of univalent functions. J. Funct. Anal. **245**(2), 475–492 (2007)
170. I. Markina, A. Vasil'ev, Virasoro algebra and dynamics in the space of univalent functions. Contemp. Math. **525**, 85–116 (2010)
171. I. Markina, A. Vasil'ev, Evolution of smooth shapes and integrable systems. ArXiv: 1108.1007 [math-ph], 2011, pp. 25
172. A. Marshakov, P. Wiegmann, A. Zabrodin, Integrable structure of the Dirichlet boundary problem in two dimensions. Comm. Math. Phys. **227**(1), 131–153 (2002)
173. D.E. Marshall, S. Rohde, The Löwner differential equation and slit mappings. J. Am. Math. Soc. **18**(4), 763–778 (2005)

174. M. Mineev-Weinstein, P.B. Wiegmann, A. Zabrodin, Integrable structure of interface dynamics. Phys. Rev. Lett. **84**(22), 5106–5109 (2000)
175. M. Mineev-Weinstein, A. Zabrodin, Whitham-Toda hierarchy in the Laplacian growth problem. J. Nonlinear Math. Phys. **8**, 212–218 (2001)
176. R.V. Moody, A new class of Lie algebras. J. Algebra **10**, 211–230 (1968)
177. J. Muñoz Porras, F. Pablos Romo, Generalized reciprocity laws. Trans. Am. Math. Soc. **360**(7), 3473–3492 (2008)
178. E. Nelson, The free Markov field. J. Funct. Anal. **12**, 221–227 (1973)
179. L. Niemeyer, L. Pietronero, H.J. Wiessmann, Fractal dimension of dielectric breakdown. Phys. Rev. Lett. **52**(12), 1033–1036 (1984)
180. J. O'Connor, E.F. Robertson, *Charles Loewner*, MacTutor History of Mathematics archive, University of St Andrews
181. K. Osterwalder, R. Schrader, Axioms for Euclidean Green's functions. Comm. Math. Phys. **31**, 83–112 (1973); **42**, 281–305 (1975)
182. E. Peschl, Zur Theorie der schlichten Funktionen. J. Reine Angew. Math. **176**, 61–94 (1936)
183. M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum Field Theory* (Perseus Books, Cambridge, 1995)
184. J.A. Pfaltzgraff, Subordination chains and univalence of holomorphic mappings in $\mathbb{C}^n$. Math. Ann. **210**, 55–68 (1974)
185. J.A. Pfaltzgraff, Subordination chains and quasiconformal extension of holomorphic maps in $\mathbb{C}^n$. Ann. Acad. Sci. Fenn. Ser. A I Math. **1**, 13–25 (1975)
186. G. Pick, Über die konforme Abbildung eines Kreises auf ein schlichtes und zugleich beschränktes Gebiet. S.-B. Kaiserl. Akad. Wiss. Wien, Math.-Natur. Kl. Abt. II a **126**, 247–263 (1917)
187. P.Ya. Polubarinova-Kochina, On a problem of the motion of the contour of a petroleum shell. Dokl. Akad. Nauk USSR **47**(4), 254–257 (1945)
188. P.Ya. Polubarinova-Kochina, Concerning unsteady motions in the theory of filtration. Prikl. Matem. Mech. **9**(1), 79–90 (1945)
189. Ch. Pommerenke, Über die Subordination analytischer Funktionen. J. Reine Angew. Math. **218**, 159–173 (1965)
190. Ch. Pommerenke, *Univalent Functions, with a Chapter on Quadratic Differentials by G. Jensen* (Vandenhoeck & Ruprecht, Göttingen, 1975)
191. Ch. Pommerenke, *Boundary Behaviour of Conformal Maps* (Springer, Berlin, 1992)
192. L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, *The Mathematical Theory of Optimal Processes* (A Pergamon Press Book. The Macmillan Co., New York, 1964)
193. T. Poreda, On generalized differential equations in Banach spaces. Diss. Math. **310**, 1–50 (1991)
194. V.I. Popov, The range of a system of functionals on the class S. Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **182**(3), 106–132 (1965)
195. V.I. Popov, L.S. Pontrjagin's maximum principle in the theory of univalent functions. Dokl. Akad. Nauk SSSR **188**, 532–534 (1969)
196. N.V. Popova, Connection between the Löwner equation and the equation $\frac{dw}{dt} = \frac{1}{w-\lambda(t)}$. Izv. Acad. Sci. Belorussian SSR **6**, 97–98 (1954)
197. A. Pressley, G. Segal, *Loop Groups*. Oxford Mathematical Monographs (Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1986)
198. D.V. Prokhorov, The geometric characterization of functions from Bazilevič subclasses. Sov. Math. (Iz. VUZ) **19**(2), 106–107 (1975)
199. D.V. Prokhorov, Optimal control method in an extremal problem on the class of univalent functions. Dokl. Akad. Nauk SSSR **275**(4), 798–801 (1984)
200. D.V. Prokhorov, Value sets for systems of functionals in the classes of univalent functions. Mat. Sb. **181**(12), 1659–1677 (1990)
201. D.V. Prokhorov, The value set of initial coefficients for bounded univalent typically real functions. Sib. Mat. Zh. **32**(5), 132–141 (1990)

202. D.V. Prokhorov, *Reachable Set Methods in Extremal Problems for Univalent Functions* (Saratov University, Saratov, 1993)
203. D.V. Prokhorov, Even coefficient estimates for bounded univalent functions. Ann. Polon. Math. **58**(3), 267–273 (1993)
204. D.V. Prokhorov, Coefficients of functions close to the identity function. Complex Variables Theor. Appl. **33**(1–4), 255–263 (1997)
205. D.V. Prokhorov, Coefficients of holomorphic functions. Complex analysis and representation theory, 2. J. Math. Sci. (New York) **106**(6), 3518–3544 (2001)
206. D.V. Prokhorov, A.A. Nikulin, Asymptotic estimates for the coefficients of bounded univalent functions. Sibirsk. Mat. Zh. **47**(5), 1128–1138 (2006)
207. D. Prokhorov, A. Vasil'ev, Univalent functions and integrable systems. Comm. Math. Phys. **262**(2), 393–410 (2006)
208. D. Prokhorov, A. Vasil'ev, Optimal control in Bombieri's and Tammi's conjectures. Georgian Math. J. **12**(4), 743–761 (2005)
209. D. Prokhorov, A. Vasil'ev, Singular and tangent slit solutions to the Löwner equation, in *Analysis and Mathematical Physics*, Trends in Mathematics (Birkhäuser, Basel, 2009), pp. 455–463
210. M. Reed, B. Simon, *Methods of Modern Mathematical Physics. I. Functional Analysis*, 2nd edn. (Academic Press, New York, 1980), p. 400
211. S. Reich, D Shoikhet, Metric domains, holomorphic mappings and nonlinear semigroups. Abstr. Appl. Anal. **3**(1–2), 203–228 (1998)
212. S. Reich, D. Shoikhet, *Nonlinear Semigroups, Fixed Points, and Geometry of Domains in Banach Spaces* (Imperial College Press, London, 2005)
213. M.S. Robertson, A remark in the odd schlicht functions. Bull. Am. Math. Soc. **42**(6), 366–370 (1936)
214. M.S. Robertson, Applications of the subordination principle to univalent functions. Pacific J. Math. **11**, 315–324 (1961)
215. R.M. Robinson, Bounded univalent functions. Trans. Am. Math. Soc. **52**, 426–449 (1942)
216. S. Rohde, O. Schramm, Basic properties of SLE. Ann. Math. **161**(2), 883–924 (2005)
217. S. Rohde, Oded Schramm: from circle packing to SLE. Ann. Probab. **39**(5), 1621–1667 (2011)
218. M. Rosenblum, J. Rovnyak, *Topics in Hardy Classes and Univalent Functions* (Birkhäuser Verlag, Basel, 1994)
219. M. Röckner, Markov property of generalized fields and axiomatic potential theory. Math. Ann. **264**, 153–177 (1983)
220. M. Röckner, Generalized Markov fields and Dirichlet forms. Acta Applicandae Math. **3**, 285–311 (1985)
221. I. Rushkin, E. Bettelheim, I.A. Gruzberg, P. Wiegmann, Critical curves in conformally invariant statistical systems. J. Phys. A Math. Theor. **40**, 2165–2195 (2007)
222. M. Sato, Y. Sato, Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold. *Nonlinear Partial Differential Equations in Applied Science Tokyo, 1982*. North-Holland Mathematics Studies, vol. 81 (North-Holland, Amsterdam, 1983), pp. 259–271
223. A. Savakis, S. Maggelakis, Models of shrinking clusters with applications to epidermal wound healing. Math. Comput. Model. **25**(6), 1–6 (1997)
224. G. Segal, G. Wilson, Loop groups and equations of KdV type. Publ. Math. IHES **61**, 5–65 (1985)
225. A.C. Schaeffer, D.C. Spencer, The coefficients of schlicht functions. II **12**, 107–125 (1945)
226. A.C. Schaeffer, D.C. Spencer, *Coefficient Regions for Schlicht Functions*. American Mathematical Society Colloquium Publications, vol. 35 (American Mathematical Society, New York, NY, 1950)
227. M. Schiffer, D.C. Spencer, *Functionals of Finite Riemann Surfaces* (Princeton University Press, Princeton, NJ, 1954)
228. M. Schiffer, O. Tammi, On the fourth coefficient of bounded univalent functions. Trans. Am. Math. Soc. **119**(1965), 67–78

229. M. Schottenloher, *A Mathematical Introduction to Conformal Field Theory*, 2nd edn. Lecture Notes in Physics, vol. 759 (Springer, Berlin, 2008)
230. O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees. Isr. J. Math. **118**, 221–288 (2000)
231. O. Schramm, S. Sheffield, Contour lines of the two-dimensional discrete Gaussian free field. Acta Math. **202**(1), 21–137 (2009)
232. O. Schramm, S. Sheffield, A contour line of the continuum Gaussian free field. arXiv: 1008.2447, 2010
233. D.-S. Shah, Le-le Fan, On the parametric representation of quasi-conformal mappings. Sci. Sin. **11**, 149–162 (1962)
234. S. Sheffield, Exploration trees and conformal loop ensembles. Duke Math. J. **147**(1), 79–129 (2009)
235. T. Sheil-Small, Some remarks on Bazilevič functions. J. Anal. Math. **43**, 1–11 (1983/1984)
236. B.M. Shein, *Private communication*, 2012
237. L. Siewierski, Sharp estimation of the coefficients of bounded univalent functions close to identity. Diss. Math. **86**, 1–153 (1971)
238. A.G. Siskakis, Semigroups of composition operators and the Cesàro operator on $H^p(D)$, PhD Thesis, University of Illinois, 1985
239. A.G. Siskakis, Semigroups of composition operators on spaces of analytic functions, a review. Contemp. Math. **213**, 229–252 (1998)
240. S. Smirnov, Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. C. R. Acad. Sci. Paris Sér. I Math. **333**, 239–244 (2001)
241. T.T. Soong, *Random Differential Equations in Science and Engineering*. Mathematics in Science and Engineering, vol. 103 (Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1973)
242. Lad. Špaček, Contribution à la théorie des fonctions univalentes. Čas. Mat. Fys. **62**(2), 12–19 (1932)
243. T. Takebe, L.-P. Teo, A. Zabrodin, Löwner equations and dispersionless hierarchies. J. Phys. A **39**(37), 11479–11501 (2006)
244. O. Tammi, On the maximalization of the coefficient a_3 of bounded schlicht functions. Ann. Acad. Sci. Fennicae. Ser. A. I. Math. Phys. **149**, 1–14 (1953)
245. O. Tammi, On optimizing parameters of the power inequality for a_4 in the class of bounded univalent functions. Ann. Acad. Sci. Fennicae. Ser. A. I. Math. Phys. **560**, 1–24 (1973)
246. O. Tammi, *Extremal Problems for Bounded Univalent Functions*, Lecture Notes in Mathematics, vol. 646 (Springer, New York, 1978)
247. O. Tammi, *Extremal Problems for Bounded Univalent Functions, II*, Lecture Notes in Mathematics, vol. 913 (Springer, New York, 1982)
248. F. Tricomi, Sull'iterazione delle funzioni di linee. Giorn. Math. Battaglini **55**, 1–8 (1917)
249. J. Tsai, The Loewner driving function of trajectory arcs of quadratic differentials. J. Math. Anal. Appl. **360**, 561–576 (2009)
250. A. Vasil'ev, Mutual change of initial coefficients of univalent functions. Matemat. Zametki **38**(1), 56–65 (1985)
251. A. Vasil'ev, Value set $\{f(r_1), f(r_2)\}$ for the class of univalent functions with real coefficients. Sib. Mat. Zh. **27**(6), 28–32 (1986)
252. A. Vasil'ev, Optimal control methods in an extremal problem on a class of solutions of the Loewner-Kufarev equation. Diff. Uravneniya **26**(3), 386–392 (1990)
253. A. Vasil'ev, From the Hele-Shaw experiment to integrable systems: a historical overview. Complex Anal. Oper. Theor. **3**(2), 551–585 (2009)
254. T. Vicsek, *Fractal Growth Phenomena* (World Scientific Publishing, Singapore, 1989)
255. M. Voda, Solution of a Loewner chain equation in several variables. J. Math. Anal. Appl. **375**(1), 58–74 (2011)
256. G.C. Wick, The evaluation of the collision matrix. Phys. Rev. **80**(2), 268–272 (1950)
257. P.B. Wiegmann, A. Zabrodin, Conformal maps and integrable hierarchies. Comm. Math. Phys. **213**(3), 523–538 (2000)

258. A.S. Wightman, *Hilberts Sixth Problem: Mathematical Treatment of the Axioms of Physics*, ed. by F.E. Browder. Proceedings of Symposia in Pure Mathematics (part 1), vol. 28 (American Mathematical Society, Providence, RI, 1976), pp. 241–268
259. E. Witten, Quantum field theory, Grassmannians, and algebraic curves. Comm. Math. Phys. **113**(4), 529–600 (1988)
260. T.A. Witten, Jr., L.M. Sander, Diffusion-limited aggregation, and kinetic critical phenomenon. Phys. Rev. Lett. **47**(2), 1400–1403 (1981)
261. T.A. Witten, L.M. Sander, Diffusion-limited aggregation. Phys. Rev. B **27**(9), 5686–5697 (1983)
262. N.J. Zabusky, M.D. Krushkal, Interaction of ï£¡solitonsï£¡ in a collisionless plasma and initial states. Phys. Rev. Lett. **15**, 240–243 (1965)
263. V.E. Zaharov, L.D. Faddeev, The Korteweg-de Vries equation is a fully integrable Hamiltonian system. Funkcional. Anal. i Prilozh. **5**(5), 18–27 (1971)



http://www.springer.com/978-3-319-01805-8

Harmonic and Complex Analysis and its Applications
Vasiliev, A. (Ed.)
2014, VIII, 358 p. 47 illus., 11 illus. in color., Hardcover
ISBN: 978-3-319-01805-8
A product of Birkhäuser Basel