Chapter 2
Cosmology

From the four types of interactions present in our world, only electromagnetism and gravity are long-range interactions. The weak and the strong interactions are so short-ranged that they cannot be responsible for the large-scale behaviour of our universe. However, despite its long range character, electromagnetism cannot be the dominant force in an electrically neutral universe. The only long range interaction that remains is gravity, whose source is energy density. Thus, a theory of the cosmos must be based on a theory of gravity. The best gravitational theory we have so far is Einstein's theory of General Relativity [9, 10]. In contrast to the Standard Model, which is a *quantum* field theory, General Relativity is a *classical* field theory. Aside from this, the distinctive feature of gravity compared to the other fundamental interactions is that it is not a theory defined *on* space-time, but a theory *of* space-time itself. The physical foundation of General Relativity is the equivalence principle, which states that gravity *uniformly* couples to *all* kind of energy density. Mathematically, space-time is described by a pseudo-Riemannian manifold $\mathcal{M}$ and the gravitational field by the metric field $g_{\mu\nu}(x)$ on $\mathcal{M}$. The gravitational interaction manifests itself geometrically as curvature of space-time.

2.1 General Relativity

General Relativity has replaced Newton's theory of gravity, but we can recover it in the limit of small velocities and vanishing curvature. General Relativity is a well-tested theory [31]. However, Einstein's theory is not only more accurate than Newton's, but also compels us to modify the fundamental concepts of space and time. The geometrical interpretation of gravity as curvature of space-time suggests that the gravitational action must contain information about the curvature. The action is a scalar functional and the simplest scalar that can be formed of the Riemannian curvature tensor $R_{\sigma\rho\mu\nu}$ is the Ricci scalar R, see Appendix A for a definition of curvature quantities. The Ricci scalar R is already of second order in the derivatives

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of the metric field $g_{\mu\nu}$. In order to obtain the correct Newtonian limit, the action must in addition be proportional to Newton's constant G. From dimensional considerations in four space-time dimensions, it follows that this prefactor must be proportional to $\kappa := 8\pi G/c^4$ or $\kappa = 1/M_\mathrm{P}^2$ in units $c = \hbar = 1$. Thus, scalar contractions of higher powers in the curvature would already correspond to higher derivative theories. For a general invariant volume element $\sqrt{-g}\,\mathrm{d}^4\mathrm{x}$, the simplest action is therefore given by

$$S_\mathrm{g} := \frac{c^4}{16\pi G} \int \mathrm{d}^4x \sqrt{-g}\; (R - 2\,\Lambda) + \text{surface terms}, \tag{2.1}$$

with the cosmological constant Λ and surface terms which we neglect in the following. This action is denoted Einstein–Hilbert action and describes the geometrical degrees of freedom. The matter content is encoded in

$$S_\mathrm{m} := \int \sqrt{-g}\,\mathrm{d}^4x\; \mathcal{L}_\mathrm{m}. \tag{2.2}$$

We define the total gravitational and matter action as

$$S := S_\mathrm{g} + S_\mathrm{m}. \tag{2.3}$$

The first functional derivative of (2.1) with respect to the metric field $g_{\mu\nu}(x)$ yields

$$\frac{\delta S_\mathrm{g}[g]}{\delta g_{\alpha\beta}} = \int \frac{\mathrm{d}^4x}{2\,\kappa} \left[\frac{\delta\sqrt{-g}}{\delta g_{\alpha\beta}}\; (R - 2\,\Lambda) + \sqrt{-g}\,\frac{\delta\, R}{\delta g_{\alpha\beta}} \right]. \tag{2.4}$$

Using the formulas derived in Appendix B.4, we find[1]

$$\frac{\delta R(x')}{\delta g_{\alpha\beta}(x)} = \Big[- R^{\mu\nu} + \nabla^\mu\nabla^\nu - g^{\mu\nu}\Box \Big] \delta^{\alpha\beta}_{\mu\nu}\, \delta(x, x'). \tag{2.5}$$

Inserting this into (2.4) and using again the formulas of Appendix B.4 yields

$$\frac{\delta S_\mathrm{g}[g(x')]}{\delta g_{\alpha\beta}(x)} = \int \mathrm{d}^4x\, \frac{\sqrt{-g}}{2\,\kappa} \left(\frac{1}{2}\, R\, g^{\mu\nu} - R^{\mu\nu} - \Lambda\, g^{\mu\nu} + \nabla^\mu\nabla^\nu - g^{\mu\nu}\Box \right) \delta^{\alpha\beta}_{\mu\nu}\, \delta(x, x'). \tag{2.6}$$

The last two terms in (2.6) are total derivatives and can be converted into surface terms. Integrating out the delta function in (2.6), the variation of (2.3) yields

$$\frac{\delta\, S}{\delta\, g_{\alpha\beta}} = \frac{\sqrt{-g}}{2\,\kappa} \left(\frac{1}{2}\, R\, g^{\alpha\beta} - R^{\alpha\beta} - \Lambda\, g^{\alpha\beta} \right) + \frac{\delta\, S_\mathrm{m}}{\delta\, g_{\alpha\beta}}. \tag{2.7}$$

[1] We use the sign convention: signature $(-1, 1, 1, 1)$, $+R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \ldots$ and $R^{\sigma}{}_{\mu\sigma\nu} = +R_{\mu\nu}$.

We define the energy-momentum tensor by

$$T^{\alpha\beta} := \frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}}{\delta g_{\alpha\beta}}. \tag{2.8}$$

The extremal condition $\delta S / \delta g_{\alpha\beta} = 0$ leads to the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \kappa T_{\mu\nu}. \tag{2.9}$$

They describe the interplay between the matter content and the geometry of space-time and form a set of ten coupled non-linear partial differential equations. With the definition of the Einstein tensor

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.10}$$

we can write the field equations (2.9) in a more compact form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.11}$$

Using the contracted Bianchi identities and the symmetries of the Riemann tensor derived in Appendix A, we find the covariant identity

$$\nabla^{\mu} G_{\mu\nu} = 0. \tag{2.12}$$

In combination with the metric compatibility relation $\nabla_{\alpha} g_{\mu\nu} = 0$, this implies the *covariant* conservation law for the energy-momentum tensor

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{2.13}$$

2.1.1 Symmetry Reduction

We are interested in solutions of (2.11) which can describe, at least approximately, our observable universe. Empirical observations suggest that the spatial part of our universe is homogeneous and isotropic averaged over large scales $\gtrsim$100 Mpc. Since there is no evidence why we should have a distinguished position in space, we should not only demand isotropy with respect to *our* position, but with respect to *all* locations in the universe. This additional assumption is called "cosmological principle". It implies that on large scales the universe looks the same for *all* observers, no matter at which *spatial* point they are placed.

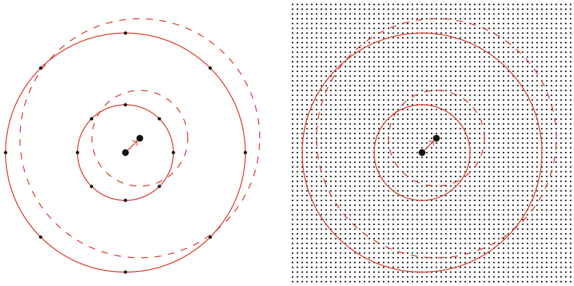

Fig. 2.1 The cosmological principle. The *black dots* represent galaxies. *Left* Isotropy only around the centre. *Right* Isotropy with respect to all points

2.2 Friedmann–Robertson–Walker Universe

By assuming a globally hyperbolic manifold with topology[2] $\mathcal{M} = \mathbb{R} \times \Sigma$, we can follow the Arnowitt–Deser–Misner-method [2] and decompose a four-dimensional manifold $\mathcal{M}$ into slices of three-dimensional hypersurfaces Σ_t at constant time t

$$\mathrm{d}s^2 = \Big[-N^2(\mathbf{x},\ t) + N_i(\mathbf{x},\ t)N^i(\mathbf{x},\ t)\Big]\,\mathrm{d}t^2 + 2\,N_i(\mathbf{x},\ t)\,\mathrm{d}t\,\mathrm{d}x^i + q_{ij}(\mathbf{x},\ t)\mathrm{d}x^i\,\mathrm{d}x^j \tag{2.14}$$

with spatial indices $i, j = 1, \ldots, 3$, the lapse function $N(\mathbf{x},\ t)$, the shift vector $N_i(\mathbf{x},\ t)$ and the three-dimensional spatial metric $q_{ij}(\mathbf{x},\ t)$ on Σ_t.

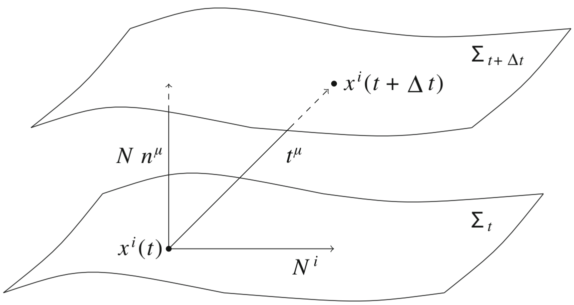

Fig. 2.2 We can decompose the full metric $g_{\mu\nu} = q_{\mu\nu} - n_\mu n_\mu$ into a part $q_{\mu\nu}$ that projects on Σ_t and a part orthogonal to Σ_t. The projector $q_{\mu\nu}$ satisfies $q_{\mu\nu}n^\nu = 0$ and $q_{\mu\nu}q^{\nu\rho} = q_\mu^{\ \rho}$, so that we can identify $q_{\mu\nu}$ with the induced three-dimensional metric q_{ij} on Σ_t. A *comoving* observer flows from $x^i \in \Sigma_t$ to $x^i \in \Sigma_{t+\Delta t}$ by means of the displacement vector t^μ. We have decomposed $t^\mu = N\,n^\mu + N^\mu$ into a component n^μ orthogonal and a component N^μ tangential to Σ_t. It is clear that N^μ is also a three-dimensional object, so that we can identify $N^\mu \equiv N^i$. The normal n^μ is a time-like unit vector $n_\mu n^\mu = -1$ so that the relation $N = -t^\mu n_\mu$ follows

[2] A globally hyperbolic manifold is equivalent to the existence of a Cauchy surface, i.e. given some initial data on a 3-hypersurface Σ_t, the evolution is uniquely determined by the equations of motion. There are also theoretical attempts to investigate whether our universe has a different topology, e.g. a torus, on the basis of experimental data, see e.g. [3, 6, 7, 18].

The symmetries of spatial homogeneity and isotropy are connected to three-dimensional translation and rotation invariance. Translation invariance means that $N(\mathbf{x},\ t)$, $N_i(\mathbf{x},\ t)$, $q_{ij}(\mathbf{x})$ in the decomposition (2.14) have to be independent of $\mathbf{x}$. Rotational invariance requires $N_i(t) \equiv 0$. Therefore, we can write (2.14) as

$$\mathrm{d}s^2 = -N^2(t)\,\mathrm{d}t^2 + q_{ij}(t)\mathrm{d}x^i\,\mathrm{d}x^j \tag{2.15}$$

Due to these symmetries, the ten independent degrees of freedom contained in the symmetric metric tensor $g_{\mu\nu}(\mathbf{x},\ t)$ reduce to one single degree of freedom, the scale factor $a(t)$ which is a function of time t only. In contrast to $a(t)$, the choice of the lapse function $N(t)$ has no physical meaning. A different choice of time can be parametrized by a different choice of the function $N(t)$. It is a pure gauge freedom, reflecting the fact that General Relativity is invariant under a reparametrization of time. Therefore, the lapse function $N(t)$ is no dynamical degree of freedom. Mathematically, it corresponds to a Lagrange multiplier in the action. Variation of the action with respect to N does not yield an equation of motion for N but a *constraint* equation instead. In fact, the homogeneous and isotropic line element (2.15) describes a maximally symmetric space-time which has $d\,(d+1)/2$ linearly independent Killing vectors, see (B.13). It is therefore a space of constant curvature parametrized by the real number $\mathfrak{K}$. The spaces of constant curvature can be divided into three different classes $\mathfrak{K} < 0,\ \mathfrak{K} = 0,\ \mathfrak{K} > 0$. These classes can be represented by the three values $\mathfrak{K} = -1,\ 0,\ 1$ for which the three-dimensional spatial slices Σ_t take the form of a three-dimensional hyperboloid, cube or sphere. According to these considerations, we can write (2.15) as

$$\mathrm{d}s^2 = -N^2(t)\,\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-\mathfrak{K}r^2} + r^2\,\mathrm{d}\Omega^2\right). \tag{2.16}$$

This line element describes a Friedmann–Robertson–Walker (FRW) space-time in spherical coordinates $(t,\ r,\ \theta,\ \phi)$, i.e. $\mathrm{d}\Omega^2 := \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.

2.2.1 Physical Versus Comoving Distance

The use of comoving quantities will simplify many considerations. We imagine for a moment a one dimensional spatially flat universe as a rubber band. The galaxies are attached to this rubber band at fixed positions, labeled by *fixed* coordinates $x_1,\ x_2, \ldots, x_n$. Changing the scale factor $a(t)$ means to stretch or shrink the rubber band, i.e., to separate or squeeze the galaxies. Since each galaxy has a fixed coordinate label x^i, homogeneity requires that the comoving *coordinate difference* $\Delta x = |x_i - x_j|$ is the same for all galaxies and remains the same for all times. The coordinates are "streaming" with the galaxies. However, the actual *physical distance* between the galaxies grows or shrinks according to the expansion or shrinking of the scale factor $a(t)$

$$D = a(t)\,\Delta x. \tag{2.17}$$

Whenever we feel that it is important to distinguish between a physical or comoving quantity, we indicate them with a subscript “p” or “c” respectively.

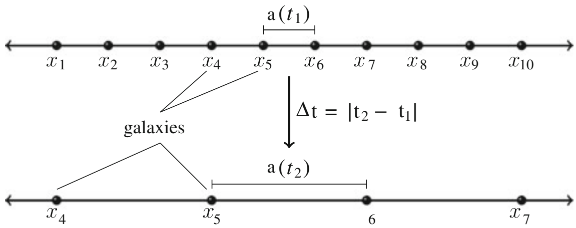

Fig. 2.3 The difference between the *comoving* coordinate difference Δx and the real *physical* distance between two neighbouring galaxies

2.2.2 Conformal Time

There is a very useful notion of time, called conformal time, which is defined by

$$\mathrm{d}\eta := \frac{N(t)\,\mathrm{d}t}{a(t)} \qquad \longleftrightarrow \qquad \eta_\mathrm{f} - \eta_\mathrm{i} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \frac{\mathrm{d}t\; N(t)}{a(t)}. \tag{2.18}$$

In terms of η, the FRW line-element (2.16) can be written as

$$\mathrm{d}s^2 = a^2(\eta)\,\left\{-\mathrm{d}\eta^2 + \left[\mathrm{d}\chi^2 + f(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]\right\}, \tag{2.19}$$

$$\text{with} \quad f(\chi) := \begin{cases} \sinh^2\chi & \text{for } \mathfrak{K} = -1 \\ \chi^2 & \text{for } \mathfrak{K} = 0 \\ \sin^2\chi & \text{for } \mathfrak{K} = +1. \end{cases} \tag{2.20}$$

In the flat case $\mathfrak{K} = 0$, we can identify χ with the radial coordinate r and (2.19) is related to the Minkowski line element by the conformal factor $a^2(\eta)$. The causal structure is fixed by null geodesics which satisfy $\mathrm{d}s^2 = 0$. Thus, for radial null geodesics we can draw the usual light cone structure as for Minkowski space.

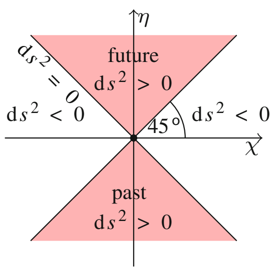

Fig. 2.4 Causal structure for radial null geodesics in conformal time η

2.2.3 Friedmann Equations

So far, we have only considered the kinematical aspects of the FRW universe. In order to describe the dynamics, we have to perform the symmetry reduction either at the level of the action (2.3) or directly for the Einstein equations (2.11). The geometrical part, including $R^{\rho}{}_{\mu\nu\sigma}$, $R_{\mu\nu}$, R and $\sqrt{-g}$, can be calculated for the line element (2.16). The square root of the determinant is given by

$$\sqrt{-g} = N\, a^3 \, \frac{r^2 \sin\theta}{\sqrt{1-\mathfrak{K}\, r^2}}. \tag{2.21}$$

In the following, no sum is performed over identical Latin indices unless it is explicitly stated. The non-vanishing components of the Christoffel symbols are

$$\Gamma^0_{00} = \frac{\dot N}{N}, \quad \Gamma^i_{i0} = \frac{\dot a}{a}, \quad \Gamma^0_{ii} = \frac{\dot a}{a\, N^2}\, q_{ii}, \tag{2.22}$$

where we have used the spatial part of the metric from the line element (2.16)

$$q_{ij} = \mathrm{diag}\left(\frac{a^2(t)}{1-\mathfrak{K}\, r^2},\ a^2(t)\, r^2,\ a^2(t)\, r^2 \sin^2\theta\right). \tag{2.23}$$

It is useful to express the curvature quantities with equal number of co- and contravariant indices to eliminate the explicit dependence on the coordinates. The only non-vanishing components of the Riemann tensor with two indices raised are

$$R^{\,0i}{}_{0i} = \frac{1}{N^2}\frac{\ddot a}{a} - \frac{\dot N}{N^2}\frac{\dot a}{a}, \quad R^{\,ij}{}_{ij} = \frac{\mathfrak{K}}{a^2} + \frac{\dot a^2}{N^2\, a^2}, \quad \text{for } i \neq j\ . \tag{2.24}$$

The non-vanishing components of the Ricci tensor with one index raised are

$$R^0{}_0 = \frac{3}{N^2}\frac{\ddot a}{a} - 3\,\frac{\dot N}{N^3}\frac{\dot a}{a}, \quad R^i{}_i = \frac{1}{N^2}\frac{\ddot a}{a} + \frac{2}{N^2}\frac{\dot a^2}{a^2} - \frac{\dot N}{N^3}\frac{\dot a}{a} + 2\,\frac{\mathfrak{K}}{a^2}. \tag{2.25}$$

By performing the sum over all components (2.25), we obtain the Ricci scalar

$$R = R^0{}_0 + \sum_{i=1,..,3}\left[R^i{}_i\right] = 6\left(\frac{1}{N^2}\frac{\ddot a}{a} + \frac{1}{N^2}\frac{\dot a^2}{a^2} - \frac{\dot N}{N^3}\frac{\dot a}{a} + \frac{\mathfrak{K}}{a^2}\right). \tag{2.26}$$

The symmetry assumptions require that the energy-momentum tensor has to acquire a special form. If we imagine the galaxies as freely streaming through the universe, this suggests that $T_{\mu\nu}$ should be the energy-momentum tensor of a perfect fluid. In the FRW universe, we can identify n^μ in Fig. 2.2, with the four-velocity u^μ, tangent to the time-like world line of a family of comoving observers. The four-velocity is defined as the four-vector $u^\mu := (\gamma,\ \mathbf{v})^T$, with the three-velocity $\mathbf{v}$ and the Lorentz factor $\gamma = 1/\sqrt{1-\mathbf{v}^2}$. A comoving observer is in his rest frame for each event so that $\gamma = 1$ and $\mathbf{v} = 0$ and $u^\mu = (1, \mathbf{0})^T = \delta^0_\mu$ for all times. With respect to the decomposition depicted in Fig. 2.2, $T_{\mu\nu}$ has to acquire the form

$$T_{\mu\nu} = \rho(t)\, u_\mu u_\nu + p(t)\, q_{\mu\nu}. \tag{2.27}$$

Using $u^\mu q_{\mu\nu} = 0$ and $u^\mu u_\mu = -1$, the coefficients $\rho(t)$ and $p(t)$ are determined by

$$\rho(t) := u^\mu\, u^\nu\, T_{\mu\nu} \quad \text{and} \quad p(t) := \frac{1}{3}\, q^{\mu\nu}\, T_{\mu\nu} \tag{2.28}$$

and can be identified as the energy density $\rho(t)$ and the pressure $p(t)$. They can only be functions of time due to the FRW symmetry. In addition, the two functions $\rho(t)$ and $p(t)$ are related by an equation of state, to be chosen independently:

$$\omega(t) := \frac{p(t)}{\rho(t)}. \tag{2.29}$$

The specific form of ω depends on the nature of the matter content described in $T_{\mu\nu}$. The energy-momentum tensor is further constrained by certain energy conditions. We will only consider the weak and strong energy conditions here. For a perfect fluid, these conditions can be parametrized by the following inequalities between ρ and p

$$\text{weak:} \quad \rho \geq 0 \quad \text{and} \quad \rho + p \geq 0 \;\Leftrightarrow\; \rho \geq 0 \quad \text{and} \quad \omega \geq -1, \tag{2.30}$$

$$\text{strong:} \quad \rho + p \geq 0 \quad \text{and} \quad \rho + 3\,p \geq 0 \;\Leftrightarrow\; \rho + p \geq 0 \quad \text{and} \quad \omega \geq -\frac{1}{3}. \tag{2.31}$$

By thinking of the galaxies as test particles flowing with the fluid, we can imagine a *comoving* observer as being attached to one of these galaxies. Inserting the symmetry-reduced expressions for the Ricci scalar and the energy-momentum tensor into the field equations (2.9), we obtain two equations, corresponding to the (00) and the (ii) components of (2.11). These are the *Friedmann equations*

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{\kappa}{3}\,\rho - \frac{\mathfrak{K}}{a^2} + \frac{\Lambda}{3}, \tag{2.32}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{\kappa}{6}\,(\rho + 3\,p) + \frac{\Lambda}{3} = -\frac{\kappa}{6}\,\rho\,(1 + 3\omega) + \frac{\Lambda}{3}. \tag{2.33}$$

We have introduced the Hubble parameter $H(t)$, defined by

$$H(t) := \frac{\dot{a}(t)}{a(t)}. \tag{2.34}$$

If we had derived the Eq. (2.32), (2.33) directly from the symmetry-reduced action, the constraint equation (2.32) would have been the result of the variation with respect to the Lagrange multiplier $N(t)$, whereas the equation of motion for $a(t)$ (2.33) would have been the result of varying with respect to the dynamical degree of freedom $a(t)$.

2.2.4 Epochs of the Universe

As a consequence of (2.12), we find by means of (2.32), (2.33)

$$\dot{\rho} + 3\,H\,(\rho + p) = \dot{\rho} + 3\,H\,\rho\,(1+\omega) = 0. \tag{2.35}$$

This has the form of a continuity equation. For $\omega = \text{const.}$, we obtain a differential equation in ρ. Using the ansatz $\rho(a) \propto a^x$, the solution in terms of a is

$$\rho(a) \propto a^{-3\,(\omega+1)}. \tag{2.36}$$

If multiple types of matter are present simultaneously, the individual densities ρ_i simply add up $\rho = \sum_i \rho_i$ and (2.36) holds individually for each type of matter. In order to obtain an explicit solution for $a(t)$ in terms of t, we substitute the solutions $\rho(a)$ into the Friedmann equations (2.32). Using again a power law ansatz $a(t) \propto t^y$, leads to

$$a(t) \propto \begin{cases} t^{\frac{2}{3\,(\omega+1)}} & \text{for } \omega \neq -1 \\ e^{H\,t} & \text{for } \omega = -1. \end{cases} \tag{2.37}$$

In order to obtain $a(\eta)$ with respect to conformal time, we use definition (2.18) and substitute (2.37). Neglecting constants of integration and prefactors yields

$$\eta \propto \begin{cases} t^{\frac{(\omega+\frac{1}{3})}{(\omega+1)}} & \text{for } \omega \neq -1 \\ -\frac{e^{-H\,t}}{H} & \text{for } \omega = -1. \end{cases} \tag{2.38}$$

Inverting this relation and re-substituting it into (2.37), we obtain the scale factor

$$a(\eta) \propto \eta^{\frac{2}{3\,(\omega+\frac{1}{3})}}. \tag{2.39}$$

For the most important choices of ω, we list the results in the following table:

In the last column we have calculated the *comoving* Hubble radius in terms of η

$$r_{\mathrm{H_c}} := (a\,H)^{-1} = \left(a\,\frac{\dot{a}}{a}\right)^{-1} = (\dot{a})^{-1} = \left(\frac{a'}{a}\right)^{-1} = \frac{a}{a'} =: \mathcal{H}^{-1} \tag{2.40}$$

Table 2.1 Overview of important quantities for different types of matter with $\omega = \text{const.}$

	ω	$\rho(a)$	$a(t)$	$a(\eta)$	$(a\,H)^{-1}$
vacuum	-1	const.	e^t	η^{-1}	$-\eta$
radiation	$1/3$	a^{-4}	$t^{1/2}$	η	η
matter	0	a^{-3}	$t^{2/3}$	η^2	$\eta/2$

The behaviour of the energy densities collected in Table 2.1 in an expanding universe and their dilution with growing scale factor are illustrated in the following two figures.

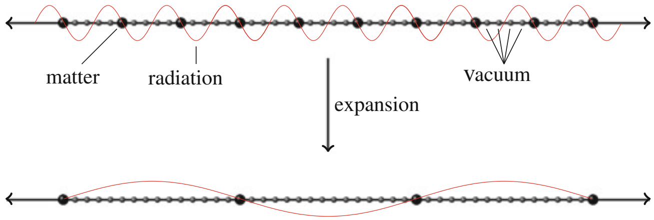

Fig. 2.5 Different types of energy densities in an expanding universe. Matter density is symbolized by the *fat black dots* and represent galaxies. Radiation density is represented by the *red wave* and vacuum density is illustrated by the constantly "re-filling" small red dots

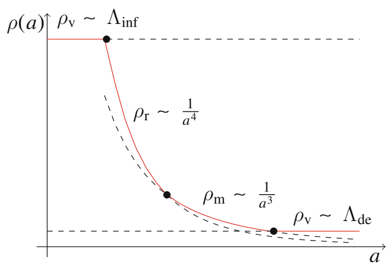

Fig. 2.6 Epochs of cosmic evolution. During the expansion of the universe, $\rho_m \sim 1/a^3$ decreases with increasing volume $V \sim a^3$, $\rho_r \sim 1/a^4$ with an additional factor of $a(t)$ due to the redshift of $\lambda_p = a(t)\,\lambda_c$ and $\rho_{\rm v}$ remains constant

The history of our universe can be divided into different epochs which are named according to the dominating energy density contribution at the respective time. In the early universe, the energy density $\rho_{\Lambda_{\rm infl}}$ was very large and nearly constant. This is the era of inflation to be discussed later. When inflaton ends, the energy is transferred into particle creation and thermic energy. This post-inflationary epoch is dominated by radiation $\rho_{\rm r} \propto 1/a^4$. The universe expands and cools down so that finally stable bound atoms can form. At some point $\rho_{\rm r} = \rho_{\rm m}$ the matter energy density $\rho_{\rm m}$ becomes the dominant contribution. In the future, $\rho_{\rm m}$ will be diluted completely. Observations of the accelerated expansion of the universe [25] suggest that we are currently at the turning point $\rho_{\rm m} = \rho_{\rm v}$, entering again into a phase dominated by a constant vacuum energy. This type of vacuum energy is called dark energy and can be described by a cosmological constant Λ_{de}. However, Λ_{de} is much smaller than the vacuum energy during inflation $\rho_{\Lambda_{\text{infl}}}$.

2.2.5 Observable Quantities

We divide (2.32) by H^2 and rewrite the first Friedmann equation in a dimensionless form by normalizing the different types of energy density with respect to a critical density $\rho_{\text{crit}} = 3H^2/\kappa$

$$1 = \Omega_\rho + \Omega_\Lambda + \Omega_{\mathfrak{K}}, \tag{2.41}$$

with

$$\Omega_\rho := \sum_i \frac{\kappa\,\rho_i}{3\,H^2} = \sum_i \frac{\rho_i}{\rho_{\text{crit}}}, \quad \Omega_\Lambda := \frac{\Lambda}{3\,H^2}, \quad \text{and} \quad \Omega_{\mathfrak{K}} := -\frac{\mathfrak{K}}{H^2\,a^2}. \tag{2.42}$$

Here, the index i labels different kinds of matter and radiation energy densities with individual equations of state. We can even subdivide the ρ_i's further by analysing different particle species separately, e.g. radiation composed of photons and neutrinos $\rho_{\text{r}} = \rho_\gamma + \rho_\nu$ and matter of baryons and cold dark matter $\rho_M = \rho_{\text{B}} + \rho_{\text{CDM}}$. On the one hand, we can consider (2.41) as a consistency relation which must be satisfied by any cosmological model. One the other hand we can make predictions for new types of energy such as dark energy. The form (2.41) is especially useful to constrain the parameters of models by experimental data. From (2.36), we can easily find the solution for a constant ω

$$\Omega_{\rho_i} = \Omega_{\rho_{i,0}} \left(\frac{H_0}{H}\right)^2 \left(\frac{a}{a_0}\right)^{-3(1+\omega)}, \tag{2.43}$$

where we have expressed Ω_{ρ_i} in terms of quantities that we measure today, indicated by a subscript "0". Using the definition of the cosmological redshift z

$$(1+z) := \frac{a(t_{\text{observation}})}{a(t_{\text{emission}})} = \frac{a_0}{a}, \tag{2.44}$$

and (2.43), we can formulate the Friedmann equation (2.41) in terms of measurable quantities

$$\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_{\rho_i,0}\,(1+z)^{3(\omega_i+1)} + \Omega_{\mathfrak{K},0}\,(1+z)^2 + \Omega_{\Lambda,0}. \tag{2.45}$$

By expanding the scale factor a in powers of $(t - t_0)$ with the value a_0 it has today

$$a(t) = a_0\Big(1 + H_0\,(t - t_0) - \frac{1}{2}\,q_0\,H_0^2\,(t - t_0)^2 + \ldots\Big), \tag{2.46}$$

we recover Hubble's law as the linear correction term with the Hubble constant H_0 observed today. Hubble's law states that the velocities of the galaxies moving away from us are proportional to their distance. The parameter of proportionality is $H(t)$. The quadratic correction describes the acceleration of the universe. By convention the sign is chosen negative, so that $q_0 > 0$ corresponds to deceleration. The deceleration parameter q_0, as measured today, is defined by

$$q_0 := -\frac{\ddot{a}}{a\,H^2}\Big|_{t=t_0}. \tag{2.47}$$

Using (2.33) and (2.42), we can re-express the deceleration parameter as

$$q_0 = \frac{1}{2}\sum_i \Omega_{\rho_i,0}\,(1 + 3\omega_i) - \Omega_{\Lambda,0}. \tag{2.48}$$

Regardless of the curvature parameter $\mathfrak{K}$, we can distinguish by a measurement of $\Omega_{\rho,0}$ and $\Omega_{\Lambda,0}$, whether our universe is accelerating or decelerating today. Of course, we could reformulate this equation in such away that it effectively depends on $\Omega_{\mathfrak{K}}$ by using the constraint (2.41). We could also include the contribution of Ω_Λ in the sum over the ordinary energy contributions in (2.48), showing again that the equation of state for the vacuum energy is $\omega = -1$. We need at least one such negative contribution in order to have a negative q_0 corresponding to the acceleration of the universe observed today.

2.3 Inflation

The general idea of inflation is that the universe underwent a phase of accelerated expansion. This picture is adequate to solve several problems of cosmology ("flatness", "horizon" and "monopole" problems) [12, 27]. However, nowadays its main success and its major experimental evidence consist of its capability to explain the emergence of the large-scale structure of our present universe from tiny quantum fluctuations in the early universe. We will first give a qualitative overview and afterwards describe how inflation can be modelled by a scalar field. Then, we analyse the space-time structure of de Sitter space and consider the scenario of slow-roll inflation [1, 19]. Finally, we investigate small quantum perturbations on the inflationary background and calculate the cosmological parameters for these fluctuations observed today [13, 14, 21, 28].

2.3.1 Qualitative Preliminary Considerations

Let us assume that the universe undergoes an exponential expansion $a(t) \sim \exp\left(H_{\mathrm{p}}\, t\right)$ with a constant physical Hubble parameter H_{p}. By investigating the Friedmann equation (2.32), the term associated with the spatial curvature $\propto \mathfrak{K}/a^2$ becomes negligible very quickly and the universe flattens out. This makes the distinction between a closed, an open and a flat universe hard to detect and can explain why we observe a nearly flat universe today. The following illustrations consider the horizon problem which lies at the very heart of the inflationary paradigm. All other problems mentioned above can be traced back to this elementary problem. The horizon problem can roughly be summarized as follows: Today, we observe a nearly isotropic cosmic microwave background radiation (CMB). If we look into the sky, we look in the past. The identical temperature of the radiation from all directions in the sky suggests that these regions must have been in causal contact at some earlier time in order to establish thermodynamical equilibrium. In order to figure out the moment of time when the radiation was sent out, we must investigate the cosmic history. As already mentioned in Sect. 2.2.4, at some point in the past the hot baryon-photon plasma has cooled down and the free electrons were caught by the ionized nucleons to form stable atoms. The mean free path of the photons, which have been scattered with the fast electrons in the hot plasma, increases after the free electrons have disappeared so that the photons decouple and are freely streaming. In this period of conversion, the surface of last scattering has been formed. The free photons emanating from this surface are the photons we detect today in the CMB. Due to the cosmic expansion, they are red-shifted so that their associated temperature is small today.

Fig. 2.7 Recombination. Via Thomson scattering the photons collide with the electrons in the hot ionized baryon-photon plasma. During the expansion of the universe the plasma cools down and it becomes energetically favourable for the electrons and the ionized nucleons to form stable atoms. The photons do not scatter anymore, their mean free path becomes nearly infinite and the universe becomes transparent

We observe an isotropic CMB with an accuracy of the temperature contrast of about $\Delta T/T \sim 10^{-5}$. In order to illustrate the problem, we can analyse the following causal diagram. Figure 2.8 shows the past light cone in terms of conformal time η. The points A and B each correspond to a sphere S^2, depicted in Fig. 2.9. Each such sphere, in turn, corresponds to a homogeneous patch of the universe. At the time of recombination η_r, the surface of last scattering (grey line in the two-dimensional illustration of Fig. 2.8) has formed and has sent out the radiation from different homogeneous patches that we measure in the CMB. The horizon problem appears when we try to explain the observed isotropic radiation of the CMB radiation. In the projection onto our sky, the two patches associated with A and B correspond to two antipodal points. Without inflation, the initial singularity would have started with the "Hot Big Bang" at $\eta = 0$. Thus, the two past light cones of A and B could have never been in causal contact and therefore would have never established thermal equilibrium to explain the observed isotropy of the CMB. This puzzle can be solved by inflation. Inflation extends the conformal time to negative infinity so that the past light cones could have intersected (large red triangle in Fig. 2.8). Thus, it is required that inflation lasts long enough, $\eta_{\text{inf}} = \eta_{\text{min}}$, in order to ensure that the past light cones of the antipodal points A and B intersect at some moment of time. It is expected that quantum gravitational effects become dominant, somewhere between the minimal η_{min}, necessary to solve the horizon problem and the initial singularity $\eta = -\infty$. The left-hand side of Fig. 2.9 shows the surface of last scattering, if there would have been no inflation. We would observe many different patches covering the sky, all with different radiation indicated by the different colours. The grey sphere corresponds to the surface of last scattering projected on our observable sky with the Earth in its centre. Each S^2 sphere lies on the grey sphere and corresponds to a local patch that would be out of causal contact without inflation. Thus, without inflation, there should be a huge number of different patches, which is, however, not observed. It is the mechanism of inflation that allows all these patches to be in causal contact at some early time $\eta < 0$. While being in causal contact, these patches could have established a thermodynamical equilibrium. This situation is depicted on the right-hand side of Fig. 2.9. The big red sphere indicates thermodynamical equilibrium and explains why we observe the same CMB from every direction in the sky.

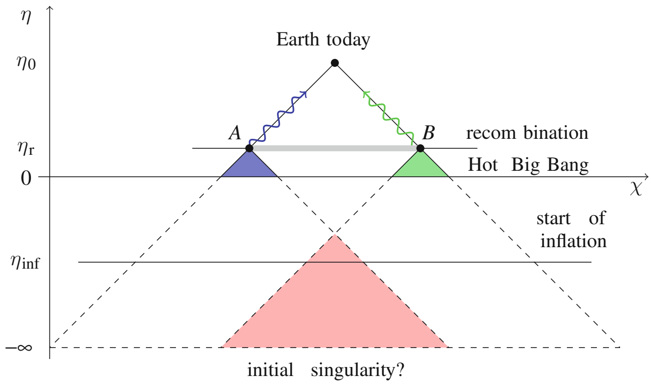

Fig. 2.8 Horizon problem. The past light cone of the Earth today in conformal time η. The isotropic CMB suggests that the patches associated with the size of the past light cones of A and B must have been in causal contact. However, this becomes possible only during a phase of inflation $\eta \to \eta_{\text{inf}}$, when the two light cones intersect

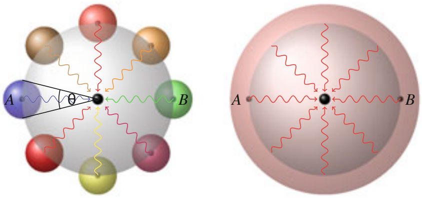

Fig. 2.9 Surface of last scattering. The *black dot* representing the Earth is surrounded by the surface of last scattering symbolized by the *grey sphere*. *Left* Without inflation we would observe radiation with different temperatures (indicated by the different colours) from different directions in the sky. Each sphere corresponds to a small homogeneous patch of the universe. *Right* All regions have been in causal contact during inflation and established thermodynamical equilibrium explaining the observed isotropy of the CMB

2.3.2 Scalar Field Model of Inflation

The qualitative overview of the last section clearly demonstrated the advantages of an inflationary phase. However, if this process is realized in nature we should be able to describe inflation at least phenomenologically by a concrete model. By looking at the definition of the scale factor in terms of conformal time η in (2.1), it is obvious that for a phase of inflation we need a scale factor $a(\eta) \propto -1/\eta$ that pushes the initial singularity from $\eta = 0$ to $\eta = -\infty$. From (2.1) we see that this corresponds to a constant equation of state $\omega = -1$ or to a vacuum energy density ρ_{v} that can be described in terms of a large cosmological constant Λ_{inf}. However, from the fact of our existence, we can conclude that inflation have had come to an end in order to allow for the evolution of structures. Thus, Λ_{inf} cannot be exactly constant. We consider a scalar field φ with the following action

$$S_{\varphi} = \int \mathrm{d}^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_{\mu}\varphi \partial_{\nu}\varphi - V(\varphi) \right). \tag{2.49}$$

From the CMB, we know that our observable universe is approximately isotropic and homogeneous which is why we assume a homogeneous scalar field $\varphi(t, \mathbf{x}) = \varphi(t)$. Assuming that the scalar field $\varphi(t) = \varphi_0$ is almost constant $\dot{\varphi}_0 \approx 0$ during the time interval of inflation implies that $\mathcal{L}_{\varphi_0} \approx V(\varphi_0) \approx \text{const}$. Therefore, the vacuum energy density can be written as

$$\rho_{\mathrm{v}} = M_{\mathrm{P}}^2 \Lambda_{\mathrm{inf}} := V(\varphi_0). \tag{2.50}$$

During inflation, we can effectively express the cosmological constant in terms of the scalar field potential $V(\varphi_0)$. The key idea is now to associate Λ_{inf} also for later times and in this way to allow a *dynamical*

$$\Lambda_\varphi(t) := \frac{V(\varphi(t))}{M_{\text{P}}^2}. \tag{2.51}$$

We know from observations that at present time scales the variation of Λ_φ is negligible and that Λ_φ is very small. In order to allow for a phase of inflation, we have to require that the potential has the appropriate shape, i.e. that it is very flat during inflation, and becomes more and more steep later on, in order to exit the phase of inflationary dynamics. This scenario is realized in the "slow-roll" approximation, which we will discuss in Sect. 2.3.4. We will now derive the Friedmann equations (2.32) and (2.33) in terms of the scalar field quantities. The energy-momentum tensor of the scalar field action can be calculated by identifying the scalar field action (2.49) with the matter action (2.2) and by using the definition of the energy-momentum tensor (2.8)

$$T^\varphi_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - \left(\frac{1}{2}\,\partial_\sigma\varphi\,\partial^\sigma\varphi + V(\varphi)\right) g_{\mu\nu}. \tag{2.52}$$

For a homogeneous scalar field $\varphi(t)$, we obtain by means of the FRW decomposition

$$T^\varphi_{\mu\nu} = \left(\frac{1}{2}\dot\varphi^2 + V(\varphi)\right) \delta^0_\mu\delta^0_\nu + \left(\frac{1}{2}\dot\varphi^2 - V(\varphi)\right) \delta^i_\mu\delta^j_\nu q_{ij}. \tag{2.53}$$

By comparison with the energy-momentum tensor of a perfect fluid (2.27), we find

$$\rho_\varphi = \frac{1}{2}\dot\varphi^2 + V(\varphi), \quad p_\varphi = \frac{1}{2}\dot\varphi^2 - V(\varphi) \quad \text{and} \quad \omega_\varphi = \frac{\frac{1}{2}\dot\varphi^2 + V(\varphi)}{\frac{1}{2}\dot\varphi^2 - V(\varphi)}. \tag{2.54}$$

Assuming in addition that the scalar-field induced Λ_φ is the only vacuum energy density, i.e. $\Lambda \equiv 0$, and is dominant, i.e $\Omega_{\Lambda_\varphi} \gg \Omega_M$, we find the Friedmann equations

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{\kappa}{3}\,\rho_\varphi - \frac{\mathfrak{K}}{a^2} = \frac{\kappa}{3}\left(\frac{1}{2}\dot\varphi^2 + V(\varphi)\right) - \frac{\mathfrak{K}}{a^2}\,, \tag{2.55}$$

$$\dot H + H^2 = \frac{\ddot a}{a} = -\frac{\kappa}{6}\,(\rho_\varphi + 3\,p_\varphi) = \frac{\kappa}{3}\left(V(\varphi) - \dot\varphi^2\right). \tag{2.56}$$

We obtain the Klein–Gordon equation by varying (2.49) with respect to $\varphi(t)$

$$\ddot\varphi + 3\frac{\dot a}{a}\,\dot\varphi + V' = 0 \quad \text{with} \quad V' := \frac{\partial V(\varphi)}{\partial\varphi}. \tag{2.57}$$

Equation (2.56) implies that for an *accelerated* expansion a positive $\ddot{a}$ is required

$$(\rho_\varphi + 3p_\varphi) < 0 \quad \leftrightarrow \quad V(\varphi) > \dot{\varphi}^2. \tag{2.58}$$

This is a necessary and sufficient condition for inflation. If we assume in addition that the weak energy condition $\rho > 0$ (2.30) is satisfied, this requires a *negative* pressure $p < 0$ during inflation. It is exactly this negative pressure which is the reason for the phase of accelerated expansion. Equation (2.58) implies that during inflation the potential should dominate over the kinetic term. If the even stronger condition $V(\varphi) \gg \dot{\varphi}^2$ holds, then we have a nearly constant φ and (2.54) becomes

$$\rho_\varphi = -p_\varphi \quad \text{and} \quad \omega_\varphi \approx -1 \quad . \tag{2.59}$$

According to Table 2.1, a constant $\omega = -1$ leads to an *exponential* phase of expansion. Based on the structure of space-time, this is denoted "exact de Sitter inflation".

2.3.3 De Sitter Space

De Sitter space $dS_{(4)}$ is a four-dimensional space with coordinates $\{x^0,\ x^1,\ x^2,\ x^3\}$ and can be described as sections of an embedded hyperboloid in a five-dimensional Minkowski space $\mathcal{M}_{(5)}$ with coordinates $\{X^0,\ X^1,\ X^2,\ X^3,\ X^4\}$. The five-dimensional line element

$$\mathrm{d}s^2 := \eta_{IJ}\, \mathrm{d}X^I\, \mathrm{d}X^J \tag{2.60}$$

is defined by the five-dimensional Minkowski metric $\eta_{IJ} := \mathrm{diag}(-1,\ 1,\ 1,\ 1,\ 1)$. De Sitter space $dS_{(4)}$ can be described by the hyperboloid equation in $\mathcal{M}_{(5)}$

$$-(X_0)^2 + (X_1)^2 + (X_2)^2 + (X_3)^2 + (X_4)^2 = C^2. \tag{2.61}$$

Topologically, $dS_{(4)}$ is equivalent to $\mathbb{R} \times \mathcal{S}^3$, corresponding to one real time $t \in \mathbb{R}$ dimension and a compact three-dimensional spatial sphere $\mathcal{S}^3$. The constant C must have the dimension of a length. In a cosmological context it therefore seems natural to identify it with the only characteristic length scale, the Hubble radius H^{-1}. De Sitter space is a maximal symmetric space with constant curvature (as four-dimensional flat Minkowski space $\mathcal{M}_{(4)}$). The generators of the isometries in $d = 4$ dimensions are the $d(d+1)/2 = 10$ linear independent Killing vectors. The Ricci scalar of $dS_{(4)}$ is (cf. (2.32) with $\rho = \mathfrak{K} = 0$)

$$R = \frac{d\,(d-1)}{C^2} = 12\, H^2. \tag{2.62}$$

$dS_{(4)}$ is a solution of the vacuum Einstein equations (2.11) with a cosmological constant

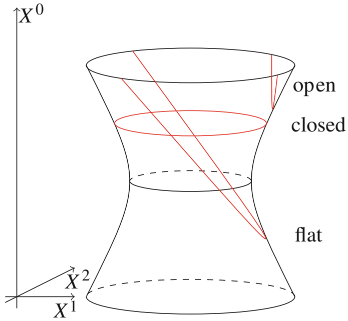

Fig. 2.10 Different slices of the de Sitter hyperboloid embedded in $\mathcal{M}_{(5)}$. The red lines correspond to hypersurfaces of constant time t

$$\Lambda = \frac{(d-2)(d-1)}{2\,C^2} = 3\,H^2. \tag{2.63}$$

The slices shown in Fig. 2.10 correspond to different choices of coordinates $X^I \in \mathcal{M}_{(5)}$ and lead to different induced metrics $\gamma_{\mu\nu}$ on $dS_{(4)}$ defined by (2.61)

$$\gamma_{\mu\nu} := \frac{\partial X^I}{\partial x^\mu}\,\frac{\partial X^J}{\partial x^\nu}\,\eta_{IJ}, \quad I,\ J, \ldots = 0, \ldots, 4. \tag{2.64}$$

This does not mean that de Sitter space is different for different coordinate choices X^I, it just means that de Sitter space is covered by different sets of coordinates x^μ. Not all choices for $X^I \in \mathcal{M}_{(5)}$ lead to a set of coordinates $x^\mu \in dS_{(4)}$ that covers the whole de Sitter space. In most of the physical applications for inflation, only a tiny patch of the complete $dS_{(4)}$ is needed and it is rather a matter of taste which different slicing in Fig. 2.10 we choose. In the flat slicing, $\gamma_{\mu\nu}$ has the simplest form. We choose the coordinates [26]

$$\begin{aligned} X^0 &:= \frac{1}{H}\sinh(H\,t) + \frac{H}{2}\,e^{H\,t}\,\mathbf{x}^2, \quad X^4 := \frac{1}{H}\cosh(H\,t) - \frac{H}{2}\,e^{H\,t}\,\mathbf{x}^2, \\ X^i &= e^{H\,t}\,x_i, \quad (i = 1, \ldots, 3,\ \mathbf{x}^2 = \sum_{i=1}^{3}(x_i)^2) \end{aligned} \tag{2.65}$$

and use (2.64) to calculate the induced metric

$$\gamma_{00} = -1, \quad \gamma_{ii} = e^{2\,H\,t}, \quad \gamma_{0i} = \gamma_{ij} = 0, \quad \text{for} \quad i \neq j\,. \tag{2.66}$$

Therefore, the de Sitter line element for the flat slicing is given by

$$\mathrm{d}s^2_{\text{flat}} = \gamma_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + e^{2\,H\,t}\,\delta_{ij}\,\mathrm{d}x^i \mathrm{d}x^j. \tag{2.67}$$

The closed slicing leads to the induced line element [26]

$$ds^2_{\text{closed}} = -dt^2 + \frac{\cosh^2(H\,t)}{H^2}\left(d\chi^2 + \sin^2 d\Omega^2\right), \tag{2.68}$$

which has the advantage that it covers the complete de Sitter space. The open slicing can be obtained in a similar way.

2.3.4 Slow-Roll Inflation

In Sect. 2.3.2, we have seen that $V \gg \dot{\varphi}^2$ was required to obtain a quasi de Sitter phase $\omega \approx -1$ of inflation. We will now make this more quantitative. From now on, we will assume a spatially flat $\mathfrak{K} = 0$ universe, which is favoured by observation. However, for $a \sim \exp(H\,t)$, the term $\propto \mathfrak{K}/a^2(t)$ in (2.55) quickly becomes negligible in any case. The *quality* of the slow-roll approximation can be quantified by the two dimensionless slow-roll parameters

$$\varepsilon := -\frac{\dot{H}}{H^2} \quad \text{and} \quad \delta := -\frac{1}{H}\frac{\ddot{\varphi}}{\dot{\varphi}}. \tag{2.69}$$

The physical meaning of these parameters becomes transparent, when expressing the Friedmann equations (2.55) and (2.56) in terms of the slow-roll parameters (2.69). Equation (2.33) takes the form

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2\,(1-\varepsilon). \tag{2.70}$$

In order to have acceleration $\ddot{a} > 0$, $\varepsilon < 1$ is required. If even $\varepsilon \ll 1$, (2.70) becomes $\ddot{a} = H^2\,a$ with the pure de Sitter solution $a = \exp(H\,t)$ in the limit $\varepsilon \to 0$. The advantage of this parametrization is that we have a *small* dimensionless parameter ε which is suited for an expansion around the pure de Sitter solution. We define the moment of time t_{end} when inflation comes to an end by the end of the phase of accelerated expansion

$$\varepsilon(t_{\text{end}}) := 1. \tag{2.71}$$

If we compare (2.70) with (2.56), we can express ε in terms of ω_φ

$$\varepsilon = 1 + \frac{\kappa\,\rho_\varphi}{6\,H^2}\,(1 + 3\,\omega_\varphi) = \frac{3}{2}\,(\omega_\varphi + 1), \tag{2.72}$$

where we have used (2.55) in the last equality. Subtracting (2.56) from (2.55) yields

$$\varepsilon = \frac{\kappa}{2}\frac{\dot{\varphi}^2}{H^2}. \tag{2.73}$$

So far, we just have reparametrized quantities in terms of ε and all expressions are still *exact*. At this point, we have to make use of the slow-roll approximation in order to write ε in terms of the potential $V(\varphi)$. Neglecting $\ddot{\varphi}$ and $\dot{\varphi}^2$ terms in the Klein–Gordon equation (2.57) and the Friedmann equation (2.55) and using $\kappa = 1/M_{\mathrm{P}}^2$ leads to

$$3\,H\,\dot{\varphi} \simeq -V' \quad \text{and} \quad H^2 \simeq \frac{1}{3}\frac{V(\varphi)}{M_{\mathrm{P}}^2}. \tag{2.74}$$

Differentiating the second relation in (2.74) with respect to time t and using the first relation in (2.74) leads to the relation

$$\dot{H} = -\frac{1}{2}\,\frac{\dot{\varphi}^2}{M_{\mathrm{P}}^2}\,. \tag{2.75}$$

Using the first relation in (2.74) in order to eliminate $\dot{\varphi}$ in (2.75) and inserting this in the definition for ε in (2.69), we can express $\varepsilon(\varphi)$ in terms of V and V'

$$\varepsilon \simeq \frac{M_{\mathrm{P}}^2}{2}\left(\frac{V'}{V}\right)^2 =: \varepsilon_{\mathrm{V}}. \tag{2.76}$$

The physical meaning of the second slow-roll parameter δ is connected with the acceleration of φ. Provided that the potential is flat enough, i.e. $\varepsilon \ll 1$, it is a measure of how long inflation lasts. It is convenient to express δ also in terms of V. Differentiating the first relation in (2.74) with respect to time t and using (2.75) to eliminate $\dot{H}$, we obtain

$$\frac{\ddot{\varphi}}{\dot{\varphi}} \simeq -\frac{V''}{3\,H} - \frac{1}{2\,M_{\mathrm{P}}^2}\,\frac{\dot{\varphi}^2}{H}\,. \tag{2.77}$$

Inserting this into the definition (2.69) for δ and using again both approximated expressions in (2.74), we obtain the second slow-roll parameter δ in terms of $V(\varphi)$

$$\delta \simeq M_{\mathrm{P}}^2\,\frac{V''}{V} - \frac{1}{2}\,M_{\mathrm{P}}^2\,\frac{V'^2}{V^2} =: \delta_{\mathrm{V}} - \varepsilon_{\mathrm{V}} \tag{2.78}$$

with

$$\delta_{\mathrm{V}} := M_{\mathrm{P}}^2\,\frac{V''}{V}\,. \tag{2.79}$$

We could define even more slow-roll parameters including higher derivatives of the potential, but for our purpose the two slow-roll parameters (2.76) and (2.79) are sufficient. We note that *only during slow-roll* the following relations between the parameters (2.69) and the parameters in terms of the potential (2.76) and (2.79) hold

$$\varepsilon \simeq \varepsilon_{\mathrm{V}} \quad \text{and} \quad \delta \simeq \delta_{\mathrm{V}} - \varepsilon_{\mathrm{V}}. \tag{2.80}$$

The slow-roll inflation ($\varepsilon \ll 1$ and $|\delta| \ll 1$) constraints the otherwise arbitrary potential $V(\varphi)$. It must be sufficiently flat for a sufficient long time in order to guarantee that φ rolls down the hill slowly as long as inflation is needed.

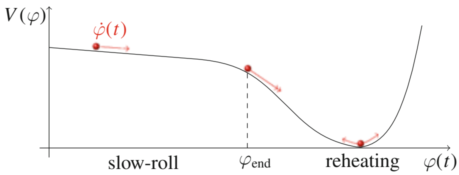

Fig. 2.11 Typical slow-roll potential

After the phase of slow-roll, $V(\varphi)$ becomes steeper, φ "speeds up", the slow-roll approximation breaks down and φ starts to oscillate around the minimum φ_0 in the harmonic potential

$$V''(\varphi)\Big|_{\varphi=\varphi_0} =: m_\varphi^2. \tag{2.81}$$

Since the universe cools down when expanding, at some point the coupling of the inflaton φ to other kinds of matter cannot be neglected any longer. Via this interaction, the inflaton transforms its energy into particle creation and thermic energy, starting a phase of reheating. Finally, the inflaton comes to rest and settles in the minimum of $V(\varphi)$.

2.4 Cosmological Perturbations

The mechanism of inflation successfully explains the observed isotropy of the CMB. However, there are two obstacles yet to overcome. First of all, more precise measurements of the CMB revealed that there are small anisotropies in the temperature contrast $\Delta T/T \sim 10^{-5}$ and we have used inflation to explain just the opposite: the smoothing-out of space-time. Thus, we have to explain the origin of these anisotropies. Secondly, we observe that matter is not homogeneously and isotropically distributed locally. We need at least some small density fluctuations to explain the gravitational clumping that ultimately leads to structure formation. We can solve both problems by taking into account quantum theory. So far, we have only considered the classical FRW evolution of the inflationary background. In quantum theory, we have to take into account quantum fluctuations of the fields. In the inflationary scenario we have considered the symmetry reduced versions $a(t)$ and $\varphi(t)$ of the metric field $g_{\mu\nu}(\mathbf{x}, t)$ and the inflaton field $\varphi(\mathbf{x}, t)$. On top of this classical (averaged) FRW background, we consider the fluctuations

$$\delta g_{\mu\nu}(\mathbf{x}, t) := g_{\mu\nu}(\mathbf{x}, t) - \bar{g}_{\mu\nu}(t) \quad \delta\varphi(\mathbf{x}, t) := \varphi(\mathbf{x}, t) - \bar{\varphi}(t). \tag{2.82}$$

The isotropic FRW metric and the homogeneous inflaton field are the background quantities, marked by a bar. The **x**-dependent fluctuations represent small inhomogeneities.

2.4.1 Generation of Density Fluctuations: Basic Mechanism

First, we will show the basic mechanism of how small quantum fluctuations can lead to density fluctuations by an illustrative example. Later we will present a more rigorous derivation. For a moment, we focus only on the fluctuations $\delta\varphi(\mathbf{x}, t)$ of the inflaton field φ. We consider again our typical slow-roll potential, but this time with a three-dimensional extension including the three spatial directions **x**.

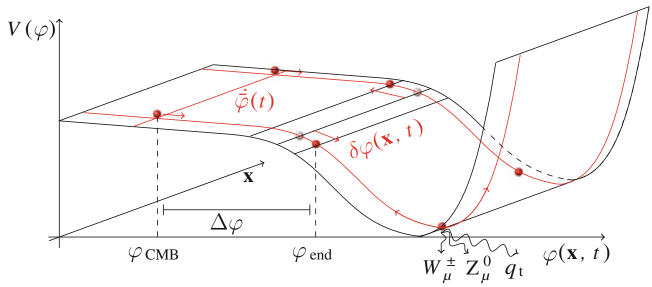

Fig. 2.12 Slow-roll potential with small perturbations of the inflaton field. The accelerated and delayed decay of φ due to small quantum corrections translates perturbations of the inflaton field $\delta\varphi$ into density perturbations $\delta\rho$

The fluctuations $\delta\varphi(\mathbf{x}, t)$ are space-dependent and should be approximately random. Therefore, it is assumed that they obey (nearly) Gaussian statistics. This in turn means that the fluctuations are at each point independent objects and thus each Fourier mode $\delta\varphi_k(t)$ represents an independent random variable. The motion of the inflaton field $\varphi(\mathbf{x}, t) = \bar{\varphi}(t) + \delta\varphi(\mathbf{x}, t)$ can be described as a superposition of the dominant classical slow-roll evolution of the background $\dot{\bar{\varphi}}(t)$ and the random perturbations characterized by small quantum oscillations $\delta\varphi(\mathbf{x}, t)$ around the mean value $\bar{\varphi}$. The small "quantum kicks" can either drive the field upwards or downwards the hill of the potential. As Fig. 2.12 suggests, the inflaton rolls down earlier in some regions **x** than in other regions **x**′. If inflation ends, the inflaton rolls down into the minimum of $V(\varphi)$ and decays. The free energy is transformed into particle creation and thermal energy, starting the phase of reheating.[3] As a consequence, the universe becomes dominated by radiation. As we know from Table 2.1 and Fig. 2.6, the energy *density* $\rho_{\mathrm{v}} \simeq V(\varphi)$ remains constant during inflation. In contrast, the matter and radiation densities dilute with an expanding scale factor. Thus, in space-time regions

[3] If $V(\varphi)$ was the Higgs potential of the Standard Model, the inflaton would settle in the electroweak vacuum, generating the mass of gauge bosons and fermions.

where inflation ends earlier, the inflaton field decays earlier and the energy density in these regions dilutes earlier, leading to an underdense region. In regions where inflation lasts longer, the field decays later, leading to slightly overdense regions. The mechanism of inflation thereby translates small quantum fluctuations of the scalar field into density perturbations. Finally, the universal attraction of gravity will amplify these small density perturbations and can explain the formation of structure.

2.4.2 Decomposition of Different Types of Fluctuations

In the last section we have roughly outlined the basic consequence of fluctuations in the inflaton field. The picture is, however, not complete. We must also include perturbations of the metric field and discuss the issue of gauge invariance. In the following sections, we will closely follow the treatment of [22] and [29].

In order to treat a general fluctuation $\delta\Phi(\mathbf{x}, \eta)$ (with Φ denoting a collection of different fields, including the metric field), we have to eliminate the spatial dependence by switching to Fourier space. Thus, we trade the infinitely many degrees of freedom associated with the space-time points $\mathbf{x}$ of $\delta\Phi(\mathbf{x}, \eta)$ for a set of infinitely many modes $\delta\Phi_{\mathbf{k}}(\eta)$, depending solely on (conformal) time and the mode $\mathbf{k}$. Further simplifications are due to the following observations: First of all, the measured CMB temperature contrast $\Delta T/T \simeq 10^{-5}$ suggests that only *small perturbations* $|\delta\Phi| \ll 1$ can be responsible for the observed anisotropy and this, in turn, justifies the approximation linear in the perturbations $\delta\Phi_{\mathbf{k}}(\eta)$. Secondly, translation invariance of the background combined with the linear evolution implies that different modes involve *individually*, i.e. couplings between different modes $\mathbf{k}$ and $\mathbf{k}'$ do not occur. Thirdly, we can classify the perturbations according to their behaviour under three-dimensional rotations in scalar-, vector- and tensor-type (S, V, T) perturbations. Linear evolution and rotational invariance of the background ensures that there will be no mixing between different types of perturbations during their evolution, i.e. the linearised Einstein equations only relate modes of equal type [4]. Ultimately, this allows us to consider each type of perturbation (S, V, T) and each mode $\mathbf{k}$ individually.

We decompose the metric perturbations into the three types

$$\delta\, g_{\mu}\nu := \delta_{\mathrm{S}}\, g_{\mu}\nu + \delta_{\mathrm{V}}\, g_{\mu}\nu + \delta_{\mathrm{T}}\, g_{\mu}\nu. \tag{2.83}$$

However, even if we can investigate the three types of perturbations individually, there still remains the issue of gauge transformations. An infinitesimal coordinate transformation

$$x^{\mu} \to \tilde{x}^{\mu} = x^{\mu} + \xi^{\mu} \tag{2.84}$$

induces a change in the perturbations. In order to illustrate the problem, we consider a perfectly homogeneous FRW universe with density $\rho = \rho(\eta)$. Performing a coordinate transformation of the time variable $\eta \to \tilde{\eta} = \eta + \xi^0(\mathbf{x}, \eta)$ will induce a change in the density perturbation which is given by the Lie dragging (see Appendix B.12) along the time direction $\xi^0(\mathbf{x},\ \eta)$

$$\delta\rho \rightarrow \widetilde{\delta\rho} = \delta\rho + \mathcal{L}_{\xi^0}\,\rho = \delta\rho + \rho'\,\xi^0(\mathbf{x},\ \eta). \tag{2.85}$$

Thus, although initially $\delta\rho = 0$, by performing a coordinate transformation of time, we induce an inhomogeneous FRW universe $\widetilde{\delta\rho} = \rho'\,\xi^0(\mathbf{x},\ \eta) \neq 0$. These perturbations are pure gauge artefacts. In order to get rid of these non-physical extra modes, we can either fix a specific gauge or we can construct and work with gauge-invariant variables exclusively. We will now investigate the three types of metric perturbations (2.83) individually. The tensor perturbations $\delta_\mathrm{T}\, g_{\mu\nu}$ are by construction gauge-invariant, which is why we can assign a direct physical meaning to them. They can be identified with gravitational waves

$$\delta_\mathrm{T}\, g_{\mu\nu} := a^2(\eta)\, h_{ij}. \tag{2.86}$$

Here, h_{ij} is a symmetric, traceless and divergence-free tensor and $i,\ j, \ldots = 1, 2, 3$. Otherwise, the divergence and trace part would contribute to vector and scalar perturbations, respectively. The tensor perturbations have two degrees of freedom according to the two different polarizations of a gravitational wave (six independent components of $h_{ij} = h_{ji}$, minus three components $h_{ij}{}^{;j} = 0$ minus one component $h_i{}^i = 0$). Similarly, the vector modes can be parametrized by two divergence-free three-vectors C_i and D_i. If they were not divergence-free, the divergence would contribute to the scalar perturbations. Counting degrees of freedom, we find six components of the vectors C_i and D_i minus two constraint equations ($C_i{}^{;i} = D_i{}^{;i} = 0$), yielding four degrees of freedom.

$$\delta_\mathrm{V}\, g_{\mu\nu} := a^2(\eta) \begin{pmatrix} 0 & -C_i \\ -C_j & D_{i,\,j} + D_{j,\,i} \end{pmatrix}. \tag{2.87}$$

However, it turns out that the $\delta_\mathrm{V}\, g_{\mu\nu}$ have only decaying solutions in an expanding universe and thus are not important for our further analysis, which is why we will neglect them in what follows. The scalar perturbations are most important, since they are the only ones that couple to matter fluctuations[4] and therefore are responsible for the formation of structure in our observable universe. The mixing between metric and matter degrees of freedom is determined by $\delta G_{\mu\nu} = \kappa\delta T_{\mu\nu}$. In order to get rid of the gauge dependence, we parametrize the scalar perturbations of $g_{\mu\nu}$ by four scalar functions ϕ, ψ, B and E

$$\delta_\mathrm{S}\, g_{\mu\nu} := a^2(\eta) \begin{pmatrix} -2\,\phi & -B_{,\,i} \\ -B_{,\,i} & 2\,(\psi\,\delta_{ij} + E_{,\,ij}) \end{pmatrix}. \tag{2.88}$$

We can therefore write the flat, scalar and tensor perturbed FRW line element as

[4] The density and pressure fluctuations contained in $\delta T_{\mu\nu}$ are scalar perturbations.

$$\begin{aligned} \mathrm{d}s^2 &= a^2(\eta)\Big\{ -(1+2\,\phi)\,\mathrm{d}\eta^2 - 2\,(\partial_i\, B)\,\mathrm{d}x^i\mathrm{d}\eta \\ &\quad + \Big[(1+2\psi)\,\delta_{ij} + 2\,\partial_i\partial_j E + h_{ij}\Big]\mathrm{d}x^i\mathrm{d}x^j\Big\}. \end{aligned} \tag{2.89}$$

Counting the total number of degrees of freedom for all types of perturbations, we end up with ten as expected. However, we have not yet taken into account the gauge transformations. Since coordinates have no physical meaning, we are free to perform a coordinate transformation eliminating four degrees of freedom. This shows that the real *physical* degrees of freedom should be six and not ten. In the following, we will focus on scalar perturbations only. Then, the result for the tensor perturbations can easily be obtained in a similar way.

First, we have to figure out how the scalar perturbations behave under gauge transformations and how we can get rid of these non-physical degrees of freedom. We decompose the generator of the infinitesimal coordinate transformation (2.84)

$$\xi^\mu(\eta,\,\mathbf{x}) = \delta^\mu_0\xi^0 + \delta^\mu_i\xi^i \tag{2.90}$$

into its temporal and spatial components. The three-vector ξ^i can be further decomposed into a transversal and a longitudinal part (a gradient of a scalar)

$$\xi^i = \xi^i_{\mathrm{T}} + \delta^{ij}\xi_{,j} \tag{2.91}$$

Decomposing the gauge degrees of freedom (2.90) according to their behaviour under spatial rotations, we notice that from the four components of ξ^μ only the time component ξ^0 and the longitudinal component ξ influence the scalar modes. The remaining two degrees of freedom are contained in ξ^i_{T} (three spatial components minus one condition of transversality $\partial_j\xi^j_{\mathrm{T}} = 0$) which only influence the vector modes. Therefore, we have to investigate how the scalar perturbations ϕ, ψ, B and E behave under the transformations

$$\eta \to \tilde{\eta} = \eta + \xi^0 \quad \text{and} \quad x^i \to \tilde{x}^i = x^i + \delta^{ij}\partial_j\xi. \tag{2.92}$$

The scalar degrees of freedom ϕ, ψ, B and E transform under (2.92) as

$$\begin{aligned} &\phi \to \tilde{\phi} = \phi + \mathcal{H}\,\xi^0 + (\xi^0)', \quad &&\psi \to \tilde{\psi} = \psi + \mathcal{H}\,\xi^0, \\ &B \to \tilde{B} = B + \xi^0 - \xi', &&E \to \tilde{E} = E + \xi. \end{aligned} \tag{2.93}$$

We can construct gauge-invariant quantities by linear combinations of (2.93) and characterize the *physical* scalar metric perturbations by the Bardeen potentials

$$\Phi := \phi - \frac{1}{a}\,[a\,(B - E')]' \quad \text{and} \quad \Psi := \psi - \mathcal{H}\,(B + E'). \tag{2.94}$$

In general, the scalar perturbations of the energy momentum tensor $\delta T^S_{\mu\nu}$ can also be characterized by four scalar functions. However, the linearised Einstein equations and the Bianchi identities yield additional constraint equations restricting the number of independent degrees of freedom. Confining ourselves to a single scalar field minimally coupled to gravity in a FRW background, only one physical scalar degree of freedom remains which fully characterizes the scalar perturbations of metric and matter (see [22, 29, 20] for a detailed treatment). An especially useful choice is the Mukhanov–Sasaki variable

$$v := a\left(\delta\varphi_{\rm gi} + \frac{\bar{\varphi}'}{\mathcal{H}}\,\Phi\right), \tag{2.95}$$

with the gauge-invariant scalar field perturbation constructed by using (2.93)

$$\delta\varphi_{\rm gi} := \delta\varphi - \bar{\varphi}'\,(B + E') \tag{2.96}$$

and the behaviour of $\delta\varphi$ under the gauge transformation (2.92)

$$\delta\varphi \to \widetilde{\delta\varphi} = \delta\varphi + \xi^0\bar{\varphi}'. \tag{2.97}$$

In [22] it has been shown that starting with the action (2.3), identifying (2.2) with (2.49) and expanding S around a FRW background up to second order in the perturbations, S_2 can be written (up to total divergences) as an effective action for the perturbation v

$$S_{\rm eff} := \int {\rm d}\eta\,{\rm d}^3x\,\mathcal{L}_{\rm eff} := \frac{1}{2}\int {\rm d}\eta\,{\rm d}^3x\left(v'^2 - \partial_i\,v\,\partial_j\,v\,\delta^{ij} + \frac{z''}{z}\,v^2\right). \tag{2.98}$$

This is the action of a free scalar field v in Minkowski space. The only relict of curved space-time is the effective *time-dependent* mass term

$$m^2_{\rm eff} := \frac{z''}{z} \quad \text{with} \quad z := \frac{a}{\mathcal{H}}\,\bar{\varphi}'. \tag{2.99}$$

The scalar v is related to a geometrically meaningful gauge-invariant quantity $\mathcal{R}$

$$v = z\,\mathcal{R}. \tag{2.100}$$

The quantity $\mathcal{R} = \psi + \mathcal{H}\,\delta\varphi/\bar{\varphi}'$ is called comoving curvature perturbation and describes the gravitational potential on spatial hypersurfaces (surfaces of constant φ). As an observable, it is especially suited, since it has a geometrical meaning. It is related to the intuitive picture for the delayed scalar field perturbations $\delta\varphi$ of Sect. 2.4.1 for the spatially flat case $\psi = 0$. The identification of $\mathcal{R}$ with the spatial curvature follows from the equation $\delta R^{(3)} = 4/a^2\,\nabla^2\,\psi$ for the perturbation of the spatial three-dimensional Ricci scalar $\delta R^{(3)}$. For super-horizon scales $\mathcal{R}$ coincides with the curvature perturbation $-\zeta = \Psi + \mathcal{H}\,\delta\rho/\bar{\rho}'$.

2.4.3 *Quantization of Fluctuations*

In order to derive the primordial power spectrum, we have to quantize the scalar field v. We follow the usual canonical quantization scheme and first calculate the conjugated momentum π_v

$$\pi_v := \frac{\partial\,\mathcal{L}_{\text{eff}}}{\partial\,v'} = v'. \tag{2.101}$$

When promoting v and π_v to operators $\hat{v}$ and $\hat{\pi}_v$, they must satisfy

$$\begin{aligned}
[\hat{v}(\mathbf{x},\,\eta),\,\hat{v}(\mathbf{x}',\,\eta')]_{\eta=\eta'} &= [\hat{\pi}_v(\mathbf{x},\,\eta),\,\hat{\pi}_v(\mathbf{x}',\,\eta')]_{\eta=\eta'} = 0,\\
[\hat{v}(\mathbf{x},\,\eta),\,\hat{\pi}_v(\mathbf{x}',\,\eta')]_{\eta=\eta'} &= i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}').
\end{aligned} \tag{2.102}$$

Now we expand the field operator $\hat{v}$ in eigenmodes of the Klein–Gordon equation for the field v. This equation can be obtained by variation of (2.98) and reads in Fourier space

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0, \tag{2.103}$$

with the time- and mode-dependent frequency

$$\omega_k(\eta) := k^2 - \frac{z''(\eta)}{z(\eta)}. \tag{2.104}$$

The field operator can be decomposed in creation and annihilation operators

$$\hat{v} = \frac{1}{2\,(2\pi)^{3/2}} \int \mathrm{d}^3k\,\left(e^{i\,\mathbf{k}\,\mathbf{x}}\,v_k^*(\eta)\,a_{\mathbf{k}} + e^{-i\,\mathbf{k}\,\mathbf{x}}\,v_k(\eta)\,a_{\mathbf{k}}^\dagger\right), \tag{2.105}$$

with the two independent solutions v_k and v_k^* which have to satisfy the reality conditions $v_k = v_{-k}^*$ (since v is a real field). If we neglect for a moment the time-dependence of (2.99), Eq. (2.103) reduces to that of a simple harmonic oscillator

$$v_k'' + k^2\,v_k = 0. \tag{2.106}$$

Since (2.106) is a second-order differential equation, we must specify two boundary conditions in order to obtain a *unique* solution. One of these conditions can be expressed by the proper normalization condition of the modes v_k, so that the commutator relations between the creation and annihilation operators $a_{\mathbf{k}}^\dagger$ and $a_{\mathbf{k}}$ acquire the standard form

$$[\hat{a}_{\mathbf{k}},\, \hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}},\, \hat{a}^\dagger_{\mathbf{k}'}] = 0, \quad [\hat{a}_{\mathbf{k}},\, \hat{a}^\dagger_{\mathbf{k}'}] = \delta^{(3)}(k - k'). \tag{2.107}$$

The correct (time-independent) choice of the Wronskian W is therefore

$$W[v_k,\, v_k^*] := i \left(v_k^*\, v_k' - v_k\, (v_k^*)' \right) = 1 = \langle v | v \rangle. \tag{2.108}$$

It can be verified directly by substitution that this choice indeed connects the commutator algebra (2.102) with (2.107). However, in order to obtain a unique solution of (2.106), we need a second condition. In terms of the annihilation and creation operators it is natural to implement this condition by fixing the state of lowest energy $|0\rangle$ by

$$\hat{a}|0\rangle := 0. \tag{2.109}$$

To see how (2.109) translates into a condition for the mode function v_k, we can express the Hamiltonian for the harmonic oscillator in terms of $\hat{a}$ and $\hat{a}^\dagger$ and perform $\hat{H}|0\rangle$

$$\hat{H}|0\rangle \propto \left[(\nu_k')^2 + \omega_k^2\, v_k^2 \right] \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} |0\rangle + E_0 |0\rangle, \tag{2.110}$$

with the ground state energy E_0. In order for $|0\rangle$ to be an eigenstate of $\hat{H}$, the combination in the bracket of the first term must vanish. This leads to the condition

$$v_k' = \pm i\, \omega_k v. \tag{2.111}$$

Substituting this back into (2.108), selects the negative solution of (2.111) for the condition of positive norm $\langle v | v \rangle \geq 0$. Integrating (2.111) with the time-independent $\omega_k = k^2$, we obtain a unique properly normalized positive frequency solution of (2.106)

$$v_k(\eta) = \frac{1}{\sqrt{2k}}\, e^{-i\, k\, \eta}. \tag{2.112}$$

So far, we have completely neglected the time-dependence of (2.104). The solution (2.112) just corresponds to a plane wave in Minkowski space. However, all interesting information about the curvature of space-time is encoded in $\omega_k(\eta)$. In the presence of the time-dependent effective mass (2.99), there is simply no unique prescription of the vacuum state as defined in (2.109) anymore [23]. This is a general feature of curved space-time and in the context of a FRW universe, the spatial expansion leads to particle creation [24]. Moreover, it can be shown that even an uniformly *accelerated* observer in a flat and empty Minkowski space-time will detect particles [8, 11, 30]. This just reflects the fact that the definition of the vacuum is an observer-dependent statement in general. Nevertheless, due to the symmetries of de Sitter space, we can at least in the inflationary scenario hope for a physically well-founded assumption that finally leads to a meaningful unique solution. We consider a comoving observer in the limit $\eta \to -\infty$ far before inflation started. All relevant comoving scales $\lambda_c \sim 1/k$ were well inside the Hubble radius H^{-1}. This is equivalent to the condition $k \gg a\, H$.

Physically, this corresponds to the fact that the modes which are deep within the Hubble radius do not "feel" the curvature (or in the FRW case, the expansion of space). Using this for exact de Sitter space $\varepsilon \to 0$ in (2.104) leads again to the time-independent frequency

$$\lim_{\substack{\varepsilon\to 0\\ \eta\to-\infty}} \omega_k(\eta) = \lim_{\substack{\varepsilon\to 0\\ \eta\to-\infty}} \left(k^2 - \frac{z''}{z}\right) = \lim_{\eta\to-\infty} \left(k^2 - \frac{2}{\eta}\right) = k^2. \tag{2.113}$$

Thus, we find again (2.112) as asymptotic solution in de Sitter space. In order to obtain a unique physically meaningful solution for the *general* equation (2.103), we *demand* that a solution must satisfy the *asymptotic boundary condition*

$$\lim_{\eta\to-\infty} v_k = \frac{1}{\sqrt{2\,k}}\, e^{-i\,k\,\eta} \tag{2.114}$$

instead of (2.111). This is called the Bunch-Davies boundary condition [5]. In general, it is difficult to solve e.q. (2.103) explicitly with the boundary conditions (2.108) and (2.114). However, we wish to describe inflation and are mainly interested in the solution for a *quasi* de Sitter case $\varepsilon \ll 1$. First, we can try to find an exact solution for exact de Sitter space $\varepsilon = 0$. Using $\lim\limits_{\varepsilon\to 0} z''/z = 2/\eta$, Eq. (2.103) becomes

$$v_k'' + \left(k^2 - \frac{2}{\eta}\right) v_k = 0. \tag{2.115}$$

Equation (2.115) can be solved exactly by the solution

$$v_k = A\,\frac{1}{\sqrt{2\,k}}\, e^{-i\,k\,\eta}\left(1 - \frac{i}{k\,\eta}\right) + B\,\frac{1}{\sqrt{2\,k}}\, e^{i\,k\,\eta}\left(1 + \frac{i}{k\,\eta}\right). \tag{2.116}$$

The boundary conditions (2.108) and (2.114) uniquely select $A = 1$ and $B = 0$

$$v_k = \frac{1}{\sqrt{2\,k}}\, e^{-i\,k\,\eta}\left(1 - \frac{i}{k\,\eta}\right). \tag{2.117}$$

2.5 Horizon Crossing and Observable Quantities

There are two complementary but equivalent ways of thinking about inflation by either considering only *comoving* quantities or by considering only *physical* quantities.

When formulating everything in comoving quantities, it is convenient to think of an underlying comoving *coordinate* grid and a constant comoving perturbation wavelength λ_c. Formulated in physical quantities, we should think of a rigid *coor-*

dinate grid and a stretched physical wavelength $\lambda_p = a(t)\,\lambda_c$. Similarly, but in a sense opposite to the wavelength or coordinate grid, we can consider the comoving and physical Hubble radius. We have required that the physical Hubble parameter H_p should be constant during the rapid expansion. This leads to a constant physical Hubble radius $r_p := {H_p}^{-1}$ (cf. Fig. 2.13). In contrast, the comoving Hubble radius $r_c := H_c^{-1} = [a(t)\,H_p]^{-1}$ decreases with growing $a(t)$. In the comoving picture inflation means to shrink the comoving horizon to a small homogeneous piece of the original patch. In the physical picture it is the underlying space-time itself that expands under the constant physical Hubble radius. Both pictures are equivalent and their use is a matter of taste, but they are often the origin of confusion.

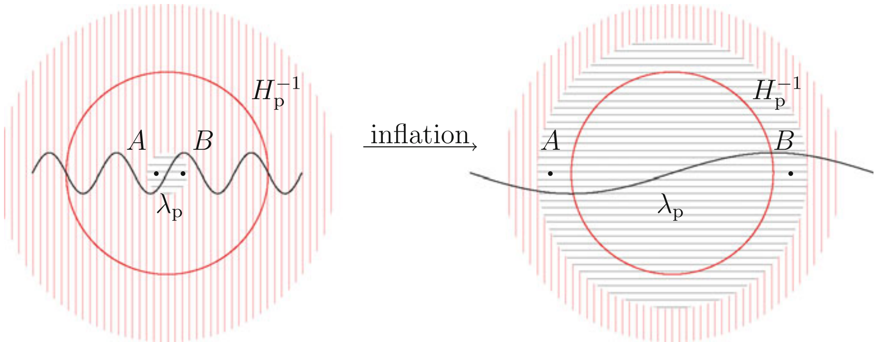

Fig. 2.13 Inflation in the physical picture. The physical Hubble radius remains constant, while the physical wavelength gets stretched. The *black horizontal pattern* represents a homogeneous patch, while the *red vertical pattern* represent different patches. The points A and B were in causal contact before inflation, have crossed the horizon during inflation and might re-enter in a post-inflationary era, when the Hubble radius grows faster than the scale factor

For modes well inside the horizon ($k/a\,H_p = r_{H_p}/\lambda_p \gg 1$), the time-dependent mass term in (2.103) can be dropped and we recover the oscillating equation of flat space-time with the plane wave solutions

$$v_k(\eta) \propto \frac{1}{\sqrt{2k}}\, e^{-i\,k\,\eta} \quad \text{for} \quad \lambda_p \ll r_{H_p}. \tag{2.118}$$

This is consistent because we would not expect curvature (expansion of space-time) to play a role for scales λ_p much smaller than r_{H_p}. For the opposite case, when the modes have left the horizon and are much bigger than the physical Hubble radius ($k \ll a\,H_p$), we can neglect the k^2 term in (2.103). Then the solution for the growing modes is

$$v_k \propto z \quad \text{for} \quad \lambda_p \gg r_{H_p}. \tag{2.119}$$

Relations (2.119) and (2.100) imply that the comoving curvature scalar remains constant

$$\mathcal{R} = \frac{v}{z} = \text{const.} \quad \text{for} \quad \lambda_\text{p} \gg r_{\text{H}_\text{p}}. \tag{2.120}$$

This aspect is crucial for the estimation of the power spectrum of the fluctuations. It guarantees that the modes outside the Hubble radius will "freeze" and are in particular independent of the physics taking place well inside the horizon (e.g. they are unaffected by the details of the reheating phase). We can therefore connect the perturbations at re-entry with the perturbations at first horizon crossing. The power spectrum of the $\mathcal{R}_k$ can be calculated by the Fourier transform

$$\hat{\mathcal{R}}(\eta, \mathbf{x}) = \frac{1}{(2\,\pi)^{3/2}} \int \mathrm{d}^3\,k\,\hat{\mathcal{R}}_\mathbf{k}(\eta)\,e^{i\,\mathbf{k}\,\mathbf{x}} \quad \text{with} \quad \hat{\mathcal{R}}_\mathbf{k}(\eta) = \frac{v_k(\eta)}{z}\,a_\mathbf{k} + \frac{v_k^*(\eta)}{z}\,a_{-\mathbf{k}}^\dagger. \tag{2.121}$$

The power spectrum is defined by the two-point correlation function

$$< 0|\hat{\mathcal{R}}_\mathbf{k}(\eta)\,\hat{\mathcal{R}}^\dagger_{\mathbf{k}'}(\eta)|0 >=: \frac{2\,\pi^2}{k^3}\mathcal{P}_\mathcal{R}\,\delta^{(3)}(\mathbf{k}-\mathbf{k}') \quad \text{with} \quad \mathcal{P}_\mathcal{R} = \frac{k^3}{2\pi^2}\,\frac{|v_k(\eta)|^2}{z^2}. \tag{2.122}$$

To illustrate the behaviour of the modes during the different epochs of cosmological evolution, we will switch to the *comoving picture*. Figure 2.14 shows the *comoving* Hubble radius $r_{\text{H}_\text{c}} = (a\,H)^{-1}$ in a plot of *comoving* length scale L_c against conformal time η.

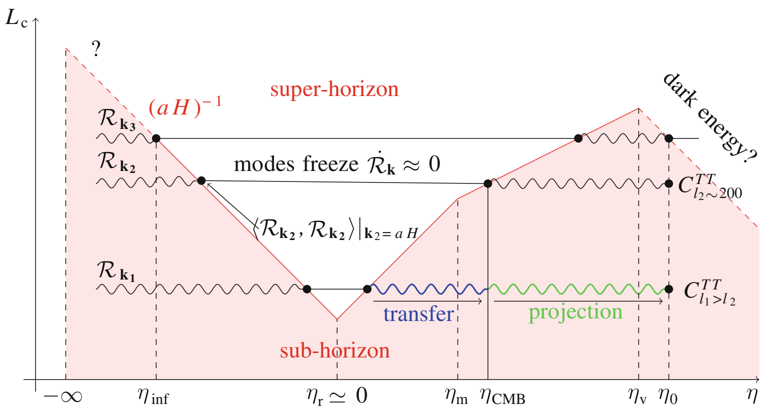

Fig. 2.14 Mode crossing. Different comoving scalar curvature modes cross the comoving Hubble radius twice at different epochs in our observable universe. The idea of this figure has been borrowed from [4]

Different curvature modes $\mathcal{R}_{\mathbf{k}^i}$, $i = 1, \ldots, 3$ with fixed comoving wavelengths $\lambda_c^i \sim 1/k^i$ cross the horizon at different times η_cross determined by the relation

$$k_{\text{cross}} := a(\eta_{\text{cross}})H(\eta_{\text{cross}}). \tag{2.123}$$

All scalar curvature modes $\mathcal{R}_{\mathbf{k}_i}$ start deep inside the sub-horizon region as free oscillating waves when the space looks like flat Minkowski space (they do not feel the expansion yet). This is the physical motivation for the Bunch-Davies vacuum condition (2.114). As soon as the modes cross the horizon at their individual time-scale k^i_{cross} and enter the super-horizon region, the modes freeze until re-entry. Decoherence can explain the quantum-to-classical transition [15–17] and once the modes re-enter the horizon, they become quasi-classical perturbations subjected to all possible physical processes within the horizon. They set the initial conditions for the density fluctuations which evolve and amplify under the influence of gravity. We can describe the post-inflationary evolution by complicated "transfer functions". The modes re-enter according to the principle "first out – last in" .

Since these small irregularities are projected onto our sky, it is useful to expand the power spectrum of perturbations in spherical harmonics in order to obtain an angular power spectrum depending on the multipole order of the spherical harmonics. The angle θ can roughly be related to the multipole l by $l \sim \pi/\theta$. Thus, small angular separations correspond to large multipoles l in the angular power spectrum. Today, we observe with increasing resolution correlations with an angular separation θ.

The mode $\mathcal{R}_{\mathbf{k}_2}$ in Fig. 2.14 re-enters the horizon at the time of recombination η_{CMB}, where the photons decoupled from the oscillating photon-baryon plasma. It corresponds to the first acoustic peak in the observed angular power spectrum and can be related to the size of the horizon at the time of recombination. It therefore provides direct information about the curvature. Different locations of the first peak would correspond to different curvature. This allows us to conclude that our universe is indeed flat up to about one percent. Modes that have re-entered at *earlier* times correspond to *larger* angular moments l or smaller angular correlations $\theta \sim \pi/l$ on the sky.

2.5.1 Slow-Roll Power Spectrum

In order to calculate the power spectrum (2.122), we have to find a solution to the mode equation (2.103) in the slow-roll regime $\varepsilon \ll 1$. In contrast to the $\varepsilon = 0$ solution with $z''/z = 2/\eta$ of pure de Sitter space, we must approximate the effective mass term (2.99) z''/z to first order in the slow-roll parameters $\varepsilon \ll 1$, $|\delta| \ll 1$. We could exactly solve (2.103) in terms of Hankel functions of the first kind $H^{(1)}_{\nu_\text{s}}(-k\,\eta)$, by expressing (2.99) as

$$\frac{z''}{z} = \frac{\nu_\text{s}^2 - 1/4}{\eta^2}, \tag{2.124}$$

where ν_s is an expression linear in the *constant* ε and δ. Let us start by re-writing the definitions (2.69) in terms of conformal expressions $\mathcal{H} = a'/a = a\,H$

$$\varepsilon = -\frac{\dot{H}}{H^2} = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \quad \text{and} \quad \delta = \varepsilon - \frac{1}{2\,\mathcal{H}}\frac{\varepsilon'}{\varepsilon} = 1 - \frac{\varphi''}{\mathcal{H}\,\varphi'}. \tag{2.125}$$

Equation (2.125) implies $\varepsilon' = 2\,\mathcal{H}(\varepsilon^2 - \varepsilon\,\delta)$ and we find that terms $\propto \varepsilon'$ and $\propto \delta'$ are of second order and can be neglected within our approximation. Using (2.99), we can calculate

$$\frac{z'}{z} = \frac{a'}{a} + \frac{\varphi''}{\varphi'} - \frac{\mathcal{H}'}{\mathcal{H}} = \mathcal{H}\left(1 + \frac{\varphi''}{\mathcal{H}}\,\varphi' - \frac{\mathcal{H}'}{\mathcal{H}^2}\right) = \mathcal{H}\,(1 + \varepsilon - \delta). \tag{2.126}$$

Differentiating once more and using (2.125), we find

$$\begin{aligned}\frac{z''}{z} &= \left(\frac{z'}{z}\right)' + \left(\frac{z'}{z}\right)^2 = \mathcal{H}'\,(1 + \varepsilon - \delta) + \mathcal{H}\,(\varepsilon' + \delta') + \mathcal{H}^2\,(1 + \varepsilon - \delta)^2 \\ &= \mathcal{H}^2\,(2 + 2\,\varepsilon - 3\,\delta) + O(2).\end{aligned} \tag{2.127}$$

We further observe that $\mathrm{d}\eta = \mathrm{d}a/a\,\mathcal{H}$, so that we can write

$$\frac{\varepsilon}{a\,\mathcal{H}}\,\mathrm{d}a = \varepsilon\,\mathrm{d}\eta = \mathrm{d}\eta - \frac{\mathcal{H}'}{\mathcal{H}^2}\,\mathrm{d}\eta = \mathrm{d}\eta + \mathrm{d}\left(\frac{1}{\mathcal{H}}\right) \quad \text{or} \quad \eta = -\frac{1}{\mathcal{H}} + \int \frac{\varepsilon}{a\,\mathcal{H}}\,\mathrm{d}a. \tag{2.128}$$

Since we have assumed that ε is approximately constant, we obtain

$$\eta = -\frac{1}{\mathcal{H}} + \varepsilon \int \mathrm{d}\eta \quad \Leftrightarrow \quad \mathcal{H} = -\frac{1}{\eta\,(1 - \varepsilon)}. \tag{2.129}$$

Inserting this into the result (2.127), we can rewrite z''/z in the desired form

$$\frac{z''}{z} \approx \frac{2 + 2\varepsilon - 3\delta}{\eta^2\,(1 - \varepsilon)^2} \approx \frac{2 + 2\varepsilon - 3\delta}{\eta^2\,(1 - 2\varepsilon)} \approx \frac{(2 + 2\varepsilon - 3\delta)(1 + 2\varepsilon)}{\eta^2} \approx \frac{2 + 6\varepsilon - 3\delta}{\eta^2}. \tag{2.130}$$

In each step we have successively approximated the expression, finally keeping only terms linear in the slow roll parameters. We substitute (2.130) in (2.124) and obtain ν_s to first order in ε and δ

$$\nu_\mathrm{s}^2 = \frac{9}{4} + 6\varepsilon - 3\delta \quad \Rightarrow \quad \nu_\mathrm{s} = \frac{3}{2} + 2\,\varepsilon - \delta + O(2). \tag{2.131}$$

The general solution of (2.103) for z''/z defined in (2.124) in terms of the Hankel function $H^{(1)}_{\nu_\mathrm{s}}(-k\,\eta)$ reads

$$v_k(\eta) = \frac{\sqrt{\pi}}{2}\,e^{i\,\pi/2\,(\nu_\mathrm{s}+1/2)}\,\sqrt{-\eta}\,H^{(1)}_{\nu_\mathrm{s}}(-k\eta). \tag{2.132}$$

Since we are interested in super-horizon scales $-k\,\eta = k/a\,H = r_{\mathrm{H_p}}/\lambda_{\mathrm{p}} \ll 1$ and we know that the scalar comoving curvature modes $\mathcal{R}_k$ freeze outside the Hubble radius $r_{\mathrm{H_p}}$, we can use $i\,H^{(1)}_{\nu_{\mathrm{s}}}(\alpha) \simeq \Gamma(\nu_{\mathrm{s}})/\pi\,(z/2)^{-\nu_{\mathrm{s}}}$ in the limit $\alpha \to 0$ [29], express $-\eta$ by (2.129) and finally find for the amplitude $|v_k(\eta)|$ the asymptotic expression

$$|v_k(\eta)| \simeq C(\nu_{\mathrm{s}})\,\frac{1}{\sqrt{2\,k}}\left(\frac{k}{a\,H}\right)^{-\nu_{\mathrm{s}}+1/2}, \quad C(\nu_{\mathrm{s}}) := 2^{\nu_{\mathrm{s}}-3/2}\,\frac{\Gamma(\nu_{\mathrm{s}})}{\Gamma(3/2)}\,(1-\varepsilon)^{\nu_{\mathrm{s}}-1/2}. \tag{2.133}$$

We can insert (2.133) into the definition of the power spectrum (2.122) and obtain

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\,\frac{|v_k(\eta)|^2}{z^2} = \frac{k^2}{4\,\pi^2}\,\frac{C^2(\nu_{\mathrm{s}})}{z^2}\left(\frac{k}{a\,H}\right)^{1-2\,\nu_{\mathrm{s}}}. \tag{2.134}$$

Since $\nu_{\mathrm{s}} \approx 3/2$ during slow-roll, we can approximate $C(\nu_{\mathrm{s}}) \approx 1$. Substituting $z = a\,\varphi'/\mathcal{H} = a\,\dot{\varphi}/H$, we finally obtain the *nearly* scale invariant power spectrum

$$\begin{aligned}\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{\asymp}(k) &\simeq \left(\frac{H}{2\,\pi}\right)^2\left(\frac{H}{\dot{\varphi}}\right)^2\left(\frac{k}{a\,H}\right)^{3-2\,\nu_{\mathrm{s}}} \\ &\simeq \left(\frac{H}{2\,\pi}\right)^2\left(\frac{H}{\dot{\varphi}}\right)^2\Big|_{k=a\,H} = \frac{V(\varphi)}{24\,\pi^2\,M_{\mathrm{P}}^4\,\varepsilon_{\mathrm{V}}}\Big|_{k=a\,H}.\end{aligned} \tag{2.135}$$

In the second identity, we have used again the fact that the comoving curvature perturbations are constant in the super-horizon regime. This allows us to evaluate them at the point where they leave the Hubble radius $k = a\,H$ since they remain unchanged until they re-enter again. In the last identity we have used the slow-roll approximation to rewrite $\mathcal{P}_{\mathcal{R}}$ in terms of the inflaton potential.

2.5.2 Cosmological Parameters During Slow-Roll

As a measure for the scale dependence of (2.135), we define the spectral index

$$n_{\mathrm{s}} - 1 := \frac{\mathrm{d}\ln\mathcal{P}_{\mathcal{R}}(k)}{\mathrm{d}\ln k} = 3 - 2\,\nu_{\mathrm{s}} = 2\delta - 4\varepsilon = 2\delta_{\mathrm{V}} - 6\varepsilon_{\mathrm{V}}. \tag{2.136}$$

In the last line, we have used relation (2.80) to write (2.136) in terms of the slow-roll parameters expressed in terms of the potential. For tensor fluctuations we can follow exactly the same steps as for the scalar perturbations and start with a similar effective action as (2.98). Since tensor perturbations are already gauge-invariant by construction, we can directly use the h_{ij} of (2.89). The Fourier decomposition reads [29]

$$h_{ij}(\mathbf{x},\,\eta) = \frac{1}{(2\,\pi)^{3/2}}\int \mathrm{d}k\,\left(\hat{h}^{+}_{\mathbf{k}}(\eta)\,\varepsilon^{+}_{ij}(\mathbf{k}) + \hat{h}^{\times}_{\mathbf{k}}(\eta)\,\varepsilon^{\times}_{ij}(\mathbf{k})\right) \tag{2.137}$$

Tensor modes are gravitational waves and have two polarizations ε^{+}_{ij} and $\varepsilon^{\times}_{ij}$.

$$\hat{v}^{+/\times}_{\mathbf{k}}(\eta) = z\, \hat{h}^{+/\times}_{\mathbf{k}}(\eta) = \frac{M_{\mathrm{P}}\, a}{\sqrt{8\,\pi}}\, \hat{h}^{+/\times}_{\mathbf{k}}(\eta), \tag{2.138}$$

with $z = a\, M_{\mathrm{P}}/\sqrt{8\,\pi}$. Thus, the effective mass (2.99) term z''/z simplifies to a''/a

$$\frac{a''}{a} = \mathcal{H}^2\,(2-\varepsilon) = \frac{(2-\varepsilon)}{\eta^2\,(1-\varepsilon)^2} = \frac{2+3\varepsilon}{\eta^2} + O(2) \tag{2.139}$$

and the corresponding index ν_{t} in the solution (2.133) is given by

$$\nu_{\mathrm{t}}^2 = \eta^2\,\frac{a''}{a} + \frac{1}{4} = \frac{9}{4} + 3\,\varepsilon \quad \Rightarrow \quad \nu_{\mathrm{t}} = \frac{3}{2} + \varepsilon + O(2). \tag{2.140}$$

Substituting this with $z = a\, M_{\mathrm{P}}/\sqrt{8\,\pi}$ in the asymptotic solution (2.133) yields

$$\begin{aligned}\mathcal{P}_h(k) &:= 4\,\mathcal{P}_{h^{+/\times}}(k) := 4\,\frac{8\,\pi}{M_{\mathrm{P}}^2\,a^2}\,\frac{k^3}{2\,\pi^2}|v_k(\eta)|^2 \simeq \frac{16}{\pi}\left(\frac{H}{M_{\mathrm{P}}}\right)^2\left(\frac{k}{a\,H}\right)^{3-2\nu_{\mathrm{t}}}\\ &\simeq \frac{16\,G}{\pi}\,H^2\Big|_{k=a\,H}.\end{aligned} \tag{2.141}$$

The tensor spectral index is defined with a different convention compared to (2.136)

$$n_{\mathrm{t}} := \frac{\mathrm{d}\ln \mathcal{P}_h(k)}{\mathrm{d}\ln k} \simeq 3 - 2\,\nu_{\mathrm{t}} = -2\,\varepsilon. \tag{2.142}$$

If the spectral indices (2.136) and (2.142) are themselves independent of the mode k, they simply describe the tilt of the power spectra. If $n_{\mathrm{s}}(k)$ and $n_{\mathrm{t}}(k)$ are functions of the mode k, this effect can be quantified by the running of the spectral indices

$$\alpha_{\mathrm{s}} := \frac{\mathrm{d}\,n_{\mathrm{s}}}{\mathrm{d}\ln k} \quad \text{and} \quad \alpha_{\mathrm{t}} := \frac{\mathrm{d}\,n_{\mathrm{t}}}{\mathrm{d}\ln k}, \tag{2.143}$$

which are however already of second order in ε and δ. The tensor to scalar ratio

$$r := \frac{\mathcal{P}_{\mathcal{R}}}{\mathcal{P}_h} = 16\,\varepsilon \tag{2.144}$$

is a direct measure for the energy scale of inflation, since $\mathcal{P}_{\mathcal{R}} \simeq 10^{-9}$ is fixed by observation and $\mathcal{P}_h \sim H^2$. We note that the measurement of $\mathcal{P}_{\mathcal{R}}$ alone would not be sufficient, since it is a ratio of the two parameters H^4 and $\dot{\varphi}^2$ with $\dot{\varphi}$ being model dependent. Combination of (2.142) and (2.144) yields the consistency relation

$$r = -8\,n_{\mathrm{t}} \tag{2.145}$$

which is a *generic* prediction of all inflationary models. A detection of primordial gravitational waves would therefore provide direct information about the absolute energy scale of inflation. We will use these cosmological parameters when considering the non-minimal Higgs inflation scenario in Chap. 6.

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