Chapter 2
Plane and Straight Line in E_3

2.1 Equations of a Straight Line in E_3

Definition 2.1 (see [2], p. 448). A **vector** is a translation of the three-dimensional space; therefore it must be studied the basics of the three-dimensional Euclidean geometry: the points, the straight lines and the planes.

Let $R = \left\{O; \bar{i}, \bar{j}, \bar{k}\right\}$ be a Cartesian reference. For $M \in E_3$, the coordinates of the point M are the coordinates of the position vector $\overline{OM}$. If

$$\overline{OM} = x\bar{i} + y\bar{j} + z\bar{k}$$

then $M(x, y, z)$.

2.1.1 *Straight Line Determined by a Point and a Nonzero Vector*

A straight line from E_3 can be determined by (see [6]):

(1) a point and a nonzero free vector;
(2) two distinct points;
(3) the intersection of the two planes.

Let be $M_0 \in E_3$, $M_0(x_0, y_0, z_0)$ and $\bar{v} \in V_3 \backslash \left\{\bar{0}\right\}$, $\bar{v} = a\bar{i} + b\bar{j} + c\bar{k}$. We intend to find the equation of the straight line determined by the point M_0 and by the nonzero vector $\bar{v}$, denoted by $d = (M_0, \bar{v})$ and represented in Fig. 2.1.

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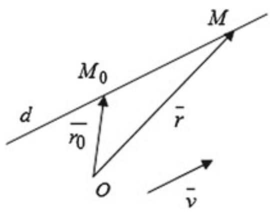
Fig. 2.1 The straight line determined by a point and a nonzero vector



We denote

$$\overline{r}_0 = \overline{\mathrm{O}M_0} = x_0\overline{i} + y_0\overline{j} + z_0\overline{k}.$$

Let be $M \in \mathrm{E}_3$, $M\,(x, y, z)$ and we denote

$$\overline{r} = \overline{\mathrm{O}M} = x\overline{i} + y\overline{j} + z\overline{k}.$$

The point $M \in d \Leftrightarrow \overline{M_0M}$ and $\overline{v}$ are collinear$\Leftrightarrow$

$$\overline{M_0M} \times \overline{v} = \overline{0}. \tag{2.1}$$

However

$$\overline{M_0M} = \overline{r} - \overline{r}_0. \tag{2.2}$$

From (2.1) and (2.2) it results (see [6]) the *vector equation* of the straight line $d = (M_0, \overline{v})$:

$$(\overline{r} - \overline{r}_0) \times \overline{v} = \overline{0};$$

$\overline{v}$ is called the *direction vector* of the straight line.

If $\overline{M_0M}$ and $\overline{v}$ are collinear then $(\exists)$ $t \in \mathbb{R}$ unique, such that

$$\overline{M_0M} = t\overline{v}.$$

Taking account the relation (2.2) it results that $\overline{r} - \overline{r}_0 = t\overline{v};$ we obtain (see [6]) the *vector parametric equation* of the straight line d:

$$\overline{r} - \overline{r}_0 = t\overline{v},\ t \in \mathbb{R}. \tag{2.3}$$

The Eq. (2.3) can be written as

$$x\overline{i} + y\overline{j} + z\overline{k} = x_0\overline{i} + y_0\overline{j} + z_0\overline{k} + ta\overline{i} + tb\overline{j} + tc\overline{k};$$

we deduce the *parametric equations* (see [4], p. 49) of the straight line d:

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}, \quad t \in \mathbb{R}. \tag{2.4}$$

If in the relation (2.4) we eliminate the parameter t we obtain (see [4], p. 49) the *Cartesian equations* of the straight line d:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{2.5}$$

In the relation (2.5), there is the following convention: if one of the denominators is equal to 0, then we shall also cancel that numerator.

2.1.2 Straight Line Determined by Two Distinct Points

Let be:

- $M_1 \in E_3,\ M_1(x_1, y_1, z_1),\ \overline{r}_1 = \overline{OM}_1$ and
- $M_2 \in E_3,\ M_2(x_2, y_2, z_2),\ \overline{r}_2 = \overline{OM}_2$.

We want to determine the equation of straight line determined by the points M_1 and M_2, denoted by $d = (M_1, M_2)$ and represented in Fig. 2.2.

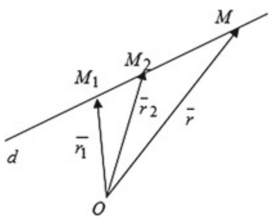


Fig. 2.2 Straight line determined by two distinct points

Let be $M \in d$, $M(x, y, z)$ and we denote $\overline{r} = \overline{OM}$. We consider that the straight line is determined by M_1 and $\overline{M_1 M_2}$.

As

$$\overline{M_1 M} \times \overline{M_1 M_2} = \overline{0}$$

it results that *vector equation* (see [6]) of the straight line d is:

$$(\overline{r} - \overline{r}_1) \times (\overline{r}_2 - \overline{r}_1) = \overline{0}.$$

Since (within the Proposition 1.1 from the Chap. 1), $(\exists)\ t \in \mathbb{R}$ unique, such that

$$\overline{M_1 M} = t\ \overline{M_1 M_2}$$

it results that

$$\bar{r} - \bar{r}_1 = t\left(\bar{r}_2 - \bar{r}_1\right), (\forall)\ t \in \mathbb{R},$$

namely the *vector parametric equation* (see [5], p. 166) is of form:

$$\bar{r} = \bar{r}_1 + t\left(\bar{r}_2 - \bar{r}_1\right), (\forall)\ t \in \mathbb{R}; \tag{2.6}$$

the *parametric equations* (see [6]) are:

$$\begin{cases} x = x_1 + t\left(x_2 - x_1\right) \\ y = y_1 + t\left(y_2 - y_1\right), \ (\forall)\ t \in \mathbb{R} \\ z = z_1 + t\left(z_2 - z_1\right) \end{cases} \tag{2.7}$$

and the *Cartesian equations* (see [4], p. 49) will be:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{2.8}$$

2.2 Plane in E_3

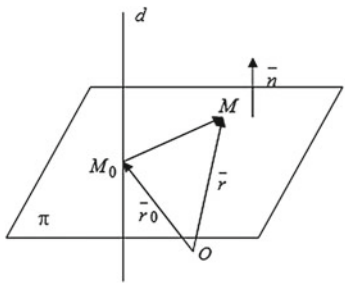
A plane can be determined (see [6]) in E_3 as follows:

(1) a point and a nonnull vector normal to plane,
(2) a point and two noncollinear vectors,
(3) three non collinear points,
(4) a straight line and a point that does not belong to the straight line,
(5) two concurrent straight lines,
(6) two parallel straight lines.

2.2.1 A a Point and a Non Zero Vector Normal to the Plane

Let be $M_0 \in \pi$, $M_0\left(x_0, y_0, z_0\right)$ and the nonnull free vector $\bar{n} = a\bar{i} + b\bar{j} + c\bar{k}$ normal to the plane (see Fig. 2.3).

Fig. 2.3 Plane determined by a point and a non zero vector normal to plane



Definition 2.2 (see [6]). The straight line d which passes through M_0 and has the direction of the vector $\overline{n}$ is called the **normal to the plane** through M_0; the vector $\overline{n}$ is the **normal vector** of the plane.

We propose to obtain the plane equation determined by the point M_0 and by the vector $\overline{n}$, denoted with $\pi = (M_0, \overline{n})$.

A point $M(x, y, z) \in \pi \Leftrightarrow \overline{M_0M}$ and $\overline{n}$ are orthogonal.

We denote

$$\overline{r} = \overline{OM} = x\overline{i} + y\overline{j} + z\overline{k}.$$

As $\overline{n}$ is perpendicular on $\overline{M_0M}$ it results

$$\overline{M_0M} \cdot \overline{n} = 0,$$

namely

$$(\overline{r} - \overline{r}_0) \cdot \overline{n} = 0;$$

from here we deduce the *normal equation* (see [6]) of the plane π:

$$\overline{r} \cdot \overline{n} - \overline{r}_0 \cdot \overline{n} = 0. \tag{2.9}$$

Writing the relation (2.9) as

$$ax + by + cz - ax_0 - by_0 - cz_0 = 0$$

we obtain the *Carthesian equation* (see [6]) of the plane π:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{2.10}$$

If we denote

$$ax_0 - by_0 - cz_0 = -d$$

then from the equation (2.10) one deduces the *general Carthesian equation* (see [6]) of the plane π:

$$ax + by + cz + d = 0. \tag{2.11}$$

2.2.2 Plane Determined by a Point and Two Noncollinear Vectors

Let $\overline{u}, \overline{v}$ be two noncollinear vectors, namely $\overline{u} \times \overline{v} \neq \overline{0}$, of the form

$$\overline{u} = l_1\overline{i} + m_1\overline{j} + n_1\overline{k}, \overline{v} = l_2\overline{i} + m_2\overline{j} + n_2\overline{k}$$

and let be $M_0 \in \pi$, $M_0(x_0, y_0, z_0)$ (see Fig. 2.4).

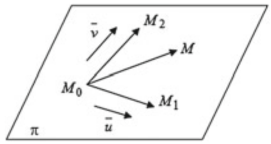


Fig. 2.4 Plane determined by a point and two non collinear vectors

We want to find the equation of the plane determined by the point M_0 and by the free vectors $\overline{u}$ and $\overline{v}$, denoted by $\pi = (M_0, \overline{u}, \overline{v})$.

Let

- $\overrightarrow{M_0M_1}$ be a representative for the free vector $\overline{u}$,
- $\overrightarrow{M_0M_2}$ be a representative for the free vector $\overline{v}$.

A point $M(x, y, z) \in \pi \Leftrightarrow \overline{M_0M}, \overline{M_0M_1}, \overline{M_0M_2}$ are coplanar. The coplanarity of these vectors can be expressed as:

(a) using the Proposition 1.21 from the Chap. 1, $(\exists)\ t_1, t_2 \in \mathbb{R}$ uniquely determined, such that

$$\overline{M_0M} = t_1\overline{M_0M_1} + t_2\overline{M_0M_2}; \tag{2.12}$$

(b) using the Proposition 1.34 from the Chap. 1, $\overline{M_0M}$ is perpendicular on $\overline{u} \times \overline{v}$, namely

$$\overline{M_0M} \cdot (\overline{u} \times \overline{v}) = 0. \tag{2.13}$$

Writing the relation (2.12) on the form:

$$\overline{r} - \overline{r}_0 = t_1\overline{u} + t_2\overline{v}$$

we deduce the *vector parametric equation* (see [6]) of the plane π:

$$\overline{r} = \overline{r}_0 + t_1\overline{u} + t_2\overline{v},\ (\forall)\ t_1, t_2 \in \mathbb{R} \tag{2.14}$$

and then the *parametric equations* of the plane π:

$$\begin{cases} x = x_0 + t_1 l_1 + t_2 l_2 \\ y = y_0 + t_1 m_1 + t_2 m_2,\ (\forall)\ t_1, t_2 \in \mathbb{R}. \\ z = z_0 + t_1 n_1 + t_2 n_2 \end{cases} \tag{2.15}$$

From the relation (2.13) we obtain the *vector equation* (see [6]) of the plane π:

$$(\overline{r} - \overline{r}_0) \cdot (\overline{u} \times \overline{v}) = 0. \tag{2.16}$$

As

$$\overline{M_0 M} = (x - x_0)\,\overline{i} + (y - y_0)\,\overline{j} + (z - z_0)\,\overline{k} \tag{2.17}$$

we have

$$\overline{M_0 M} \cdot (\overline{u} \times \overline{v}) = \begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix};$$

so from the relation (2.13) we deduce the *Carthesian equations* (see [6]) of the plane π:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0. \tag{2.18}$$

2.2.3 Plane Determined by Three Noncollinear Points

Let $M_0, M_1, M_2 \in$ E$_3$ be three noncollinear points, $M_0\,(x_0, y_0, z_0)$, $M_1\,(x_1, y_1, z_1)$, $M_2\,(x_2, y_2, z_2)$. It results that $\overline{M_0 M_1}, \overline{M_0 M_2}$ are noncollinear. We propose to obtain the equation of the plane determined by these points, which is represented in Fig. 2.5 and it is denoted by $\pi = (M_0, M_1, M_2)$.

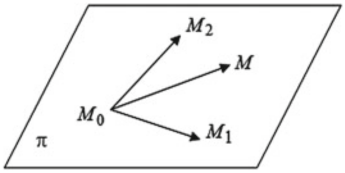


Fig. 2.5 Plane determined by three noncollinear points

We note that $\pi = (M_0, M_1, M_2)$ coincides with $\pi_1 = \left(M_0, \overline{M_0 M_1}, \overline{M_0 M_2}\right)$, namely we are in the case presented in the previous paragraph. We have

$$\begin{aligned} \overline{M_0 M_1} &= (x_1 - x_0)\,\overline{i} + (y_1 - y_0)\,\overline{j} + (z_1 - z_0)\,\overline{k}, \\ \overline{M_0 M_2} &= (x_2 - x_0)\,\overline{i} + (y_2 - y_0)\,\overline{j} + (z_2 - z_0)\,\overline{k}. \end{aligned}$$

A point $M\,(x, y, z) \in \pi \Leftrightarrow \overline{M_0 M}, \overline{M_0 M_1}, \overline{M_0 M_2}$ are coplanar, namely

$$\overline{M_0 M} \cdot \left(\overline{M_0 M_1} \times \overline{M_0 M_2}\right) = 0.$$

Using $\overline{M_0 M}$ from (2.17) we obtain the following *Cartesian equation* (see [6]) of the plane π:

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ x_2-x_0 & y_2-y_0 & z_2-z_0 \end{vmatrix} = 0. \tag{2.19}$$

2.2.4 Plane Determined by a Straight Line and a Point that Doesn't Belong to the Straight Line

Let $d \subset E_3$ and a point $M_0 \notin d$ (see Fig. 2.6).

We want to obtain the equation of the plane determined by the straight line d and by the point M_0, denoted by $\pi = (M_0, d)$.

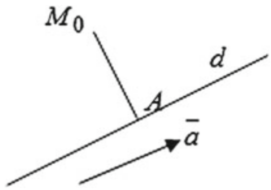


Fig. 2.6 Plane determined by a straight line and a point that doesn't belong to straight line

Let be $A \in d$, hence we have $d = (A, \overline{a})$. We note that $\pi = (M_0, d)$ coincides with $\pi_1 = \left(M_0, \overline{a}, \overline{M_0 A}\right)$. If $\overline{r}_0$ is the position vector of the point M_0 (denoted with $M_0\,(\overline{r}_0)$), $A\,(\overline{r}_A)$ and $M\,(x, y, z) \in \pi$ then the *vector equation* (see [6]) of the plane π is:

$$(\overline{r} - \overline{r}_0) \cdot [\overline{a} \times (\overline{r}_A - \overline{r}_0)] = 0 \tag{2.20}$$

and the *Carthesian equation* (see [6]) of the plane π:

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ a_1 & a_2 & a_3 \\ x_A-x_0 & y_A-y_0 & z_A-z_0 \end{vmatrix} = 0, \tag{2.21}$$

where $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$ and $A\,(x_A, y_A, z_A)$.

2.2.5 Plane Determined by Two Concurrent Straight Lines

Let $d_1 \cap d_2 = \{P\}$, see Fig. 2.7; the straight line

- d_1 is the straight line which passes through P and has the direction vector $\overline{a}_1$, $d_1 = (P, \overline{a}_1)$,
- d_2 is the straight line which passes through P and has the direction vector $\overline{a}_2$, $d_2 = (P, \overline{a}_2)$.

We want to find the equation of the plane determined by the straight lines d_1 and d_2.

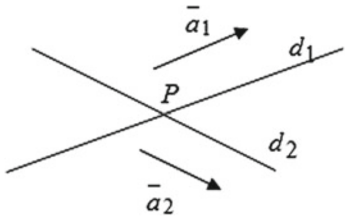


Fig. 2.7 Plane determined by two concurrent straight lines

Noting that $\pi = (d_1, d_2)$ coincides with $\pi = (P, \overline{a}_1, \overline{a}_2)$, i.e. with the plane which passes through P and has the direction vectors $\overline{a}_1$ and $\overline{a}_2$. If $M(x, y, z) \in \pi$ we deduce that the *vector equation* (see [6]) of the plane is:

$$(\overline{r} - \overline{r}_P) \cdot (\overline{a}_1 \times \overline{a}_2) = 0; \tag{2.22}$$

the *Cartesian equation* of the plane π will be:

$$\begin{vmatrix} x - x_P & y - y_P & z - z_P \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0, \tag{2.23}$$

where: $\overline{a}_1 = l_1\overline{i} + m_1\overline{j} + n_1\overline{k}$, $\overline{a}_2 = l_2\overline{i} + m_2\overline{j} + n_2\overline{k}$, $P(x_P, y_P, z_P)$.

Example 2.3 Check if the following straight lines are concurrent:

$$\begin{cases} d_1 : \frac{x-1}{2} = \frac{y-7}{1} = \frac{z-5}{4} \\ d_2 : \frac{x-6}{3} = \frac{y+1}{-2} = \frac{z}{1} \end{cases}$$

and then write the equation of the plane which they determine.

Solution

We note that the direction vectors of the two straight lines are: $\overline{a}_1 = 2\overline{i} + \overline{j} + 4\overline{k}$ and respectively $\overline{a}_2 = 3\overline{i} - 2\overline{j} + \overline{k}$. As

$$\overline{a}_1 \times \overline{a}_2 = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 2 & 1 & 4 \\ 3 & -2 & 1 \end{vmatrix} = 9\overline{i} + 10\overline{j} - 7\overline{k} \neq \overline{0}$$

it results that the vectors $\overline{a}_1$ and $\overline{a}_2$ are noncollinear, namely $d_1 \cap d_2 \neq \emptyset$.

Let $P = d_1 \cap d_2$. Since:

- $P \in d_1$ we obtain $x_P - 1 = 2y_P - 14$;
- $P \in d_2$ we obtain $-2x_P + 12 = 3y_P + 3$.

Solving the system

$$\begin{cases} x_P - 1 & = 2y_P - 14 \\ -2x_P + 12 & = 3y_P + 3 \end{cases}$$

we obtain: $x_P = -3$, $y_P = 5$, $z_P = -3$.

The plane determined by the straight lines d_1 and d_2 will have the equation

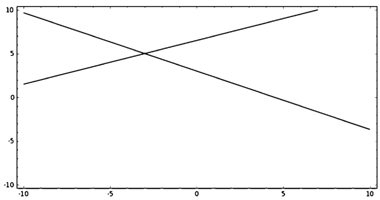
$$\pi : \ 9(x+3) + 10(y-5) - 7(z+3) = 0,$$

namely

$$\pi : \ 9x + 10y - 7z - 44 = 0.$$

We shall present the solution of this problem in Sage, too:

```
sage: a1=vector([2,1,4]);a2=vector([3,-2,1])
sage: var("x y z")
(x, y, z)
sage: p=implicit_plot((x-1)/2 ==y-7==(z-5)/4, (x,-10,10), (y,-10,10))
sage: q=implicit_plot((x-6)/3 ==(y+1)/(-2)==z, (x,-10,10), (y,-10,10))
sage: (p+q).show(aspect_ratio=0.5)
```




```
sage: a1.cross_product(a2)
(9, 10, -7)
sage: eqn=[x-1==2*y-14, -2*x+12==3*y+3]
sage: s = solve(eqn, x,y); s
[[x == -3, y == 5]]
sage: M=matrix(SR,3,[x+3, y-5, z+3, 2,1,4,3,-2,1])
sage: M
[x + 3 y - 5 z + 3]
[    2     1     4]
[    3    -2     1]
sage: M.det() ==0
9*x + 10*y - 7*z - 44 == 0
```

2.2.6 Plane Determined by Two Parallel Straight Lines

Let $d_1, d_2 \in$E$_3$, $d_1 || d_2$,see Fig. 2.8; the straight line:

- d_1 is the straight line which passes through A_1 and has the direction vector $\overline{a}$, $d_1 = (A_1, \overline{a})$,

- d_2 is the straight line which passes through A_2 and has the direction vector $\overline{a}$, $d_2 = (A_2, \overline{a})$.

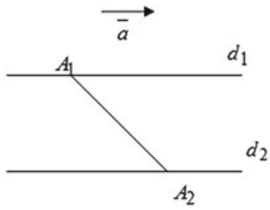


Fig. 2.8 Plane determined by two parallel straight lines

The plane determined by d_1 and d_2 is the plane determined by A_1 and the two non collinear vectors $\overline{a}$ and $\overline{A_1A_2}$.

If $M(x, y, z) \in \pi$ then the *vector equation* (see [6]) of the plane π is:

$$\left(\overline{r} - \overline{r}_{A_1}\right) \cdot \left(\overline{a} \times \overline{A_1A_2}\right) = 0. \tag{2.24}$$

The *Cartesian equation* of the plane π:

$$\begin{vmatrix} x - x_{A_1} & y - y_{A_1} & z - z_{A_1} \\ a_1 & a_2 & a_3 \\ x_{A_2} - x_{A_1} & y_{A_2} - y_{A_1} & z_{A_2} - z_{A_1} \end{vmatrix} = 0, \tag{2.25}$$

where: $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$, $A_1\left(x_{A_1}, y_{A_1}, z_{A_1}\right)$, $A_2\left(x_{A_2}, y_{A_2}, z_{A_2}\right)$.

2.2.7 The Straight Line Determined by the Intersection of Two Planes

We consider $\pi_1, \pi_2 \in$ E$_3$ (see Fig. 2.9) having the equations:

$$\begin{cases} \pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \\ \pi_2 : a_2x + b_2y + c_2z + d_2 = 0. \end{cases}$$

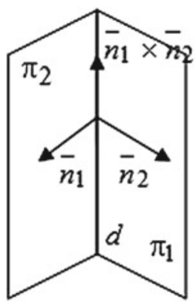


Fig. 2.9 Straight line determined by the intersection of two planes

The intersection of the planes π_1 and π_2 is the set of solutions of the system of equations determined by the equations of π_1 and π_2.

We denote

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$

If rank $(A) = 2$ it results a compatible system which is undetermined simple and the intersection of the two planes is a straight line. If rank $(A) \neq$ rank $\left(\overline{A}\right)$, $\overline{A}$ being the extend matrix of the system it results an incompatible system, therefore $\pi_1 \cap \pi_2 = \emptyset$, namely $\pi_1 || \pi_2$.

Let

- $\overline{n}_1$be the normal to π_1, $\overline{n}_1 = a_1\overline{i} + b_1\overline{j} + c_1\overline{k}$,
- $\overline{n}_2$ be the normal to π_2, $\overline{n}_2 = a_2\overline{i} + b_2\overline{j} + c_2\overline{k}$.

We have

$$\left.\begin{array}{l} d \subset \pi_1 \Rightarrow \overline{n}_1 \perp d \\ d \subset \pi_2 \Rightarrow \overline{n}_2 \perp d \end{array}\right\} \Rightarrow \text{The straight line } d \text{ has the direction vector } \overline{n}_1 \times \overline{n}_2.$$

We denote

$$\overline{u} = \overline{n}_1 \times \overline{n}_2, \overline{u} = l\overline{i} + m\overline{j} + n\overline{k}.$$

We have

$$\overline{n}_1 \times \overline{n}_2 = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1c_2 - b_2c_1)\,\overline{i} + (a_2c_1 - a_1c_2)\,\overline{j} + (a_1b_2 - a_2b_1)\,\overline{k}.$$

We deduce

$$\begin{cases} l = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ m = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} \\ n = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \end{cases}$$

The equation of the straight line is (see [6]):

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}, \tag{2.26}$$

(x_0, y_0, z_0) being a solution of the system.

Example 2.4 Write the equation of a plane which:

(a) passes through the point $M(-2, 3, 4)$ and is parallel with the vectors $\overline{v}_1 = \overline{i} - 2\overline{j} + \overline{k}$ and $\overline{v}_2 = 3\overline{i} + 2\overline{j} + 4\overline{k}$;
(b) passes through the point $M(1, -1, 1)$ and is perpendicular on the planes $\pi_1 : x - y + z - 1 = 0$ and $\pi_2 : 2x + y + z + 1 = 0$.

Solution

(a) The vector equation of the plane is

$$\pi : (\overline{r} - \overline{r}_M) \cdot (\overline{v}_1 \times \overline{v}_2) = 0.$$

Where as

- $\overline{v}_1 \times \overline{v}_2 = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 1 & -2 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -10\overline{i} - \overline{j} + 8\overline{k},$
- $\overline{r} - \overline{r}_M = (x + 2)\,\overline{i} + (y - 3)\,\overline{j} + (z - 4)\,\overline{k}$

we obtain

$$\pi : -10\,(x + 2) - (y - 3) + 8\,(z - 4) = 0$$

or

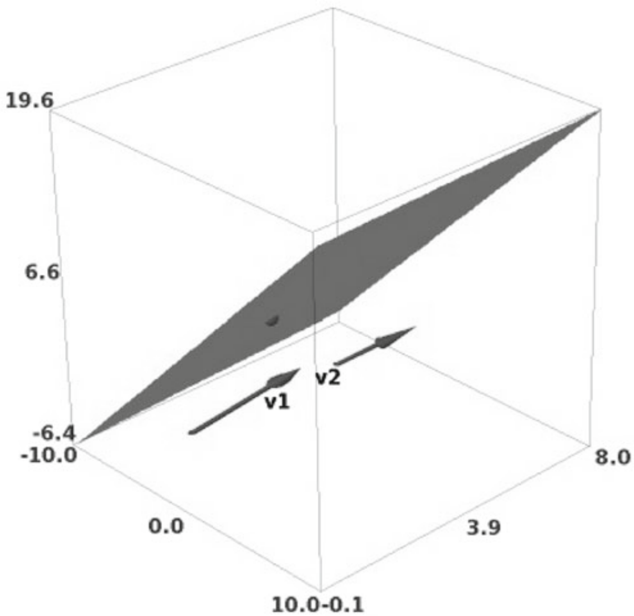
$$\pi : -10x - y + 8z - 49 = 0.$$

This equation can be determined with Sage, too:

```
sage: v1=vector([1,-2,1]);v2=vector([3,2,4])
sage: vp= v1.cross_product(v2)
sage: var("x y z")
(x, y, z)
sage: r=vector([x,y,z]);M=vector([-2,3,4])
sage: (r-M).dot_product(vp)==0
-10*x - y + 8*z - 49 == 0
```

and it can also be plotted:

```
sage: var("x y z")
(x, y, z)
sage: pl=plot3d((49+10*x+y)/8,(x,-10,10),(y,0,8),rgbcolor="lightblue")
sage: A=point3d((-2,3,4),color='red',size=20)
sage: v1=arrow3d((3,3,3),(4,5,4),6,color='red')+text3d("v1",(1.2,2,0.4))
sage: v2=arrow3d((0,0,0),(3,2,4),6,color='red')+text3d("v2",(3.3,2.7,2.5))
sage: A+pl+v1+v2
```



(b) The normal of the required plane is $\overline{n} = \overline{n}_1 \times \overline{n}_2$, where $\overline{n}_1 = \overline{i} - \overline{j} + \overline{k}$ and $\overline{n}_2 = 2\overline{i} + \overline{j} + \overline{k}$; therefore $\overline{n} = -2\overline{i} + \overline{j} + 3\overline{k}$.

The Cartesian equation of the plane will be

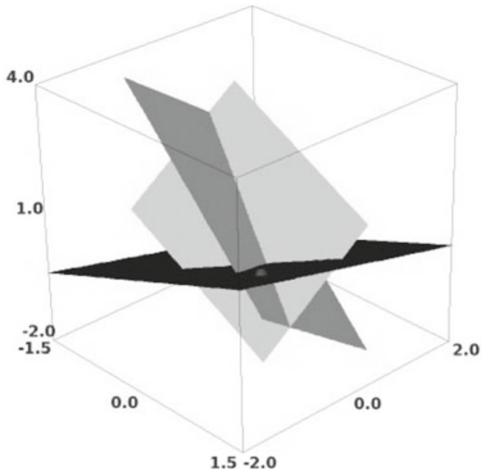
$$\pi : (-2)(x-1) + (y+1) + 3(z-1) = 0 \Leftrightarrow \pi : -2x + y + 3z = 0.$$

We can also find this equation in Sage:

```
sage: n1=vector([1,-1,1]);n2=vector([2,1,1]);
sage: n=n1.cross_product(n2)
sage: var("x y z")
(x, y, z)
sage: M1=vector([1,-1,1]);r=vector([x,y,z]);
sage: (r-M1).dot_product(n)==0
-2*x + y + 3*z == 0
```

A graphical solution in Sage is:

```
sage: var("x y z")
(x, y, z)
sage: M=point3d((1,-1,1),color='red',size=20)
sage: p1=plot3d(1-x+y,(x,-1,1),(y,-1,1),rgbcolor="lightblue")
sage: p2=plot3d(1-2*x-y,(x,-1,1),(y,-1,1),rgbcolor="lightblue")
sage: p=plot3d((2*x-y)/3,(x,-1.5,1.5),(y,-2,2),rgbcolor="blue")
sage: M+p1+p2+p
```



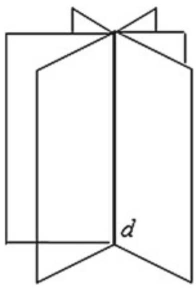
2.3 Plane Fascicle

Let be $d = \pi_1 \cap \pi_2$

$$\begin{cases} \pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \\ \pi_2 : a_2x + b_2y + c_2z + d_2 = 0. \end{cases}$$

Definition 2.5 (see [1], p. 62 and [2], p. 681). The set of the planes which contain the straight line d is called a **plane fascicle** of axis d (see Fig. 2.10). The straight line d is called the **fascicle axis** and π_1, π_2 are called the **base planes** of the fascicle.

Fig. 2.10 Plane fascicle



An arbitrary plane of the fascile has the equation of the form:

$$\pi : a_1x + b_1y + c_1z + d_1 + \lambda (a_2x + b_2y + c_2z + d_2) = 0, \ \lambda \in \mathbb{R}^*.$$

Example 2.6 Determine a plane which passes through the intersection of the planes $\pi_1 : x + y + 5z = 0$ and $\pi_2 : x - z + 4 = 0$ and which forms with the plane $\pi : x - 4y - 8z + 12 = 0$ an angle $\varphi = \frac{\pi}{2}$.

Solution

Let be $d = d_1 \cap d_2$. A plane of the plane fascicle of axis d has the equation

$$\pi' : x + 5y + z + \lambda (x - z + 4) = 0,$$

namely

$$\pi' : (1 + \lambda) x + 5y + (1 - \lambda) z + 4\lambda = 0.$$

It results that

$$\overline{n'} = (1 + \lambda) \overline{i} + 5\overline{j} + (1 - \lambda) \overline{k}.$$

As $\overline{n} = \overline{i} - 4\overline{j} - 8\overline{k}$ we shall deduce

$$\cos \prec \left(\pi, \pi'\right) = \cos \frac{\pi}{2} = \frac{\overline{n} \cdot \overline{n'}}{\|\overline{n}\| \cdot \left\|\overline{n'}\right\|} = \frac{-27 + 9\lambda}{9\sqrt{27 + 2\lambda^2}} \Leftrightarrow \lambda = 3.$$

The equation of the required plane is

$$\pi' : (1 + 3) x + 5y + (1 + 3) z + 4 \cdot 3 = 0$$

or

$$\pi' : 4x + 5y - 2z + 12 = 0.$$

The solution in Sage of this problem is:

```
sage: var("la x y z");n=vector([1,-4,-8])
(la, x, y, z)
sage: np=vector([1+la,5,1-la]);nnp=expand(sqrt((1+la)^2+25+(1-la)^2))
sage: u=np.dot_product(n)/(n.norm()*nnp)
sage: solve(u==cos(pi/2),la)
[la == 3]
sage: eq=(1+la)*x+5*y+(1-la)*z+4*la
sage: eq.substitute(la=3)==0
4*x + 5*y - 2*z + 12 == 0
```

2.4 Distances in E_3

2.4.1 Distance from a Point to a Straight Line

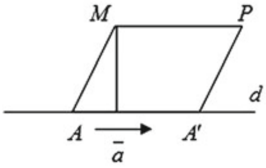
Let be $d = (A, \overline{a})$ with $A (x_A, y_A, z_A)$, $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$ and $M \in E_3$.

Let $\overrightarrow{AA'}$ be a representative for $\overline{a}$. The equation of the straight line is:

$$\frac{x - x_A}{a_1} = \frac{y - y_A}{a_2} = \frac{z - z_A}{a_3}.$$

We build the parallelogram $AA'PM$ (see Fig. 2.11).

Fig. 2.11 The distance from a point to a straight line



We know that

$$\mathrm{A}_{AA'PM} = \left\| \overline{AA'} \times \overline{MA} \right\|. \tag{2.27}$$

However

$$\mathrm{A}_{AA'PM} = \left\| \overline{AA'} \right\| \cdot \rho\,(M, d)\,. \tag{2.28}$$

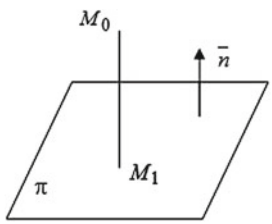
From (2.27) and (2.28) it results that the *distance formula from a point to a straight line* is [6]

$$\rho\,(M, d) = \frac{\left\| \overline{AA'} \times \overline{MA} \right\|}{\left\| \overline{AA'} \right\|} = \frac{\left\| \overline{a} \times \overline{MA} \right\|}{\left\| \overline{a} \right\|}. \tag{2.29}$$

2.4.2 Distance from a Point to a Plane

We consider the plane $\pi : ax + by + cz + d = 0$ and the point $M_0\,(x_0, y_0, z_0)$, $M_0 \notin \pi$. Let M_1 be the projection of M_0 on the plane π, $M_1\,(x_1, y_1, z_1)$, see Fig. 2.12.

Fig. 2.12 The distance from a point to a plane



The distance from the point M_0 to the plane π is

$$\rho(M_0, \pi) = \left\| \overline{M_0 M_1} \right\|.$$

Let $d = (M_0, \overline{n})$ be the normal line to the plane which passes through M_0, $\overline{n} = a\overline{i} + b\overline{j} + c\overline{k}$. The equation of this straight line is

$$d : \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

or

$$d : \begin{cases} x = x_0 + ta \\ y = y_0 + tb, \ t \in \mathbb{R}. \\ z = z_0 + tc \end{cases}$$

As $M_1 \in d$ we deduce

$$d : \frac{x_1 - x_0}{a} = \frac{y_1 - y_0}{b} = \frac{z_1 - z_0}{c} = t \Rightarrow$$

$$\begin{cases} x_1 = x_0 + ta \\ y_1 = y_0 + tb, \ t \in \mathbb{R}. \\ z_1 = z_0 + tc \end{cases} \tag{2.30}$$

Since $M_1 \in \pi \Rightarrow ax_1 + by_1 + cz_1 + d = 0 \Rightarrow$

$$ax_1 + by_1 + cz_1 = -d \tag{2.31}$$

Multiplying the first equation of (2.31) with a, the second with b and the third with c we have:

$$\begin{cases} ax_1 = ax_0 + ta^2 \\ by_1 = by_0 + tb^2, \ t \in \mathbb{R}. \\ cz_1 = cz_0 + tc^2 \end{cases} \tag{2.32}$$

Adding the three equations of (2.32) it results

$$ax_1 + by_1 + cz_1 = ax_0 + by_0 + cz_0 + t\left(a^2 + b^2 + c^2\right). \tag{2.33}$$

Substituting (2.31) into (2.33) we deduce

$$ax_0 + by_0 + cz_0 + d = -t\left(a^2 + b^2 + c^2\right),$$

namely

$$t = -\frac{ax_0 + by_0 + cz_0 + d}{a^2 + b^2 + c^2}. \tag{2.34}$$

We have

$$\overline{M_0M_1} = (x_1 - x_0)\,\overline{i} + (y_1 - y_0)\,\overline{j} + (z_1 - z_0)\,\overline{k};$$

therefore

$$\left\|\overline{M_0M_1}\right\| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2} \overset{(2.30)}{=} \sqrt{t^2\left(a^2 + b^2 + c^2\right)} \Rightarrow$$

$$\left\|\overline{M_0M_1}\right\| = |t|\sqrt{a^2 + b^2 + c^2}. \tag{2.35}$$

Substituting (2.34) into (2.35) we can deduce [6] the *distance formula from a point to a plane*:

$$\rho\left(M_0, \pi\right) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{2.36}$$

Example 2.7 (see [6]). One gives:

- the plane $\pi : x + y - z + 2 = 0$,
- the straight line

$$d : \begin{cases} x - y - 1 = 0 \\ x + 2y + z - 4 = 0 \end{cases}$$

- the point $A = (1, 1, 2)$.

 (a) Compute the distance from the point A to the plane π.
 (b) Find the distance from the point A to the straight line d.

Solution

(a) Using the formul (2.36) we achieve:

$$\rho\left(A, \pi\right) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

(b) The distance from the point A to the straight line d (Fig. 2.13) is computed using the formula

$$\rho\left(A, d\right) = \frac{\left\|\overline{MA'} \times \overline{AM}\right\|}{\left\|\overline{MA'}\right\|} = \frac{\left\|\overline{a} \times \overline{AM}\right\|}{\left\|\overline{a}\right\|}.$$

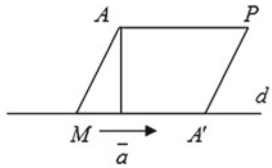


Fig. 2.13 The distance from a point to a straight line

The direction vector of the straight line d is

$$\bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -\bar{i} - \bar{j} + 3\bar{k}.$$

We obtain: $\|\bar{a}\| = \sqrt{11}$.
From $M \in d$ we have

$$\begin{cases} x_M - y_M - 1 = 0 \\ x_M + 2y_M + z_M - 4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_M - y_M - 1 = 0 \\ x_M + 2y_M = 4 - z_M . \end{cases}$$

Denoting $z_M = u \in \mathbb{R}$ we deduce

$$3y_M = 3 - u \Leftrightarrow y_M = 1 - \frac{1}{3}u$$

and

$$x_M = y_M + 1 = 2 - \frac{1}{3}u.$$

We can suppose that $u = 0$; we obtain: $\begin{cases} x_M = 2 \\ y_M = 1 \\ z_M = 0 \end{cases} \Longrightarrow M(2, 1, 0)$. We have: $\overline{AM} = \bar{i} - \overline{2k}$ and

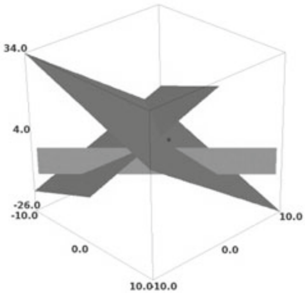
$$\bar{a} \times \overline{AM} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & -1 & 3 \\ 1 & 0 & -2 \end{vmatrix} = 2\bar{i} + \bar{j} + \bar{k};$$

therefore $\left\|\bar{a} \times \overline{AM}\right\| = \sqrt{6}$. We shall obtain

$$\rho(A, d) = \frac{\sqrt{6}}{\sqrt{11}} = 0.739.$$

Solving this problem with Sage, we shall have:

```
sage: var("x y z");
(x, y, z)
sage: plane=plot3d(x+y+2, (x, -10, 0), (y, -10, 10));
sage: A = point3d((1, 1, 2), color = 'red', size = 10)
sage: p1=implicit_plot3d(x-y-1==0, (x, -10, 10), (y, -10, 10),(z, -10, 0),rgbcolor="orange");
sage: p2=plot3d(4-x-2*y, (x, -10, 10), (y, -10, 10),rgbcolor="orange");
sage: p1+p2+plane+A
```




```
sage: a=1;b=1;c=-1;d=2;
sage: f=a*x+b*y+c*z+d
sage: (f.subs(x=1,y=1,z=2)).abs()/sqrt(a^2+b^2+c^2)
2/3*sqrt(3)
sage: a1=vector([1,-1,0]);a2=vector([1,2,1]);
sage: aa=a1.cross_product(a2); aa
(-1, -1, 3)
sage: na=aa.norm(); na
sqrt(11)
sage: eqn=[x-y-1==0,x+2*y+z-4==0];
sage: s=solve(eqn,x,y,z);s
[[x == -1/3*r3 + 2, y == -1/3*r3 + 1, z == r3]]
sage: s[0][0].subs(r2=0);s[0][1].subs(r2=0);s[0][2].subs(r2=0);
x == -1/3*r3 + 2
y == -1/3*r3 + 1
z == r3
sage: A=vector([1,1,2]); M=vector([2,1,0])
sage: AM=M-A; AM
(1, 0, -2)
sage: cp=aa.cross_product(AM); cp
(2, 1, 1)
sage: ncp=cp.norm(); ncp
sqrt(6)
sage: rho=ncp/na; rho.n(digits=3)
0.739
```

2.4.3 Distance Between Two Straight Lines

Let d_1, d_2 be two noncoplanar straight lines (see Fig. 2.14).

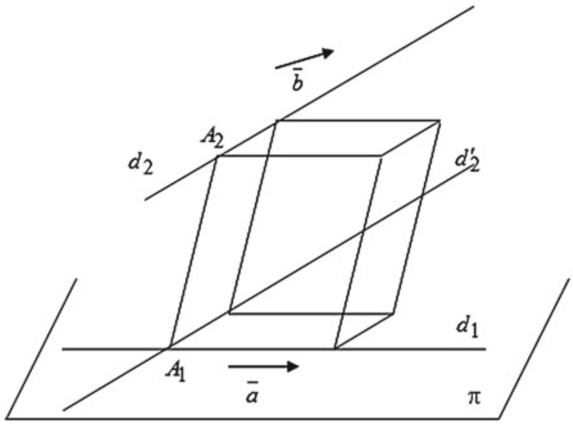


Fig. 2.14 the distance between two straight lines

The distance between the straight lines d_1 and d_2 is

$$\rho\left(d_1, d_2\right) = \rho\left(A_1, A_2\right) = \rho\left(A_2, \pi\right),$$

where:

- π is the plane which passes through d_1 and it is parallel with d_2,
- $\rho\left(A_2, \pi\right)$ is the height which corresponds to the vertex A_2 of the oblique parallelepiped built on the vectors $\overline{a}$, $\overline{b}$, $\overline{A_1A_2}$.

Therefore, the *distance formula between two straight lines* is [6]:

$$\rho\left(d_1, d_2\right) = \frac{\mathrm{V}_{parallelepiped}}{\mathrm{A}_{base}} = \frac{\left|\overline{a}\cdot\left(\overline{b}\times\overline{A_1A_2}\right)\right|}{\left\|\overline{a}\times\overline{b}\right\|}. \tag{2.37}$$

Definition 2.8 (see [6]). The support straight line corresponding to the segment which represents the distance between two straight lines is called the **common perpendicular** of the two straight lines.

Let δ be the straight line which represents the common perpendicular. To determine the equations of the straight line δ:

1. find the direction of the the common perpendicular $\overline{n} = \overline{a}\times\overline{b}$, $\overline{a}$ and $\overline{b}$ being the direction vectors of the two straight lines;
2. write the equation of a plane π_1, which passes through d_1 and contains $\overline{n}$;
3. write the equation of a plane π_2, which passes through d_2 and contains $\overline{n}$;
4. $\delta = \pi_1 \cap \pi_2$ is the common perpendicular searched by us.

If

- $d_1 = \left(A_1, \overline{a}\right),\ A_1\left(x_1, y_1, z_1\right),$
- $d_2 = \left(A_2, \overline{a}\right),\ A_2\left(x_2, y_2, z_2\right),$

- $\overline{n} = n_1\overline{i} + n_2\overline{j} + n_3\overline{k}$,
- $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$,
- $\overline{b} = b_1\overline{i} + b_2\overline{j} + b_3\overline{k}$,

then the *equations of the common perpendicular* δ are [6]:

$$\delta : \begin{cases} \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & a_2 & a_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0 \\ \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ b_1 & b_2 & b_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0. \end{cases} \tag{2.38}$$

Example 2.9 Let be the straight lines

$$d_1 : \begin{cases} x = 1 + 2r \\ y = 3 + r \\ z = -2 + r \end{cases} \text{ and } d_2 : \begin{cases} x = 1 + s \\ y = -2 - 4s, r, s \in \mathbb{R}. \\ z = 9 + 2s \end{cases}$$

Find:

(a) the angle between these straight lines;
(b) the equation of the common perpendicular;
(c) the distance between the two straight lines.

Solution

(a) We have

$$\cos \prec (d_1, d_2) = \cos \prec \left(\overline{a}, \overline{b}\right) = \frac{\overline{a} \cdot \overline{b}}{\|\overline{a}\| \left\|\overline{b}\right\|},$$

where

- $d_1 = (A, \overline{a})\,,\ A\,(1, 3, -2)\,,\ \overline{a} = 2\overline{i} + \overline{j} + \overline{k}$,
- $d_2 = \left(B, \overline{b}\right),\ B\,(1, -2, 9)\,,\ \overline{b} = \overline{i} - 4\overline{j} + 2\overline{k}$.

We obtain

$$\cos \prec (d_1, d_2) = 0 \Rightarrow \prec (d_1, d_2) = \frac{\pi}{2}.$$

(b) The direction of the common perpendicular is

$$\overline{n} = \overline{a} \times \overline{b},$$

namely

$$\overline{n} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 2 & 1 & 1 \\ 1 & -4 & 2 \end{vmatrix} = 6\overline{i} - 3\overline{j} - 9\overline{k}.$$

As

- the equation of the plane which passes through d_1 and contains $\overline{n}$ is

$$\pi_1 : \begin{vmatrix} x-1 & y-3 & z+2 \\ 2 & 1 & 1 \\ 6 & -3 & -9 \end{vmatrix} = 0 \Leftrightarrow \pi_1 : x - 4y + 2z + 15 = 0;$$

- the equation of the plane which passes through d_2 and contains $\overline{n}$ is

$$\pi_2 : \begin{vmatrix} x-1 & y+2 & z-9 \\ 1 & -4 & 2 \\ 6 & -3 & -9 \end{vmatrix} = 0 \Leftrightarrow \pi_2 : 2x + y + z - 9 = 0.$$

The equations of the common perpendicular will be:

$$\begin{cases} x - 4y + 2z + 15 = 0 \\ 2x + y + z - 9 = 0. \end{cases}$$

(c) Using (2.37), we have

$$\rho(d_1, d_2) = \frac{\left|\overline{a} \cdot \left(\overline{b} \times \overline{AB}\right)\right|}{\left\|\overline{a} \times \overline{b}\right\|},$$

where

- $\overline{AB} = (x_B - x_A)\,\overline{i} + (y_B - y_A)\,\overline{j} + (z_B - z_A)\,\overline{k} = -5\overline{j} + 11\overline{k}$,
- $\overline{a} \cdot \left(\overline{b} \times \overline{AB}\right) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -4 & 2 \\ 0 & -5 & 11 \end{vmatrix} = -84$,
- $\left\|\overline{a} \times \overline{b}\right\| = \sqrt{6^2 + (-3)^2 + (-9)^2} = \sqrt{126}$;

therefore

$$\rho(d_1, d_2) = \frac{84}{\sqrt{126}} = 2\sqrt{14}.$$

We shall solve this problem using Sage:

```
sage: a=vector([2,1,1]);b=vector([1,-4,2]);var("x y z")
(x, y, z)
sage: a.dot_product(b)
0
sage: n=a.cross_product(b); n
(6, -3, -9)
sage: A=vector([1,3,-2]);B=vector([1,-2,9]);
sage: p1=matrix(SR,3,[x-A[0],y-A[1],z-A[2],a[0],a[1],a[2],n[0],n[1],n[2]]);
sage: pp1=p1.determinant();pp1;pp1/(-6)==0
-6*x + 24*y - 12*z - 90
x - 4*y + 2*z + 15 == 0
sage: p2=matrix(SR,3,[x-B[0],y-B[1],z-B[2],b[0],b[1],b[2],n[0],n[1],n[2]]);
sage: pp2=p2.determinant();pp2;pp2/21==0
42*x + 21*y + 21*z - 189
2*x + y + z - 9 == 0
sage: AB=B-A;
sage: u=a.dot_product(b.cross_product(AB))
sage: rho=u.abs()/n.norm();rho
2*sqrt(14)
```

2.5 Angles in E_3

2.5.1 Angle Between Two Straight Lines

Definition 2.11 (see [3], p. 112). Let d_1, d_2 be two straight lines, which have the direction vectors $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$ and respectively $\overline{b} = b_1\overline{i} + b_2\overline{j} + b_3\overline{k}$. The **angle between the straight lines** d_1 and d_2 is the angle between the vectors $\overline{a}$ and $\overline{b}$ (see Fig. 2.15).

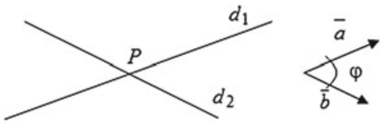


Fig. 2.15 The angle between two straight lines

Hence

$$\cos\varphi = \frac{\overline{a}\cdot\overline{b}}{\|\overline{a}\|\,\|\overline{b}\|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}},\ \varphi \in [0, \pi]\,. \tag{2.39}$$

Remark 2.12 (see [3], p. 112).

(1) $\varphi = \frac{\pi}{2} \Rightarrow$the straight lines are perpendicular $\Leftrightarrow$

$$\overline{a}\cdot\overline{b} = 0 \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0.$$

(2) $\varphi = 0 \Rightarrow$the straight lines are parallel$\Leftrightarrow \overline{a}\times\overline{b} = \overline{0} \Leftrightarrow$

$$\begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \overline{0} \Leftrightarrow (a_2b_3 - a_3b_2)\,\overline{i} + (a_3b_1 - a_1b_2)\,\overline{j} + (a_1b_2 - a_2b_1)\,\overline{k} = \overline{0} \Leftrightarrow$$

$$\begin{cases} a_2b_3 - a_3b_2 = 0 \\ a_3b_1 - a_1b_2 = 0 \\ a_1b_2 - a_2b_1 = 0 \end{cases} \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}. \tag{2.40}$$

Therefore $d_1||d_2$ the relation (2.40) occurs.

2.5.2 Angle Between Two Planes

Let be

- $\pi_1 : a_1x + b_1y + c_1z + d_1 = 0$ and the normal vector $\overline{n}_1 = a_1\overline{i} + b_1\overline{j} + c_1\overline{k}$,

- $\pi_2 : a_2x + b_2y + c_2z + d_2 = 0$ and the normal vector $\overline{n}_2 = a_2\overline{i} + b_2\overline{j} + c_2\overline{k}$.

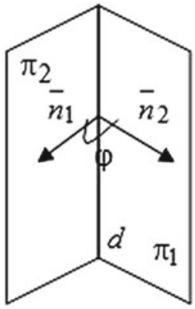


Fig. 2.16 The angle between two planes

Definition 2.13 (see [3], p. 113). The **angle** φ **between the planes** π_1 and π_2 is the angle between the vectors $\overline{n}_1$ and $\overline{n}_2$ (see Fig. 2.16).

Hence

$$\cos\varphi = \frac{\overline{n}_1 \cdot \overline{n}_2}{\|\overline{n}_1\|\,\|\overline{n}_2\|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}},\ \varphi \in [0, \pi]\,. \tag{2.41}$$

Remark 2.14 (see [3], p. 113).

(1) $\pi_1 || \pi_2 \Longleftrightarrow \overline{n}_1$ and $\overline{n}_2$ are collinear $\Longleftrightarrow \overline{n}_1 = t\overline{n}_2,\ t \in \mathbb{R}$; therefore

$$\pi_1 || \pi_2 \Longleftrightarrow \overline{a}_1 = t\overline{a}_2, \overline{b}_1 = t\overline{b}_2, \overline{c}_1 = t\overline{c}_2.$$

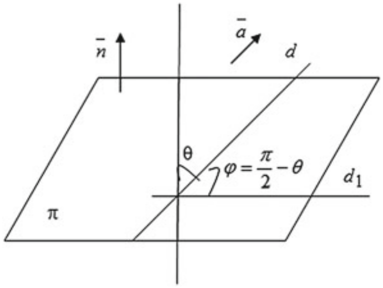
(2) $\pi_1 \perp \pi_2 \Longleftrightarrow \overline{n}_1 \cdot \overline{n}_2 = 0 \Longleftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0.$

2.5.3 Angle Between a Straight Line and a Plane

Let d be the straight line with the direction vector $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$ and the plane π having the normal vector $\overline{n} = n_1\overline{i} + n_2\overline{j} + n_3\overline{k}$.

Definition 2.15 (see [3], p. 113). The angle φ between the straight line d and the plane π is the angle between the straight line d and the projection of this straight line on the plane π (see Fig. 2.17).

Fig. 2.17 The angle between a straight line and a plane



The angle between the straight line d and the plane π is related to the angle θ, the angle of the vectors $\overline{a}$ and $\overline{n}$, through the relations: $\theta = \frac{\pi}{2} \pm \varphi$ as the vectors are on the same side of the or in different parts. Hence:

$$\cos\theta = \cos\left(\frac{\pi}{2} \pm \varphi\right) = \pm\sin\varphi,\ \theta \in [0, \pi] \Rightarrow \varphi \in \left[0, \frac{\pi}{2}\right].$$

As

$$\cos\theta = \frac{\overline{n} \cdot \overline{a}}{\|\overline{n}\|\ \|\overline{a}\|},\ \theta \in [0, \pi]$$

it results that

$$\sin\varphi = \frac{|a_1 n_1 + a_2 n_2 + a_3 n_3|}{\sqrt{n_1^2 + n_2^2 + n_3^2}\sqrt{a_1^2 + a_2^2 + a_3^2}},\ \varphi \in \left[0, \frac{\pi}{2}\right]. \tag{2.42}$$

Remark 2.16 (see [3], p. 113).

(1) $d || \pi \Leftrightarrow \overline{n} \cdot \overline{a} = 0 \Leftrightarrow a_1 n_1 + a_2 n_2 + a_3 n_3 = 0.$
(2) $d \perp \pi \Rightarrow \varphi = \frac{\pi}{2} \Rightarrow \theta = 0 \Rightarrow \overline{n} || \overline{a} \overset{(2.40)}{\Longrightarrow} \frac{n_1}{a_1} = \frac{n_2}{a_2} = \frac{n_3}{a_3}.$

Example 2.17 Are given

- the planes $\pi_1 : 2x - y + 7 = 0$ and $\pi_2 : x - 5y + 3z = 0$,
- the straight line $d : \frac{x-1}{2} = \frac{y+3}{-1} = \frac{z}{5}$.

Compute:

(a) the angle of these planes;
(b) the angle between the straight line and the plane π_1.

Solution

(a) We have $\overline{n}_1 = (2, -1, 0)$, $\overline{n}_2 = (1, -5, 3)$; hence

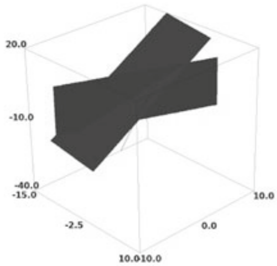
$$\cos \prec (\pi_1, \pi_2) = \cos\varphi \overset{(2.41)}{=} \frac{\overline{n}_1 \cdot \overline{n}_2}{\|\overline{n}_1\|\ \|\overline{n}_2\|} = \frac{\sqrt{7}}{5} = 0.529.$$

(b) As the direction vector of the straight line d is $\overline{a} = (2, -1, 5)$ we obtain

$$\sin \prec (d, \pi_1) = \sin \varphi \overset{(2.42)}{=} \frac{|a_1 n_1 + a_2 n_2 + a_3 n_3|}{\sqrt{n_1^2 + n_2^2 + n_3^2}\sqrt{a_1^2 + a_2^2 + a_3^2}} = \frac{1}{\sqrt{6}}.$$

We need the following code in Sage to solve this problem:

```
sage: var("x y z")
(x, y, z)
sage: pl1=implicit_plot3d(2*x-y+7==0, (x, -10, 10), (y, -10, 10),(z, -10, 10),rgbcolor="green")
sage: pl2=plot3d((5*y-x)/3, (x, -10, 0), (y, -10, 10),rgbcolor="purple")
sage: l=line3d([(7,-6,15), (-15,5,-40)], color='blue')
sage: pl1+pl2+l
```




```
sage: n1=vector([2,-1,0]);n2=vector([1,-5,3]);
sage: (n1.dot_product(n2)/(n1.norm()*n2.norm())).n(digits=3)
0.529
sage: a=vector([2,-1,5]);
sage: (a.dot_product(n1)).abs()/(n1.norm()*a.norm()).n(digits=3)
0.408
```

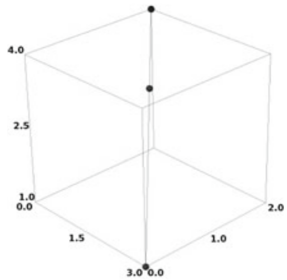
2.6 Problems

1. Check if the points $M_1\,(3, 0, 1)$, $M_2\,(0, 2, 4)$, $M_3\left(1, \frac{4}{3}, 3\right)$ are collineare.

Solution

Using Sage we shall have:

```
sage: var("x y z")
(x, y, z)
sage: M1=vector([3,0,1]);M2=vector([0,2,4]);M3=vector([1,4/3,3])
sage: (M3[0]-M1[0])/(M2[0]-M1[0])==(M3[1]-M1[1])/(M2[1]-M1[1])==(M3[2]-M1[2])/(M2[2]-M1[2])
True
sage: po1=point3d((3,0,1),size=20,color='red')
sage: po2=point3d((0,2,4),size=20,color='blue')
sage: po3=point3d((1,4/3,3),size=20,color='green')
sage: l=line3d([(3,0,1),(0,2,4)])
sage: po1+po2+po3+l
```

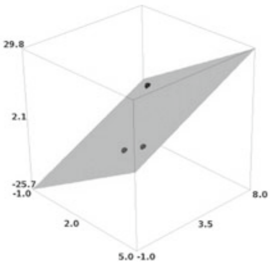


2. Write the equation of the plane determined by the points: $M_1\,(3, 1, 0)$, $M_2\,(0, 7, 2)$, $M_3\,(4, 1, 5)$.

Solution

We shall give a solution in Sage:

```
sage: var("x y z")
(x, y, z)
sage: M1=vector([3,1,0]);M2=vector([0,7,2]);M3=vector([4,1,5])
sage: M=matrix(SR,3,[x-M1[0],y-M1[1],z-M1[2],M2[0]-M1[0],M2[1]-M1[1],M2[2]-M1[2],M3[0]-M1[0],M3[1]-M1[1],M3[2]-M1[2]])
sage: M.determinant()==0
30*x + 17*y - 6*z - 107 == 0
sage: po1=point3d((3,1,0),size=20,color='red')
sage: po2=point3d((0,7,2),size=20,color='blue')
sage: po3=point3d((4,1,5),size=20,color='green')
sage: pl=plot3d((30*x+17*y-107)/6, (x, -1, 5), (y, -1, 8),rgbcolor="lightblue")
sage: po1+po2+po3+pl
```



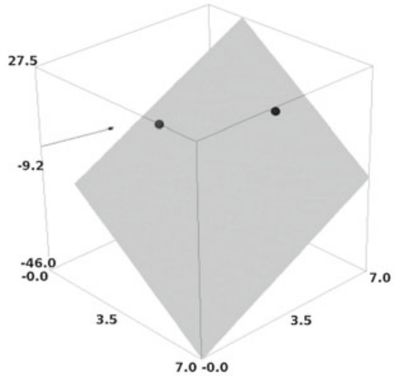
3. Write the equation of a plane perpendicular on the planes $\pi_1 : x - y + z - 1 = 0$ and $\pi_2 : 2x + y + z + 1 = 0$ and which passes through the point $M_0\,(1, -1, 1)$.
4. Write the equation of a plane which passes through the points $M_1\,(3, 1, 2)$, $M_2\,(4, 6, 5)$ and is parallel with the vector $\overline{v} = \overline{i} + 2\overline{j} + 3\overline{k}$.

Solution

Solving this problem in Sage we have:

```
sage: var("x y z")
(x, y, z)
sage: M1=vector([2,3,4]);M2=vector([4,6,5])
sage: M1M2=M2-M1;M1M2
(2, 3, 1)
sage: r=vector([x,y,z]);v=vector([1,2,3])
sage: u=M1M2.cross_product(v);u
(7, -5, 1)
sage: w=r-M1;w
(x - 2, y - 3, z - 4)
sage: u.dot_product(w)==0
7*x - 5*y + z - 3 == 0
```

```
sage: a=arrow3d((0,0,0), (1,2,3),color='blue');
sage: po1=point3d((2,3,4),size=20,color='red')
sage: po2=point3d((4,6,5),size=20,color='blue')
sage: pl=plot3d(-7*x+5*y+3, (x, 1.5, 7), (y, 0, 7),rgbcolor="lightblue")
sage: a+pl+po1+po2
```



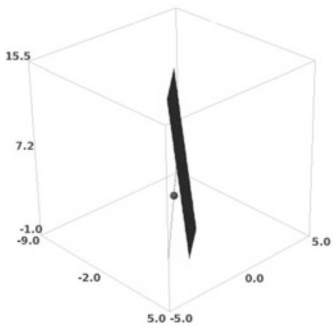
5. Let d be the straight line determined by the point $P_0\,(2, 0, -1)$ and the direction vector $\overline{v} = \overline{i} - \overline{j}$. Compute the distance from the point $P\,(1, 3, -2)$ to the straight line d.
6. Write the ecuation of the perpendicular from the point $M\,(-2, 0, 3)$ on the plane $\pi : 7x - 5y + z - 11 = 0$.

Hint. The perpendicular from a point to a plane is the straight line which passes through that point and has the normal vector of the plane as a direction vector.

Solution

Solving this problem with Sage, we shall have:

```
sage: var("x y z")
(x, y, z)
sage: n=vector([7,-5,1]);M=vector([-2,0,3])
sage: eq1=[(x-M[0])/n[0]==(y-M[1])/n[1]];eq2=[(y-M[1])/n[1]==(z-M[2])/n[2]];eq1+eq2
[1/7*x + 2/7 == -1/5*y, -1/5*y == z - 3]
sage: l=line3d([(-2,0,3),(-9,5,2),(5,-5,4)],color="purple")
sage: po1=point3d((-2,0,3),size=20,color='red')
sage: pl=plot3d(11-7*x+5*y, (x, -1, 1), (y, -1, -0.5),rgbcolor="blue")
sage: po1+pl+l
```



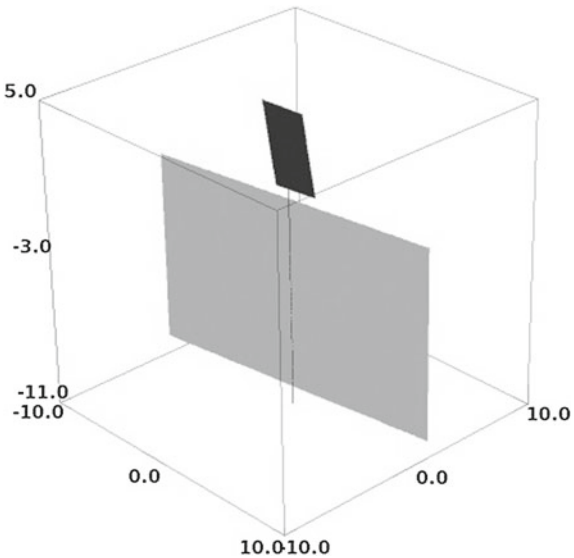
7. It gives a tetrahedron $ABCD$ defined by the points $A\,(3, 0, 0)$, $B\,(2, 4, 0)$, $C\,(-3, -1, 0)$, $D\,(0, 0, 5)$. Write the equations of its faces, the edge equations and the equations corresponding to the heights of the tetrahedron $ABCD$.
8. Write the equation of the plane which passes through Oz and is perpendicular on the plane $\pi : 8x + y + 2z - 1 = 0$.

Hint. The equation of the plane which passes through Oz is: $ax + by = 0$.

Solution

We shall present the solution in Sage:

```
sage: var("a b x y z")
(a, b, x, y, z)
sage: n1=vector([a,b,0]);n2=vector([8,1,2])
sage: eqn=[n1.dot_product(n2)==0];
sage: s=solve(eqn,a,b);s
([a == -1/8*b], [1])
sage: n1=vector([s[0][0].subs(b==1),1,0]);n1
(a == (-1/8), 1, 0)
sage: l=line3d([(0,0,1),(0,0,-1),(0,0,-11)],color="purple")
sage: p1=implicit_plot3d(-x/8+y==0, (x, -10, 10), (y, -10, 10),(z, -10, 0),rgbcolor="lightblue")
sage: p2=plot3d((1-8*x-y)/2, (x, -1, 0), (y, -1, 2),rgbcolor="red")
```



9. Determine the projection equation of the straight line having the equations

$$d : \begin{cases} x - 3z + 1 = 0 \\ y - 2z - 3 = 0 \end{cases}$$

on the plane $\pi : x - y + 2z - 1 = 0$.

10. Let be the straight lines:

$$d_1 : \begin{cases} x - 2y + z + 1 = 0 \\ y - z = 0 \end{cases}$$

and

$$d_2 : \frac{x-1}{2} = \frac{y+3}{1} = \frac{z}{8}.$$

(a) Find the equation of the common perpendicular.
(b) Compute the distance between the two straight lines.

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