

Chapter 2

Nine Years of $f(R)$ Gravity and Cosmology

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Abstract $f(R)$ gravity was reintroduced in cosmology as an alternative to dark energy in explaining the current acceleration of the universe. This area of research is reviewed with emphasis on the theoretical viability of $f(R)$ gravity, stability, weak-field limit, and recent developments, as well as some recent work on scalar-tensor and $f(R)$ black holes which are the endpoint of gravitational collapse.

2.1 Introduction

$f(R)$ gravity was introduced already in the early days of General Relativity (“GR”) to explore possible alternatives [1, 2] and was studied sporadically by a few authors [3] until attempts to renormalize GR [4] showed the need of higher order corrections to the Einstein-Hilbert action, while approaches to quantum gravity introduced higher order terms and scalar fields coupling non-minimally with the spacetime curvature [5]. With the 1998 discovery of the acceleration of the universe using type Ia supernovae [6], dark energy was introduced in the standard cosmological model to account for 73% of the energy content of the universe [7]. It is difficult to conceive of a cosmological constant Λ whose energy density is fine-tuned by ~ 120 orders of magnitude to account for the present acceleration. Attempts to explain the cosmic acceleration in the context of GR and without dark energy, using the backreaction of inhomogeneities on the cosmic dynamics [8] or by postulating that we live near the centre of a giant void in a dust-dominated universe [9], have so far been unconvincing. An alternative is to abandon GR and modify the Einstein-Hilbert action by replacing the Ricci scalar R with a non-linear function $f(R)$ [10, 11],

$$\begin{aligned} S_{EH} &= \frac{1}{2\kappa} \int d^4x \sqrt{-g} R + S^{(m)} \\ \longrightarrow S &= \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S^{(m)}, \end{aligned} \quad (2.1)$$

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where $\kappa \equiv 8\pi G$ and $S^{(m)}$ is the matter part of the action. Quantum corrections to the Einstein-Hilbert Lagrangian of the type $f(R) = R + \alpha R^2$ were already present in the first inflationary model of the early universe at large curvatures [12] and extra motivation for the $f(R)$ theories used more recently in cosmology comes from stringy physics [13] and from the asymptotic safety scenario [14].

The first model of $f(R)$ gravity attempting to explain the present-day cosmic acceleration was of the form $f(R) = R - \mu^4/R$ with a mass scale $\mu \sim H_0^{-1} \approx 10^{-33}$ eV [10–15]. This particular model was soon ruled out because of a catastrophic instability [16] and because it violates the post-Newtonian tests of GR [17], but the $f(R)$ proposal flourished. In principle one could include in the action other curvature invariants, as in $f(R, R_{ab}R^{ab}, R_{abcd}R^{abcd}, \dots)$. However, the theory then generally contains ghosts [18–20]. In principle, the metric tensor g_{ab} contains six degrees of freedom which appear when the Einstein-Hilbert Lagrangian is modified to $\sqrt{-g} f(R, R_{ab}R^{ab}, R_{abcd}R^{abcd}, \dots)$. In $f(R)$ gravity we will be concerned only with the usual massless spin 2 modes of GR, plus a massive scalar mode. The $f(R)$ proposal for cosmology has been taken rather seriously by the cosmological community, judging from the approximately 1100 papers which appeared on this subject since 2003 (see [21–24], for reviews and [25] for short introductions). Although the activity in this area seems to have peaked, this is still an active area of research. In this talk I will consider $f(R)$ gravity as a proof of principle that modifying gravity can explain the cosmic acceleration without dark energy and as a relatively simple but interesting alternative to GR keeping in mind that, in the road to quantization or to scenarios in which GR emerges from more fundamental constituents, GR will certainly fail somewhere, and that accompanying *infrared* corrections to the Einstein-Hilbert action are expected.

2.2 Versions of $f(R)$ Gravity

Many works in the literature focus on specific choices of the function $f(R)$ but even if the $f(R)$ proposal were ultimately the correct answer to the puzzle of the cosmic acceleration (which we do not claim), there is no indication on the form of the function $f(R)$, except for the fact that it must be very close to $f(R) = R$ at Solar System scales. Therefore, we will consider only the general features of $f(R)$ gravity. There are three versions of $f(R)$ modified gravity: metric (or second order) formalism, Palatini (or first order) formalism, and metric-affine gravity.

2.2.1 Metric $f(R)$ Gravity

In the metric formalism [10, 11] one varies the action

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S^{(m)} \quad (2.2)$$

only with respect to the (inverse) metric g^{ab} , i.e., the connection is the metric one. The resulting field equations are

$$f'(R)R_{ab} - \frac{f(R)}{2}g_{ab} = \nabla_a \nabla_b f'(R) - g_{ab} \square f'(R) + \kappa T_{ab}, \quad (2.3)$$

where a prime denotes differentiation with respect to R , $\square = g^{ab} \nabla_a \nabla_b$, and ∇_a is the covariant derivative of g_{ab} . The field equations are clearly of fourth order. Tracing eq. (2.3) yields

$$3\square f'(R) + Rf'(R) - 2f(R) = \kappa T, \quad (2.4)$$

which makes it clear that $f'(R)$ is a propagating degree of freedom (by contrast, the trace equation of GR is simply the algebraic equation $R = -\kappa T$). Eq. (2.3) can be rewritten as the effective Einstein equation

$$G_{ab} = \frac{\kappa}{f'} \left(T_{ab} + T_{ab}^{(eff)} \right), \quad (2.5)$$

where

$$T_{ab}^{(eff)} = \frac{1}{\kappa} \left[\frac{f(R) - Rf'(R)}{2} g_{ab} + \nabla_a \nabla_b f'(R) - g_{ab} \square f'(R) \right]. \quad (2.6)$$

The positivity of the effective gravitational coupling $G_{eff} \equiv G/f'(R)$ requires $f'(R) > 0$ (the graviton carries positive kinetic energy).

In a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe with line element $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$, the field equations become

$$H^2 = \frac{1}{3f'(R)} \left[\kappa \rho^{(m)} + \frac{Rf'(R) - f(R)}{2} - 3H\dot{R}f''(R) \right], \quad (2.7)$$

$$2\dot{H} + 3H^2 = -\frac{1}{f'(R)} \left[\kappa P^{(m)} + f'''(R)(\dot{R})^2 + 2H\dot{R}f''(R) + \ddot{R}f''(R) + \frac{f(R) - Rf'(R)}{2} \right], \quad (2.8)$$

where an overdot denotes differentiation with respect to t and $H \equiv \dot{a}/a$ is the Hubble parameter. The corresponding phase space is a 2-dimensional rather complicated subset of a 3-dimensional space [26].

2.2.2 Palatini $f(R)$ Gravity

The Palatini (or first order) formalism of $f(R)$ gravity was also introduced to explain the cosmic acceleration without dark energy [15] The action is

$$S_{Palatini} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(\tilde{R}) + S^{(m)}[g_{ab}, \psi^{(m)}]. \quad (2.9)$$

There are two Ricci tensors: R_{ab} , which is constructed with the metric connection of g_{ab} ; and $\tilde{R}_{ab} = \tilde{R}_{ab}[\Gamma_{\mu\nu}^\rho]$, constructed with the non-metric connection $\Gamma_{\mu\nu}^\rho$ (and with contraction $\tilde{R} = g^{ab}\tilde{R}_{ab}$). That is, it is not assumed that the connection is the metric connection: the metric g_{ab} and the connection $\Gamma_{\mu\nu}^\alpha$ are treated as independent variables. When $f(R)$ is linear (the case of GR with a cosmological constant), this point of view is inconsequential and the Palatini variation produces the same field equations (the Einstein equations) as the metric variation. However, when $f(R)$ is non-linear, the variation with respect to g^{ab} yields

$$f'(\tilde{R})\tilde{R}_{ab} - \frac{f(\tilde{R})}{2} g_{ab} = \kappa T_{ab}, \quad (2.10)$$

which are second order equations (there are no second derivatives $\nabla_a \nabla_b f'$, $\square f'$). The variation with respect to the (non-metric) connection yields

$$\frac{1}{\sqrt{-g}} \tilde{\nabla}_d \left(\sqrt{-g} f'(\tilde{R}) g^{ab} \right) - \tilde{\nabla}_d \left(\sqrt{-g} f'(\tilde{R}) g^{d(a} \right) \delta_c^{b)} = 0, \quad (2.11)$$

where $\tilde{\nabla}_a$ is the covariant derivative of the non-metric connection Γ .

Palatini $f(R)$ gravity has been shown to contain a non-dynamical scalar field $f'(\tilde{R})$ and to run into problems when building Newtonian polytropic stars [27] and with the Cauchy problem in matter [28]. Because of these problems, we will no longer consider the Palatini version of $f(R)$ gravity here.

2.2.3 Metric-affine $f(R)$ Gravity

In metric-affine $f(R)$ gravity [29], not only the metric g_{ab} and the connection $\Gamma_{\mu\nu}^\rho$ are independent variables, but the matter part of the action is allowed to depend explicitly on this connection. The action has the form

$$S_{\text{affine}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(\tilde{R}) + S^{(m)}[g_{ab}, \Gamma_{\mu\nu}^\rho, \psi^{(m)}]. \quad (2.12)$$

The possibility of a non-symmetric connection and of non-vanishing torsion is rather natural in this class of theories, which may lead to a revival of the torsion theories popular in the 1970's. Work on metric-affine gravity has been quite limited [29] because of the complication of the field equations and because cosmological accelerating dynamics can be obtained with less effort in simpler theories (which should not dismiss fundamental motivation of a different nature for these theories).

There are also “hybrid” formalisms which interpolate between the metric and the Palatini versions [30]. In the following we focus exclusively on metric $f(R)$ gravity.

2.3 Equivalence with Scalar-Tensor Gravity

If $f''(R) \neq 0$, metric and Palatini $f(R)$ gravities can be recast as $\omega = 0$ and $\omega = -3/2$ Brans-Dicke theories with a special potential, respectively [31]. Therefore, these classes of theories are nothing but scalar-tensor gravity, albeit of a form which had not been studied in depth before 2003 (see [32] for reviews of scalar-tensor gravity; this fact testifies of how little was really understood in scalar-tensor gravity).

Let us see how the reduction works for metric $f(R)$ gravity. Beginning with the action (2.2), consider the extra scalar field $\phi = R$ and the equivalent action

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\psi(\phi)R - V(\phi)] + S^{(m)}, \quad (2.13)$$

where $\psi(\phi) = f'(\phi)$ and

$$V(\phi) = \phi f'(\phi) - f(\phi). \quad (2.14)$$

If $\phi = R$, the action (2.13) reduces to (2.2). Vice-versa, varying (2.13) with respect to g^{ab} yields

$$R_{ab} - \frac{1}{2} g_{ab} R = \frac{1}{\psi} \left(\nabla_a \nabla_b \psi - g_{ab} \square \psi - \frac{V}{2} g_{ab} \right) + \frac{\kappa}{\psi} T_{ab}, \quad (2.15)$$

while variation with respect to ϕ yields $R \frac{d\psi}{d\phi} - \frac{dV}{d\phi} = (R - \phi) f''(\phi) = 0$ and $\phi = R$ if $f'' \neq 0$. Then, the massive scalar $\phi = R$ or, alternatively, $\psi = f'$ is dynamical and satisfies the trace equation

$$3\square\psi + 2U(\psi) - \psi \frac{dU}{d\psi} = \kappa T. \quad (2.16)$$

In terms of ψ , the action is

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\psi R - U(\psi)] + S^{(m)}, \quad (2.17)$$

where the potential is defined implicitly [33] by $U(\psi) = V(\phi(\psi)) - f(\phi(\psi))$. This is the action of a Brans-Dicke theory with Brans-Dicke parameter $\omega = 0$. The condition $f'' \neq 0$ ensures that the change of variable $R \rightarrow \psi(R)$ is invertible.

2.4 Criteria for Viability

Modifying gravity is *a priori* dangerous because the successes of GR and even of Newtonian gravity could easily be spoiled. Criteria for viability include a correct cosmological dynamics, stability, the absence of ghosts, correct Newtonian and post-Newtonian limit, a well-posed Cauchy problem, and cosmological perturbations compatible with the cosmic microwave background and large-scale structure.

Correct cosmological dynamics includes an early inflationary era (or an epoch replacing inflation with equal success), followed by a radiation era (well constrained by primordial nucleosynthesis) and by a matter era during which galaxies and other structures can grow, followed by the present accelerated era. The transitions between consecutive eras must be smooth. In the past, problems were been pointed out with the exit from the radiation era in some models [34], but these have now been solved. One can prescribe an arbitrary expansion history $a(t)$ and integrate an ODE for the function $f(R)$ which produces it (“designer $f(R)$ gravity”) [35]. The function $f(R)$ thus determined is not unique and, in general, does not assume a simple form of the kind encountered in the simple models.

Stability. The prototype $f(R)$ model given by the choice $f(R) = R - \mu^4/R$ suffers from the now notorious Dolgov-Kawasaki instability [16], the study of which has been generalized to arbitrary metric $f(R)$ gravity [36] and extended to more general theories [37]. Parametrize the deviations from GR as

$$f(R) = R + \epsilon \varphi(R), \quad (2.18)$$

where ϵ is a positive smallness parameter and $f''(R) \neq 0$. Then the trace equation gives

$$\square R + \frac{\varphi'''}{\varphi''} \nabla^c R \nabla_c R + \left(\frac{\epsilon \varphi' - 1}{3\epsilon \varphi''} \right) R = \frac{\kappa T}{3\epsilon \varphi''} + \frac{2\varphi}{3\varphi''}. \quad (2.19)$$

Expanding locally the metric and the Ricci scalar as $g_{ab} = \eta_{ab} + h_{ab}$ and $R = -\kappa T + R_1$ (where η_{ab} is the Minkowski metric and h_{ab} and R_1 are perturbations) one obtains, to first order

$$\begin{aligned} \ddot{R}_1 - \nabla^2 R_1 - \frac{2\kappa \varphi'''}{\varphi''} \dot{T} \dot{R}_1 + \frac{2\kappa \varphi'''}{\varphi''} \vec{\nabla} T \cdot \vec{\nabla} R_1 \\ + \frac{1}{3\varphi''} \left(\frac{1}{\epsilon} - \varphi' \right) R_1 = \kappa \ddot{T} - \kappa \nabla^2 T - \frac{(\kappa T \varphi^2 + 2\varphi)}{3\varphi''}. \end{aligned} \quad (2.20)$$

The coefficient of R_1 yields the effective mass squared $m^2 \simeq \frac{1}{3\epsilon \varphi''}$ and, therefore, the scalar degree of freedom is stable if $f''(R) > 0$ and unstable if $f''(R) < 0$. For example, the prototype model $f(R) = R - \mu^4/R$ has $f'' < 0$ and the instability manifests itself on the time scale (dictated by the value of $\mu \simeq H_0^{-1}$) $t \sim 10^{-26}$ s [16]. A physical interpretation of this stability criterion is the following [38]: the effective gravitational coupling is $G_{\text{eff}} = G/f'(R) > 0$. If $dG_{\text{eff}}/dR = -f''G/(f')^2 > 0$, G_{eff} increases with R and, at large curvatures, causes gravity to become stronger, which causes an even larger R , in a positive feedback. *Vice-versa*, if $dG_{\text{eff}}/dR < 0$ then G_{eff} does not increase as the curvature increases and there is stability.

The previous stability analysis assumes a flat background and, therefore, is only valid in the limit of small wavelengths. A stability analysis can be performed analytically for de Sitter space (which is often a late-time attractor of the cosmological dynamics). A study with a covariant and gauge-invariant formalism (necessary for

long-wavelength cosmological perturbations) [39] yields the first order stability criterion

$$\frac{(f'_0)^2 - 2f_0 f''_0}{f'_0 f''_0} \geq 0, \quad (2.21)$$

where the zero subscript describes the fact that quantities are evaluated in the background de Sitter space of constant curvature R_0 .

Beyond the linear approximation, certain metric $f(R)$ models have been found to suffer from a non-linear instability, which makes it difficult to construct relativistic stars in the presence of strong gravity because of a singularity developing at large curvature [40]. However, the problem may be cured by adding a small quadratic term to $f(R)$ [41] and it appears to be related to the complication of working with an implicit potential which becomes ill-defined in the scalar-tensor description of metric $f(R)$ gravity [42].

Weak-field limit. The weak-field limit of metric $f(R)$ gravity was studied in [43–45], following early work on particular models [21, 22]. The idea is to compute the PPN parameter γ through the PPN expansion of the line element

$$\begin{aligned} ds^2 = & - [1 + 2\Psi(r) - H_0^2 r^2] dt^2 \\ & + [1 + 2\Phi(r) + H_0^2 r^2] dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \end{aligned} \quad (2.22)$$

in Schwarzschild coordinates, where $|\Psi(r)|, |\Phi(r)| \ll 1$ and $H_0 r \ll 1$, $R(r) = R_0 + R_1$. The PPN parameter γ is given by $\gamma = -\Phi(r)/\Psi(r)$ [46]. Under the assumptions that [43] $f(R)$ is analytical at R_0 , $mr \ll 1$ (where m is the effective mass of the scalar degree of freedom), and that the matter composing the spherical body satisfies $P \simeq 0$, $T = T_0 + T_1 \simeq -\rho$, the trace equation yields

$$\nabla^2 R_1 - m^2 R_1 = \frac{-\kappa \rho}{3f''_0}, \quad (2.23)$$

where

$$m^2 = \frac{(f'_0)^2 - 2f_0 f''_0}{3f'_0 f''_0} \quad (2.24)$$

(the same expression obtained with gauge-invariant analysis of de Sitter space or by calculating the propagator). If $mr \ll 1$, the solution is

$$\Psi(r) = \frac{-\kappa M}{6\pi f'_0} \frac{1}{r}, \quad \Phi(r) = \frac{\kappa M}{12\pi f'_0} \frac{1}{r}, \quad (2.25)$$

from which one obtains the PPN parameter $\gamma = -\Phi(r)/\Psi(r) = 1/2$ in violent contrast with the Cassini limit $|\gamma - 1| < 2.3 \cdot 10^{-5}$ [47]. This result would be the end of metric $f(R)$ gravity if it wasn't for the fact that mr is not always small. Due to the *chameleon effect*, the range of the scalar field degree of freedom and its effective

mass m depend on the curvature and the energy density of the environment. This range λ can be small ($m > 10^{-3}$ eV and $\lambda < 0.2$ mm) at Solar System densities and large (of the order of the Hubble radius H_0^{-1}) at cosmological densities [20, 48]. The chameleon effect is not imposed by hand but is built into $f(R)$ gravity models. It is analogous to the chameleon mechanism of quintessence models with potentials $V(\phi) \approx 1/\phi^\alpha$ (with $\alpha > 0$) [49]. Several explicit forms of the function $f(R)$ are now known to exhibit an efficient chameleon mechanism sheltering the Solar System tests from modified gravity, which shows its effects at large scales [21–23].

Cosmological perturbations Most theories of gravity alternative to GR admit FLRW solutions [23], therefore, obtaining the correct dynamics for a FLRW universe does not discriminate between competing theories of gravity and cosmology. However, the growth history of non-linear cosmological perturbations is sensitive to the theory of gravity and its study has the potential to discriminate between different models in the near future. Recently, work on $f(R)$ gravity has moved to the study of perturbations and its comparison with large-scale structure surveys.

By assuming a cosmic evolution $a(t)$ typical of the Λ CDM model it is found (e.g., [50]) that, to lowest order, vector and tensor modes are unaffected by $f(R)$ corrections to the Einstein-Hilbert action. The condition $f''(R) \geq 0$ for the stability of scalar modes is recovered in this context, and $f(R)$ corrections do affect the scalar modes. Large angle anisotropies in the cosmic microwave background are produced, as well as different correlations (compared with the case of Einstein gravity) between cosmic microwave background and galaxy surveys.

The Cauchy problem A well-posed initial value problem is necessary in order for a physical theory to have predictive power. The Cauchy problem for metric and Palatini $f(R)$ gravity was studied in [28] (see also the early work of [51, 52]). Using the equivalence between $f(R)$ and scalar-tensor gravity, the Cauchy problem was reduced to the analogous one for Brans-Dicke gravity and, taking advantage of previous results for scalar-tensor gravity [53], the following was established. The initial value problem for metric $f(R)$ gravity is well-posed *in vacuo* and for “reasonable” forms of matter (meaning, the same for which the Cauchy problem of GR is well-posed [54]). The Cauchy problem is also well-posed for Palatini $f(R)$ gravity *in vacuo* (in which case the theory reduces to GR with a cosmological constant), but is extremely unlikely to be well-posed or well-formulated in matter, a problem that is reminiscent of the difficulties in constructing matter configurations [27] and is ultimately due to the structure of the field equations—this adds to the problems of Palatini $f(R)$ gravity [21, 55, 56] (but see [57] for a different view).

2.5 Scalar-Tensor and $f(R)$ Black Holes

In any theory of gravity it is important to understand the structure of its spherically symmetric solutions and, in particular, of its black holes. In GR stationary black holes, which are the endpoint of gravitational collapse, must be axisymmetric [58].

In scalar-tensor (and, therefore, in $f(R)$ gravity) Birkhoff's theorem is lost and spherical, asymptotically flat, black holes are not forced to be static. Let us consider Brans-Dicke theory in the Jordan frame, described by the action

$$S_{BD} = \int d^4x \sqrt{-\hat{g}} \left[\varphi \hat{R} - \frac{\omega_0}{\varphi} \hat{\nabla}^\mu \varphi \hat{\nabla}_\mu \varphi + L_m(\hat{g}_{\mu\nu}, \psi) \right] \quad (2.26)$$

A 1972 theorem by Hawking [58] states that the endpoint of axisymmetric collapse in this theory must be a GR black hole. This result was generalized (for spherical symmetry only) in [59]. A simple proof for more general stationary spacetimes became available recently [60], which extends Hawking's result to the general scalar-tensor theory described by the action

$$S_{ST} = \int d^4x \sqrt{-\hat{g}} \left[\varphi \hat{R} - \frac{\omega(\varphi)}{\varphi} \hat{\nabla}^\mu \varphi \hat{\nabla}_\mu \varphi - V(\varphi) + L_m(\hat{g}_{\mu\nu}, \psi) \right]. \quad (2.27)$$

We assume 1) asymptotic flatness (realistic astrophysical collapse occurs on scales much smaller than H_0^{-1}), therefore the Brans-Dicke scalar field $\varphi \rightarrow \varphi_0$ as $r \rightarrow +\infty$, $V(\varphi_0) = 0$, and $\varphi_0 V'(\varphi_0) = 2 V(\varphi_0)$. 2) Stationarity (the black hole is the endpoint of gravitational collapse).

Using the Einstein frame variables $\hat{g}_{\mu\nu} \rightarrow g_{\mu\nu} = \varphi \hat{g}_{\mu\nu}$, $\varphi \rightarrow \phi$ with $d\phi = \sqrt{\frac{2\omega(\varphi)+3}{16\pi}} \frac{d\varphi}{\varphi}$ ($\omega \neq -3/2$), the action is cast into the form

$$S_{ST} = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi} - \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - U(\phi) + L_m(g_{\mu\nu}, \psi) \right] \quad (2.28)$$

where $U(\phi) = V(\varphi)/\varphi^2$. The Einstein frame field equations are

$$\begin{aligned} \hat{R}_{\mu\nu} - \frac{1}{2} \hat{R} \hat{g}_{\mu\nu} &= \frac{\omega(\varphi)}{\varphi^2} \left(\hat{\nabla}_\mu \varphi \hat{\nabla}_\nu \varphi - \frac{1}{2} \hat{g}_{\mu\nu} \hat{\nabla}^\lambda \varphi \hat{\nabla}_\lambda \varphi \right) \\ &+ \frac{1}{\varphi} \left(\hat{\nabla}_\mu \hat{\nabla}_\nu \varphi - \hat{g}_{\mu\nu} \hat{\square} \varphi \right) - \frac{V(\varphi)}{2\varphi} \hat{g}_{\mu\nu}, \end{aligned} \quad (2.29)$$

$$(2\omega + 3) \hat{\square} \varphi = -\omega' \hat{\nabla}^\lambda \varphi \hat{\nabla}_\lambda \varphi + \varphi V' - 2V. \quad (2.30)$$

Since the conformal factor depends only on φ , the Einstein frame symmetries are the same as in the Jordan frame and there exists a timelike Killing vector ξ^μ describing stationarity and a spacelike Killing vector ζ^μ which describes axial symmetry. Consider, *in vacuo*, a 4-volume $\mathcal{V}$ bounded by the horizon $\mathcal{H}$, two Cauchy hypersurfaces $\mathcal{S}_1, \mathcal{S}_2$, and a timelike 3-surface at infinity. Multiply the Einstein frame field equation $\hat{\square} \varphi = U'(\phi)$ by U' and integrate the result over $\mathcal{V}$. This yields

$$\int_{\mathcal{V}} d^4x \sqrt{-g} U'(\phi) \hat{\square} \varphi = \int_{\mathcal{V}} d^4x \sqrt{-g} U'^2(\phi), \quad (2.31)$$

which can be rewritten as

$$\begin{aligned} & \int_{\mathcal{V}} d^4x \sqrt{-g} [U''(\phi) \nabla^\mu \phi \nabla_\mu \phi + U'^2(\phi)] \\ &= \int_{\partial \mathcal{V}} d^3x \sqrt{|h|} U'(\phi) n^\mu \nabla_\mu \phi \end{aligned} \quad (2.32)$$

where n^μ is the normal to the boundary and h is the determinant of the induced metric $h_{\mu\nu}$ on this boundary. Now split the boundary into its constituents, $\int_{\mathcal{V}} = \int_{S_1} + \int_{S_2} + \int_{\text{horizon}} + \int_{r=\infty}$. We have $\int_{S_1} = -\int_{S_2}$, $\int_{r=\infty} = 0$, and

$$\int_{\text{horizon}} d^3x \sqrt{|h|} U'(\phi) n^\mu \nabla_\mu \phi = 0 \quad (2.33)$$

because of the symmetries. As a result, it is

$$\int_{\mathcal{V}} d^4x \sqrt{-g} [U''(\phi) \nabla^\mu \phi \nabla_\mu \phi + U'^2(\phi)] = 0. \quad (2.34)$$

Since $U'^2 \geq 0$, $\nabla^\mu \phi$ (orthogonal to both ξ^μ, ζ^μ on H) is spacelike or zero, and $U''(\phi)$ must be non-negative for stability (the black hole is the endpoint of collapse!), then it must be $\nabla_\mu \phi \equiv 0$ in $\mathcal{V}$ and $U'(\phi_0) = 0$. Since for $\phi = \text{const.}$, the theory reduces to GR, black holes must be of the Kerr type.

Now, metric $f(R)$ gravity is a special Brans-Dicke theory with $\omega = 0$ and $V \neq 0$ and is covered by the previous discussion. For $\omega = -3/2$, the vacuum theory reduces to GR and Hawking's theorem applies (Palatini $f(R)$ gravity is a special Brans-Dicke theory with $\omega = -3/2$ and $V \neq 0$). Exceptions not covered by our proof include theories in which $\omega \rightarrow \infty$ somewhere, and theories in which ϕ diverges at infinity or on the horizon (an example is the maverick solution of [61], which is however unstable). The proof is extended immediately to the case of electrovacuum or any conformal matter with energy-momentum trace $T = 0$ because the Brans-Dicke scalar couples only to the trace T and the equations used in the proof do not change by including such forms of matter.

Even though Birkhoff's theorem is lost, black holes which are the endpoint of axisymmetric gravitational collapse and are asymptotically flat in *general* scalar-tensor gravity are the same as in GR (*i.e.*, Kerr-Newman black holes). The exceptions are unphysical or unstable solutions which cannot be the endpoint of collapse, or do not satisfy the weak/null energy condition. Asymptotic flatness is a technical requirement, but realistic gravitational collapse in an astrophysical context occurs on scales which are much smaller than the Hubble time and the influence of an asymptotically FLRW structure during this short timescale is completely negligible. Exceptions are situations involving the formation of primordial black holes in the very early universe when the collapse and the Hubble time scales are comparable.

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