Real Options with Competition and Incomplete Markets

Alain Bensoussan and SingRu (Celine) Hoe

Abstract Ever since the first attempts to model capital investment decisions as options, financial economists have sought more accurate, more realistic real options models. Strategic interactions and market incompleteness are significant challenges that may render existing classical models inadequate to the task of managing the firm's capital investments. The purpose of this paper is to address these challenges. The issue of incompleteness comes in for the valuation of payoffs due to absence of a unique martingale measure. One approach is to valuate assets by considering a rational utility-maximizing consumer/investor's joint decisions with respect to portfolio investment strategy and consumption rule. In our situation, we add the stopping time as an additional decision. We employ variational inequalities (V.I.s) to solve the optimal stopping problems corresponding to times to invest. The regularity of the obstacle (payoffs received at the decision time) is a major element for defining the optimal strategy. Due to the lack of smoothness of the obstacle raised by the game problem, the optimal strategy is a two-interval solution, characterized by three thresholds.

Keywords Stackelberg leader-follower game · Utility maximization · Bellman equation · Optimal stopping

Mathematics Subject Classification (2010) 91G80 · 91A30 · 91A15

A. Bensoussan (✉)
International Center for Decision and Risk Analysis, School of Management, University of Texas at Dallas, 800 West Cambell Rd, SM30, Richardson, TX 75080-3021, USA
e-mail: axb046100@utdallas.edu

A. Bensoussan
City University of Hong Kong, Hong Kong, China

S.(C.) Hoe
Texas A&M University-Commerce, Commerce, TX 75429, USA
e-mail: hoceline02@yahoo.com

Y. Kabanov et al. (eds.), *Inspired by Finance*, DOI 10.1007/978-3-319-02069-3_2,

1 Investment Game Problems and General Model Assumptions

We consider a Stackelberg leader-follower game for exploiting an irreversible investment opportunity with payoffs of a continuous stochastic income stream $Y(t)$ for a fixed cost K. We limit the flexibility in the investment decisions to the times when to invest. The roles of leader and follower are predetermined by regulations. Each firm chooses its individual stopping time to invest over an infinite horizon with the constraint that the follower be forbidden to undertake the investment until the leader has already done so. By investing K, the leader receives $\delta_1 Y(t)$ per unit time till the follower's entry. Once both have entered, each gets a continuous cashflow stream $\delta_2 Y(t)$ per unit time, with $\delta_2 < \delta_1$.

Consider a probability space $(\Omega, \mathscr{F}, \mathrm{Q})$ with $W^*(t) = (W(t), W^0(t))^T$ a standard Wiener process. The asset S representing the market and the cashflow process Y evolve as follows:

$$dS(t) = rS(t)dt + \sigma S(t)\left(\lambda dt + dW(t)\right), \tag{1}$$

$$dY(t) = Y(t)\left(\alpha dt + \varsigma\left(\rho dW(t) + \sqrt{1-\rho^2}dW^0(t)\right)\right), \tag{2}$$

where $W(t)$ and $W^0(t)$ are independent Wiener processes, $\rho^2 < 1$ is the correlation coefficient between market uncertainty and the cashflow process uncertainty, and r (risk-free rate), $\sigma, \lambda, \alpha, \varsigma$ are all constants. The market is incomplete since the market asset S can span only the portion of the stochastic cashflow risk driven by the Wiener process $W(t)$, leaving the remaining risk driven by $W^0(t)$ unhedgeable. There is no unique martingale measure, so the risk-neutral pricing is no longer appropriate, and an alternative must be developed in this framework.

We adopt utility-based pricing in which a risk averse investor/firm maximizes the expected utility of consumption. We assume that the investor's risk preferences are characterized by a constant absolute risk aversion utility function

$$U(C) = -\frac{1}{\gamma}e^{-\gamma C} \tag{3}$$

where the argument C is the investor's consumption, and γ is his/her risk aversion parameter, $\gamma > 0$.

Remark 1 We allow for negative consumption. For $C \in \mathbf{R}$, U increases from $-\infty$ to 0. As $C \to -\infty$, it leads to huge negative values. We interpret this effect as a penalty to the utility maximization investor. We could of course impose the constraint of non-negative consumption. However, imposing non-negativity on the consumption would rule out the analytical solutions for further developments, a property we would like to retain for the full analysis. Therefore, we choose to accept for negative consumption which could lead huge negative utility values (big penalties for our utility maximization investor) instead of imposing the non-negativity constraint on the consumption. We also note that the negative consumption occurs when x becomes very negative and we cannot avoid this situation since $x \in \mathbf{R}$.

Each firm maximizes its expected discounted utility from consumption over an infinite horizon, subject to choice over investment timing, consumption, hedge position in the market asset, and allocation in the riskless bond. Thus, each firm considers undertaking the investment as an additional decision besides portfolio investment and consumption decisions. The decision remains a stopping time, for which the right approach is that of variational inequality (V.I.) [1, 5]. Our duopoly game requires us to solve two V.I.s corresponding to the leader's and the follower's optimal stopping respectively. As such, we will need two obstacles corresponding to each V.I. We obtain the obstacles from solving continuous control problems, i.e., portfolio investment and consumption decisions, and we call this as solutions to postinvestment utility maximization. Employing the obstacles obtained, we then form V.I.s to solve the optimal stopping problems, and we call this as solutions to preinvestment utility maximization.

One point to note is that we need to consider an auxiliary problem of which the cashflow process (2) hits zero; the problem will then be reduced to classical investment-consumption portfolio decisions. We next summarize the general notations used in the paper to facilitate reading:

- τ for the follower's stopping time and θ for the leader's stopping time;
- $F^1(x, y)$ for the follower's obstacle, i.e., solution to follower's postinvestment utility maximization, and $F(x, y)$ for the follower's solution to the V.I., i.e., solution to the follower's preinvestment utility maximization;
- $L^1(x, y)$ for the leader's obstacle, i.e., solution to leader's postinvestment utility maximization, and $L(x, y)$ for the leader's solution to the V.I., i.e., solution to the leader's preinvestment utility maximization;
- $F(x)$ for the solution to the classical investment-consumption utility maximization, i.e., no augmented stochastic income stream $Y(t)$.

We detail follower's problem and solution in Sect. 2 and the leader's in Sect. 3. We conclude in Sect. 4. We omit most of the proofs except the main result.

2 Follower's Problem and Solution

We start with the follower's investment problem. Given the initial wealth, x, the follower optimizes his portfolio by dynamically choosing allocations in the market asset S, the riskless bond, and the consumption rate, C. The follower's wealth, X, evolves as follows:

$$\begin{cases} dX(t) = \pi(t)X(t)\sigma(\lambda dt + dW(t)) + rX(t)dt - C(t)dt, \quad t < \tau, \\ X(\tau) = X(\tau - 0) - K, \\ dX(t) = \pi(t)X(t)\sigma(\lambda dt + dW(t)) + rX(t)dt - C(t)dt + \delta_2 Y(t)dt, \quad t > \tau, \\ dY(t) = Y(t)\left(\alpha dt + \varsigma\left(\rho dW(t) + \sqrt{1-\rho^2}dW^0(t)\right)\right), \\ X(0) = x, \quad Y(0) = y, \end{cases} \tag{4}$$

where $\pi(t)$ is the proportion of wealth invested in asset S, $C(t)$ is the consumption rate, and τ is the stopping time to undertake the investment, chosen optimally by the follower. The wealth process is discontinuous at τ. From (4), we observe that the wealth process has two possible evolution regimes. To facilitate further exposition, we introduce the processes X^0 and X^1 (regime 0 and regime 1, respectively):

$$dX^0(t) = \pi(t)X^0(t)\sigma(\lambda dt + dW(t)) + rX^0(t)dt - C(t)dt, \tag{5}$$

$$dX^1(t) = \pi(t)X^1(t)\sigma(\lambda dt + dW(t)) + rX^1(t)dt - C(t)dt + \delta_2 Y(t)dt. \tag{6}$$

The follower's problem is to maximize his expected discounted utility from consumption by choosing stopping time τ, consumption rate C, and investment strategy π. We have to solve the problem in two steps, beginning with the utility maximization after τ (postinvestment utility maximization) and then solving the complete utility maximization prior to τ (preinvestment utility maximization). The rationale behind this two-step procedure is because we need a clearly defined obstacle function when solving the stopping time problem.

2.1 Postinvestment Utility Maximization

After τ, the follower solves his utility maximization as a control problem of portfolio selections and consumption rules augmented by a stochastic cashflow stream $\delta_2 Y(t)$ per unit time.

To facilitate representation, for $\mathscr{F}_t$-adapted processes $\pi(t)$, $C(t)$, we introduce the local integrability conditions

$$I^i = \begin{cases} \mathrm{E}\int_0^T \big(\pi(t)X^i(t)\big)^2 dt < \infty, & \forall\, T, \\ \mathrm{E}\int_0^T (C(t))^2\, dt < \infty, & \forall\, T, \end{cases} \tag{7}$$

and define

$$\tau_N^i = \inf\{t:\ X^i(t) < -N\}, \qquad i = 0, 1. \tag{8}$$

The follower reveals his preference through his expected discounted utility of consumption, and so, to the pair $(C(\cdot), \pi(\cdot))$, we introduce the objective function

$$J\big(C(\cdot)\big) = \mathrm{E}\int_0^\infty e^{-\mu t} U\big(C(t)\big)dt, \tag{9}$$

where μ, a constant, is the discount rate. This function is well-defined, but it may take the value $-\infty$. Since the follower can manage his investment-consumption portfolio, we consider the following control problem:

$$F^1(x, y) = \sup_{\{\pi(\cdot), C(\cdot)\} \in \mathscr{U}^1_{x,y}} J\big(C(\cdot)\big), \tag{10}$$

where

$$\mathcal{U}^1_{x,y} = \Big\{(\pi, C):\ I^1;\ \tau^1_N \uparrow \infty \text{ as } N \uparrow \infty;\ e^{-\mu T}\mathrm{E}e^{-r\gamma\left(X^1(T)+f(Y(T))\right)} \to 0, \\ \text{as } T \uparrow \infty\Big\},$$

and $f(y)$ is a positive function of linear growth with $f(0)=0$ which will be made precise later (cf. (17), (16)).

We associate the value function $F^1(x,y)$ with the Bellman equation:

$$-\mu F^1 \frac{\partial F^1}{\partial x}(rx+\delta y) + \frac{\partial F^1}{\partial y}\alpha y + \frac{1}{2}\frac{\partial^2 F^1}{\partial y^2} y^2 \varsigma^2 + \sup_C\left\{U(C) - C\frac{\partial F^1}{\partial x}\right\} \\ + \sup_\pi\left\{\pi x\sigma\left(\lambda\frac{\partial F^1}{\partial x} + y\varsigma\rho\frac{\partial^2 F^1}{\partial xy}\right) + \frac{1}{2}\frac{\partial^2 F^1}{\partial x^2}\pi^2 x^2\sigma^2\right\} = 0. \tag{11}$$

The domain is $x \in \mathbf{R},\ y > 0$.

We note that if $y=0$, then $Y(t)=0$ for all t. The problem reduces to the classical investment-consumption problem with the solution given by:

$$F(x) = -\frac{1}{r\gamma}\exp\left\{-r\gamma x + 1 - \frac{\mu + \frac{\lambda^2}{2}}{r}\right\}. \tag{12}$$

We thus have:

$$F^1(x,0) = F(x). \tag{13}$$

We look for a solution of (10) in the form

$$F^1(x,y) = -\frac{1}{r\gamma}\exp\left\{-r\gamma(x+f(y)) + 1 - \frac{\mu + \frac{\lambda^2}{2}}{r}\right\} \tag{14}$$

in which, by (13),

$$f(0) = 0. \tag{15}$$

By (14), defining the optimal feedback,

$$\widehat{C}(x,y) = -\frac{1}{\gamma}\ln\frac{\partial F^1}{\partial x}$$

and

$$\widehat{\pi}(x,y) = -\frac{\lambda\frac{\partial F^1}{\partial x} + y\varsigma\rho\frac{\partial^2 F^1}{\partial x\partial y}}{\frac{\partial^2 F^1}{\partial x^2}\sigma x},$$

we reduce the Bellman equation (11) to:

$$\frac{1}{2}y^2\varsigma^2 f'' + (\alpha - \lambda\varsigma\rho)yf' - \frac{1}{2}r\gamma y^2\varsigma^2(1-\rho^2)f'^2 - rf + \delta_2 y = 0. \tag{16}$$

Proposition 1 *The value function,*

$$f(y) = \inf_{\{v(\cdot)\in\mathscr{U}_y\}} \mathrm{E}\int_0^\infty e^{-rt}\left(\delta_2 Y_y(t) + \frac{1}{2}v^2(t)\right)dt \tag{17}$$

with

$$\begin{cases} dY_y(t) = Y_y(t)\big[\alpha - \lambda\varsigma\rho + \varsigma\sqrt{r\gamma(1-\rho^2)}v(t)\big]dt + Y_y(t)\varsigma dW(t), \;\; Y_y(0) = y\,, \\ \mathscr{U}_y = \{v(\cdot):\; E\int_0^\infty e^{-rt}v^2(t)dt < \infty,\; e^{-rT}EY_y(T) \to 0 \; as \; T \to \infty\}\,, \end{cases} \tag{18}$$

is the unique function in $C^2(0,\infty)$ *solving* (16), (15) *on the interval* $[0, \varepsilon y + M_\varepsilon]$,[1] *and such that* $f(y) \uparrow \infty$ *as* $y \uparrow \infty$.

Proposition 2 *The function* $f'(y)$ *is bounded.*

We now state the result that the value function given by (10) is indeed of the form (14).

Theorem 1 *The function* $F^1(x, y)$ *given by* (14) *coincides with the value function given by* (10).

2.2 Preinvestment Utility Maximization

We now turn to the problem of optimal stopping with the obstacle defined by $F^1(x, y)$, the solution to the postinvestment utility maximization. Before the stopping time τ, the wealth process is governed by (5) and the cashflow process evolves as (2). Set $\theta^0 = \inf\{t:\; Y(t) = 0\}$. At time $\tau \wedge \theta^0$ the follower stops. If $\theta^0 \le \tau$, the investment never takes place and the follower receives $F(X^0(\theta^0))$, where $F(x)$ is given by (12). If $\tau < \theta^0$, the follower receives $F^1(X^0(\tau) - K, Y(\tau))$ at the stopping time τ, where $F^1(x, y)$ is given by (14). Therefore, the objective function is:

$$\begin{aligned} &J_{x,y}\big(C(\cdot), \pi(\cdot), \tau\big) \\ &\quad = \mathrm{E}\bigg[\int_0^{\tau\wedge\theta^0} e^{-\mu t}U\big(C(t)\big)dt + F^1\big(X^0(\tau) - K, Y(\tau)\big)e^{-\mu\tau}\mathbf{1}_{\tau<\theta^0} \\ &\qquad + F\big(X^0(\theta^0)\big)e^{-\mu\theta^0}\mathbf{1}_{\theta^0\le\tau}\bigg], \end{aligned} \tag{19}$$

[1] M_ε is defined as: $M_\varepsilon = \dfrac{\left(\frac{\delta_2}{\varepsilon} - r + \alpha - \lambda\varsigma\rho\right)^2}{2\varsigma^2 r^2\gamma(1-\rho^2)}$. If $r + \lambda\varsigma\rho - \alpha > 0$, we can take $M_\varepsilon = 0$, hence $\varepsilon = \frac{\delta_2}{r-\alpha+\lambda\varsigma\rho}$. Note that ε can be arbitrarily small.

and we define the associated value function:

$$F(x,y)=\sup_{\{\pi(\cdot),C(\cdot),\tau\}\in\mathscr{U}^0_{xy}} J_{x,y}\big(C(\cdot),\pi(\cdot),\tau\big), \tag{20}$$

where

$$\mathscr{U}^0_{x,y}=\Big\{(C,\pi,\tau):\ I^0;\ \tau\wedge\theta^0<\infty \text{ a.s.};\ \tau^*=\lim\uparrow\tau^0_N\geq\tau\wedge\theta^0 \text{ a.s.}\Big\}.$$

As a consequence of Dynamic Programming, assuming sufficient smoothness of the function $F(x,y)$, we may write the strong formulation of V.I. that $F(x,y)$ must satisfy as follows:

$$\begin{cases} -\mu F+\frac{\partial F}{\partial x}rx+\frac{\partial F}{\partial y}\alpha y+\frac{1}{2}\frac{\partial^2 F}{\partial y^2}\varsigma^2 y^2+\sup_C\big(U(C)-C\frac{\partial F}{\partial x}\big)\\ \quad +\sup_\pi\big[\pi x\sigma\big(\lambda\frac{\partial F}{\partial x}+y\varsigma\rho\frac{\partial^2 F}{\partial x\partial y}\big)+\frac{1}{2}\pi^2x^2\sigma^2\frac{\partial^2 F}{\partial x^2}\big]\leq 0,\\ F(x,y)\geq F^1(x-K,y),\\ \big(F(x,y)-F^1(x-K,y)\big)\big[-\mu F+\frac{\partial F}{\partial x}rx+\frac{\partial F}{\partial y}\alpha y+\frac{1}{2}\frac{\partial^2 F}{\partial y^2}\varsigma^2y^2\\ \qquad +\sup_C\big(U(C)-C\frac{\partial F}{\partial x}\big)+\sup_\pi\big[\pi x\sigma\big(\lambda\frac{\partial F}{\partial x}+y\varsigma\rho\frac{\partial^2 F}{\partial x\partial y}\big)+\frac{1}{2}\pi^2x^2\sigma^2\frac{\partial^2 F}{\partial x^2}\big]\big]\\ \quad =0. \end{cases} \tag{21}$$

We have the boundary condition:

$$F(x,0)=F(x). \tag{22}$$

We look for a solution of the form:

$$F(x,y)=-\frac{1}{r\gamma}\exp\left[-r\gamma\big(x+g(y)\big)+1-\frac{\mu+\frac{\lambda^2}{2}}{r}\right]. \tag{23}$$

Using (23) and (14) and defining the optimal feedback

$$\widehat{C}(x,y)=-\frac{1}{\gamma}\ln\frac{\partial F}{\partial x}$$

and

$$\widehat{\pi}(x,y)=-\frac{\lambda\frac{\partial F}{\partial x}+y\varsigma\rho\frac{\partial^2 F}{\partial x\partial y}}{\frac{\partial^2 F}{\partial x^2}\sigma x},$$

we transform V.I. (21) to the form:

$$\begin{cases} \frac{1}{2}y^2\varsigma^2g''+g'y(\alpha-\lambda\varsigma\rho)-\frac{1}{2}y^2\varsigma^2r\gamma(1-\rho^2)g'^2-rg\leq 0,\\ g(y)\geq f(y)-K\,,\\ \big(g(y)-f(y)+K\big)\Big[\frac{1}{2}y^2\varsigma^2g''+g'y(\alpha-\lambda\varsigma\rho)-\frac{1}{2}y^2\varsigma^2r\gamma(1-\rho^2)g'^2-rg\Big]\\ =0,\\ g(0)=0\,. \end{cases} \tag{24}$$

This V.I. cannot be interpreted as a control problem because the non-linear operator is connected to a minimization problem, while the inequalities are connected to a maximization problem. So, $g(y)$ is more appropriately the value function of a differential game rather than of a control problem. Define

$$u(y) = g(y) - f(y) + K.$$

Then (24) becomes (using the equation of $f(y)$ (cf.(16)):

$$\begin{cases} -\frac{1}{2}y^2\varsigma^2 u'' - yu'\big(\alpha - \lambda\varsigma\rho - yf'\varsigma^2 r\gamma(1-\rho^2)\big) + \frac{1}{2}y^2\varsigma^2 r\gamma(1-\rho^2)u'^2 + ru \\ \quad \geq -\delta_2 y + rK, \\ u \geq 0, \\ u\big[-\frac{1}{2}y^2\varsigma^2 u'' - yu'\big(\alpha - \lambda\varsigma\rho - yf'\varsigma^2 r\gamma(1-\rho^2)\big) + \frac{1}{2}y^2\varsigma^2 r\gamma(1-\rho^2)u'^2 \\ \quad + ru + \delta_2 y - rK\big] = 0, \\ u(0) = K. \end{cases} \tag{25}$$

We study (25) by the threshold approach. Let $\hat{y}$ be fixed, to be determined below. We consider the Dirichlet problem

$$\begin{cases} -\frac{1}{2}y^2\varsigma^2 u'' - yu'\big(\alpha - \lambda\varsigma\rho - yf'\varsigma^2 r\gamma(1-\rho^2)\big) + \frac{1}{2}y^2\varsigma^2 r\gamma(1-\rho^2)u'^2 + ru \\ \quad = -\delta_2 y + rK, \ \ 0 < y < \hat{y}, \\ u(0) = K, \quad u(\hat{y}) = 0. \end{cases} \tag{26}$$

For $\hat{y}$ fixed, this problem is a classical Bellman equation. Similar to Proposition 1, equation (26) is a Bellman equation of the following control problem with the controlled diffusion:

$$\begin{cases} dY_y(t) = Y_y(t)\big[\alpha - \lambda\varsigma\rho - Y_y(t)f'(Y_y(t))\varsigma^2 r\gamma(1-\rho^2) \\ \qquad\qquad + \varsigma\sqrt{r\gamma(1-\rho^2)}v(t)\big]dt + Y_y(t)\varsigma dW(t), \\ Y_y(0) = y, \quad 0 < y < \hat{y}, \end{cases} \tag{27}$$

and the value function

$$u(y) = \inf_{v(\cdot)} E\left[\int_0^{\theta_y(v(\cdot))} e^{-rt}\left[-\delta_2 Y_y(t) + rK + \frac{1}{2}v^2(t)\right]dt \right.$$
$$\left. + e^{-r\theta_y(v(\cdot))}K\mathbf{1}_{Y_y(\theta_y(v(\cdot)))=0}\right], \tag{28}$$

where $\theta_y(v(\cdot)) = \inf\{t : \ Y_y(t) \text{ is outside } (0, \hat{y})\}$ and it is finite (a.s.). Obviously, $u(y) > 0$ if $\hat{y} < \frac{Kr}{\delta_2}$, and we also have $u(y) \leq K$.[2]

[2]For $v(t)$, we can take the same class as in problem (17)–(18).

Theorem 2 *There exists a unique value $\hat{y}$ such that*

$$u'(\hat{y}) = 0, \quad \hat{y} \geq \frac{Kr}{\delta_2}. \tag{29}$$

The value function $u(y)$ (cf. (28)) extended by zero beyond $\hat{y}$ is the unique solution of V.I. (25). It is C^1 and piecewise C^2.

Referring back to (24), from Theorem 2, we have obtained that there exists a unique solution of (24) such that $g(y) \in C^1$ and piecewise C^2. There exists a unique $\hat{y}$ such that

$$\begin{cases} -\frac{1}{2}y^2\varsigma^2 g'' - g'y(\alpha - \lambda\varsigma\rho) + \frac{1}{2}y^2\varsigma^2 r\gamma(1-\rho^2)g'^2 + rg = 0, \quad y < \hat{y}, \\ g(y) = f(y) - K, \quad y \geq \hat{y}\,, \\ g'(\hat{y}) = f'(\hat{y}), \\ g(0) = 0\,. \end{cases} \tag{30}$$

Note that $g(y) \geq 0$ since $u(y) \geq -f(y) + K$. We generate the main result that the value function given by (20) is indeed of the form (23).

Theorem 3 *The function $F(x, y)$ defined by (23) coincides with the value function given by (20).*

2.3 Follower's Optimal Stopping Rule

We next define the optimal stopping rule as:

$$\hat{\tau}(y) = \inf\{t : \; Y_y(t) \geq \hat{y}\}, \tag{31}$$

where $Y_y(t)$ is the process defined in (2) and $\hat{y}$ is the unique value defined by the V.I. (29) (the smooth matching point). We must note that the follower's stopping time $\hat{\tau}(y)$ is the follower's optimal entry if he can enter in the market at time zero. Since the follower can enter only after the leader (who starts at time θ), for finite θ, the follower will enter at time:[3]

$$\hat{\tau}_\theta = \theta + \hat{\tau}\big(Y_y(\theta)\big). \tag{33}$$

[3] For any test function $\Psi(x, s)$, we have the formula:

$$\mathrm{E}\left[\Psi(Y_y(\hat{\tau}_\theta), \hat{\tau}_\theta)|\mathscr{F}_\theta\right] = \Psi(Y_y(\theta), \theta)\mathbf{1}_{Y_y(\theta)\geq\hat{y}} + \mathbf{1}_{Y_y(\theta)<\hat{y}}\mathrm{E}\left[\Psi(\hat{y}, t + \hat{\tau}(y))\right]|_{y=Y_y(\theta), t=\theta}\,. \tag{32}$$

3 Leader's Problem and Solution

After solving the follower's optimal policy, we are now ready to solve the leader's problem, which is complicated by the fact that he must share the market (project value) upon the follower's optimal entry at $\hat{\tau}_\theta$. Thus, by investing K, the leader expects to receive a continuous cash flow $\delta_1 Y(t)$ per unit time prior to the follower's entry, and $\delta_2 Y(t)$ per unit time afterwards. The leader's wealth evolves according to the following system of stochastic equations:

$$\begin{cases} dX(t) = \pi(t)X(t)\sigma\big(\lambda dt + dW(t)\big) + rX(t)dt - C(t)dt, \quad t < \theta\ , \\ X(\theta) = X(\theta - 0) - K, \\ dX(t) = \pi(t)X(t)\sigma\big(\lambda dt + dW(t)\big) + rX(t)dt + \delta_1 Y(t)dt - C(t)dt, \\ \quad \theta < t < \hat{\tau}_\theta, \\ dX(t) = \pi(t)X(t)\sigma\big(\lambda dt + dW(t)\big) + rX(t)dt + \delta_2 Y(t)dt - C(t)dt, \quad t > \hat{\tau}_\theta, \\ dY(t) = Y(t)\Big(\alpha dt + \varsigma\big(\rho dW(t) + \sqrt{1-\rho^2}dW^0(t)\big)\Big), \\ X(0) = x, \quad Y(0) = y, \quad \text{with } x \in \mathbf{R},\ y \geq 0, \end{cases} \tag{34}$$

where θ and $\hat{\tau}_\theta$ are stopping times chosen optimally by the leader and the follower, respectively. The leader's problem is to maximize his expected discounted utility from consumption by choosing stopping time θ, consumption rate C, and investment strategy π. As in the follower's case, we have to solve leader's complete utility maximization problem in two steps.

3.1 Postinvestment Utility Maximization

Suppose that $\theta = 0$, the leader's wealth is x, and the cash flow $y > 0$; then the leader's wealth becomes immediately $x - K$ since he must pay the fixed cost of entry, K. The leader must share the market upon follower's entry at $\hat{\tau}(y)$. Thus, for a generic initial wealth x, the leader's wealth evolves as follows:

$$\begin{cases} dX^{L1}(t) = \pi(t)X^{L1}(t)\sigma\big(\lambda dt + dW(t)\big) + \big(rX^{L1}(t) + \delta_1 Y(t) - C(t)\big)dt, \\ \quad t < \hat{\tau}(y), \\ X^{L1}(0) = x, \\ dX^2(t) = \pi(t)X^2(t)\sigma\big(\lambda dt + dW(t)\big) + rX^2(t)dt + \delta_2 Y(t)dt - C(t)dt, \\ \quad t > \hat{\tau}(y), \\ X^2\big(\hat{\tau}(y)\big) = X^{L1}\big(\hat{\tau}(y)\big). \end{cases} \tag{35}$$

If $\theta = 0$ and $y \geq \hat{y}$, the follower enters immediately, and the leader's problem is identical to the follower's, i.e., (10). So, we consider the function

$$L^2(x, y) = -\frac{1}{r\gamma} e^{-r\gamma\left(x+f(y)\right)+1-\frac{\mu+\frac{\lambda^2}{2}}{r}}, \tag{36}$$

where f is the solution of (16), (15) on the interval $[0, \varepsilon y + M_\varepsilon]$.[4]

If $\theta = 0$ and $y < \hat{y}$, the leader's problem is described as follows. The wealth process is described by X^{L1} in (35) and the cash flow process follows (2). Recall that $\theta^0 = \inf\{t: \ Y_y(t) = 0\}$ and $\hat{\tau}(y) = \inf\{t: \ Y_y(t) \geq \hat{y}\}$.[5] If $\theta^0 < \hat{\tau}(y)$, the follower never invests, and the leader's value function at time θ^0 corresponds to $F(X^{L1}(\theta^0))$ (cf. (12)). If $\hat{\tau}(y) < \theta^0$, the leader's value function corresponds to $L^2(X^{L1}(\hat{\tau}(y)), Y(\hat{\tau}(y)))$, at the follower's entry time, $\hat{\tau}(y)$. Thus, to a pair of $(C(\cdot), \pi(\cdot))$, we associate the objective function

$$\begin{aligned} J\big(C(\cdot), \pi(\cdot)\big) = E\bigg[\int_0^{\hat{\tau}(y)\wedge\theta^0} & e^{-\mu t} U(C(t))dt + F\big(X^{L1}(\theta^0)\big)e^{-\mu\theta^0}\mathbf{1}_{\theta^0\leq\hat{\tau}(y)} \\ & + L^2\big(X^{L1}\big(\hat{\tau}(y)\big), Y\big(\hat{\tau}(y)\big)\big)e^{-\mu\hat{\tau}(y)}\mathbf{1}_{\hat{\tau}(y)<\theta^0}\bigg], \end{aligned} \tag{37}$$

and we consider the value function:

$$L^1(x, y) = \sup_{\{\pi(\cdot), C(\cdot)\}\in\mathscr{U}^1_{x,y}} J\big(C(\cdot), \pi(\cdot)\big), \tag{38}$$

where

$$\mathscr{U}^1_{x,y} = \{(\pi, C): \ I^1; \ \tau^* = \lim \uparrow \tau^1_N \geq \hat{\tau}(y) \wedge \theta^0 \text{ a.s.}\}$$

with I^1 and τ^1_N defined in (7) and (8) by replacing X^1 with X^{L1} respectively. We associate the value function with the Bellman equation:

$$\begin{cases} -\mu L^1 + \frac{\partial L^1}{\partial x}(rx + \delta_1 y) + \frac{\partial L^1}{\partial y}\alpha y + \frac{1}{2}\frac{\partial^2 L^1}{\partial y^2}\varsigma^2 y^2 + \sup_C\left(U(C) - C\frac{\partial L^1}{\partial x}\right) \\ \quad + \sup_\pi\left[\pi x\sigma\left(\lambda\frac{\partial L^1}{\partial x} + \varsigma\rho y\frac{\partial^2 L^1}{\partial x\partial y}\right) + \frac{1}{2}\pi^2 x^2\sigma^2\frac{\partial^2 L^1}{\partial x^2}\right] = 0\,, \\ L^1(x, \hat{y}) = L^2(x, \hat{y})\,. \end{cases} \tag{39}$$

We study the Bellman equation (39) for $y \in]0, \hat{y}[$ and we define:

$$L^1(x, y) = L^2(x, y), \quad \text{if} \quad y > \hat{y},$$

where $L^2(x, y)$ is defined in (36). The extension is continuous but not C^1. Also, we note that for $y = 0$, then $Y(t) = 0$ for all t, the problem then reduces to the

[4]See footnote 1 for the definition of M_ε.

[5]Here $Y_y(t)$ is the process defined in (2).

classical investment-consumption portfolio optimization problem; thus we have the boundary condition:

$$L^1(x,0) = F(x). \tag{40}$$

We look for a solution of the form

$$L^1(x,y) = -\frac{1}{r\gamma} e^{-r\gamma\left(x+q(y)\right)+1-\frac{\mu+\frac{\lambda^2}{2}}{r}} \tag{41}$$

with q solving the problem

$$\begin{cases} \frac{1}{2}y^2\varsigma^2 q'' + (\alpha - \lambda\varsigma\rho)yq' - \frac{1}{2}r^2\gamma^2 y^2\varsigma^2(1-\rho^2)q'^2 - rq + \delta_1 y = 0, \\ \quad 0 < y < \hat{y}, \\ q(0) = 0, \quad q(\hat{y}) = f(\hat{y}). \end{cases} \tag{42}$$

where $f(y)$ is the solution of (16), (15) on the interval $[0, \varepsilon y + M_\varepsilon]$. We extend $q(y)$ by $f(y)$ for $y > \hat{y}$.

The function $L^1(x,y)$ is continuous but not C^1. The study of (42) is similar to (16), but it is simpler because it is defined on a bounded interval. Similar to the study of (16), we can show that $q(y)$ may be interpreted as a function of a control problem, and there exists a unique solution which is $C^2(0,\hat{y})$. For $\delta_1 > \delta_2$, we have:

$$q(y) \geq f(y)\,. \tag{43}$$

Theorem 4 *The function $L^1(x,y)$ defined by (41) coincides with the value function given in (38).*

3.1.1 The Leader's Pre-investment Utility Maximization

We now turn to the leader's optimal stopping problem (i.e., choice of θ) with obstacle defined by $L^1(x,y)$, the solution to the postinvestment utility maximization. Before the stopping time θ, the leader's wealth and the cashflow process evolve as (5) and (2) respectively.

At time $\theta \wedge \theta^0$, the leader stops. If $\theta^0 \leq \theta$, the leader never takes the investment and receives $F(X^0(\theta^0))$ (cf. (12)). If $\theta < \theta^0$, the leader receives

$$L^1(X^0(\theta) - K, Y(\theta))$$

(cf. (41)) at θ. Therefore, the objective function is

$$J_{x,y}\big(C(\cdot),\pi(\cdot),\theta\big) = \mathrm{E}\bigg[\int_0^{\theta\wedge\theta^0} U\big(C(t)\big)e^{-\mu t}dt + L^1\big(X^0(\theta) - K, Y(\theta)\big)e^{-\mu\theta}\mathbf{1}_{\theta<\theta^0} + F\big(X^0(\theta^0)\big)e^{-\mu\theta^0}\mathbf{1}_{\theta^0\leq\theta}\bigg]. \tag{44}$$

We define the value function

$$L(x, y) = \sup_{\{\pi(\cdot), C(\cdot), \theta\} \in \mathscr{U}^0_{x,y}} J_{x,y}\big(C(\cdot), \pi(\cdot), \theta\big), \tag{45}$$

where

$$\mathscr{U}^0_{x,y} = \{(C, \pi, \theta) : \ I^0; \ \theta \wedge \theta^0 < \infty \text{ a.s.}; \ \tau^* = \lim \uparrow \tau^0_N \geq \theta \wedge \theta^0 \text{ a.s.}\}$$

with I^0 and τ^0_N defined in (7) and (8) respectively. As a consequence of Dynamic Programming, assuming sufficient smoothness of L, we can associate the strong formulation of V.I. to the value function $L(x, y)$ as:

$$\begin{cases} -\mu L + rx\frac{\partial L}{\partial x} + \alpha y\frac{\partial L}{\partial y} + \frac{1}{2}\varsigma^2 y^2 \frac{\partial^2 L}{\partial y^2} + \sup_c\left(U(C) - C\frac{\partial L}{\partial x}\right) \\ \quad + \sup_\pi\left[\pi\sigma x\left(\lambda\frac{\partial L}{\partial x} + \varsigma\rho y\frac{\partial^2 L}{\partial x\partial y}\right) + \frac{1}{2}\pi^2 x^2\sigma^2\frac{\partial^2 L}{\partial x^2}\right] \leq 0, \\ L(x, y) \geq L^1(x - K, y), \\ \left(L(x, y) - L^1(x - K, y)\right)\left[-\mu L + rx\frac{\partial L}{\partial x} + \alpha y\frac{\partial L}{\partial y} + \frac{1}{2}\varsigma^2 y^2\frac{\partial^2 L}{\partial y^2}\right. \\ \quad \left. + \sup_C\left(U(C) - C\frac{\partial L}{\partial x}\right) + \sup_\pi\left[\pi\sigma x\left(\lambda\frac{\partial L}{\partial x} + \varsigma\rho y\frac{\partial^2 L}{\partial x\partial y}\right) + \frac{1}{2}\pi^2x^2\sigma^2\frac{\partial^2 L}{\partial x^2}\right]\right] = 0. \end{cases} \tag{46}$$

We have the boundary condition:

$$L(x, 0) = F(x). \tag{47}$$

We look for a solution of the form

$$L(x, y) = -\frac{1}{r\gamma}e^{-r\gamma\left(x + h(y)\right) + 1 - \frac{\mu + \frac{\lambda^2}{2}}{r}}, \tag{48}$$

with $h(y)$ satisfying the following V.I.:

$$\begin{cases} \frac{1}{2}y^2\varsigma^2 h'' + h'y(\alpha - \lambda\varsigma\rho) - \frac{1}{2}y^2\varsigma^2 r\gamma(1 - \rho^2)h'^2 - rh \leq 0\,, \\ h(y) \geq q(y) - K, \\ \left(h(y) - q(y) + K\right)\left[\frac{1}{2}y^2\varsigma^2 h'' + h'y(\alpha - \lambda\varsigma\rho) - \frac{1}{2}y^2\varsigma^2 r\gamma(1 - \rho^2)h'^2 - rh\right] \\ \quad = 0, \\ h(0) = 0. \end{cases} \tag{49}$$

We encounter a new difficulty that does not occur in the follower's problem. We observe that the leader's obstacle $q(y) - K$ is C^0 but not C^1. We cannot as in (25) consider $u(y) = h(y) - q(y) + K$ since $q(y)$ is not sufficiently smooth. We will consider nonetheless the function

$$u(y) = h(y) - f(y) + K$$

which satisfies the following problem:

$$\begin{cases} -\frac{1}{2}y^2\varsigma^2u'' - y\big(\alpha - \lambda\varsigma\rho - y\varsigma^2 r\gamma(1-\rho^2)f'\big)u' + \frac{1}{2}\varsigma^2 r\gamma y^2(1-\rho^2)u'^2 + ru \\ \quad \geq -\delta_2 y + rK, \\ u \geq m, \\ (u-m)\big[-\frac{1}{2}y^2\varsigma^2u'' - y\big(\alpha - \lambda\varsigma\rho - y\varsigma^2 r\gamma(1-\rho^2)f'\big)u' \\ \quad + \frac{1}{2}\varsigma^2 r\gamma y^2(1-\rho^2)u'^2 + ru + \delta_2 y - rK\big] = 0, \\ u(0) = K. \end{cases} \tag{50}$$

In (50), the function $m = q(y) - f(y)$ is the solution of the problem

$$\begin{cases} -\frac{1}{2}y^2\varsigma^2m'' - y\big(\alpha - \lambda\varsigma\rho - y\varsigma^2 r\gamma(1-\rho^2)f'\big)m' + \frac{1}{2}\varsigma^2 r\gamma y^2(1-\rho^2)m'^2 + ry \\ \quad = (\delta_1 - \delta_2)y, \quad 0 < y < \hat{y}, \\ m(0) = m(\hat{y}) = 0, \end{cases} \tag{51}$$

and $m(y)$ is extended by 0 for $y > \hat{y}$. The function m is continuous but its derivative is discontinuous at $\hat{y}$. The difficulty is that one cannot interpret $u(y)$ as the value function of a control problem. Instead, it is, more appropriately, the value function of a stochastic differential game.

Theorem 5 *We assume* $\frac{r-\alpha+\lambda\varsigma\rho}{\gamma\varsigma^2(1-\rho^2)} > \delta_1\hat{y}$. *There exists a unique* $u(y) \in C^1(0,\infty)$, *piecewise* C^2, *solving* (50). *This function vanishes for* y *sufficiently large. Moreover, it is the value function given by*

$$u(y) = \inf_{v(\cdot)}\sup_{\theta} J_y\big(v(\cdot),\theta\big) \tag{52}$$

with the controlled diffusion and objective function given by

$$\begin{cases} dY_y(t) = Y_y(t)\big(\alpha - \lambda\varsigma\rho - \varsigma^2 r\gamma(1-\rho^2)Y_y(t)f'(Y_y(t)) \\ \qquad\qquad + v(t)\varsigma\sqrt{r\gamma(1-\rho^2)}\big)dt + \varsigma Y_y(t)dW(t), \\ Y_y(0) = y, \\ J_y\big(v(\cdot),\theta\big) = \mathrm{E}\big[\int_0^{\theta^0\wedge\theta}\big(-\delta_2 Y_y(t) + rK + \frac{1}{2}v^2(t)\big)e^{-rt}dt + Ke^{-r\theta^0}\mathbf{1}_{\theta^0<\theta} \\ \qquad\qquad + m\big(Y_y(\theta)\big)e^{-r\theta}\mathbf{1}_{\theta<\theta^0}\big], \end{cases} \tag{53}$$

where $\theta^0 = \inf\{t :\ Y_y(t) = 0\}$, *and* m *is the solution of* (51) *extended by zero for* $y > \hat{y}$.[6]

We next state that the solution of (50) is characterized by two intervals.

[6]For $v(t)$, we can take the same class as in problem (17)–(18).

Theorem 6 *The solution $u(y)$ of (50) is of the form*

$$-\frac{1}{2}y^2\varsigma^2u'' - y\big(\alpha - \lambda\varsigma\rho - y\varsigma^2 r\gamma(1-\rho^2)f'\big)u' + \frac{1}{2}\varsigma^2 r\gamma y^2(1-\rho^2)u'^2 + ru$$
$$= -\delta_2 y + rK, \quad 0 < y < y_1 \quad \textit{and} \quad y_2 < y < y_3, \tag{54}$$

with the value matching and smooth pasting conditions:

$$\begin{cases} u(y_1) = m(y_1), & u'(y_1) = m'(y_1), \\ u(y_2) = m(y_2), & u'(y_2) = m'(y_2), \\ u(y_3) = 0, & u'(y_3) = 0, \end{cases} \tag{55}$$

where $m(y) = g(y) - f(y)$, the solution of (51) and extended by 0 for $y > \hat{y}$. There exists a unique triple $y_1,\ y_2,\ y_3$ with $0 < y_1 < y_2 < \hat{y} < y_3$ such that (54), (55) hold.

Proof We know that u, the solution to (50), vanishes for $\bar{y} > \hat{y}$, $\bar{y}$ sufficiently large. Since $u(0) > m(0)$ and $u(\bar{y}) = m(\bar{y}) = 0$, there exists a first point $y_1 < \bar{y}$ such that $u(y_1) = m(y_1)$. We must have $y_1 < \hat{y}$. Otherwise, $y_1 = \bar{y}$ and u coincides with the solution of (25), i.e., the same system (50) with $m = 0$. But then $\bar{y} = \hat{y}$, hence $y_1 = \hat{y}$. In this case, $\tilde{u} = u - m$ satisfies the equation

$$-\frac{1}{2}y^2\varsigma^2\tilde{u}'' - y\big(\alpha - \lambda\varsigma\rho - y(f' + m')\varsigma^2 r\gamma(1-\rho^2)\big)\tilde{u}'$$
$$+ \frac{1}{2}y^2\varsigma^2 r\gamma(1-\rho^2)(\tilde{u}')^2 + r\tilde{u} = -\delta_1 y + rK \tag{56}$$

with the boundary conditions

$$\tilde{u}(0) = K, \qquad \tilde{u}(\hat{y}) = 0,$$

and since $u'(\hat{y}) = 0$, $\tilde{u}'(\hat{y} - 0) = -m'(\hat{y} - 0)$ which implies $\tilde{u}'(\hat{y} - 0) > 0$. It follows that $\tilde{u}(y) < 0$ for y close to $\hat{y}$, which is impossible since it must be positive. Therefore, $y_1 < \hat{y}$. We claim also that $\delta_1 y_1 \geq rK$. Indeed, set $\tilde{u}(y) = u(y) - m(y)$, then it satisfies (56) with the boundary conditions

$$\tilde{u}(0) = K, \qquad \tilde{u}(y_1) = 0, \qquad \tilde{u}'(y_1) = 0. \tag{57}$$

The matching of the derivatives comes from the fact that $\tilde{u}(y)$ is C^1 and $\tilde{u}(y) > 0$, $\tilde{u}(y_1) = 0$. So y_1 is a local minimum, hence $\tilde{u}'(y_1) = 0$.

Suppose $\delta_1 y_1 < rK$, then using (56), we see that $\tilde{u}''(y_1 - 0) < 0$; hence, $\tilde{u}(y) < 0$ for $y < y_1$, close to y_1. This is impossible.

Since $u(\hat{y}) > m(\hat{y}) = 0$, there exists an interval in which $\hat{y}$ is contained and such that the equation holds on this interval. One of the extremities of this interval is $y_3 = \bar{y}$. Call y_2 the other extremity, such that $u(y_2) = m(y_2)$. Therefore, $y_1 \leq y_2 < \hat{y}$. Necessarily, $y_2 > y_1$. Otherwise, u will be the solution of the equation on $(0, y_3)$, which is the case studied at the beginning of the proof, which is impossible. But then we have $u(y_2) = m(y_2),\ u'(y_2) = m'(y_2)$.

On the other hand, on the interval (y_1, y_2), m satisfies (51) and the right-hand side $(\delta_1 - \delta_2)y > -\delta_2 y + rK$, since $\delta_1 y > rK$, by virtue of $y > y_1$ and $\delta_1 y_1 > rK$.

Thus, m satisfies all conditions on $(y_1,\ y_2)$. Therefore, $u = m$ on $(y_1,\ y_2)$. By the uniqueness of u (Theorem 5), the triple $y_1,\ y_2,\ y_3$ is necessarily unique. □

We note the property that $u(y) \geq -f(y) + K$, which implies $h(y) \geq 0$. It remains to show that $L(x, y)$ defined by (48) is the value function (45).

Theorem 7 *The function $L(x, y)$ defined by* (48) *coincides with the value function* (44).

3.2 *Leader's Optimal Stopping Rule*

The optimal stopping rule for the leader is defined as:

$$\hat{\theta}(y) = \begin{cases} \inf\{t:\ Y_y(t) \geq y_1\}, & \text{if } 0 < y < y_1, \\ 0, & \text{if } y_1 \leq y \leq y_2, \\ \inf\{t:\ Y_y(t) \leq y_2 \text{ or } Y_y(t) \geq y_3\}, & \text{if } y_2 < y < y_3, \\ 0, & \text{if } y \geq y_3, \end{cases} \tag{58}$$

where $Y_y(t)$ is the process defined in (2).

4 Conclusion

We study a problem similar to the one presented in Bensoussan et al. [2]. Although we consider the investment payoffs governed by a geometric Brownian motion dynamics like the lump-sum payoff case in Bensoussan et al. [2], we do not encounter additional regularity issues encountered in the lump-sum payoff case, which results from indifference consideration for overcoming the comparison of gains and losses at different times in the incomplete markets. On the contrary, we are able to characterize a two-interval solution for the leader's optimal investment rule as the arithmetic Brownian motion cashflow payoff case presented in Bensoussan et al. [2]. The choice of a geometric Brownian motion cashflow process is motivated by the specification of an uncertain payoff arising from a stochastic demand process for the project's output, common in the financial economics literature (see, for example, Dixit and Pindyck [3] and Grenadier [4]). We note that to study cashflow process in terms of a geometric Brownian motion process rather than an arithmetic Brownian motion process invokes additional nontrivial mathematical consideration. Comparing with the arithmetic Brownian motion cashflow payoff case, the current study requires additional absorbing barrier consideration as well as an additional intermediate study of non-linear 2nd order differential equation, which turns out to be a solution to a minimization problem.

The economic interpretation of the leader's two-interval solution for the Stackelberg game is interesting. Below the lower threshold, neither player will invest because the output value is too low. Above the upper threshold, both players invest as soon as possible because output value is very high. Around the middle threshold, output value is attractive to the follower, who invests as soon as possible. As a result the leader will have little or no time to exploit their monopoly position in the output market. Since output value is below the upper threshold, the leader prefers to invest at a lower threshold value, thus decreasing the follower's interest. This allows the leader to maintain a monopoly position in the output market for a longer time. This result, understandable but not necessarily intuitive, can be revealed only through the mathematics of the V.I.

Acknowledgement The first author acknowledges support of National Science Foundation DMS-1303775 and of the Research Grants Council of HKSAR (CityU 500113).

References

1. Bensoussan, A.: Applications of Variational Inequalities in Stochastic Control. Elsevier/North-Holland, Amsterdam (1978)
2. Bensoussan, A., Diltz, J.D., Hoe, S.: Real options games in complete and incomplete markets with several decision makers. SIAM J. Financ. Math. **1**(1), 666–728 (2010)
3. Dixit, A., Pindyck, R.S.: Investment Under Uncertainty. Princeton University Press, Princeton (1994)
4. Grenadier, S.: The strategic exercise of options: development cascades and overbuilding in real estate markets. J. Finance **51**(5), 1653–1679 (1996)
5. Kinderlenher, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, San Diego (1980)

http://www.springer.com/978-3-319-02068-6

Inspired by Finance
The Musiela Festschrift
Kabanov, Y.; Rutkowski, M.; Zariphopoulou, T. (Eds.)
2014, XXIII, 543 p. 33 illus., 14 illus. in color., Hardcover
ISBN: 978-3-319-02068-6