Preface

This monograph grew out of my Ph.D. thesis, which I wrote at Bielefeld University between 2008 and 2012. Its main objective is the analysis of numerical methods, which approximate solutions to stochastic evolution equations (SEEq) on infinite dimensional state spaces. I therefore invite the reader to join me at the meeting point of several fascinating mathematical fields: numerical analysis, probability theory and stochastic analysis, PDE theory, and functional analysis.

For me the study of numerical methods always has the convenient advantage that I can investigate my object of interest not only theoretically but also during computer experiments. In moments of doubts, a numerical simulation may motivate me to look harder for a theoretical proof or it helps avoid wasting time on a dead end if I am on the wrong track. Computer experiments also serve as a constant well of inspiration by making me stumbling over unexpected phenomena, whose understanding often raises interesting theoretical questions.

To this respect numerical methods for SEEqs are a very grateful study object. Even without changing the initial parameters one may be able to see many different kinds behaviors just by the presence of the random forcing terms. A very interesting task is then to puzzle out which phenomena are due to the underlying stochastic evolution equation and which stems from the application of a numerical method and their inevitable errors.

For this it is of course necessary to first have a closer look at stochastic evolution equations and their properties. As this book is mainly written from the viewpoint of a numerical analyst, we concentrate on basic questions which are important for the design and the behavior of numerical schemes. In particular, we study existence, uniqueness, and spatio-temporal regularity of solutions to SEEqs.

After having made ourselves familiar with the object which we want to approximate, we next have to think about the aim of the approximation. Shall our numerical scheme give a good pathwise approximation of the solution or shall it just closely reproduce certain statistical properties? The two possibilities are related to the two different kinds of errors which we study in this monograph.

On the one hand, there is the so-called *strong error of convergence*, which is measured in terms of the L^p-norm for random variables. This error is small if

for a given initial state and a given sample path of the forcing random noise the corresponding trajectories of the exact solution and the numerical scheme are close to each other.

On the other hand, the *weak error of convergence* is already small if the laws of the exact solution and the numerical scheme almost coincide. Usually, a strongly convergent scheme is also weakly convergent and the strong order of convergence is a lower bound for the weak order of convergence. However, a well-known rule of thumb states that for many numerical schemes the weak order is actually twice the strong order of convergence.

While the analysis of the strong error often relies on similar techniques as for the deterministic problem, the weak error analysis is often much more demanding and usually builds upon the solution to Kolmogorov's equation associated with the SEEq. Consequently, the theory concerning the weak error analysis is by far not so mature and much more restrictive as for the strong error.

To some degree the same holds true for this monograph: While the strong error analysis is carried out for more general SEEqs with multiplicative noise, we only consider linear SEEqs with additive noise in the discussion of the weak error. But for the latter, my main intention is to introduce the reader to the beautiful theory of the Malliavin Calculus. I hope that the introduction to this topic is particularly accessible to readers who have already gained some familiarity with Wiener processes and appreciate a more functional analytical approach as in the standard literature.

The Malliavin Calculus is then used to open a new path to the weak error analysis, which completely bypasses Kolmogorov's equation. This new approach enjoys several advantages as, for example, the possibility to consider the weak error for SEEqs with non-autonomous stochastic coefficients. But concerning the treatment of equations with multiplicative noise much more research needs to be done.

I tried to do my best in order to keep this monograph mostly self-contained. However, I did not succeed everywhere: At the very least the reader should have a basic knowledge of standard techniques in functional analysis. It is also helpful if the reader is familiar with numerical methods for finite dimensional stochastic differential equations. More advanced topics such as Wiener processes on Hilbert spaces, strongly continuous semigroups, and Galerkin finite elements are provided without proof.

Notation

In order not to hide easily understandable ideas behind a complicated notation, I tried to keep everything as standard and simple as possible. However, since these notes are written in the intersection of several different mathematical fields, some collisions were inevitable.

For example, in stochastic analysis it is very common to denote the time evaluation of a stochastic process by an index, that is $X(t) = X_t$. However, this habit collides with the notation in numerical analysis, where the spatial discretization of a function u is usually denoted by u_h. From my point of view, it generates much less confusion if I stick to the classical notation in analysis to write $X(t)$ for the time evaluation of a stochastic process and I do not change the notation for the numerical methods to something nonstandard. Please accept my apologies in advance if this is the cause of new confusion which I was not able to foresee.

Further, throughout this book I apply the concept of *generic constants*. Usually denoted by the letter C, I understand by a generic constant a positive and finite entity, which may vary from occurrence to occurrence and depends on exogenous parameters such as the final time T, the Lipschitz constants of the nonlinearities f and g, or the norm of the initial value X_0. However, C is always chosen in a way that it is independent of endogenous parameters of the numerical method such as the spatial and temporal step sizes h and k.

By this means, perhaps interesting relationships between the exogenous parameters may remain hidden and it is unclear, in which way the constant C behaves, if exogenous parameters are varied. On the other hand, the derivation of explicit representations of all constants would have complicated many proofs and possibly distracted the reader from the underlying ideas and techniques.

Otherwise, new notation is usually explained at first appearance or found in the symbol list.

Acknowledgments

This text would not have been possible without the invaluable support and guidance of so many people.

First of all, I would like to mention my Ph.D. supervisor Prof. W.-J. Beyn (Bielefeld University). I profited very much from his experience and his ability to ask the right questions. He always gave me the freedom to follow my own ideas, but he also did a very good job in preventing me from getting lost in the process.

I also would like to express my sincere gratitude to Prof. S. Larsson (Chalmers University of Technology), who was my patient guide into the finite element world. But perhaps even more importantly, he broadened my horizons in many aspects of life, not just mathematics.

Another constant source of inspiration was Prof. M. Röckner (Bielefeld University), who taught me the fundamentals of stochastic analysis and always knew a precise reference to someone who already solved the problems I just stumbled over.

There are many more colleagues whom I would like to thank for their helpful and interesting discussions, advice, and support. In particular, I mention Prof. A. Jentzen (ETH Zürich), who gave many helpful comments to several aspects of these notes. Many of my results proceed on the basis of his work. Further, I would like to

mention A. Andersson and F. Lindgren (both Chalmers University of Technology) who inspired my interest in Malliavin Calculus and weak convergence for SPDEs.

I also say thank you to Dr. S. Cox, D. Holtz and Dr. T. Hüls for helping me to track down an uncountable number of typos. Without their help the manuscript would be much worse as it is now. The same is true for D. Otten who supported me in my teaching duties, especially during the last weeks of writing.

Most importantly, I would like to thank my girlfriend Christine, my family, and all my friends and colleagues for their encouragement, backing, and understanding, in particular during the time when I was too consumed by my own work. I hope to get the chance to return the favor.

Last but not least, I would like to thank an anonymous referee, whose comments and insights were very helpful in improving [50], and consequently, also the results in Sects. 2.5 and 2.6. His comments also strongly helped to improve the classification of these results into the existing literature in Sect. 1.1.

Most of the research in Sects. 2.5 and 2.6 and Chap. 3 has been carried out during an extended stay at Chalmers University of Technology, Gothenburg, Sweden. I would like to thank my host Prof. S. Larsson and his working group for the generous hospitality and the inspiring environment they created for me. In this matter, my thanks also go to Prof. W.-J. Beyn and the German Academic Research Service (DAAD) for making this wonderful stay possible.

Since my Ph.D. thesis was financially supported by the CRC 701 "Spectral Structures and Topological Methods in Mathematics" and the DFG-IGK 1132 "Stochastics and Real World Models," the same holds true for this monograph.

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August 2013

http://www.springer.com/978-3-319-02230-7

Strong and Weak Approximation of Semilinear Stochastic Evolution Equations

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2014, XIV, 177 p. 4 illus., Softcover
ISBN: 978-3-319-02230-7