Chapter 1
Introduction

In this chapter we familiarize ourselves with semilinear stochastic evolution equations, which are the starting point of this monograph. Then we present our main results and give a glimpse into the most important methods and techniques which are used in this text.

Let H be a separable Hilbert space and let the linear operator $A \colon \mathrm{dom}(A) \subset H \to H$ be positive definite, self-adjoint with compact inverse. For simplicity, we assume throughout this chapter that H consists of all square-integrable real-valued mappings on the unit interval, that is $H = L^2([0,1], \mathscr{B}([0,1]), \mathrm{d}x; \mathbb{R})$, and that $-A$ denotes the Laplace operator on $[0,1]$ with Dirichlet boundary conditions. Consequently, $-A$ is the generator of an analytic semigroup of contractions $(E(t))_{t\in[0,T]}$.

By $(\Omega, \mathscr{F}, \mathbf{P})$ we denote a complete probability space with filtration $(\mathscr{F}_t)_{t\in[0,T]}$. Then, we consider a semilinear stochastic evolution equation of the form

$$\begin{aligned} \mathrm{d}X(t) + \big[AX(t) + f(t, X(t))\big]\,\mathrm{d}t &= g(t, X(t))\,\mathrm{d}W(t), \quad \text{for } 0 \le t \le T, \\ X(0) &= X_0. \end{aligned} \tag{1.1}$$

Here, the solution $X \colon [0,T] \times \Omega \to H$ is a Hilbert space valued stochastic process, W denotes an H-valued Wiener process and the nonlinearities $f \colon [0,T] \times \Omega \times H \to H$ and $g \colon [0,T] \times \Omega \times H \to L(H)$ satisfy certain Lipschitz conditions. In order to keep this chapter as focused on our main results as possible, all details concerning the assumptions on f, g, W and X_0 are deferred to Sect. 2.3.

We treat (1.1) in terms of the semigroup approach as it is found in the book [18, Chap. 7] by Da Prato and Zabczyk. That is, we understand X as a *mild solution* (see Definition 2.18) and we also apply this methodology to our numerical approximation schemes.

Further, let a nonlinear functional $\varphi \colon H \to \mathbb{R}$ be given, which we always assume to be as smooth as needed. Our primary task, which we want to accomplish in this book, is the numerical approximation of the real number

$$\mathbf{E}\big[\varphi(X(T))\big]. \tag{1.2}$$

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In this monograph we omit all questions concerning the computation of the expectation by Monte Carlo methods and we focus on the discretization of the solution $X(T)$ itself. For a more computational treatment of Monte Carlo algorithm we refer to [29] and references therein.

If one wants to write a computer program which computes an approximation of (1.2), by the discrete and finite nature of the computer's memory, it is inevitable to represent the solution X on finite partitions in space and time. In this monograph we are therefore concerned with discretizations of the time interval $[0, T]$ and the Hilbert space H.

In addition, for the implementation of the Wiener noise W one also needs to consider a discretization of its covariance operator. However, this question is not addressed in this book and we refer to [3, Sect. 4] and the references therein.

As it is common in numerical analysis of deterministic partial differential equations, we first analyze the so called spatially semidiscrete approximation of (1.1), that is, we only discretize with respect to the Hilbert space H. For this we apply the well-established theory of Galerkin finite element methods from [66]. This theory has the advantage that it is suitable to treat different Galerkin methods in an unified setting.

In particular, let us stress that the theory of Galerkin finite element methods does not use any explicit knowledge of the eigenfunctions and eigenvalues of the operator A. This important feature is inherited by our results which are therefore applicable in a variety of situations.

To be more precise, let us denote by $(S_h)_{h\in(0,1]}$ a family of finite dimensional subspaces of H by $(S_h)_{h\in(0,1]}$, where we assume that S_h consists of spatially regular functions. The parameter $h \in (0,1]$ governs the granularity of the spatial approximation, that is we expect to obtain a better approximation of elements in H if h gets smaller. In our example with $H = L^2([0,1];\mathbb{R})$, S_h may be a standard finite element space or the linear span of finitely many eigenfunctions of A (see Examples 3.6 and 3.7).

The spatially semidiscrete approximation $X_h\colon [0,T] \times \Omega \to S_h$ of the mild solution to (1.1) is given by the stochastic evolution equation

$$\begin{aligned} \mathrm{d}X_h(t) + [A_h X_h(t) + P_h f(t, X_h(t))]\,\mathrm{d}t &= P_h g(t, X_h(t))\,\mathrm{d}W(t), \\ &\quad \text{for } 0 \le t \le T, \\ X_h(0) &= P_h X_0, \end{aligned} \tag{1.3}$$

where P_h denotes the orthogonal projector onto S_h and $A_h\colon S_h \to S_h$ is a discrete version of the operator A which will be defined in Sect. 3.2.

Next, we also introduce a spatio-temporal discretization of the stochastic evolution equation (1.1). Let $k \in (0,1]$ denote a fixed time step which defines a time grid $t_j = jk$, $j = 0, 1, \ldots, N_k$, with $N_k k \le T < (N_k + 1)k$.

Further, by X_h^j we denote the approximation of the mild solution X at time t_j. We combine the Galerkin methods with a linearly implicit Euler–Maruyama scheme and obtain the recursion, for $j = 1, \ldots, N_k$,

$$\begin{aligned} X_h^j - X_h^{j-1} + k\big(A_h X_h^j + P_h f(t_{j-1}, X_h^{j-1})\big) &= P_h g(t_{j-1}, X_h^{j-1})\Delta W^j, \\ X_h^0 &= P_h X_0, \end{aligned} \tag{1.4}$$

with the Wiener increments $\Delta W^j := W(t_j) - W(t_{j-1})$ which are $\mathscr{F}_{t_j}$-adapted, H-valued random variables. Consequently, X_h^j is itself an $\mathscr{F}_{t_j}$-adapted random variable which takes values in S_h.

Coming back to our computational task, it is therefore important to analyze the error

$$\big|\mathbf{E}\big[\varphi(X(T))\big] - \mathbf{E}\big[\varphi(X_h^{N_k})\big]\big|, \tag{1.5}$$

where the real-valued function $\varphi\colon H \to \mathbb{R}$ varies over a sufficiently large set of smooth test functions.

The error in (1.5) is called *weak error* and we say that the numerical approximation $X_h^{N_k}$ converges weakly to $X(T)$ for $h, k \to 0$, if the weak error vanishes for sufficiently many φ. In particular, if the weak error is supposed to be small, this forces the distribution of $X_h^{N_k}$ to be close to the distribution of $X(T)$.

While the approximation of the real number in (1.2) clearly corresponds to the weak convergence of the approximation method, the *strong error* plays an important role as well in this book. One reason for this is that it holds true by the mean-value theorem for Fréchet differentiable mappings that

$$\begin{aligned} &\big|\mathbf{E}\big[\varphi(X(T)) - \varphi(X_h^{N_k})\big]\big| \\ &\quad = \Big| \int_0^1 \mathbf{E}\big[\big(\varphi'\big(X_h^{N_k} + s(X(T) - X_h^{N_k})\big), X(T) - X_h^{N_k}\big)\big]\,\mathrm{d}s \Big| \\ &\quad \le C\big(1 + \|X(T)\|_{L^p(\Omega;H)}^{p-1} + \|X_h^{N_k}\|_{L^p(\Omega;H)}^{p-1}\big)\big\|X(T) - X_h^{N_k}\big\|_{L^p(\Omega;H)}, \end{aligned} \tag{1.6}$$

where we applied Hölder's inequality under the assumption that φ' satisfies a polynomial growth condition of the form $\|\varphi'(x)\| \le C(1 + \|x\|^{p-1})$. Here, the so called *strong error*

$$\big\|X(T) - X_h^{N_k}\big\|_{L^p(\Omega;H)} \tag{1.7}$$

is measured with respect to the norm in $L^p(\Omega; H)$. Altogether, this shows that every strongly convergent approximation is also weakly convergent and the order of convergence of the strong error is always a lower bound for the order of the weak error.

Further, a strongly convergent scheme ensures that important features of single sample paths of the solution are reproduced by its approximation. In particular, strong convergence also implies the pathwise convergence analyzed in [15, 42, 43].

Finally, as it was shown by Giles [27, 28], the strong order of convergence is also essential for developing efficient multilevel Monte Carlo methods for applications where the weak approximation is of interest in the first place. SPDE related references are [4, 5].

The following three sections give a more detailed overview of the results covered in this monograph. The first is concerned with the spatial and temporal regularity of the mild solution X to (1.1) and presents the main idea behind the technique used in Sects. 2.5 and 2.6. In the remaining two sections we sketch our strong and weak convergence results.

1.1 Optimal Regularity Results

Before we introduce our discretization schemes it is important to develop a deeper understanding of the properties of X itself. We will do this in all details in the Sects. 2.3–2.6. Of particular importance for the error analysis of numerical approximations is the spatial and temporal regularity of the mild solution, since this constitutes a natural bound on the order of convergence. Therefore, in order to obtain an optimal result on the order of convergence it is crucial to have a precise knowledge of the regularity.

The spatial and temporal regularity of the mild solution to (1.1) has already been extensively studied in the literature. For example, we refer to [18] for results on the stochastic convolution. Often, these results are derived under the assumption that $-A$ in (1.1) is the generator of an analytic semigroup and that the nonlinearity g satisfies a global Lipschitz assumption on H. In this situation it is shown that the mild solution takes values in $\dot{H}^{1-\epsilon}$ for every $\epsilon \in (0,1]$, where $\dot{H}^s := \mathrm{dom}(A^{\frac{s}{2}})$, $s \geq 0$, denotes the domain of fractional powers of the operator A (see Appendix B.2).

Under our additional assumption, that A is self-adjoint, positive definite and with compact inverse, it is also well-known, for example in [17, 18, Chap. 6.3] and [67], that the ϵ can be removed from the spatial regularity result and that the mild solution takes indeed values in $\dot{H}^1 = \mathrm{dom}(A^{\frac{1}{2}})$ with probability one.

On the other hand, in the recent paper [41] the authors work in the more general situation, where $-A$ is only assumed to be the generator of an analytic semigroup. But, in addition to the global Lipschitz condition, they impose a linear growth bound on the nonlinearity $g\colon H \to L(H)$ of the form $\|A^{\frac{r}{2}} g(x)\|_{L_2^0} \leq C(1 + \|A^{\frac{r}{2}} x\|)$ for some $r \in [0,1)$ and all $x \in \dot{H}^r = \mathrm{dom}(A^{\frac{r}{2}})$, where $\|\cdot\|_{L_2^0}$ denotes the Hilbert–Schmidt operator norm (see (2.10) and Assumption 2.20). In this case, the authors show that the mild solution takes values in $\dot{H}^{1+r-\epsilon}$ for every $\epsilon \in (0, 1+r]$ with probability one.

While a corresponding global Lipschitz condition on g as a mapping on $\dot{H}^r$ may force g to be affine linear, the linear growth bound is also satisfied by several Nemytskii operators as it is shown in [41, Sect. 4]. Therefore, the linear growth bound has proven itself to be useful in order to obtain sharper estimates of the spatial regularity for semilinear stochastic evolution equations.

In our regularity analysis in Sects. 2.5 and 2.6 we work under both additional assumptions, that is we study the linear growth bound from [41] under the assumption that A is self-adjoint and positive definite with compact inverse. As our first truly new result, we show in Sects. 2.5 and 2.6 that the ϵ can also be removed in this situation. Hence, $X(t)$ takes values in $\dot{H}^{1+r}$ with probability one (see Theorem 2.27).

The two main ingredients of our regularity analysis are sharp integral versions of the smoothing property of the semigroup (see Lemma B.9*(iii)*) which are explicitly tailored to estimate the regularity of the convolution. Secondly, we employ the following technique in order to estimate the norms of stochastic convolutions. Let $p \geq 2$. After applying a Burkholder–Davis–Gundy-type inequality (see Proposition 2.12) we obtain

$$\begin{aligned}&\Big\| A^{\frac{1+r}{2}} \int_0^t E(t-\sigma) g(\sigma, X(\sigma))\, \mathrm{d}W(\sigma) \Big\|_{L^p(\Omega;H)} \\ &\quad \leq C \Big(\mathbf{E}\Big[\Big(\int_0^t \big\| A^{\frac{1+r}{2}} E(t-\sigma) g(\sigma, X(\sigma)) \big\|_{L_2^0}^2 \, \mathrm{d}\sigma \Big)^{\frac{p}{2}} \Big] \Big)^{\frac{1}{p}}.\end{aligned}$$

At this point Lemma B.9*(iii)* is not directly applicable since $g(\sigma, X(\sigma))$ is obviously not independent of σ. However, since

$$\begin{aligned}&\Big(\mathbf{E}\Big[\Big(\int_0^t \big\| A^{\frac{1+r}{2}} E(t-\sigma) g(\sigma, X(\sigma)) \big\|_{L_2^0}^2 \, \mathrm{d}\sigma \Big)^{\frac{p}{2}} \Big] \Big)^{\frac{1}{p}} \\ &\quad \leq \Big(\mathbf{E}\Big[\Big(\int_0^t \big\| A^{\frac{1+r}{2}} E(t-\sigma) \big(g(\sigma, X(\sigma)) - g(t, X(t)) \big) \big\|_{L_2^0}^2 \, \mathrm{d}\sigma \Big)^{\frac{p}{2}} \Big] \Big)^{\frac{1}{p}} \\ &\qquad + \Big(\mathbf{E}\Big[\Big(\int_0^t \big\| A^{\frac{1+r}{2}} E(t-\sigma) g(t, X(t)) \big\|_{L_2^0}^2 \, \mathrm{d}\sigma \Big)^{\frac{p}{2}} \Big] \Big)^{\frac{1}{p}}\end{aligned}$$

Lemma B.9*(iii)* is applicable to the second summand and the first can by estimated by Lemma B.9*(i)* together with the Hölder continuity of the mapping $\sigma \mapsto g(\sigma, X(\sigma))$. For details we refer to the proof of Lemma 2.29.

Our technique is already known in the literature for Bochner integrals consisting of a convolution with an analytic semigroup, for example in [19, Prop. 3] and [65, p. 157]. However, to the best of our knowledge, this technique appeared for the first time in [50] in order to derive regularity estimates of stochastic convolutions.

In this book, we slightly extend the results of [50] to nonlinearities f and g which are allowed to depend on $t \in [0, T]$ and $\omega \in \Omega$.

1.2 Optimal Strong Error Estimates

After we have established the regularity estimates, our second main result is concerned with the optimal order of convergence of the strong error for the spatially semidiscrete approximation (1.3) and the spatio-temporal discretization

(1.4) of (1.1). For the last 15 years this has been a very active field of research and for an extensive list of references we refer to the review article [38].

We basically work under the same assumptions as for the results on optimal regularity in Sects. 2.5 and 2.6. That is, there exists a parameter $r \in [0,1)$ which, in particular, controls the spatial regularity of the mild solution X such that $X(t) \in \dot{H}^{1+r}$ with probability one for all $t \in [0,T]$.

In this situation, we prove in Sect. 3.4 that the spatially semidiscrete approximation (1.3) is strongly convergent and for all $p \geq 2$ it holds for a constant C, which is independent of h, that

$$\|X_h(t) - X(t)\|_{L^p(\Omega;H)} \leq C h^{1+r}, \text{ for all } t \in (0,T].$$

Therefore, in our example of a standard finite element semidiscretization, the approximation X_h converges with order $1+r$ to the mild solution X. Since this rate coincides with the spatial regularity of X it is called optimal (see [66, Chap. 1]).

We stress that, to the best of our knowledge, in all articles, which deal with the numerical approximation of semilinear stochastic partial differential equations, the obtained order of convergence is of the suboptimal form $1+r-\epsilon$ for any $\epsilon > 0$ (see [70] or [34], where also stronger Lipschitz assumptions have been imposed on f, g) or the error estimates contain a logarithmic term of the form $\log(t/h)$ as in [44].

A similar result (see Theorem 3.14) also holds for the spatio-temporal discretization (1.4) of X, namely, there exists a constant C, independent of $k, h \in (0,1]$, such that

$$\|X_h^j - X(t_j)\|_{L^p(\Omega;H)} \leq C(h^{1+r} + k^{\frac{1}{2}}), \quad \text{for all } j = 1, \dots, N_k.$$

Again, we obtain the optimal order of convergence with respect to the spatio-temporal discretization.

It is a well-known fact [13], that any one-step method, which only uses the information of the driving Wiener process provided by the increments $\Delta W^j = W(t_j) - W(t_{j-1})$, has the maximum strong order of convergence $\frac{1}{2}$ with respect to the time step k. It is possible to overcome this barrier by considering Itô–Taylor approximations as in [39]. In particular, Milstein-like schemes are discussed in the recent papers [3, 40, 48, 51].

In principle, these optimal error estimates are derived by a similar technique as sketched above for the optimal regularity results. Indeed, in Sects. 3.3 and 3.5 we present several lemmas which play the same crucial role as Lemma B.9*(iii)* and *(iv)* in the proof of the optimal spatial regularity. All these lemmas are concerned with the spatial semidiscretization and the fully discrete approximation of the deterministic homogeneous equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) + Au(t) = 0, \quad t > 0, \quad \text{with } u(0) = x. \tag{1.8}$$

In particular, these lemmas are tailored to estimate the integral of the square-norm of the discretization error, as it appears in estimates of stochastic integrals. For the fully discrete approximation these lemmas are to some extend new and have first appeared in [49].

1.3 A New Approach to Weak Convergence

In the context of SPDEs, the study of the weak error of convergence has gained broad attention only recently and the literature is much less extensive compared to the analysis of the strong error. Without giving a complete list, important contributions were achieved in [21, 22, 26, 35, 45, 46] and the references therein.

So far, the usual approach in the literature to analyze the weak error involves the study of the solution to the *Kolmogorov equation* associated to the stochastic evolution equation. Unlike the finite dimensional case, which leads to a deterministic partial differential equation on a finite dimensional spatial domain, the Kolmogorov equation associated to a stochastic evolution equation on a Hilbert space H is in fact a PDE on the infinite dimensional space H, see for example [63, 64] and the references therein.

In particular for semilinear stochastic evolution equations, the analysis of its Kolmogorov equation turns out to be involved. Since the derivation of growth bounds for various derivatives of the solution to the Kolmogorov equation is crucial in the weak error analysis, one usually needs to invoke assumptions on f and g which are much more restrictive compared to the analysis of the strong error.

For example, in [21] the author assumes that $f: H \to H$, $g: H \to L(H)$ are deterministic, time independent and three times continuously Fréchet differentiable with bounded derivatives. Under a further, more technically motivated assumption on g, namely that

$$\|g''(x)[y, y]\|_{L(H)} \leq C\, \|A^{-\frac{1}{4}} y\|^2 \quad \text{for all } x, y \in H,$$

the author proves that a temporal semidiscrete approximation based on the implicit Euler scheme converges weakly with order $\frac{1}{2} - \epsilon$, $\epsilon > 0$ arbitrary small, to the mild solution of (1.1). For this, the Wiener process is assumed to be a cylindrical white noise with $Q = \mathrm{Id}_H$ and, besides the Kolmogorov equation, the author also applies techniques from the Malliavin calculus in order to obtain this result in [21].

Note that in the same situation, but under much less severe regularity assumptions on f and g, the strong error is known to converge with order $\frac{1}{4} - \epsilon$, for every $\epsilon > 0$ arbitrary small.

In these notes we propose a new approach to analyze the weak error of convergence, which solely builds on methods from the Malliavin calculus and completely avoids the solution to the Kolmogorov equation. By these means we hope to obtain weak convergence results with optimal order in much more general situations, in particular if f and g are also allowed to depend on $\omega \in \Omega$.

However, so far we are only able to apply our ansatz to linear stochastic evolution equations with additive noise, that is

$$\begin{aligned} \mathrm{d}X(t) + \big[AX(t) + f(t)\big]\,\mathrm{d}t &= g(t)\,\mathrm{d}W(t), \qquad \text{for } 0 \le t \le T,\\ X(0) &= X_0. \end{aligned} \tag{1.9}$$

This equation is understood in the same sense as (1.1), but now the inhomogeneities $f\colon [0,T]\times\Omega \to H$ and $g\colon [0,T]\times\Omega \to L(H)$ are stochastic processes which do not directly depend on $X(t)$. Our more detailed assumptions are given in Sect. 5.2.

The starting point of our ansatz is again (1.6), which this time is considered for the semidiscrete approximation $X_h(T)$. Just before we applied the Hölder inequality in (1.6) we have

$$\begin{aligned} &\big|\mathbf{E}\big[\varphi(X_h(T)) - \varphi(X(T))\big]\big|\\ &\quad = \Big|\int_0^1 \mathbf{E}\big[\big(\varphi'\big(X(T) + s(X_h(T) - X(T))\big), X_h(T) - X(T)\big)\big]\,\mathrm{d}s\Big|. \end{aligned} \tag{1.10}$$

Here we insert the variation of constants formula for $X(T)$ and $X_h(T)$, which read

$$X(T) = E(T)X_0 - \int_0^T E(T-\sigma)f(\sigma)\,\mathrm{d}\sigma + \int_0^T E(T-\sigma)g(\sigma)\,\mathrm{d}W(\sigma)$$

and

$$X_h(T) = E_h(T)X_0 - \int_0^T E_h(T-\sigma)P_h f(\sigma)\,\mathrm{d}\sigma + \int_0^T E_h(T-\sigma)P_h g(\sigma)\,\mathrm{d}W(\sigma),$$

where $(E(t))_{t\in[0,T]}$ and $(E_h(t))_{t\in[0,T]}$ denote the strongly continuous semigroups generated by $-A$ and $-A_h$, respectively. Then, we get

$$\begin{aligned} &\mathbf{E}\big[\varphi(X_h(T)) - \varphi(X(T))\big]\\ &\quad = \int_0^1 \mathbf{E}\big[\big(\varphi'\big(X(T) + s(X_h(T) - X(T))\big), X_h(T) - X(T)\big)\big]\,\mathrm{d}s\\ &\quad = \int_0^1 \mathbf{E}\Big[\Big(\varphi'\big(X(T) + s(X_h(T) - X(T))\big), F_h(T)X_0\\ &\qquad\quad - \int_0^T F_h(T-\sigma)f(\sigma)\,\mathrm{d}\sigma\Big)\Big]\,\mathrm{d}s\\ &\qquad + \int_0^1 \mathbf{E}\Big[\Big(\varphi'\big(X(T) + s(X_h(T) - X(T))\big), \int_0^T F_h(T-\sigma)g(\sigma)\,\mathrm{d}W(\sigma)\Big)\Big]\,\mathrm{d}s, \end{aligned}$$

where $F_h(t) := E_h(t)P_h - E(t)$ denotes the error operator of the spatial discretization. If we apply Hölder's inequality to the first summand, we obtain the optimal order of weak convergence. However, the estimate of the second summand is more delicate and the Malliavin calculus comes into play. The idea is to apply Theorem 4.13, the so called *Bismut's integration by parts formula*, which yields

$$\mathbf{E}\Big[\Big(F, \int_0^T \Psi(t)\,\mathrm{d}W(t)\Big)\Big] = \mathbf{E}\big[\big(DF, \Psi\big)_{L^2([0,T];L_2^0)}\big]. \tag{1.11}$$

Here, $F\colon \Omega \to H$ is smooth in the sense of the Malliavin calculus and D denotes the Malliavin derivative. In Chap. 4 we review the most important properties of D and present a proof of (1.11).

We follow this idea in Sect. 5.3 and derive a representation formula of the weak error (see Theorems 5.9 and 5.10) which is then used to prove the weak order of convergence for the spatially semidiscrete approximation and the spatio-temporal discretization of $X(T)$. We validate the well-known rule of thumb for the weak approximation of SODEs, that under quite general assumptions on f and g the order of convergence of the weak error is almost twice the order of the strong error.

Let us remark that a similar error representation formula is given in [45], which in contrast to our formula contains derivatives of the solution to the Kolmogorov equation. However, both formulas yield the same error estimates if applied to linear stochastic partial differential equations such as the stochastic heat equation, the stochastic wave equation or the linearized stochastic Cahn–Hilliard equation as it is shown in [45].

1.4 Outline

In this section we give a short overview of the remaining parts of this book.

Chapter 2 is mainly concerned with the study of the stochastic evolution equation (1.1). In the first two sections we recall some useful facts on Wiener processes and stochastic Itô-integrals which take values in separable Hilbert spaces. The content of these sections is primarily based on [18, Chaps. 3 and 4] and [61, Chap. 2]. In Sect. 2.3 we describe all assumptions and details which are concerned with the semilinear stochastic evolution equation (1.1). This section also includes several examples.

In the remainder of this chapter we prove an existence and uniqueness result in Sect. 2.4, where our proof is a slightly generalized and simplified version of the proof of [41, Th. 1]. In Sects. 2.5 and 2.6 we present our optimal spatial and temporal regularity results. The final section of Chap. 2 contains some indications to possible generalizations of our results.

Chapter 3 is devoted to the analysis of the strong error. After some preliminaries in the first section, we provide a brief introduction into the Galerkin finite element methods. In particular, we introduce the discrete analogue of A and provide two more concrete examples of spatial discretizations.

Sections 3.3 and 3.4 are concerned with the spatially semidiscrete approximation of the stochastic evolution equation. First, we derive sharp integral versions of well-known convergence results from [66] and prove some extensions to non-smooth initial data. Then, we give the proof of our result on the optimal order of strong convergence for the spatial semidiscretization. The Sects. 3.5 and 3.6 follow the same path but are concerned with the strong convergence of the spatio-temporal discretization.

In Chap. 4 we give an overview of the fundamentals of the Malliavin Calculus for Hilbert space valued stochastic processes. This section is based on [32, 54]. As a nice application of Bismut's integration by parts formula, which is presented in Sect. 4.2, we give a short proof of the well-known stochastic Fubini theorem [18, Th. 4.18] in Sect. 4.3.

Chapter 5 deals with the weak order of convergence of the spatially semidiscrete approximation and spatio-temporal discretization of the linear stochastic evolution equation (1.9). After some preliminaries and the precise formulation of the assumptions, we first derive for both approximation methods a representation formula of the weak error in Sect. 5.3. In Sect. 5.4 we apply this formula in order to analyze the weak error for the inhomogeneous heat equation.

The final Chap. 6 contains four numerical experiments, where we illustrate the validity of our theoretical findings for the spatially semidiscrete approximation. In particular, our experiments consider the stochastic heat equation with and without inhomogeneities in Sects. 6.2 and 6.3. Further, in Sects. 6.4 and 6.5 we are concerned with the geometric Brownian motion in infinite dimensions, again with and without inhomogeneities.

The appendix contains some auxiliary results, where Appendix A is concerned with several variations of the classical Gronwall-Lemma. In Appendix B we collect several results on semigroups with unbounded infinitesimal generator. The perhaps most important results are the integral versions of the smoothing property in Lemma B.9, which have many consequences throughout this monograph. In the final appendix, we cite a generalized version of Lebesgue's dominated convergence theorem from [2, 1.23].

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