Chapter 4
Systems with Nondegenerate Characteristics

Abstract In this chapter we introduce the notion of nondegenerate multiple characteristics. Simple characteristics are nondegenerate characteristics of order 1. A double characteristic ρ of L is nondegenerate if and only if the rank of the Hessian at ρ of the determinant of $L(x, \xi)$ is maximal. We prove that every hyperbolic system which is close to a hyperbolic system with nondegenerate multiple characteristic has a nondegenerate characteristic of the same order nearby. This implies that hyperbolic systems with a nondegenerate multiple characteristic can not be approximated by strictly hyperbolic systems which contrasts with the case of scalar hyperbolic operators. We also prove that if every multiple characteristic of the system L is nondegenerate then there exists a smooth symmetrizer and hence the Cauchy problem for L is C^∞ well posed for any lower order term. Finally we discuss about the stability of symmetric systems in the space of hyperbolic systems.

4.1 Nondegenerate Characteristics

Let $P(x)$ be an $m \times m$ matrix valued smooth function defined near $\bar{x} \in \mathbb{R}^n$. We assume that $P(x)$ is a polynomial in x_1 so that

$$P(x) = \sum_{j=0}^{q} A_j(x') x_1^{q-j} \tag{4.1}$$

where $x' = (x_2, \ldots, x_n)$. We adapt the definitions of hyperbolicity and characteristics in Chap. 1 to $P(x)$.

Definition 4.1. We say $P(x)$ is hyperbolic near $\bar{x}$ with respect to $\theta = (1, 0, \ldots, 0) \in \mathbb{R}^n$ if $\det A_0(x') \neq 0$ near $x' = \bar{x}'$ and

$$\det P(x + \lambda\theta) = 0 \Longrightarrow \lambda \text{ is real} \tag{4.2}$$

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for any x near $\bar{x}$. We say that $\bar{x}$ is a characteristic of order r of $P(x)$ if

$$\partial_x^\alpha(\det P)(\bar{x}) = 0, \quad \forall|\alpha| < r, \quad \partial_x^\alpha(\det P)(\bar{x}) \neq 0, \quad \exists|\alpha| = r. \tag{4.3}$$

We now define nondegenerate characteristics. To do so we first define the localization of $P(x)$ at a characteristic.

Definition 4.2. Let $\bar{x}$ be a characteristic of $P(x)$ verifying

$$\mathrm{Ker} P(\bar{x}) \cap \mathrm{Im}\, P(\bar{x}) = \{0\}. \tag{4.4}$$

Set $\dim \mathrm{Ker} P(\bar{x}) = r$. Let $\{v_1, \ldots, v_r\}$ be a basis for $\mathrm{Ker} P(\bar{x})$ and let $\{\ell_1, \ldots, \ell_r\}$ be the dual basis vanishing on $\mathrm{Im}\, P(\bar{x})$, that is

$$\ell_i(\mathrm{Im}\, P(\bar{x})) = 0, \quad \ell_i(v_j) = \delta_{ij}$$

where δ_{ij} is the Kronecker's delta. Then we define the localization of P at $\bar{x}$, a linear transformation on $\mathrm{Ker}\, P(\bar{x})$, defined by a $r \times r$ matrix $P_{\bar{x}}(x)$ with respect to the basis $\{v_1, \ldots, v_r\}$

$$\big(\ell_i(P(\bar{x} + \mu x)v_j)\big)_{1 \le i,j \le r} = \mu[P_{\bar{x}}(x) + O(\mu)]. \tag{4.5}$$

Remark. Let $\{\tilde{v}_j\}$ be another basis for $\mathrm{Ker} P(\bar{x})$ where $\tilde{v}_j = \sum t_{kj} v_k$ with a non singular $r \times r$ matrix $T = (t_{ij})$ and let $\{\tilde{\ell}_i\}$ be the dual basis vanishing on $\mathrm{Im}\, P(\bar{x})$. Define $\tilde{P}_{\bar{x}}(x)$ by (4.5) with $\{\tilde{v}_j\}$ and $\{\tilde{\ell}_i\}$ then it is clear that $\tilde{P}_{\bar{x}}(x) = T^{-1} P_{\bar{x}}(x) T$ and hence $P_{\bar{x}}(x)$ is a well defined linear map on $\mathrm{Ker} P(\bar{x})$.

Let us denote

$$P_{\bar{x}} = \{P_{\bar{x}}(x) \mid x \in \mathbb{R}^n\} \subset M_r(\mathbb{C}) \tag{4.6}$$

which is a linear subspace of $M_r(\mathbb{C})$.

Definition 4.3. We call $\dim_{\mathbb{R}} P_{\bar{x}}$, the dimension of the linear subspace $\{P_{\bar{x}}(x) \mid x \in \mathbb{R}^n\}$ over $\mathbb{R}$, the real reduced dimension of $P_{\bar{x}}(x)$.

We first show

Lemma 4.1. *Let $T(x)$ be a smooth non singular $m \times m$ matrix near $\bar{x}$ and let $\tilde{P}(x) = T^{-1}(x)P(x)T(x)$. Then if $\bar{x}$ is a characteristic of order r of $P(x)$ verifying (4.4) then $\bar{x}$ is also a characteristic of order r of $\tilde{P}(x)$ verifying (4.4) and there is a non singular $r \times r$ matrix such that*

$$\tilde{P}_{\bar{x}} = T^{-1} P_{\bar{x}} T.$$

Proof. Since $\mathrm{Ker}\, \tilde{P}(\bar{x}) = T^{-1}(\mathrm{Ker}\, P(\bar{x}))$ and $\mathrm{Im}\, \tilde{P}(\bar{x}) = T^{-1}(\mathrm{Im}\, P(\bar{x}))$ with $T = T(\bar{x})$ it is easy to see

$$\tilde{\ell}_i\big(\tilde{P}(\bar{x}+\mu x)\tilde{v}_j\big) = \ell_i\big(P(\bar{x}+\mu x)v_j\big) + O(\mu^2)$$

where $\tilde{\ell}_i(\cdot) = \ell_i(T\cdot)$ and $\tilde{v}_j = T^{-1}v_j$. This proves the assertion. □

Lemma 4.2. *Assume that $P(x)$ is hyperbolic near $\bar{x}$. Let $\bar{x}$ be a characteristic verifying (4.4) with* $\dim \operatorname{Ker} P(\bar{x}) = r$. *Then we have*

$$\det P(\bar{x}+\mu x) = \mu^r\big(c \det P_{\bar{x}}(x) + O(\mu)\big) \tag{4.7}$$

with $c \neq 0$. Assume further that $\det P_{\bar{x}}(x) \not\equiv 0$ *then*

$$\det P_{\bar{x}}(\theta) \neq 0, \tag{4.8}$$

$$\det P_{\bar{x}}(x+\lambda\theta) = \det\big(P_{\bar{x}}(x) + \lambda P_{\bar{x}}(\theta)\big) = 0 \Longrightarrow \lambda \in \mathbb{R}, \forall x \in \mathbb{R}^n. \tag{4.9}$$

Proof. In view of (4.4) we can choose a non singular constant matrix T so that

$$T^{-1}P(\bar{x})T = \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}$$

where G is a non singular $(m-r)\times(m-r)$ matrix. With $\tilde{P}(x) = T^{-1}P(x)T$ we write

$$\tilde{P}(\bar{x}+\mu x) = \tilde{P}(\bar{x}) + \mu\hat{P}(x) + O(\mu^2).$$

Denoting

$$\hat{P}(x) = \begin{bmatrix} \hat{P}_{11}(x) & \hat{P}_{12}(x) \\ \hat{P}_{21}(x) & \hat{P}_{22}(x) \end{bmatrix}$$

it is clear $\tilde{P}_{\bar{x}}(x) = \hat{P}_{11}(x)$ which follows from the definition. Since $\det \tilde{P}_{\bar{x}} = \det P_{\bar{x}}$ by Lemma 4.1 we have

$$\det P_{\bar{x}}(x) = \det \hat{P}_{11}(x). \tag{4.10}$$

Note that

$$\det P(\bar{x}+\mu x) = \det \tilde{P}(\bar{x}+\mu x) = \mu^r\big(\det G \det \hat{P}_{11}(x) + O(\mu)\big) \tag{4.11}$$

which shows the first assertion. To prove the second assertion suppose $\det P_{\bar{x}}(\theta) = 0$ so that $\det P(\bar{x}+\mu\theta) = o(\mu^r)$ by (4.7). This implies that $(\partial/\partial x_1)^j \det P(\bar{x}) = 0$ for $j = 0,\dots,r$. Since $\det P(x)$ is hyperbolic in the sense (4.2) it follows from Lemma 1.9 that

$$\big(\partial_x^\alpha \det P\big)(\bar{x}) = 0, \quad \forall|\alpha| \le r.$$

This implies $\det P_{\bar{x}}(x) \equiv 0$ which is a contradiction. We turn to the third assertion. Since

$$\det P(\bar{x} + \mu(x + \lambda\theta)) = \mu^r \big(c \det P_{\bar{x}}(x + \lambda\theta) + O(\mu)\big)$$

if $\det P_{\bar{x}}(x + \lambda\theta) = 0$ has a non real root λ, then taking $\mu \neq 0$ sufficiently small the equation

$$c \det P_{\bar{x}}(x + \lambda\theta) + O(\mu) = 0$$

admits a non real root. This contradicts (4.2). □

Definition 4.4. Denote by $M_r^h(\mathbb{C})$ the set of all $r \times r$ Hermitian matrices and by $M_r^s(\mathbb{R})$ the set of all real $r \times r$ symmetric matrices. Then r^2 and $r(r+1)/2$ is the dimension of $M_r^h(\mathbb{C})$ and $M_r^s(\mathbb{R})$ over $\mathbb{R}$ respectively.

Definition 4.5. We say that $\bar{x}$ is a nondegenerate characteristic of order r of $P(x)$ if the following conditions are verified;

$$\mathrm{Ker} P(\bar{x}) \cap \mathrm{Im}\, P(\bar{x}) = \{0\}, \tag{4.12}$$

$$\dim_{\mathbb{R}} P_{\bar{x}} = r^2 = \dim_{\mathbb{R}} M_r^h(\mathbb{C}), \quad (r = \dim \mathrm{Ker}\, P(\bar{x})), \tag{4.13}$$

$$\det P_{\bar{x}}(\theta) \neq 0, \quad P_{\bar{x}}(\theta)^{-1} P_{\bar{x}}(x) \text{ is diagonalizable } \forall x \in \mathbb{R}^n. \tag{4.14}$$

When $P(x)$ is real valued then we say that $\bar{x}$ is a nondegenerate characteristic of order r if

$$\mathrm{Ker}\, P(\bar{x}) \cap \mathrm{Im}\, P(\bar{x}) = \{0\}, \tag{4.15}$$

$$\dim_{\mathbb{R}} P_{\bar{x}} = r(r+1)/2 = \dim_{\mathbb{R}} M_r^s(\mathbb{R}), \quad (r = \dim \mathrm{Ker}\, P(\bar{x})), \tag{4.16}$$

$$\det P_{\bar{x}}(\theta) \neq 0, \quad P_{\bar{x}}(\theta)^{-1} P_{\bar{x}}(x) \text{ is diagonalizable } \forall x \in \mathbb{R}^n. \tag{4.17}$$

Example 4.1. Simple characteristics verify (4.12)–(4.14) with $r = 1$ and hence a simple characteristic is a nondegenerate characteristic of order 1.

Example 4.2. Let $q = 1$ and $m = 2$ so that $P(x) = x_1 + A_1(x')$ where $A_1(x')$ is a real valued 2×2 matrix with $A_1(0) = O$. As we will see in the next section that if the rank of the Hessian of $\det P(x)$ at $x = 0$ is 3 then $x = 0$ is a nondegenerate characteristic of order 2.

Example 4.3. Let us consider

$$P(x) = \xi_1 I + \sum_{j=2}^{d} F_j \xi_j$$

where $\{I, F_2, \ldots, F_d\}$ span $M_m^s(\mathbb{R})$ and $d = m(m+1)/2$. Then every characteristic of P is nondegenerate. We check this. Let $\bar{\xi}$ be a characteristic of order r of $P(\xi)$ so that 0 is an eigenvalue of $P(\bar{\xi})$ of multiplicity r. Take an orthogonal matrix T such that

$$T^{-1}P(\bar{\xi})T = \begin{bmatrix} O & O \\ O & G \end{bmatrix}$$

where G is a $(m-r)\times(m-r)$ non singular matrix. Denoting

$$\tilde{P}(\xi) = T^{-1}P(\xi)T = (\varphi_{ij}(\xi))_{1\le i,j\le m}$$

we note that $\varphi_{ij}(\xi) = \varphi_{ji}(\xi)$ and $\varphi_{ij}(\xi)$, $i \le j$ are linearly independent. Writing

$$\tilde{P}(\xi) = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix}$$

it is clear that $\tilde{P}_{\bar{\xi}}(\xi) = \tilde{P}_{11}(\xi)$ and $\dim \tilde{P}_{\bar{\xi}} = r(r+1)/2$ because φ_{ij}, $i \le j$ are linearly independent. Since $\tilde{P}_{11}(\xi)$ is symmetric for every ξ then (4.14) is also obvious. Thus by Lemma 4.1 we conclude that $\bar{\xi}$ is a nondegenerate characteristic of P.

To study $P(x)$ we consider the following $mq \times mq$ matrix valued function

$$\mathscr{P}(x) = x_1 I + \begin{bmatrix} 0 & -I_m & & & \\ 0 & 0 & -I_m & & \\ & & & \ddots & \\ & & & & -I_m \\ A_q(x') & \cdots & & \cdots & A_1(x') \end{bmatrix} = x_1 I + \mathscr{A}(x')$$

where I and I_m are the $mq \times mq$ and $m \times m$ identity matrix respectively. It is clear that

$$\det\mathscr{P}(x) = \det P(x). \tag{4.18}$$

Then the condition (4.2) implies that all eigenvalues of $\mathscr{A}(x')$ are real, equivalently

$$\text{all eigenvalues of } \mathscr{P}(x) \text{ are real.} \tag{4.19}$$

In the rest of this section we prove

Proposition 4.1.1 *Let $\bar{x}$ be a nondegenerate characteristic of order r of $P(x)$. Then $\bar{x}$ is also a nondegenerate characteristic of order r of $\mathscr{P}(x)$ and vice versa.*

Proof. Assume that $\bar{x}$ is a nondegenerate characteristic of order r of $P(x)$ and show that $\bar{x}$ is also a nondegenerate characteristic of order r of $\mathscr{P}(x)$. We first check

$$\Big[\frac{\partial P}{\partial x_1}(\bar{x})\mathrm{Ker}P(\bar{x})\Big] \oplus \mathrm{Im}\ P(\bar{x}) = \mathbb{C}^m. \tag{4.20}$$

Let $\{v_1, \ldots, v_r\}$ be a basis for $\mathrm{Ker}P(\bar{x})$ and take $\{\ell_i\}$ so that $\ell_i(\mathrm{Im}\ P(\bar{x})) = 0$ and $\ell_i(v_j) = \delta_{ij}$. Then by definition we have

$$P_{\bar{x}}(x) = \left(\ell_i\big((\sum_{k=1}^{n} \frac{\partial P}{\partial x_k}(\bar{x})x_k)v_j\big)\right) = \sum_{k=1}^{n}\left(\ell_i\big((\frac{\partial P}{\partial x_k}(\bar{x}))v_j\big)\right)x_k$$

and hence

$$P_{\bar{x}}(\theta) = \left(\ell_i\big((\frac{\partial P}{\partial x_1}(\bar{x}))v_j\big)\right).$$

Then $\det P_{\bar{x}}(\theta) \neq 0$ implies that

$$\Big[\frac{\partial P}{\partial x_1}(\bar{x})\mathrm{Ker}P(\bar{x})\Big] \cap \mathrm{Im}\ P(\bar{x}) = \{0\}$$

and hence (4.20).

We note that

$$\mathrm{Ker}\mathscr{P}(x) = \{{}^t(u, x_1u, \ldots, x_1^{q-1}u) \mid u \in \mathrm{Ker}\ P(x)\}$$

and $\dim \mathrm{Ker}\mathscr{P}(\bar{x}) = r$. We next describe $\mathrm{Im}\ \mathscr{P}(x)$. Write

$$\phi_k(x) = \sum_{j=0}^{q-k} A_j(x')x_1^{q-j-k}$$

then it is easy to see that

$$\mathrm{Im}\ \mathscr{P}(x) = \{{}^t(w^{(1)}, \ldots, w^{(q-1)}, P(x)v - \sum_{k=1}^{q-1}\phi_k(x)w^{(k)}) \mid w^{(1)}, \ldots, w^{(q-1)}, v \in \mathbb{C}^m\}.$$

We now show that

$$\mathrm{Ker}\mathscr{P}(\bar{x}) \cap \mathrm{Im}\ \mathscr{P}(\bar{x}) = \{0\}. \tag{4.21}$$

Let ℓ be a linear form on $\mathbb{C}^{mq}$. Writing $v = {}^t(v^{(1)}, \ldots, v^{(q)}) \in \mathbb{C}^{mq}$ with $v^{(j)} \in \mathbb{C}^m$ one can write

$$\ell(v) = \sum_{j=1}^{q} \ell^{(j)}(v^{(j)})$$

where $\ell^{(j)}$ are linear forms on $\mathbb{C}^m$. Assume $\ell(\mathrm{Im}\,\mathscr{P}(\bar{x})) = 0$. This implies that

$$\ell^{(j)}(\cdot) = \ell^{(q)}(\phi_j(\bar{x})\cdot), \quad 1 \le j \le q-1, \quad \ell^{(q)}(\mathrm{Im}\,P(\bar{x})) = 0 \tag{4.22}$$

and then we have

$$\begin{aligned}\sum_{j=1}^{q} \ell^{(j)}(\bar{x}_1^{j-1}u) &= \sum_{j=1}^{q-1} \ell^{(q)}(\bar{x}_1^{j-1}\phi_j(\bar{x})u) + \ell^{(q)}(\bar{x}_1^{q-1}u) \\ &= \ell^{(q)}(\sum_{j=1}^{q-1} \bar{x}_1^{j-1}\phi_j(\bar{x})u + \bar{x}_1^{q-1}u) = 0.\end{aligned} \tag{4.23}$$

From this, noting the identity

$$\sum_{j=1}^{q-1} x_1^{j-1}\phi_j(x) + x_1^{q-1} = \frac{\partial P}{\partial x_1}(x)$$

one gets

$$\ell^{(q)}(\frac{\partial P}{\partial x_1}(\bar{x})u) = 0, \quad \forall u \in \mathrm{Ker}\,P(\bar{x}). \tag{4.24}$$

From (4.20) and (4.22) it follows that $\ell^{(q)} = 0$ and hence $\ell = 0$. This proves that

$$\mathrm{Ker}\,\mathscr{P}(\bar{x}) + \mathrm{Im}\,\mathscr{P}(\bar{x}) = \mathbb{C}^{mq} \tag{4.25}$$

and hence (4.21).

We next examine (4.13), (4.14) for $\mathscr{P}(x)$. Let $U = {}^t(u, \bar{x}_1 u, \ldots, \bar{x}_1^{q-1}u) \in \mathrm{Ker}\,\mathscr{P}(\bar{x})$ where $u \in \mathrm{Ker}\,P(\bar{x})$. Consider $\mathscr{P}(x)U$

$$\begin{aligned}\mathscr{P}(x)U &= {}^t((x_1-\bar{x}_1)u, (x_1-\bar{x}_1)\bar{x}_1 u, \ldots, (x_1-\bar{x}_1)\bar{x}_1^{q-2}u, v) \\ &= {}^t(w^{(1)}, w^{(2)}, \ldots, w^{(q-1)}, v)\end{aligned}$$

where the last component v is

$$\begin{aligned}v &= P(\bar{x}_1, x')u + (x_1\bar{x}_1^{q-1} - \bar{x}_1^q)u \\ &= P(x)u + [P(\bar{x}_1, x') - P(x_1, x')]u + \bar{x}_1^{q-1}(x_1-\bar{x}_1)u.\end{aligned}$$

Now it is easy to see that this is equal to

$$P(x)u - \sum_{k=1}^{q-1} \phi_k(\bar{x}) w^{(k)} + O((x_1 - \bar{x}_1)^2). \tag{4.26}$$

Let ℓ be a linear form on $\mathbb{C}^{mq}$ with $\ell(\mathrm{Im}\,\mathscr{P}(\bar{x})) = 0$. From (4.26) it follows that

$$\begin{aligned} \ell(\mathscr{P}(x)U) &= \sum_{j=1}^{q-1} \ell^{(j)}(w^{(j)}) + \ell^{(q)}\big(P(x)u \\ &\qquad - \sum_{k=1}^{q-1} \phi_k(\bar{x}) w^{(k)}\big) + O((x_1 - \bar{x}_1)^2) \\ &\qquad = \ell^{(q)}(P(x)u) + O((x_1 - \bar{x}_1)^2) \end{aligned} \tag{4.27}$$

by (4.22). Let us take $U_j = {}^t(u_j, \bar{x}_1 u_j, \ldots, \bar{x}_1^{q-1} u_j) \in \mathrm{Ker}\,\mathscr{P}(\bar{x})$ where $\{u_j\}$ is a basis for $\mathrm{Ker}\,P(\bar{x})$. Then one can write

$$\frac{\partial P}{\partial x_1}(\bar{x}) u_j - \sum_{k=1}^{r} a_{jk} u_k \in \mathrm{Im}\,P(\bar{x}),$$

thanks to (4.20) with a non singular $A = (a_{jk})$. Take $\tilde{\ell}_i$ so that

$$\tilde{\ell}_i(\mathrm{Im}\,P(\bar{x})) = 0, \quad \tilde{\ell}_i(u_j) = \delta_{ij}.$$

Let us take $\ell_i^{(q)}$

$$\ell_i^{(q)} = \sum_{k=1}^{r} b_{ik} \tilde{\ell}_k, \quad B = (b_{ik}) = {}^t A^{-1}$$

so that

$$\ell_i^{(q)}\Big(\frac{\partial P}{\partial x_1}(\bar{x}) u_j\Big) = \sum_{k=1}^{r} b_{ik} \sum_{p=1}^{r} a_{jp} \tilde{\ell}_k(u_p) = \delta_{ij}.$$

We now define linear forms ℓ_i on $\mathbb{C}^{mq}$ by

$$\ell_i(w^{(1)}, \ldots, w^{(q)}) = \sum_{t=1}^{q-1} \ell_i^{(q)}(\phi_t(\bar{x}) w^{(t)}) + \ell_i^{(q)}(w^{(q)})$$

then we have

$$\ell_i(\operatorname{Im}\mathscr{P}(\bar{x})) = 0, \quad \ell_i(U_j) = \delta_{ij} \tag{4.28}$$

as observed above. From (4.27) it follows that

$$\begin{aligned}\ell_i(\mathscr{P}(\bar{x}+\mu x)U_j) &= \ell_i^{(q)}(P(\bar{x}+\mu x)u_j) + O(\mu^2)\\ &= \sum_{k=1}^{r} b_{ik}\tilde{\ell}_k(P(\bar{x}+\mu x)u_j) + O(\mu^2)\\ &= \mu\big(BP_{\bar{x}}(x) + O(\mu)\big).\end{aligned}$$

Since $B = ({}^tA)^{-1} = P_{\bar{x}}(\theta)^{-1}$ we conclude that

$$\mathscr{P}_{\bar{x}}(x) = P_{\bar{x}}(\theta)^{-1}P_{\bar{x}}(x). \tag{4.29}$$

Since $\mathscr{P}_{\bar{x}}(\theta) = I$ then (4.13) and (4.14) for $\mathscr{P}_{\bar{x}}(x)$ follow immediately.

Conversely assume (4.25). Let $\ell^{(q)}$ be a linear form on $\mathbb{C}^m$ with $\ell^{(q)}(\operatorname{Im}P(\bar{x})) = 0$, $\ell^{(q)}(\operatorname{Ker}P(\bar{x})) = 0$ and define $\ell^{(j)}$, $1 \le j \le q-1$ by (4.22). Then we have $\ell(\operatorname{Im}\mathscr{P}(\bar{x})) = 0$ and moreover (4.22) shows $\ell(\operatorname{Ker}\mathscr{P}(\bar{x})) = 0$ and hence $\ell = 0$ by (4.25). Thus we have $\ell^{(q)} = 0$ which proves $\operatorname{Ker}P(\bar{x}) \oplus \operatorname{Im}P(\bar{x}) = \mathbb{C}^m$ and hence (4.12). To check (4.13), (4.14) for $P(x)$ we note that $\operatorname{Ker}\mathscr{P}(\bar{x}) \cap \operatorname{Im}\mathscr{P}(\bar{x}) = \{0\}$ implies that

$$u \in \operatorname{Ker}P(\bar{x}),\ \frac{\partial P}{\partial x_1}(\bar{x})u \in \operatorname{Im}P(\bar{x}) \Longrightarrow u = 0.$$

Hence we have (4.20) again and thus (4.29). Then the rest of the proof is clear. □

Remark. Assume that $q = 1$ and $A_1(x')$ is symmetric in (4.1). Then (4.12) and (4.14) are always verified.

Remark. By definition, the order of nondegenerate characteristics never exceed m, the size of the matrix whatever q is.

4.2 Nondegenerate Double Characteristics

Nondegenerate double characteristics have a special feature.

Lemma 4.3. *Let $\bar{x}$ be a double characteristic. Then $\bar{x}$ is nondegenerate if and only if* $\dim\operatorname{Ker}P(\bar{x}) = 2$ *and the rank of the Hessian of* $\det P(x)$ *at $\bar{x}$ is* 4. *When $P(x)$ is real valued then $\bar{x}$ is nondegenerate if and only if* $\dim\operatorname{Ker}P(\bar{x}) = 2$ *and the rank of the Hessian of* $\det P(x)$ *at $\bar{x}$ is* 3.

To prove the lemma we first note

Lemma 4.4. *Let* A_j *be* 2×2 *constant matrices with* $\operatorname{Tr} A_j = 0$, $1 \le j \le m$. *Assume that the quadratic form*

$$Q(x) = \det\Big(\sum_{j=1}^{m} A_j x_j\Big)$$

is real nonpositive definite in $\mathbb{R}^m$. *Then the rank of* $Q(x)$ *is at most* 3 *and if* $\operatorname{rank} Q = 3$ *then there is a constant matrix* N *such that*

$$N^{-1} A_j N$$

is an Hermitian matrix for all j. *If all* A_j *are real then* $\operatorname{rank} Q \le 2$ *and if* $\operatorname{rank} Q = 2$ *then there is a real constant matrix* N *such that all*

$$N^{-1} A_j N$$

are real symmetric.

Proof. With a non singular real matrix $T = (t_{ij})$ one can assume

$$Q(Tx) = \det\Big(\sum_{j=1}^{m} H_j x_j\Big) = -\sum_{j=1}^{k} x_j^2, \quad \operatorname{Tr} H_j = 0 \tag{4.30}$$

where $H_j = \sum_{i=1}^{m} t_{ji} A_i$ and $\operatorname{rank} Q = k$. If $k \le 2$ then nothing to be proved. Thus we assume $k \ge 3$. Since $\det H_1 = -1$, $\operatorname{Tr} H_1 = 0$, one can diagonalize H_1

$$H_1' = N_1^{-1} H_1 N_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Denoting $H_2' = N_1^{-1} H_2 N_1 = (h_{ij})$ and taking $x_j = 0$, $j \ge 3$ it follows from (4.30) that $h_{11} = h_{22} = 0$, $h_{12} h_{21} = 1$. Setting

$$N_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & h_{12} \end{bmatrix}$$

it follows that

$$N_2^{-1} H_1' N_2 = H_1', \quad N_2^{-1} H_2' N_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let us put $N = N_1 N_2$ and $N^{-1} H_j N = H_j' = (h_{pq}^{(j)})$, $j \geq 3$. Take $x_j = 0$ unless $j = 1, 3$ then we get $h_{11}^{(3)} = h_{22}^{(3)} = 0$, $h_{12}^{(3)} h_{21}^{(3)} = 1$ and taking $x_j = 0$ unless $j = 2$, 3 we get $h_{12}^{(3)} + h_{21}^{(3)} = 0$. Thus we conclude $h_{12}^{(3)} = \pm i$. The same procedure gives

$$H_j' = \epsilon_j \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad (\epsilon_j = 1 \text{ or } -1), \quad 3 \leq j \leq k.$$

Repeating similar arguments we obtain $H_j = O$ for $j > k$. We summarize

$$\begin{aligned} N^{-1}(\sum_{j=1}^{k} H_j x_j) N &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_2 \\ &\quad + \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} (\sum_{j=3}^{k} \epsilon_j x_j), \quad H_j = O, \quad j > k \end{aligned} \tag{4.31}$$

and from (4.30)

$$-\det(\sum_{j=1}^{k} H_j x_j) = x_1^2 + x_2^2 + (\sum_{j=3}^{k} \epsilon_j x_j)^2 = \sum_{j=1}^{k} x_j^2. \tag{4.32}$$

The identity (4.32) holds only if $k = 3$ and all $N^{-1} H_j N$ are Hermitian. Since T is real then $N^{-1} A_j N$ are also Hermitian. This proves the assertion. If all A_j are real, we can take N real and the proof is similar. □

Proof of Lemma 4.3. Take T so that

$$T^{-1} P(\bar{x}) T = \begin{bmatrix} A & O \\ O & G \end{bmatrix} \tag{4.33}$$

where G is a non singular matrix of order $m - 2$ and all eigenvalues of A are zero. Assume that $\dim \mathrm{Ker}\, P(\bar{x}) = 2$. Then it follows that $A = O$ and hence $\mathrm{Ker}\, P(\bar{x}) \cap \mathrm{Im}\, P(\bar{x}) = \{0\}$.

Assume that $\mathrm{rank\,Hess}_{\bar{x}} \det P = 4$ and hence $\det P_{\bar{x}}(x) \not\equiv 0$ by Lemma 4.2. From Lemma 4.2 again we have $\det P_{\bar{x}}(\theta) \neq 0$ and $P_{\bar{x}}(\theta)^{-1} P_{\bar{x}}(x)$ has only zero eigenvalues for every x. Then writing

$$\begin{aligned} P_{\bar{x}}(\theta)^{-1} P_{\bar{x}}(x) &= x_1 I_2 + \sum_{j=2}^{n} A_j x_j \\ &= \Big(x_1 - \frac{1}{2} \mathrm{Tr}\, (\sum_{j=2}^{n} A_j x_j)\Big) I_2 + \sum_{j=2}^{n} \tilde{A}_j x_j \end{aligned}$$

it follows that $\det(\sum_{j=2}^{n}\tilde{A}_j x_j)$ is a real nonpositive quadratic form on $\mathbb{R}^{n-1}$ of which rank is 3 since the rank of the real quadratic form $\det(P_{\bar{x}}(\theta)^{-1}P_{\bar{x}}(x))$ is 4. Note $\mathrm{Tr}\,\tilde{A}_j = 0$. From Lemma 4.4 there exists a constant 2×2 matrix T such that $T^{-1}\tilde{A}_j T$ is Hermitian for every j so that one can write

$$\begin{aligned} &T^{-1}(P_{\bar{x}}(\theta)^{-1}P_{\bar{x}}(x))T \\ &\quad = \phi_1(x)\begin{bmatrix}1&0\\0&1\end{bmatrix} + \phi_2(x)\begin{bmatrix}1&0\\0&-1\end{bmatrix} + \phi_3(x)\begin{bmatrix}0&1\\1&0\end{bmatrix} + \phi_4(x)\begin{bmatrix}0&i\\-i&0\end{bmatrix}\end{aligned} \tag{4.34}$$

with real linear forms $\phi_i(x)$ and obviously $P_{\bar{x}}(\theta)^{-1}P_{\bar{x}}(x)$ is diagonalizable for every x. Since $\phi_i(x)$, $i = 1,2,3,4$ are linearly independent it is clear that $\dim_{\mathbb{R}} P_{\bar{x}} = 4$.

Conversely we assume that a double characteristic $\bar{x}$ is nondegenerate. Take T so that (4.33) holds. From $\mathrm{Ker}P(\bar{x}) \cap \mathrm{Im}P(\bar{x}) = \{0\}$ it follows that $A = O$ and hence $\dim\mathrm{Ker}\,P(\bar{x}) = 2$. Assume $\dim P_{\bar{x}} = 4$ and $\det P_{\bar{x}}(\theta) \neq 0$. Let us write

$$P_{\bar{x}}(\theta)^{-1}P_{\bar{x}}(x) = (x_1 - \psi(x))I_2 + \sum_{j=2}^{4} A_j x_j$$

where $\mathrm{Tr}\,A_j = 0$ and $\{I_2, A_2, A_3, A_4\}$ are linearly independent by assumption. Since $P_{\bar{x}}(\theta)^{-1}P_{\bar{x}}(x)$ has only real eigenvalues for every x then $\det(\sum_{j=2}^{4} A_j x_j)$ is nonpositive definite so that one can write

$$\det(\sum_{j=2}^{4} A_j x_j) = -\sum_{i=1}^{k} \ell_j(x)^2$$

with linearly independent $\ell_j(x)$ where $k \le 3$ by Lemma 4.4. Assume that $\ell_i(x) = 0$, $i = 1,\dots,k$ then $\sum_{j=2}^{4} A_j x_j$ has only zero eigenvalues because $\mathrm{Tr}\,A_j = 0$. Since $\sum_{j=2}^{4} A_j x_j$ is diagonalizable by assumption then we conclude that $\sum_{j=2}^{4} A_j x_j = O$ so that

$$\sum_{j=2}^{4} A_j x_j = \sum_{i=1}^{k} H_j \ell_j(x)$$

which proves $k = 3$. Thus $\det(P_{\bar{x}}(\theta)^{-1}P_{\bar{x}}(x))$ has rank 4 and from Lemma 4.2 it follows that $\mathrm{rank}\,\mathrm{Hess}_{\bar{x}}\det P = 4$. This proves the assertion.

The case that $P(x)$ is real valued, the proof is just a repetition with obvious modifications. □

Proposition 4.1. *Let $m = 2$ and $q = 1$. Assume that $P(\bar{x}) = O$ and the rank of* $\mathrm{Hess}\det P$ *is* 4 *at* $\bar{x}$ (3 *if $P(x)$ is real valued*)*. Then $\Sigma = \{x \mid \partial_x^{\alpha}(\det P)(x) = 0,$*

$|\alpha| \leq 1\}$ *is a* C^∞ *manifold near* $\bar{x}$ *with* $\operatorname{codim}\Sigma = \operatorname{rank}\operatorname{Hess}_{\bar{x}} \det P$ *on which* $P(x) = O$.

In fact, in Sect. 4.5, we prove this proposition in much more generality (Proposition 4.3). The smoothness of the characteristic set is closely related to the existence of smooth symmetrizers (see [48]). Indeed we have

Proposition 4.2 ([17, 48]). *Let* $m = 2$ *and* $q = 1$. *Assume that* $P(\bar{x}) = O$ *and the rank of* $\operatorname{Hess}\det P$ *is* 4 *at* $\bar{x}$ *(*3 *if* $P(x)$ *is real valued). Then* $P(x)$ *has a smooth symmetrizer near* $\bar{x}$*, that is there is a smooth* 2×2 *matrix valued* $S(x')$ *defined near* $\bar{x}'$ *such that*

$$\begin{aligned} S^*(x') &= S(x') \ \text{ and } S(x') \text{ is positive definite,} \\ S(x')P(x) &= P^*(x)S(x') \end{aligned}$$

where $P^*(x)$ *denotes the adjoint matrix of* $P(x)$.

Example 4.4. Let us consider second order differential operator $P(D) = (p_{ik}(D))$ with 3×3 constant matrix coefficients

$$p_{ik}(\tau, \xi) = (\tau^2 - \sigma_i|\xi|^2)\delta_{ik} - (1 - \sigma_i)\xi_i\xi_k$$

which is called the modified elasticity operator in [25] where $\xi = (\xi_1, \xi_2, \xi_3)$ and

$$0 < \sigma_1 < \sigma_2 < \sigma_3 < 1.$$

Note that the excluded case where $\sigma_1 = \sigma_2 = \sigma_3$ yields the elasticity equations. We follow the arguments in John [25]. Let $Q(\tau, \xi) = \det P(\tau, \xi)$ then $Q(\tau, \xi)$ can be written

$$Q(\tau, \xi) = (\tau^2 - q_0(\xi))(\tau^4 - 2q_1(\xi)\tau^2 + q_0(\xi)q_2(\xi))$$

where q_0, q_1, q_2 are the definite quadratic forms given by

$$q_0 = |\xi|^2, \ \ q_2 = \sigma_1\sigma_2\sigma_3\sum_{j=1}^{3}\frac{1}{\sigma_j}\xi_j^2, \ \ q_1 = \frac{1}{2}(\sigma_1 + \sigma_2 + \sigma_3)|\xi|^2 - \frac{1}{2}\sum_{j=1}^{3}\sigma_j\xi_j^2.$$

Taking the homogeneity into account we consider multiple characteristics (τ, ξ) with $|\xi| = 1$. It is shown in [25] that (τ, ξ), $|\xi| = 1$ is a multiple characteristic if and only if

$$D(\xi) = 4(q_1^2 - q_0q_2) = 0, \quad |\xi| = 1$$

which gives 4 points

$$\pm(\beta_3/\beta_2, 0, \beta_1/\beta_2), \ \ \pm(-\beta_3/\beta_2, 0, \beta_1/\beta_2) \tag{4.35}$$

where $\beta_1 = (\sigma_3 - \sigma_2)^{1/2}$, $\beta_2 = (\sigma_3 - \sigma_1)^{1/2}$, $\beta_3 = (\sigma_2 - \sigma_1)^{1/2}$. If we set

$$D^*(\xi) = D(\xi) + \beta_1^2\beta_3^2(q_0(\xi) - 1)^2$$

then we have at a double characteristic (4.35) which we denote $\hat{\xi}$

$$\frac{\partial^2 D^*(\hat{\xi})}{\partial\xi_i \xi_k} = 8\beta_1^2\beta_3^2(\delta_{ik} - \frac{1}{2}(\delta_{i1}\delta_{k3} + \delta_{k1}\delta_{i3})\hat{\xi}_i\hat{\xi}_k)$$

and hence Hessian of D^* is positive definite. This shows that the Hessian of $D(\xi)$ has at least rank 2 and then the Hessian of $(\tau^2 - q_1)^2 + D(\xi)$ has rank 3 which proves that the double characteristic (4.35) are nondegenerate.

We find similar second order differential operators $P(D) = (p_{ik}(D))$ with 3×3 constant matrix coefficients in [64] in the studies of relativistic elastodynamics.

Example 4.5. We have

Theorem 4.1 ([22]). *In the set $\mathscr{P}$ of all positive definite real symmetric 3×3 matrix valued quadratic forms*

$$A(\xi) = \sum_{j,k=1}^{3} A_{jk}\xi_j\xi_k$$

the subset for which the characteristics of $\det(\tau^2 I - A(\xi))$ *are at most double and the double characteristics are nondegenerate is an open and dense subset.*

We have also

Theorem 4.2 ([22]). *One can choose a positive definite real symmetric 3×3 matrix valued quadratic form A such that the characteristics of* $\det(\tau^2 I - A(\xi))$ *are at most double, the double characteristics are nondegenerate, and there are at least* 12 *of them.*

4.3 Symmetrizability (Special Case)

We first note that, considering $-A_0(x')^{-1}P(x)$, we may assume that $P(1, 0, \ldots, 0) = -I_2$ so that

$$P(x) = -x_1 I_2 + A'(x'), \quad A'(x') \in C^\infty(\Omega, M_2(\mathbb{C}))$$

which is also written

$$P(x) = -(x_1 - \frac{1}{2}\mathrm{Tr}\, A'(x'))I_2 + A(x'),\ \mathrm{Tr}\, A(x') = 0.$$

Note that

$$g(x') = \det A(x') \leqq 0$$

and Tr $A'(x')$ is real which follows from the hyperbolicity of det $P(x)$. Let us denote

$$A(x') = \begin{bmatrix} a(x') & b(x') \\ c(x') & -a(x') \end{bmatrix}.$$

We denote by $da(x')$ the differential of a at $\bar{x}'$ so that $a(x'+\bar{x}') = da(x') + O(|x'-\bar{x}'|^2)$ and by Re a and Im a the real part and the imaginary part of a respectively.

We first assume that $P(x)$ is real valued and rank $\mathrm{Hess}_{\bar{x}}\det P = 3$. The assumption is reduced to rank $\mathrm{Hess}_{\bar{x}} g = 2$. From Proposition 4.1 it follows that $\Sigma' = \{x' \mid g(x') = 0\}$ is a smooth manifold of codimension 2. Then there are $\ell_i(x'), i = 1, 2$ such that $\Sigma' = \{\ell_1(x') = 0, \ell_2(x') = 0\}$ and

$$A(x') = H_1(x')\ell_1 + H_2(x')\ell_2, \quad g(x') = -\ell_1^2 - \ell_2^2$$

where $d\ell_i(x')$ are linearly independent. Let K_1 be the restriction of H_1 to $\ell_2 = 0$ then it is clear that det $K_1 = -1$ and Tr $K_1 = 0$. Hence there is a real 2×2 matrix $N(x')$ such that

$$N^{-1}K_1N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and then we have

$$N^{-1}AN = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}(\ell_1 + \alpha\ell_2) + \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix}\ell_2.$$

From the Taylor expansion of det $A(x')$ around $\bar{x}'$ it is easy to see that $\alpha(\bar{x}') = 0$, $\beta(\bar{x}')\gamma(\bar{x}') = 1$ and consequently the matrix

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1/\gamma(x') \end{bmatrix}$$

is well defined near $\bar{x}'$. Putting $T(x') = N(x')M(x')$ and writing $\beta(x')\gamma(x') = 1 + \psi$ we have

$$T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}(\ell_1 + \alpha\ell_2) + \begin{bmatrix} 0 & 1+\psi \\ 1 & 0 \end{bmatrix}\ell_2.$$

We now define S by

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1+\psi \end{bmatrix}.$$

Since $\psi(\bar{x}') = 0$ it is easy to see that S is a desired symmetrizer of $T^{-1}AT$. Since the symmetrizability is invariant under similar transformations we get the desired assertion.

We next prove the proposition assuming that $\operatorname{rank}\operatorname{Hess}_{\bar{x}}\det P = 4$. Since the hypothesis $\operatorname{rank}\operatorname{Hess}_{\bar{x}}\det P = 4$ reduces to $\operatorname{rank}\operatorname{Hess}_{\bar{x}} g = 3$ we may assume that

$$Q = (d\operatorname{Re} a)^2 - (d\operatorname{Im} a)^2 + (d\operatorname{Re} b)(d\operatorname{Re} c) - (d\operatorname{Im} b)(d\operatorname{Im} c)$$

is nonnegative definite and has rank 3. Here we note that a real quadratic form Q which is nonnegative definite can not vanish on a linear subspace V unless $\operatorname{codim} V \geqq \operatorname{rank} Q$. We first remark that $d\operatorname{Re} a \neq 0$. If it were not true we would have

$$\begin{aligned} 0 \ll Q &= -(d\operatorname{Im} a)^2 + (d\operatorname{Re} b)(d\operatorname{Re} c) - (d\operatorname{Im} b)(d\operatorname{Im} c) \\ &\ll (d\operatorname{Re} b)(d\operatorname{Re} c) - (d\operatorname{Im} b)(d\operatorname{Im} c). \end{aligned}$$

It is clear that there is a linear subspace V $(\subset \mathbb{R}^{n-1})$ with $\operatorname{codim} V \leqq 2$ on which Q vanishes and hence $\operatorname{rank} Q \leqq 2$. This contradicts the assumption.

Set $\varphi = \operatorname{Re} a$ and denote by $b|_{\varphi=0}$ the restriction of b to the surface $\{\varphi = 0\}$.

Lemma 4.5. *Let $b = \beta\varphi + \tilde{b}$, $c = \gamma\varphi + \tilde{c}$ with $\tilde{b} = b|_{\varphi=0} = \tilde{b}_1 + i\tilde{b}_2$, $\tilde{c} = c|_{\varphi=0} = \tilde{c}_1 + i\tilde{c}_2$ where $\tilde{b}_i, \tilde{c}_i$ are real. Then we have*

$$d\tilde{b}_i \neq 0, \;\; d\tilde{c}_i \neq 0 \textit{ at } \bar{x}', \; i = 1, 2.$$

Proof. Denoting $\operatorname{Im} a = \alpha\varphi + \tilde{\alpha}$ with $\tilde{\alpha} = a|_{\varphi=0}$ one can write

$$A(x') = \varphi \begin{bmatrix} (1+i\alpha) & \beta \\ \gamma & -(1+i\alpha) \end{bmatrix} + \begin{bmatrix} i\tilde{\alpha} & \tilde{b}_1 + i\tilde{b}_2 \\ \tilde{c}_1 + i\tilde{c}_2 & -i\tilde{\alpha} \end{bmatrix}.$$

From the non-positivity of g on $\{\varphi = 0\}$ it follows that

$$\tilde{b}_1\tilde{c}_1 - \tilde{b}_2\tilde{c}_2 - \tilde{\alpha}^2 \geqq 0, \tag{4.36}$$

$$\tilde{b}_1\tilde{c}_2 + \tilde{b}_2\tilde{c}_1 = 0 \tag{4.37}$$

near $\bar{x}'$. Suppose, for instance, that $d\tilde{b}_1(\bar{x}') = 0$ and hence $d\tilde{b}_2 = 0$ or $d\tilde{c}_1 = 0$ (at $\bar{x}'$) by (4.37). If $d\tilde{b}_2 = 0$ then $d\tilde{\alpha} = 0$ by (4.36) and then Q vanishes on $\{x' \mid d\varphi(x') = 0\}$ because $da = (1 + i\alpha)d\varphi$ at $\bar{x}'$. This is a contradiction. The other cases will be proved similarly. □

Lemma 4.6. *$d\tilde{b}_1$ is not proportional to $d\tilde{b}_2$ at $\bar{x}'$. There is a positive function $m(x')$ defined near $\bar{x}'$ such that*

$$\tilde{c}_1(x') = m(x')\tilde{b}_1(x'), \quad \tilde{c}_2(x') = -m(x')\tilde{b}_2(x').$$

Proof. Suppose that $d\tilde{b}_2 = kd\tilde{b}_1$ at $\bar{x}'$ with some $k \in \mathbb{R}$ and hence $d\tilde{c}_2 = -kd\tilde{c}_1$ by (4.37) at $\bar{x}'$. Since from (4.36) we see

$$d\tilde{b}_1 d\tilde{c}_1 - d\tilde{b}_2 d\tilde{c}_2 - d\tilde{\alpha} d\tilde{\alpha} = (1+k^2) d\tilde{b}_1 d\tilde{c}_1 - d\tilde{\alpha} d\tilde{\alpha} \gg 0,$$

and hence $d\tilde{b}_1$ and $d\tilde{c}_1$ must be proportional to $d\tilde{\alpha}$ at $\bar{x}'$ if $d\tilde{\alpha} \neq 0$. Then it is clear that Q vanishes on $\{x' \mid d\tilde{\alpha}(x') = d\varphi(x') = 0\}$ which is a contradiction. If $d\tilde{\alpha} = 0$ (at $\bar{x}'$) then Q vanishes on $\{x' \mid d\varphi(x') = d\tilde{c}_1(x') = 0\}$ which also gives a contradiction. This proves the first assertion. The second assertion easily follows from the first one and (4.36), (4.37). □

We can put A in a special form.

Lemma 4.7. *Let $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$, β_i, γ_i real. Set $\psi_i = \tilde{b}_i + \beta_i\varphi$ $(i = 1, 2)$, $B = \gamma_2 + m\beta_2$, $C = \gamma_1 - m\beta_1$. Then we have*

$$A = \varphi \begin{bmatrix} 1 & 0 \\ C + iB & -1 \end{bmatrix} + \psi_1 \begin{bmatrix} -iB/2 & 1 \\ m & iB/2 \end{bmatrix} + \psi_2 \begin{bmatrix} -iC/2 & i \\ -im & iC/2 \end{bmatrix}.$$

Moreover $d\varphi, d\psi_i$ are linearly independent at $\bar{x}'$ and the set $\{x' \mid A(x') = O\}$ is given by

$$S = \{x' \mid \varphi(x') = \psi_1(x') = \psi_2(x') = 0\}.$$

Proof. Recall that

$$A = \varphi \begin{bmatrix} 1 + i\alpha & \beta \\ \gamma & -(1+i\alpha) \end{bmatrix} + \begin{bmatrix} i\tilde{\alpha} & \tilde{b}_1 + i\tilde{b}_2 \\ m(\tilde{b}_1 - i\tilde{b}_2) & -i\tilde{\alpha} \end{bmatrix}.$$

We observe the imaginary part of g

$$\mathrm{Im}\, g = 2\alpha\varphi^2 + 2\tilde{\alpha}\varphi + \mathrm{Im}\,(\beta\gamma)\varphi^2 + \mathrm{Im}\,(\gamma + \beta m)\varphi\tilde{b}_1 + \mathrm{Re}\,(\gamma - \beta m)\varphi\tilde{b}_2.$$

Since $\mathrm{Im}\, g = 0$ near $\bar{x}'$ and $d\varphi \neq 0$ at $\bar{x}'$ it follows that

$$2\alpha\varphi + 2\tilde{\alpha} + \mathrm{Im}\,(\beta\gamma)\varphi + \mathrm{Im}\,(\gamma + \beta m)\tilde{b}_1 + \mathrm{Re}\,(\gamma - \beta m)\tilde{b}_2 = 0 \tag{4.38}$$

near $\bar{x}'$. Now we set

$$D = \mathrm{Im}\,(\beta\gamma), \quad B = \mathrm{Im}\,(\gamma + \beta m) = \gamma_2 + \beta_2 m,$$
$$C = \mathrm{Re}\,(\gamma - \beta m) = \gamma_1 - \beta_1 m.$$

Noticing $D = \beta_1 B + \beta_2 C$ it follows from (4.38) that

$$(\alpha\varphi + \tilde{\alpha}) = -\frac{1}{2}(\psi_1 B + \psi_2 C) \tag{4.39}$$

which shows that $a = (1 + i\alpha)\varphi + i\tilde{\alpha} = \varphi - i(\psi_1 B + \psi_2 C)/2$. On the other hand it is easy to see

$$m(\tilde{b}_1 - i\tilde{b}_2) + \gamma\varphi = (C + iB)\varphi + m(\psi_1 - i\psi_2), \quad \tilde{b}_1 + i\tilde{b}_2 + \beta\varphi = \psi_1 + i\psi_2$$

because $\gamma_1 = C + m\beta_1$ and $\gamma_2 = B - m\beta_2$. These prove the first part. The rest of the assertion is obvious. □

Lemma 4.8. *We have*

$$4m - (B^2 + C^2) > 0 \ \ at \ \ \bar{x}'.$$

Proof. Let us set $\tilde{B} = B|_{\varphi=0}$, $\tilde{C} = C|_{\varphi=0}$. From (4.39) it follows that

$$\tilde{\alpha} = -(\tilde{B}\tilde{b}_1 + \tilde{C}\tilde{b}_2)/2.$$

On the other hand (4.36) and Lemma 4.6 give that

$$m(\tilde{b}_1^2 + \tilde{b}_2^2) - \tilde{\alpha}^2 \geqq 0 \ \text{ near } \ \bar{x}'.$$

Since the quadratic form $m((d\tilde{b}_1)^2 + (d\tilde{b}_2)^2) - (\tilde{B}d\tilde{b}_1 + \tilde{C}d\tilde{b}_2)^2/4$ is the restriction of Q to $\{x' \mid d\varphi(x') = 0\}$ this must have rank 2 and then positive definite. This shows that $4m - (\tilde{B}^2 + \tilde{C}^2) > 0$ at $\bar{x}'$ and hence the result. □

To finish the proof of Proposition 4.2 we give a required smooth symmetrizer $S(x')$ for P by

$$S(x') = \begin{bmatrix} 2m(x') & -C(x') + iB(x') \\ -C(x') - iB(x') & 2 \end{bmatrix}$$

which satisfies $S(x') = S^*(x')$ clearly. Using Lemma 4.7 it is easy to check that $S(x')A(x') = A^*(x')S(x')$ and hence

$$S(x')P(x) = P^*(x)S(x').$$

The positivity of S follows from Lemma 4.8.

4.4 Stability and Smoothness of Nondegenerate Characteristics

In this section we discuss the stability of nondegenerate characteristics and the smoothness of nondegenerate characteristic set.

Theorem 4.3. *Assume that $P(x)$ is an $m \times m$ (resp. real) matrix valued smooth function of the form (4.1) verifying (4.2) in a neighborhood U of $\bar{x}$ and let $\bar{x}$ be a nondegenerate characteristic of order r of P. Let $\tilde{P}(x)$ be another $m \times m$ (resp. real) matrix valued smooth function of the form (4.1) verifying (4.2) which is sufficiently close to $P(x)$ in C^{q+2}, then $\tilde{P}(x)$ has a nondegenerate characteristic of the same order close to $\bar{x}$. Moreover, near $\bar{x}$, the characteristics of order r are nondegenerate and they form a smooth manifold of codimension r^2 (resp. $r(r+1)/2$). In particular, near $\bar{x}$ the set of characteristics of order r of $P(x)$ itself consists of nondegenerate ones which form a smooth manifold of codimension r^2(resp. $r(r+1)/2$).*

To prove Theorem 4.3, taking Proposition 4.1.1 into account, we study $P(x)$ of the form

$$P(x) = x_1 I + P^{\#}(x') \tag{4.40}$$

where we assume that

$$\det P(x) = 0 \Longrightarrow x_1 \text{ is real near } x' = \bar{x}'. \tag{4.41}$$

This is equivalent to say that all eigenvalues of $P^{\#}(x')$ are real. Now to prove Theorem 4.3 it suffices to prove

Proposition 4.3. *Assume that $P(x)$ is an $m \times m$ (resp. real) matrix valued smooth function of the form (4.40) verifying (4.41) and $\bar{x}$ is a nondegenerate characteristic of order r of $P(x)$. Let $\tilde{P}(x)$ be another $m \times m$ (resp. real) matrix valued smooth function of the form (4.40) verifying (4.41) which is sufficiently close to $P(x)$ in C^2 near $\bar{x}$. Then $\tilde{P}(x)$ has a nondegenerate characteristic of the same order close to $\bar{x}$. Moreover, near $\bar{x}$, the characteristics of order r of $\tilde{P}(x)$ are nondegenerate and form a smooth manifold of codimension r^2 (resp. $r(r+1)/2$). In particular, the characteristics of order r of $P(x)$ itself consists of nondegenerate ones which form a smooth manifold of codimension r^2 (resp. $r(r+1)/2$).*

The rest of this section is devoted to the proof of Proposition 4.3. We first show that the proof is reduced to the case that P and $\tilde{P}$ are $r \times r$ matrix valued function. Without restrictions we may assume that $\bar{x} = 0$. As in the previous section, we take T so that one has

$$T^{-1} P(0) T = \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}$$

where G is non singular. Denote $T^{-1}P(x)T$ and $T^{-1}\tilde{P}(x)T$ by $P(x)$ and $\tilde{P}(x)$ again. Writing

$$P(x) = \begin{bmatrix} P_{11}(x) & P_{12}(x) \\ P_{21}(x) & P_{22}(x) \end{bmatrix}$$

we have

$$P_{11}(x) = x_1 I + \sum_{j=2}^{n} A_j x_j + O(|x|^2) = P_0(x) + O(|x|^2). \tag{4.42}$$

From the assumption $P_0(x)$ is diagonalizable for every x and $\{I, A_2, \ldots, A_n\}$ span a r^2 (resp. $r(r+1)/2$) dimensional subspace over $\mathbb{R}$ in $M_r(\mathbb{C})$ (resp. $M_r(\mathbb{R})$). By Lemma 4.2 all eigenvalues of $P_0(x)$ are real then one can apply

Lemma 4.9 ([52, 66, 68, 69]). *Let us consider*

$$L(x) = \sum_{j=1}^{n} A_j x_j, \quad A_1 = I$$

where A_j are $r \times r$ constant matrices. Assume that the real reduced dimension of $L(x)$, that is the dimension of the space spanned by $\{A_j\}$ over $\mathbb{R}$, is at least $r^2 - 2$ ($(r(r+1)/2) - 1$ if all A_j are real) and $L(x)$ is diagonalizable with real eigenvalues for every x. Then there is a constant matrix T such that

$$T^{-1}L(x)T$$

is Hermitian (symmetric) for every $x \in \mathbb{R}^n$.

Thus we conclude that there is a constant matrix S such that

$$S^{-1}(x_1 + \sum_{j=2}^{n} A_j x_j)S = x_1 + \sum_{j=2}^{n} \tilde{A}_j x_j$$

where $\tilde{A}_j$ are Hermitian (resp. symmetric) and $\{I, \tilde{A}_2, \ldots, \tilde{A}_n\}$ span $M_r^h(\mathbb{C})$ (resp. $M_r^s(\mathbb{R})$). We still denote

$$\begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} P(x) \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}, \quad \begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} \tilde{P}(x) \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}$$

by $P(x)$ and $\tilde{P}(x)$ again so that writing

$$P(x) = \begin{bmatrix} P_{11}(x) & P_{12}(x) \\ P_{21}(x) & P_{22}(x) \end{bmatrix}$$

we may assume that

$$P_{11}(x) = x_1 I + \sum_{j=2}^{n} A_j x_j + O(|x|^2) \tag{4.43}$$

where

$$\{I, A_2, \ldots, A_n\} \text{ span } M_r^h(\mathbb{C}) \quad (\text{resp. } M_r^s(\mathbb{R})). \tag{4.44}$$

Let $\{F_1, F_2, \ldots, F_k\}$, $F_1 = I$ be a basis for $M_r^h(\mathbb{C})$ (resp. $M_r^s(\mathbb{R})$) where $k = r^2$ (resp. $k = r(r+1)/2$). Writing

$$x_1 I + \sum_{j=2}^{n} A_j x_j = \sum_{j=1}^{k} F_j \ell_j(x)$$

we make a linear change of coordinates $\tilde{x}_j = \ell_j(x)$, $j = 1, \ldots, n$ so that denoting $x_j = \tilde{x}_j$, $1 \le j \le k$ again and $(\tilde{x}_{k+1}, \ldots, \tilde{x}_n) = (y_1, \ldots, y_l)$ we have

$$P_{11}(x, y) = \sum_{j=1}^{k} F_j x_j + O((|x| + |y|)^2). \tag{4.45}$$

Note that the coefficient of x_1 in $\tilde{P}_{11}(x, y)$ is the identity matrix I. We now prepare the next lemma.

Lemma 4.10. *Let $P(x)$ be an $m \times m$ matrix valued C^∞ function defined near $x = 0$. With a blocking*

$$P(0) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

assume that A_{11} and A_{22} has no common eigenvalue. Then there is $\epsilon = \epsilon(A_{11}, A_{22}) > 0$ such that if $\|A_{21}\| + \|A_{12}\| < \epsilon$ then one can find a smooth matrix $T(x)$ defined in $|x| < \epsilon$ such that

$$T(x)^{-1} P(x) T(x) = \begin{bmatrix} \hat{P}_{11}(x) & 0 \\ 0 & \hat{P}_{22}(x) \end{bmatrix}$$

where $T(x) = I + T_1(x)$ and $\|T_1(0)\| \to 0$ as $\|A_{21}\| + \|A_{12}\| \to 0$.

Proof. We first show that there are G_{12}, G_{21} such that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & G_{12} \\ G_{21} & I \end{bmatrix} = \begin{bmatrix} I & G_{12} \\ G_{21} & I \end{bmatrix} \begin{bmatrix} A_{11}+X_{11} & 0 \\ 0 & A_{22}+X_{22} \end{bmatrix} \tag{4.46}$$

provided $\|A_{12}\| + \|A_{21}\|$ is small. Equation (4.46) is written as

$$\begin{bmatrix} A_{11}+A_{12}G_{21} & A_{11}G_{12}+A_{12} \\ A_{21}+A_{22}G_{21} & A_{21}G_{12}+A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}+X_{11} & G_{12}A_{22}+G_{12}X_{22} \\ G_{21}A_{11}+G_{21}X_{11} & A_{22}+X_{22} \end{bmatrix}.$$

This gives $A_{12}G_{21} = X_{11}$, $A_{21}G_{12} = X_{22}$. Plugging these relations into the remaining two equations we have

$$\begin{aligned} A_{12} + A_{11}G_{12} &= G_{12}A_{22} + G_{12}A_{21}G_{12}, \\ A_{21} + A_{22}G_{21} &= G_{21}A_{11} + G_{21}A_{12}G_{21}. \end{aligned}$$

Let us set

$$\begin{aligned} F_1(G_{12}, G_{21}, A_{12}, A_{21}) &= G_{12}A_{22} - A_{11}G_{12} + G_{12}A_{21}G_{12} - A_{12}, \\ F_2(G_{12}, G_{21}, A_{12}, A_{21}) &= G_{21}A_{11} - A_{22}G_{12} + G_{21}A_{12}G_{21} - A_{21} \end{aligned}$$

then the equations become

$$\begin{cases} F_1(G_{12}, G_{21}, A_{12}, A_{21}) = 0, \\ F_2(G_{12}, G_{21}, A_{12}, A_{21}) = 0. \end{cases} \tag{4.47}$$

It is well known that (see [71] for example)

$$\frac{\partial(F_1, F_2)}{\partial(G_{12}, G_{21})}(0, 0, 0, 0)$$

is non singular if A_{11} and A_{22} have no common eigenvalue. Then by the implicit function theorem there exist smooth $G_{12}(A_{12}, A_{21})$ and $G_{21}(A_{12}, A_{21})$ defined for small $\|A_{12}\| + \|A_{21}\|$ with $G_{12}(0,0) = 0$, $G_{21}(0,0) = 0$ verifying (4.47). This proves the assertion.

We next look for $T(x)$ in the form

$$T(x) = T_0 + T_1(x), \quad T_0(x) = \begin{bmatrix} I & G_{12} \\ G_{21} & I \end{bmatrix}, \quad T_1(0) = 0.$$

The equation which is verified by $T(x)$ is

$$(P_0 + P_1(x))(T_0 + T_1(x)) = (T_0 + T_1(x))(\tilde{P}_0 + \tilde{P}_1(x)) \tag{4.48}$$

where $P_0 = P(0)$, $P_0 T_0 = T_0 \tilde{P}_0$ and

$$\tilde{P}_1(x) = \begin{bmatrix} \tilde{P}_{11}(x) & 0 \\ 0 & \tilde{P}_{22}(x) \end{bmatrix}.$$

Recall that

$$\tilde{P}_0 = \begin{bmatrix} A_{11} + A_{12}G_{21} & 0 \\ 0 & A_{22} + A_{21}G_{12} \end{bmatrix}, \quad P_1(x) = \begin{bmatrix} P_{11}(x) & P_{12}(x) \\ P_{21}(x) & P_{22}(x) \end{bmatrix}.$$

Look for $T_1(x)$ in the form

$$T_1(x) = \begin{bmatrix} 0 & T_{12}(x) \\ T_{21}(x) & 0 \end{bmatrix}.$$

Equating the off diagonal entries of both sides of (4.48) we get

$$\begin{cases} A_{11}T_{12} + P_{12}(x) + P_{11}(x)G_{12} + P_{11}(x)T_{12} \\ \qquad\qquad = (G_{12} + T_{12})\tilde{P}_{22}(x) + T_{12}(A_{22} + A_{21}G_{12}), \\ A_{22}T_{21} + P_{21}(x) + P_{22}(x)G_{21} + P_{22}(x)T_{21} \\ \qquad\qquad = (G_{21} + T_{21})\tilde{P}_{11}(x) + T_{21}(A_{11} + A_{12}G_{21}). \end{cases} \tag{4.49}$$

On the other hand, equating the diagonal entries of both sides we have

$$\begin{cases} \tilde{P}_{11}(x) = A_{12}T_{21} + P_{11}(x) + P_{12}(x)(G_{21} + T_{21}), \\ \tilde{P}_{22}(x) = A_{21}T_{12} + P_{22}(x) + P_{21}(x)(G_{12} + T_{12}). \end{cases} \tag{4.50}$$

Plugging (4.50) into (4.49) we obtain

$$\begin{aligned} f_1(T_{12}, x) = {} & A_{11}T_{12} - T_{12}(A_{22} + A_{21}G_{12}) \\ & + P_{11}(x)G_{12} + P_{12}(x) + P_{11}(x)T_{12} \\ & - (G_{12} + T_{12})(A_{21}T_{12} + P_{21}(x)(G_{12} + T_{12}) + P_{22}(x)) = 0 \end{aligned}$$

and

$$\begin{aligned} f_2(T_{21}, x) = {} & A_{22}T_{21} - T_{21}(A_{11} + A_{12}G_{21}) \\ & + P_{22}(x)G_{21} + P_{21}(x) + P_{22}(x)T_{12} \\ & - (G_{21} + T_{21})(A_{12}T_{21} + P_{12}(x)(G_{21} + T_{21}) + P_{11}(x)) = 0. \end{aligned}$$

Since

$$f_1(T_{12}, 0) = A_{11}T_{12} - T_{12}A_{22}, \quad f_2(T_{21}, 0) = A_{22}T_{21} - T_{21}A_{11}$$

when $A_{21} = 0$, $A_{12} = 0$, $x = 0$, it is clear that

$$\frac{\partial f_1}{\partial T_{12}}(0,0), \quad \frac{\partial f_2}{\partial T_{21}}(0,0)$$

are non singular if $\|A_{12}\| + \|A_{21}\|$ is small. Then by the implicit function theorem there exist smooth $T_{12}(x)$ and $T_{21}(x)$ with $T_{12}(0) = 0$, $T_{21}(0) = 0$ such that

$$f_1(T_{12}(x), x) = 0, \quad f_2(T_{21}(x), x) = 0.$$

This proves the assertion. □

We return to the proof of Proposition 4.3. Since $\tilde{P}(x, y)$ is sufficiently close to $P(x, y)$ and

$$P(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}, \quad \det G \neq 0$$

one can apply Lemma 4.10 to $\tilde{P}(x, y)$ and find a $G(x, y)$ such that

$$G(x,y)^{-1}\tilde{P}(x,y)G(x,y) = \begin{bmatrix} \tilde{P}_{11}(x,y) & 0 \\ 0 & \tilde{P}_{22}(x,y) \end{bmatrix}. \tag{4.51}$$

Denote $G(x,y)^{-1}P(x,y)G(x,y)$ and $G(x,y)^{-1}\tilde{P}(x,y)G(x,y)$ by $P(x,y)$ and $\tilde{P}(x,y)$ again. We summarize our arguments in

Proposition 4.4. *Assume that P_{orig} and $\tilde{P}_{orig}$ verify the assumption in Proposition 4.3. Then we may assume that P_{orig} and $\tilde{P}_{orig}$ have the form*

$$\tilde{P}(x,y) = \begin{bmatrix} \tilde{P}_{11}(x,y) & 0 \\ 0 & \tilde{P}_{22}(x,y) \end{bmatrix}, \quad P(x,y) = \begin{bmatrix} P_{11}(x,y) & P_{12}(x,y) \\ P_{21}(x,y) & P_{22}(x,y) \end{bmatrix}$$

with

$$P_{11}(x,y) = \sum_{j=1}^{k} A_j x_j + \sum_{j=1}^{l} B_j y_j + R(x,y), \quad R(x,y) = O(|(x,y)|^2)$$

where the following properties are verified; for any neighborhood U of the origin there is a neighborhood $W \subset U$ of the origin such that for any $\epsilon > 0$ one can find $\tilde{\epsilon} > 0$ so that if $|\tilde{P}_{orig} - P_{orig}|_{C^2(U)} < \tilde{\epsilon}$ then we have

$$|\tilde{P}_{11}(x,y) - P_{11}(x,y)|_{C^2(W)} < \epsilon, \tag{4.52}$$

$$|\sum_{j=1}^{k} A_j x_j + \sum_{j=1}^{l} B_j y_j - \sum_{j=1}^{k} F_j x_j| < C\epsilon(|x| + |y|). \tag{4.53}$$

Moreover one has

$$\det(\lambda + \tilde{P}_{11}(x, y)) = 0 \Longrightarrow \lambda \textit{ is real}.$$

Proof. Since $P(x, y)$ and $\tilde{P}(x, y)$ are obtained from P_{orig} and $\tilde{P}_{orig}$ by a smooth change of basis and a linear change of coordinates then (4.52) is clear. Let us recall

$$G(x, y) = \begin{bmatrix} I & G_{12}(x, y) \\ G_{21}(x, y) & I \end{bmatrix}$$

which verifies (4.51) where $\|G_{12}(0, 0)\| + \|G_{21}(0, 0)\|$ becomes as small as we please if $\tilde{\epsilon}$ is small. Hence $G(x, y)$ is enough close to the identity and then (4.53) follows from (4.45). Note that

$$\det(\lambda + \tilde{P}_{orig}) = \det(\lambda + \tilde{P}_{11}(x, y))\det(\lambda + \tilde{P}_{22}(x, y)).$$

Then the last assertion follows immediately. □

We proceed to the next step. Write

$$\tilde{P}_{11}(x, y) = \tilde{P}_{11}(0, y) + (\tilde{\phi}^i_j(x, y))_{1 \le i, j \le r} \tag{4.54}$$

so that $\tilde{\phi}^i_j(0, y) = 0$. Let us define $t^i_j(x, y)$ by

$$\tilde{\phi}^i_j(x, y) = \phi^i_j(x) + t^i_j(x, y)$$

where

$$F(x) = \sum_{j=1}^{k} F_j x_j = (\phi^i_j(x))_{1 \le i, j \le r}.$$

Lemma 4.11. *Assume that* $|\tilde{P}_{11}(x, y) - P_{11}(x, y)|_{C^2(W)} < \epsilon$ *and* $\{(x, y) \mid |x|, |y| < \epsilon\} \subset W$. *Then for* $|x|, |y| < \epsilon$ *we have*

$$|t^i_j(x, y)| \le C|x|, \quad |\partial_{x_\mu} t^i_j(x, y)| \le C\epsilon, \quad \mu = 1, \dots, k.$$

Proof. Write

$$\tilde{P}_{11}(x, y) = \tilde{P}_{11}(0, y) + \sum_{j=1}^{k} \tilde{A}_j(y) x_j + \tilde{R}(x, y), \quad \tilde{R}(x, y) = O(|x|^2)$$

so that

$$T = (t^i_j(x,y)) = \sum_{j=1}^{k} \tilde{A}_j(y)x_j - \sum_{j=1}^{k} F_j x_j + \tilde{R}(x,y). \tag{4.55}$$

Noting $\partial_{x_j}\tilde{P}_{11}(0,y) = \tilde{A}_j(y)$, $\partial_{x_j}P_{11}(0,y) = A_j + \partial_{x_j}R(0,y)$ and

$$|\partial_{x_j}R(0,y)| \le C|y| \le C\epsilon \quad \text{if } |y| < \epsilon$$

with C independent of $\tilde{P}$, one gets

$$|\tilde{A}_j(y) - A_j| \le C\epsilon \quad \text{if } |y| < \epsilon. \tag{4.56}$$

Now it is clear that

$$|\tilde{A}_j(y) - F_j| \le C'\epsilon \quad \text{if } |y| < \epsilon \tag{4.57}$$

because of (4.53) and (4.56). On the other hand from

$$P_{11}(0,y) = \sum_{j=1}^{l} B_j y_j + R(0,y)$$

and (4.53) it follows that

$$|P_{11}(0,y)| \le C\epsilon|y| + C|y|^2 \le C\epsilon|y| \quad \text{if } |y| < \epsilon.$$

Moreover $|\tilde{P}_{11}(0,y) - P_{11}(0,y)|_{C^2(W)} < \epsilon$ shows

$$|\tilde{P}_{11}(0,y)| < \epsilon + C\epsilon|y| < C'\epsilon \quad \text{if } |y| < \epsilon. \tag{4.58}$$

We now estimate $T(x,y) = (t^i_j(x,y))$ and $\partial_{x_j}T(x,y)$. Note that $|\partial_{x_j}\tilde{R}(x,y)| \le C|x|$ since $\partial_{x_j}\tilde{R}(0,y) = 0$ and $|\partial_x^\alpha \tilde{R}(x,y)| \le C$ for $|\alpha| = 2$ with C independent of $\tilde{P}$. Then by (4.55) and (4.57) one sees

$$\begin{cases} |T(x,y)| \le C\epsilon|x| + C|x|^2 \le C'\epsilon|x| & \text{if } |x| < \epsilon, \\ |\partial_{x_j}T(x,y)| \le C\epsilon + C|x| \le C'\epsilon & \text{if } |x|, |y| < \epsilon \end{cases} \tag{4.59}$$

which proves the assertion. □

Recall

$$F(x) = \sum_{j=1}^{k} F_j x_j = (\phi^i_j(x))_{1\le i,j\le r}$$

where $F_1 = I$ and $\{F_1, \ldots, F_k\}$ be a basis for $M_r^h(\mathbb{C})$ over $\mathbb{R}$ (resp. $M_r^s(\mathbb{R})$) and hence $k = r^2$ (resp. $k = r(r+1)/2$).

Proposition 4.5. *Assume that $P(x)$ is a $r \times r$ matrix valued smooth function defined in a neighborhood of the origin of $\mathbb{R}^n$. Assume that all eigenvalues of $P(x)$ are real and*

$$\sum_{j=1}^{n} \frac{\partial P}{\partial x_j}(0) x_j \tag{4.60}$$

is sufficiently close to $F(x)$ in C^1. Then there is a $\delta > 0$ such that $P(x)$ is diagonalizable for every x with $|x| < \delta$.

Proof. Let us write

$$P(\omega + x) = P(\omega) + Q(x, \omega)$$

so that $Q(0, \omega) = 0$. For $T \in U(r)$, a unitary matrix of order r we consider

$$\begin{aligned} T^* P(\omega + x) T &= T^* P(\omega) T + T^* Q(x, \omega) T \\ &= P^T(\omega) + Q^T(x, \omega) = P^T(\omega) + (\phi_j^i(x, \omega; T))_{1 \le i, j \le m}. \end{aligned}$$

We show that there exist a $\delta > 0$ and a neighborhood W of the origin of $\mathbb{R}^k$ such that with $x = (x_a, x_b)$, $x_a = (x_1, \ldots, x_k)$, $x_b = (x_{k+1}, \ldots, x_n)$ the map

$$W \ni x_a \mapsto \big((\mathrm{Re}\, \phi_j^i(x, \omega; T))_{i \ge j}, (\mathrm{Im}\, \phi_j^i(x, \omega; T))_{i > j} \big) \in \mathbb{R}^k$$

is a diffeomorphism from W into $\{y \in \mathbb{R}^k \mid |y| < \delta\}$ for every $T \in U(r)$ and every x_b, ω with $|x_b|, |\omega| < \delta$. To see this we write

$$\begin{aligned} Q(x, \omega) &= P(x + \omega) - P(\omega) = \sum_{j=1}^{n} \frac{\partial P}{\partial x_j}(\omega) x_j + \tilde{R}(x, \omega) \\ &= \sum_{j=1}^{k} F_j x_j + \sum_{j=1}^{k} \left(\frac{\partial P}{\partial x_j}(\omega) - F_j \right) x_j + \sum_{j=k+1}^{n} \frac{\partial P}{\partial x_j}(\omega) x_j + \tilde{R}(x, \omega) \\ &= \sum_{j=1}^{k} F_j x_j + R(x, \omega) \end{aligned}$$

then it is clear that for any $\epsilon > 0$ one can find $\delta' > 0$ such that

$$\|R(x, \omega)\| \le \epsilon |x| \tag{4.61}$$

if $|x|$, $|\omega| < \delta'$ and if (4.60) is sufficiently close to $F(x)$. Let us study

$$Q^T(x,\omega) = \sum_{j=1}^{k} F_j^T x_j + R^T(x,\omega) = \sum_{j=1}^{k} \ell_j(x_a;T)F_j + R^T(x,\omega)$$

where $\ell_j(x_a;T)$ are linear in x_a. Since $U(r) \subset \mathbb{R}^{r^2}$ is compact it is clear that we have

$$\left|\frac{\partial(\ell_1,\ldots,\ell_k)}{\partial(x_1,\ldots,x_k)}(x_a;T)\right| \geq c > 0$$

with some $c > 0$ for every $T \in U(r)$. In view of (4.61), taking $\epsilon > 0$ so small we conclude that

$$\left|\frac{\partial\big((\mathrm{Re}\,\phi_j^i)_{i\geq j}, (\mathrm{Im}\,\phi_j^i)_{i>j})\big)}{\partial(x_a)}(0,0,0;T)\right| \geq c' > 0$$

with some $c' > 0$ for every $T \in U(r)$. By the implicit function theorem and the compactness of $U(r)$ there exists a smooth $x_a(y_a, x_b, \omega; T)$ defined in $|y_a|$, $|x_b|$, $|\omega| < \delta''$ and $T \in U(r)$ such that

$$\begin{cases} \mathrm{Re}\,\phi_j^i(x_a(y_a,x_b,\omega;T),x_b,\omega;T) = y_j^i & \text{for} \quad i \geq j, \\ \mathrm{Im}\,\phi_j^i(x_a(y_a,x_b,\omega;T),x_b,\omega;T) = \tilde{y}_j^i & \text{for} \quad i > j \end{cases}$$

where we have set $y_a = \big((y_j^i)_{i\geq j}, (\tilde{y}_j^i)_{i>j}\big) \in \mathbb{R}^k$. This proves the assertion.

We now show that $P(\omega)$ is diagonalizable for every $\omega \in \mathbb{R}^n$ with $|\omega| < \delta = \min\{\delta', \delta''\}$. Take $T \in U(r)$ so that

$$P^T(\omega) = (\bigoplus_{i=1}^{s} \lambda_i I_{r_i}) + (A_{ij})_{1\leq i,j\leq s} \tag{4.62}$$

where $\{\lambda_i\}$ are different from each other and A_{ij} are $r_i \times r_j$ matrices such that $A_{ij} = 0$ if $i > j$ and A_{ii} are upper triangular with zero diagonal entries. Let us set

$$J = \bigcup_{p=1}^{s-1} \{(i,j) \mid r_p < i \leq m, r_{p-1} < j \leq r_p\}$$

where $r_0 = 0$. As observed above one can take $((y_j^i)_{i\geq j}, (\tilde{y}_j^i)_{i>j}, x_b)$ as a new system of local coordinates around the origin of $\mathbb{R}^n$. Denote

$$y_{II} = \Big((y_j^i)_{(i,j)\in J}, (\tilde{y}_j^i)_{((i,j)\in J, i>j)}\Big), \quad y_a = (y_I, y_{II})$$

and, putting $y_{II}=0$, $x_b=0$, consider

$$\det(\lambda+P(\omega+x))=\det(\lambda+P^T(\omega+x))=\prod_{i=1}^{s}\det(\lambda+K_i(y_I,\omega;T))$$

where

$$K_i(y_I,\omega;T)=\lambda_i I_{r_i}+A_{ii}+(\phi_q^p(y_I,\omega;T))_{s_{i-1}<p,q\le s_i}$$

with $s_i=r_1+\cdots+r_i$, $s_0=0$. Note that we have

$$\begin{cases}\phi_q^p(y_I,\omega;T)=y_q^p+i\,\tilde{y}_q^p \quad \text{if} \quad p>q,\\ \phi_p^p(y_I,\omega;T)=y_p^p+\operatorname{Im}\phi_p^p(y_I,\omega;T).\end{cases}$$

We will conclude $A_{ii}=0$ repeating the same arguments proving the next lemma.

Lemma 4.12. *Let A be a constant matrix of order r such that $A=\alpha I_r+\tilde{A}$ where α is a real constant and $\tilde{A}$ is upper triangular with zero diagonal entries. Let $P(x)=A+(\phi_j^i(x))$ where $\phi_j^i(x)$ are linear in x and* $\operatorname{Re}\phi_j^i(x)$, *$i\ge j$,* $\operatorname{Im}\phi_j^i(x)$, *$i>j$ are linearly independent over $\mathbb{R}$. Suppose that all eigenvalues of $P(x)$ are real. Then A is necessarily diagonal matrix.*

Proof. Let us set $y_a=(y_j^i)=(\operatorname{Re}\phi_j^i)_{i\ge j}$, $y_b=(\operatorname{Im}\phi_j^i)_{i>j}$ and let (y_a,y_b,y_c) is a new system of local coordinates of $\mathbb{R}^n$ which is related to x by a non singular linear transformation. Let $A=(a_{pq})$ and we first show that $a_{p,p+1}=0$ for $p=1,\ldots,r-1$. Take $y_j^i=0$ for $i\ge j$ unless $(i,j)=(p+1,p)$ and $y_b=0$, $y_c=0$. Then it is clear that

$$\begin{aligned}&\det(\lambda+A+(\phi_j^i))\\ &\quad=\prod_{j\ne p,p+1}(\lambda+\alpha+\operatorname{Im}\phi_j^j)\\ &\qquad\times\Big((\lambda+\alpha+\operatorname{Im}\phi_p^p)(\lambda+\alpha+\operatorname{Im}\phi_{p+1}^{p+1})-y_p^{p+1}(a_{p,p+1}+\phi_{p+1}^p)\Big).\end{aligned}$$

Since Im $\phi_i^i(x)$ and $\phi_{p+1}^p(x)$ are constant times y_p^{p+1} then we see

$$\begin{aligned}&(\lambda+\alpha+\operatorname{Im}\phi_p^p)(\lambda+\alpha+\operatorname{Im}\phi_{p+1}^{p+1})-y_p^{p+1}(a_{p,p+1}+\phi_{p+1}^p)\\ &\quad=(\lambda+\alpha)^2+O(|y_p^{p+1}|)(\lambda+\alpha)-y_p^{p+1}a_{p,p+1}+O(|y_p^{p+1}|^2)=0\end{aligned}$$

would have a non-real root for small y_p^{p+1} unless $a_{p,p+1}=0$.

We now proceed by induction on $q-p$. Suppose that

$$a_{pq}=0 \quad \text{for} \quad p+1\le q\le p+r-1.$$

Let $q = p + r$ and take $y^i_j = 0$ for $i \geq j$ unless $(i, j) = (q, p)$ and $y_b = 0$, $y_c = 0$. We note that

$$\det(\lambda + A + (\phi^i_j)) = \prod_{j \neq p, p+1, \ldots, q} (\lambda + \alpha + \phi^j_j) Q(\lambda)$$

where $Q(\lambda)$ has the form

$$(\lambda + \alpha)^{r+1} - (\lambda + \alpha)^{r-1} y^q_p a_{pq} + \sum_{j=0}^{r-1} O(|y^q_p|^{r-1-j})(\lambda + \alpha)^j.$$

If we have set $\lambda + \alpha = \sqrt{|y^q_p|} z$ then $Q(\lambda) = 0$ is reduced to

$$z^{r+1} - z^{r-1} a_{pq} + O(|y^q_p|) R(z, y^q_p) = 0 \tag{4.63}$$

where R is a polynomial in z of degree $r - 1$. Thus if $a_{pq} \neq 0$ then (4.63) has a non real root for small y^q_p and hence $Q(\lambda) = 0$ would have a non real root. This proves that $a_{pq} = 0$. By induction we get the desired assertion.

For the real matrix case the proof is similar. □

Since ω, $|\omega| < \delta$ is arbitrary to prove Proposition 4.5 it suffices to show that $P^T(\omega)$ is diagonalizable which follows from the next lemma.

Lemma 4.13. *Let $A = (A_{ij})_{1 \leq i, j \leq s}$ be a constant matrix of order m where A_{ij} are $r_i \times r_j$ matrices. Suppose that $A_{ii} = \lambda_i I_{r_i}$ where $\lambda_i \neq \lambda_j$ if $i \neq j$ and $A_{ij} = O$ if $i > j$. Then A is diagonalizable.*

Proof. It suffices to construct S so that $S^{-1} A S = D = \lambda_1 I_{r_1} \oplus \cdots \oplus \lambda_s I_{r_s}$. Let us set $S = (S_{ij})$ where the blocking corresponds to that of A and $S_{ij} = O$ if $i > j$ and $S_{ii} = I_{r_i}$. From $AS = SD$ it follows that

$$(\lambda_i - \lambda_j) S_{ij} = - \sum_{k \geq i+1} A_{ik} S_{kj} \quad (i < j).$$

In particular $S_{r-1,r} = -(\lambda_{r-1} - \lambda_r)^{-1} A_{r-1,r}$ is determined by the above equation. Inductively S_{ir} are determined for $1 \leq i \leq r - 1$. Then we proceed to

$$(\lambda_i - \lambda_{r-1}) S_{i,r-1} = - \sum_{k \geq i+1} A_{ik} S_{k,r-1}.$$

Repeating this argument we obtain $S_{i,r-1}$ and hence the desired assertion. □

We now prove that near $(0, 0)$ the set of characteristics of order r of $\tilde{P}(x, y)$ is a smooth manifold. We first show that near $y = 0$ there is a unique smooth $g(y)$ such that

$$\tilde{P}_{11}(g(y), y) = O.$$

To see this let us study the map

$$\Phi : B_a \ni x \mapsto \big((\mathrm{Re}\,\tilde{\phi}^i_j(x,y))_{i\geq j}, (\mathrm{Im}\,\tilde{\phi}^i_j(x,y))_{i>j}\big) \in \mathbb{R}^k$$

where $B_a = \{x \in \mathbb{R}^k \mid |x| \leq a\}$. Let

$$A : \mathbb{R}^k \ni x \mapsto \big((\mathrm{Re}\,\phi^i_j(x))_{i\geq j}, (\mathrm{Im}\,\phi^i_j(x))_{i>j}\big) \in \mathbb{R}^k$$

which is a linear transformation on $\mathbb{R}^k$. Since $(\mathrm{Re}\,\phi^i_j(x))_{i\geq j}, (\mathrm{Im}\,\phi^i_j(x))_{i>j}$ are linearly independent then A is non singular. From Lemma 4.11 one can choose $\epsilon > 0$ so that

$$|A^{-1}\Phi'_x(x,y) - I| < 1/2 \quad \text{if } |x|, |y| < \epsilon.$$

Let us write $\tilde{P}_{11}(0,y) = (b^i_j(y))$ and note that $|b^i_j(y)| \leq C'\epsilon$ for $|y| < \epsilon$ by (4.58). Then choosing $\epsilon > 0$ sufficiently small we can apply the implicit function theorem to conclude that there exists a unique smooth $g(y,\theta,\kappa)$ defined in $|(\theta,\kappa)| \leq \epsilon$, $|y| \leq \epsilon$ such that

$$\begin{cases} \mathrm{Re}\,\tilde{\phi}^i_j(g(y,\theta,\kappa),y) = \theta^i_j - \mathrm{Re}\,b^i_j(y), \; i \geq j, \\ \mathrm{Im}\,\tilde{\phi}^i_j(g(y,\theta,\kappa),y) = \kappa^i_j - \mathrm{Im}\,b^i_j(y), \; i > j \end{cases} \tag{4.64}$$

and in the real case

$$\tilde{\phi}^i_j(g(y,\theta),y) = \theta^i_j - b^i_j(y), \quad i \geq j \tag{4.65}$$

such that

$$|g(y,\theta,\kappa)| < C\epsilon. \tag{4.66}$$

Set

$$(\psi^i_j(y,\theta,\kappa)) = \tilde{P}_{11}(g(y,\theta,\kappa),y) \tag{4.67}$$

then from (4.54) and (4.64) it follows that

$$\begin{cases} \mathrm{Re}\,\psi^i_j(y,\theta,\kappa) = \theta^i_j, \;\; i \geq j, \\ \mathrm{Im}\,\psi^i_j(y,\theta,\kappa) = \kappa^i_j, \;\; i > j. \end{cases} \tag{4.68}$$

Let us write

$$\psi^i_j(y,\theta,\kappa) = c^i_j(y) + \chi^i_j(y,\theta,\kappa)$$

where $c^i_j(y) = \psi^i_j(y,0,0)$ and $\chi^i_j(y,\theta,\kappa) = O(|(\theta,\kappa)|)$. Let us put $h(\lambda) = \det(\lambda + \tilde{P}_{11}(g(y,\theta,\kappa),y))$. From Proposition 4.4 it follows that $h(\lambda) = 0$ implies

λ is real. Repeating the same arguments as in the proof of Lemma 4.12 we conclude that $c_q^p(y) = 0$ for $p < q$ and $\operatorname{Im} c_p^p(y) = 0$. This, together with (4.68), implies

$$\tilde{P}_{11}(g(y, 0, 0), y) = O. \tag{4.69}$$

The proof for the real case is similar.

We now prove that near $(0, 0)$ the set of characteristics of order r of $\tilde{P}(x, y)$ is a smooth manifold given by $x = g(y, 0, 0)$. Let $(\bar{x}, \bar{y})$ be a characteristic of order r of $\tilde{P}(x, y)$ close to $(0, 0)$. Then it is clear that $(\bar{x}, \bar{y})$ is a characteristic of the same order for $\tilde{P}_{11}(x, y)$ because $\det \tilde{P}_{22}(x, y) \neq 0$ near $(0, 0)$. Recalling that $\tilde{P}_{11}(x, y)$ has the form

$$\tilde{P}_{11}(x, y) = x_1 + \tilde{P}_{11}^{\#}(x', y), \quad x' = (x_2, \ldots, x_k)$$

we see that $\det \tilde{P}_{11}(x_1, \bar{x}', \bar{y}) = (x_1 - \bar{x}_1)^r$ and hence

$$\det(\lambda + \tilde{P}_{11}(\bar{x}, \bar{y})) = \lambda^r.$$

Thus the zero is an eigenvalue of multiplicity r of $\tilde{P}_{11}(\bar{x}, \bar{y})$. On the other hand Proposition 4.4 gives

$$\left|\frac{\partial \tilde{P}_{11}}{\partial x_j}(0) - F_j\right|, \quad \left|\frac{\partial \tilde{P}_{11}}{\partial y_j}(0)\right| < C\epsilon. \tag{4.70}$$

Then one can apply Proposition 4.5 to conclude that $\tilde{P}(\bar{x}, \bar{y})$ is diagonalizable. This shows that

$$\tilde{P}_{11}(\bar{x}, \bar{y}) = O$$

and hence one gets $\bar{x} = g(\bar{y}, 0, 0)$.

Finally we show that the characteristics $(g(y, 0, 0), y)$ are nondegenerate. From (4.69) we have

$$\tilde{P}(g(y, 0, 0), y) = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}_{22}(g(y, 0, 0), y) \end{bmatrix}$$

and hence

$$\operatorname{Ker} \tilde{P}(g(y, 0, 0), y) \cap \operatorname{Im} \tilde{P}(g(y, 0, 0), y) = \{0\}. \tag{4.71}$$

It is also clear that $\tilde{P}_{(g(y,0,0),y)}(x, y)$ is given by

$$\sum_{j=1}^{k} \frac{\partial \tilde{P}_{11}}{\partial x_j}(g(y, 0, 0), y) x_j + \sum_{j=1}^{l} \frac{\partial \tilde{P}_{11}}{\partial y_j}(g(y, 0, 0), y) y_j.$$

On the other hand since $|\tilde{P}_{11} - P_{11}|_{C^2(W)} < \epsilon$ it follows from Proposition 4.5 and (4.66) that

$$\left|\frac{\partial \tilde{P}_{11}}{\partial x_j}(g(y,0,0),y) - F_j\right|, \quad \left|\frac{\partial \tilde{P}_{11}}{\partial y_j}(g(y,0,0),y)\right| < C\epsilon \tag{4.72}$$

if $|y| < \epsilon$. This clearly shows that

$$\dim \tilde{P}_{(g(y,0,0),y)} = r^2. \tag{4.73}$$

To finish the proof, taking $\tilde{P}_{(g(y,0,0),y)}(\theta) = I$ into account, it is enough to show that $\tilde{P}_{(g(y,0,0),y)}(x,y)$ is diagonalizable for every (x,y). Note that from Lemma 4.2 all eigenvalues of $\tilde{P}_{(g(y,0,0),y)}(x,y)$ are real. Then from Proposition 4.5 and (4.72) it follows that $\tilde{P}_{(g(y,0,0),y)}(x,y)$ is diagonalizable for every (x,y) near $(0,0)$ and hence for all (x,y).

The proof for the real case is similar. Thus the proof of Proposition 4.3 is completed. □

Example 4.6. Consider a second order differential operator $P(x,D) = (p_{ik}(x,D))$ with 3×3 matrix coefficients

$$p_{ik}(x,\tau,\xi) = (\tau^2 - \sigma_i(x)|\xi|^2)\delta_{ik} - (1 - \sigma_i(x))\xi_i\xi_k$$

where $\sigma_i(x)$ are real smooth and close to σ_i in Example 4.4. We assume that $Q(x,\tau,\xi) = \det P(x,\tau,\xi) = 0$ has only real roots for any x and ξ. Then from Theorem 4.3 it follows that every characteristic of $P(x,\xi)$ are at most double and the double characteristics are nondegenerate.

Example 4.7. Let $A(\xi) = \sum_{j,k=1}^{3} A_{jk}\xi_j\xi_k$ be one of them discussed in Example 4.5, that is the characteristics of $\det(\tau^2 I - A(\xi))$ are at most double and the double characteristics are nondegenerate. Let $A_{jk}(x)$ be real smooth 3×3 matrices which are close to A_{jk} and set

$$A(x,\xi) = \sum_{j,k=1}^{3} A_{jk}(x)\xi_j\xi_k$$

and assume that $\det(\tau^2 I - A(x,\xi)) = 0$ has only real roots for any x and ξ. Then from Theorem 4.3 we see that every characteristic of $\tau^2 I - A(x,\xi)$ are at most double and the double characteristics are nondegenerate.

Example 4.8. As in Example 4.3 take $P(\xi) = \xi_1 I + \sum_{j=2}^{d} F_j\xi_j$ where $\{I, F_2, \ldots, F_d\}$ is a basis for $M_m^s(\mathbb{R})$. Consider

$$P(x,\xi) = \xi_1 I + \sum_{j=2}^{d} A_j(x)\xi_j$$

where $A_j(x)$ are real smooth $m \times m$ matrices which are enough close to F_j in C^2 and we assume that $P(x, \xi)$ has only real eigenvalues for any x and any ξ. Then from Theorem 4.3 it follows that every multiple characteristic of $P(x, \xi)$ is nondegenerate.

4.5 Symmetrizability (General Case)

In this section to simplify notations let us write $\mathscr{P}(x)$, $x = (x_0, x_1, \dots, x_n)$ which is a real analytic $m \times m$ matrix valued function defined near the origin of $\mathbb{R}^{n+1}$. We assume that all eigenvalues of $\mathscr{P}(x)$ are real near $x = 0$. We also denote by d_m the dimension of $M_m^h(\mathbb{C})$ (resp. $M_m^s(\mathbb{R})$) over $\mathbb{R}$, that is

$$d_m = m^2 \quad (resp.\ \ d_m = m(m+1)/2).$$

Our main concern in this section is to prove

Theorem 4.4. *Assume that all eigenvalues of $\mathscr{P}(x)$ are real near a nondegenerate characteristic $x = 0$ of order m and $\mathscr{P}_0(\Theta) = I_m$ with some $\Theta \in \mathbb{R}^{n+1}$. Then there is a real analytic symmetrizer near $x = 0$, that is there is a real analytic positive definite $H(x)$, $H^*(x) = H(x)$, defined near $x = 0$ such that*

$$\mathscr{P}(x)H(x) = H(x)\mathscr{P}^*(x).$$

Corollary 4.1. *Assume that $\mathscr{P}(x)$ has the form $x_0 I + P(x')$ with $x' = (x_1, \dots, x_n)$ and all eigenvalues of $P(x')$ are real near $x' = 0$. Suppose that $x = 0$ is a nondegenerate characteristic of order m of $\mathscr{P}(x)$. Then there is a real analytic positive definite $H(x')$, $H^*(x') = H(x')$, defined near $x' = 0$ such that*

$$P(x')H(x') = H(x')P^*(x').$$

We first give another proof, based on Theorem 4.4, for that the set of nondegenerate characteristics is a smooth manifold of codimension d_m.

Proposition 4.6. *Assume the same assumptions as in Theorem 4.4. Then we can choose a new system of local coordinates X and a real analytic $T(X)$ defined near $X = 0$ so that*

$$T(X)^{-1}\mathscr{P}(x(X))T(X) = \sum_{j=0}^{k-1} F_j X_j$$

with $k = d_m$ where $F_0 = I$ and $\{F_j\}$ span $M_m^h(\mathbb{C})$ over $\mathbb{R}$ (resp. $M_m^s(\mathbb{R})$).

Proof. From Theorem 4.4 there is a positive definite $H(x)$ such that $\mathscr{P}(x)H(x) = H(x)\mathscr{P}^*(x)$. This shows that

$$S(x) = H(x)^{-1/2}\mathscr{P}(x)H(x)^{1/2}$$

is Hermitian. Let us write $S(x) = (\phi^i_j(x))$ and hence $\phi^i_j(0) = 0$. With $\phi^i_j(x) = d\phi^i_j(x) + O(|x|^2)$ we note that $\{(\mathrm{Re}\, d\phi^i_j(x))_{i\geq j}, (\mathrm{Im}\, d\phi^i_j(x))_{i>j}\}$ are linearly independent over $\mathbb{R}$. Then taking a new system of local coordinates X so that $X_0 = \mathrm{Re}\,\phi^1_1(x)$, $X_i = \mathrm{Re}\,\phi^i_i(x) - \mathrm{Re}\,\phi^1_1(x)$, $2 \leq i \leq m$, $(X_{m+1}, \ldots, X_{p-1}) = (\mathrm{Re}\,\phi^i_j(x))_{i>j}$, $(X_p, \ldots, X_{k-1}) = (\mathrm{Im}\,\phi^i_j(x))_{i>j}$ we get the assertion with $T(X) = H(x(X))^{1/2}$. □

From Proposition 4.6 it is clear that, near $x = 0$, the set $\mathscr{P}(x) = O$ is given by $\Sigma = \{X_j = 0 \mid j = 0, \ldots, d_m-1\}$ which is a smooth manifold of codimension d_m. It is also clear that for $x \in \Sigma$ the properties (4.13) and (4.14) hold, that is Σ consists of nondegenerate characteristics. On the other hand let $\bar{x}$ be a characteristic of order m for $\mathscr{P}(x)$ so that 0 is the eigenvalue of $\mathscr{P}(\bar{x})$ of multiplicity m. Then 0 is the eigenvalue of $\sum_{j=0}^{d_m-1} F_j\bar{X}_j$ of multiplicity m where $\bar{x} = x(\bar{X})$. Since $\sum_{j=0}^{d_m-1} F_j\bar{X}_j$ is Hermitian we see $\sum_{j=0}^{d_m-1} F_j\bar{X}_j = O$ and hence $\bar{X}_j = 0$ for $j = 0, \ldots, d_m-1$. Thus we conclude $\bar{X} \in \Sigma$.

We start to prove Theorem 4.4. Choosing a system of local coordinates so that $\Theta = (1, 0, \ldots, 0)$ we can assume that $\mathscr{P}_0(x)$ verifies the assumption of Lemma 4.9. Then one can assume that $T^{-1}\mathscr{P}_0(x)T$ is Hermitian (resp. symmetric) for every x with some constant matrix T. By a linear change of coordinates x one may assume that

$$T^{-1}\mathscr{P}_0(x)T = x_0 I + \sum_{j=1}^{k} F^j x_j$$

with $k = d_m - 1$ where $\{I, F^j\}$ span the space $M^h_m(\mathbb{C})$ (resp. $M^s_m(\mathbb{R})$) over $\mathbb{R}$. Since $\mathscr{P}(x) = \mathscr{P}_0(x) + R(x)$, $R(x) = O(|x|^2)$ as $x \to 0$, to prove Theorem 4.4, writing $\mathscr{P}(x) = x_0 I + P(x)$, it is enough to show the following theorem.

Theorem 4.5. *Let $P(x) = \sum_{j=1}^{k} F^j x_j + R(x)$ where $x = (x_0, \ldots, x_n)$, and $R(x)$ is real analytic near the origin so that $R(x) = O(|x|^2)$ as $x \to 0$. Assume that $\{F^j\}$ are Hermitian (resp. symmetric) $l \times l$ constant matrices such that $\{I, F^j\}$ span the space $M^h_l(\mathbb{C})$ (resp. $M^s_l(\mathbb{R})$) over $\mathbb{R}$ and $k = d_l - 1$. Suppose that all eigenvalues of $P(x)$ are real near the origin. Then there is a positive definite real analytic $G(x)$ with $G(0) = I$ defined near the origin verifying*

$$P(x)G(x) = G(x)P^*(x), \quad G^*(x) = G(x). \tag{4.74}$$

Remark. Assume, for instance, that a positive definite $G(x)$ verifying (4.74) exists. Expanding both sides of (4.74) in the Taylor expansions around the origin and equating the first order terms we see that

$$\sum_{j=1}^{k} F^j G(0) x_j = \sum_{j=1}^{k} G(0) F^j x_j$$

so that $G(0)$ commutes with all Hermitian (resp. symmetric) matrices of order m and hence $G(0) = \alpha I$ with $\alpha \neq 0$. Since $G(0)$ is positive definite and hence $\alpha > 0$ we may suppose that $G(0) = I$ considering $\alpha^{-1} G(x)$ which also verifies (4.74).

To prove Theorem 4.5, we proceed by induction on the size of matrices. When $l = 2$, since $\mathscr{P}(0) = O$ and $x = 0$ is a nondegenerate double characteristic thanks to Proposition 4.2 there is a real analytic symmetrizer $G(x)$ verifying (4.74). Let the assumption of Theorem 4.5 be verified for $l < m$. Since $\{I, F^j\}$ span $M_l^h(\mathbb{C})$ (resp. $M_l^s(\mathbb{R})$), choosing a new system of local coordinates x we may suppose that the Hermitian (resp. symmetric) part of $R(x)$ can be removed so that

$$\mathscr{P}(x) = x_0 I + \sum_{j=1}^{k} F^j x_j + R(x)$$

with $k = d_l - 1$ where $R(x)$ is anti-Hermitian (resp. anti-symmetric). Since the all eigenvalues of $\mathscr{P}(x)$ are real it follows that

$$R(x_0, 0, \ldots, 0, x_{k+1}, \ldots, x_n) = O.$$

Changing notations slightly we write $x = (x_1, x_2, \ldots, x_k)$, $y = (x_0, x_{k+1}, \ldots, x_n)$ with $k = d_l - 1$ and

$$P(x, y) = \sum_{j=1}^{k} F^j x_j + R(x, y)$$

so that $P(0, y) = O$. We divide the proof of the assertion for $l = m$ into two steps. In the first step, introducing the polar coordinates $x = r\omega$, we blow up $P(x, y)$ at $x = 0$ so that

$$Q(r, \omega, y) = r^{-1} P(r\omega, y)$$

will be studied. We prove

Proposition 4.7. *Suppose that the assertion of Theorem 4.5 holds for $l < m$. Let $P(x, y) = \sum_{j=1}^{k} F^j x_j + R(x, y)$, $k = d_m - 1$ be a real analytic $m \times m$ matrix valued function with real eigenvalues near the origin such that $R(x, y) = O(|(x, y)|^2)$ as $(x, y) \to 0$ and $R(0, y) = O$. Assume that $\{I, F^j\}$ span $M_m^h(\mathbb{C})$ (resp. $M_m^s(\mathbb{R})$). Then for every $\omega \neq 0$ there is a positive definite $H(r, \phi, y)$ with diagonal entries* 1 *which is real analytic near* $(0, \omega, 0)$ *such that*

$$P(r\phi, y) H(r, \phi, y) = H(r, \phi, y) P^*(r\phi, y), \quad H^*(r, \phi, y) = H(r, \phi, y). \tag{4.75}$$

Thus we can construct a symmetrizer with diagonal entries 1 of the blown up $P(r\phi, y)$ in a neighborhood of every $(0, \omega, 0)$ with $\omega \neq 0$. In the second step we first observe that such symmetrizers can be continued analytically to a neighborhood of $\{0\} \times S^{k-1} \times \{0\}$.

Lemma 4.14. *Suppose that at every* $(0, \omega, 0)$ *with* $\omega \neq 0$ *there is a positive definite real analytic symmetrizer* $H(r, \phi, y)$ *with diagonal entries* 1 *verifying (4.75). Then there is* $H(r, \phi, y)$ *with diagonal entries* 1 *which is real analytic in* $I \times S^{k-1} \times J$ *such that*

$$P(r\phi, y)H(r, \phi, y) = H(r, \phi, y)P^*(r\phi, y), \quad H^*(r, \phi, y) = H(r, \phi, y) \tag{4.76}$$

holds for $(r, \phi, y) \in I \times S^{k-1} \times J$ *where* I, J *are open intervals containing the origin.*

We next show that the symmetrizer obtained in Lemma 4.14 is the blown up of a real analytic $G(x, y)$ defined near the origin $(x, y) = (0, 0)$.

Proposition 4.8. *Assume that* $H(r, \phi, y)$ *verifies (4.76) where* $H(r, \phi, y)$ *is real analytic in* $I \times S^{k-1} \times J$ *with diagonal entries* 1. *Then* $H(r, \phi, y)$ *is a blown up of a real analytic* $G(x, y)$*, that is*

$$H(r, \phi, y) = G(r\phi, y).$$

In particular we have

$$P(x, y)G(x, y) = G(x, y)P^*(x, y), \quad G^*(x, y) = G(x, y).$$

Combining Propositions 4.7 and 4.8, Theorem 4.5 follows immediately by induction on l.

First step: We prove Proposition 4.7. Assume that the assertion of Theorem 4.5 holds for $l < m$. We study the case $l = m$. Let us recall

$$P(x, y) = L(x) + R(x, y), \quad L(x) = \sum_{j=1}^{k} F^j x_j$$

where $k = d_m - 1$ and $\{I, F^j\}$ span the space $M_m^h(\mathbb{C})$ (resp. $M_m^s(\mathbb{R})$) over $\mathbb{R}$. Let $S(a) = \epsilon I + \mathrm{diag}(a_1, \ldots, a_m)$, $|a_i| < \epsilon$ where $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and set

$$P_1(x, y, a) = S(a)^{-1} P(x, y) S(a).$$

Introducing the polar coordinates $x = r\omega$ we study

$$\tilde{P}(r, \omega, y, a) = r^{-1} P_1(r\omega, y, a)$$

near $(r, \omega, y, a) = (0, \omega, 0, 0)$.

Lemma 4.15. *All eigenvalues of* $\tilde{P}(r,\omega,y,a)$ *are real near* $(0,\omega,0,0)$ *with* $\omega \neq 0$. *The multiplicity of eigenvalues of* $\tilde{P}(0,\omega,0,0)$ *are less than m if* $\omega \neq 0$.

Proof. The first assertion is clear. Recall that

$$\tilde{P}(r,\omega,y,a) = S(a)^{-1}\left(L(\omega) + P^2(\omega;r,y) + O(|(r,y)|^2)\right)S(a)$$

with some $m \times m$ matrix $P^2(\omega;r,y)$ which is linear in (r,y) so that

$$\tilde{P}(0,\omega,0,0) = L(\omega). \tag{4.77}$$

If $L(\omega)$, $\omega \neq 0$ has an eigenvalue $\lambda \in \mathbb{R}$ of multiplicity m then it follows that $L(\omega) - \lambda I = O$ because $L(\omega)$ is Hermitian (resp. symmetric). This contradicts the fact that $\{I, F^j\}$ are linearly independent. Hence the assertion. □

We fix $\omega \neq 0$ and choose a unitary (resp. an orthogonal) T_0 so that

$$T_0^{-1}L(\omega)T_0 = \oplus_{j=1}^p \lambda_j I_{s_j}$$

where λ_j are different from each other and $p \geq 2$ as was seen above. Taking into account

$$\begin{aligned}
&S(a) = \epsilon I + O(|a|), \quad S^{-1}(a) = \epsilon^{-1}I + O(|a|),\\
&P^2(\omega+\theta;r,y) = P^2(\omega;\theta,y) + O(|(r,\theta,y)|^2),\\
&L(\omega+\theta) = L(\omega) + L(\theta)
\end{aligned}$$

we set

$$\begin{aligned}
Q(r,\theta,y,a) &= r^{-1}T_0^{-1}P_1(r(\omega+\theta),y,a)T_0 = T_0^{-1}\tilde{P}(r,\omega+\theta,y,a)T_0\\
&= \tilde{L}(\omega) + \tilde{L}(\theta) + \tilde{P}^2(\omega;r,y,a) + O(|(r,\theta,y,a)|^2)
\end{aligned}$$

where $\tilde{L}(\omega) = \oplus\lambda_j I_{s_j}$, $\tilde{L}(\theta) = T_0^{-1}L(\theta)T_0$ and $\tilde{P}^2(\omega;r,y,a)$ is linear in (r,y,a). It is also clear that with $\tilde{L}(\theta) = \sum_{j=1}^k \tilde{F}^j\theta_j$, the matrices $\{I,\tilde{F}^j\}$ span $M_m^h(\mathbb{C})$ (resp. $M_m^s(\mathbb{R})$).

Note that the coefficients of a_j in $\tilde{P}^2(\omega;r,y,a)$ are anti-Hermitian (resp. anti-symmetric) although the fact is not used in the sequel.

Set $Q(r,\theta,y,a) = (Q_{ij}(r,\theta,y,a))$ then it is well known that there is a real analytic $T(r,\theta,y,a)$ defined near the origin with $T(0) = I$ such that

$$QT = T(\oplus_{j=1}^p \tilde{Q}_j) \tag{4.78}$$

(see for example [72]). We need a little bit more information on $\tilde{Q}_j$. Let $T = (T_{ij})$ with $T_{ii} = I_{s_i}$ then (4.78) yields

$$\sum Q_{ik}T_{ki} = \tilde{Q}_i, \quad \sum Q_{ik}T_{kj} = T_{ij}\tilde{Q}_j, \ i \neq j. \tag{4.79}$$

Plugging the first term of (4.79) into the second, we get

$$Q_{ii}T_{ij} - T_{ij}Q_{jj} = \sum_{k\neq j} T_{ij}Q_{jk}T_{kj} - \sum_{k\neq i} Q_{ik}T_{kj}$$

and hence for $i \neq j$

$$(\lambda_i - \lambda_j)T_{ij} = -Q_{ij} + O(|(r,\theta,y,a)|\sum_{k\neq l}|T_{kl}|). \tag{4.80}$$

By the implicit function theorem one can solve (4.80) so that $T_{ij} = T_{ij}(r,\theta,y,a)$, $T_{ij}(0) = 0$. Plugging T_{ij} into (4.79) we get $\tilde{Q}_i$ to be

$$\tilde{Q}_i(r,\theta,y,a) = Q_{ii}(r,\theta,y,a) + O(|(r,\theta,y,a)|^2).$$

We summarize what we have proved; there is a real analytic $T(r,\theta,y,a)$ defined near the origin with $T(0) = I$ such that

$$Q(r,\theta,y,a)T(r,\theta,y,a) = T(r,\theta,y,a)(\oplus_{j=1}^{p}\tilde{Q}_j(r,\theta,y,a))$$

where $\tilde{Q}_j(r,\theta,y,a)$ verifies

$$\tilde{Q}_j(r,\theta,y,a) = \lambda_j I_{s_j} + \tilde{L}_{jj}(\theta) + \tilde{P}_{jj}^2(\omega;r,y,a) + O(|(r,\theta,y,a)|)^2.$$

Here we have written $\tilde{L}(\theta) = (\tilde{L}_{ij}(\theta))$, $\tilde{P}^2(\omega;r,y,a) = (\tilde{P}_{ij}^2(\omega;r,y,a))$ and the blocking corresponds to that of $\oplus\lambda_j I_{s_j}$.

Lemma 4.16. *All eigenvalues of $\tilde{Q}_j(r,\theta,y,a)$ are real near $(r,\theta,y,a) = (0,0,0,0)$. In a new system of local coordinates (r,ψ,y,a), where ψ is linear in (r,θ,y,a), $\tilde{Q}_j$ takes the form*

$$\tilde{Q}_j(r,\psi,y,a) = (\lambda_j + b_j(r,\psi,y,a))I_{s_j} + \sum_{i=1}^{r_j}\tilde{F}_{jj}^i\psi_i + O(|(r,\psi,y,a)|^2)$$

with $r_j = d_{s_j} - 1$ where $b_j(r,\psi,y,a)$ is linear in (r,ψ,y,a) and $\{I_{s_j},\tilde{F}_{jj}^i\}$ span $M_{s_j}^h(\mathbb{C})$ (resp. $M_{s_j}^s(\mathbb{R})$).

Proof. It is clear that all eigenvalues of $\tilde{Q}_j(r,\theta,y,a)$ are real near the origin because so are those of $Q(r,\theta,y,a)$ by Lemma 4.15. We next show that $\tilde{P}^2_{jj}(\omega;r,y,a)$ is Hermitian (resp. symmetric). Recall that

$$\tilde{L}_{jj}(\theta) = \sum_i \tilde{F}^i_{jj}\theta_i$$

where $\{\tilde{F}^i_{jj}\}$ are Hermitian (resp. symmetric) matrices and, together with I_{s_j}, span the space $M^h_{s_j}(\mathbb{C})$ (resp. $M^s_{s_j}(\mathbb{R})$) over $\mathbb{R}$ since $\tilde{F}^i = T_0^{-1}F^iT_0$ and I span the $M^h_m(\mathbb{C})$ (resp. $M^s_m(\mathbb{R})$). Take $\bar{\theta}$, $\tau \in \mathbb{R}$ so that

$$\tilde{L}_{jj}(\bar{\theta}) + \tilde{P}^2_{jj}(\omega;r,y,a) = \tilde{P}^{2(ah)}_{jj}(\omega;r,y,a) + \tau I_{s_j}$$

where $\tilde{P}^{2(ah)}_{jj}(\omega;r,y,a)$ denotes the anti-Hermitian (resp. anti-symmetric) part of $\tilde{P}^2_{jj}(\omega;r,y,a)$. Then we have

$$\tilde{Q}_j(\mu r,\mu\bar{\theta},\mu y,\mu a) = (\lambda_j+\mu\tau)I_{s_j} + \mu\tilde{P}^{2(ah)}_{jj}(\omega;r,y,a) + O(\mu^2).$$

If $\tilde{P}^{2(ah)}_{jj}(\omega;r,y,a) \neq O$ then $\tilde{Q}_j(\mu r,\mu\bar{\theta},\mu y,\mu a)$ has non-real eigenvalues, taking μ small enough, and hence a contradiction. Thus we can write

$$\tilde{P}^2_{jj}(\omega;r,y,a) = \sum_i c_i(r,y,a)\tilde{F}^i_{jj} + c_0(r,y,a)I_{s_j}$$

where $c_i(r,y,a)$ are linear functions of (r,y,a) so that

$$\tilde{Q}_j(r,\theta,y,a) = (\lambda_j + c_0(r,y,a))I_{s_j} + \sum_i \tilde{F}^i_{jj}(\theta_i + c_i(r,y,a)) + O(|(r,\theta,y,a)|^2).$$

Renumbering $\{\tilde{F}^i_{jj}\}$, if necessary, we may suppose that $\{I,\tilde{F}^1_{jj},\ldots,\tilde{F}^{r_j}_{jj}\}$ are linearly independent so that

$$\sum_i \tilde{F}^i_{jj}(\theta_i + c_i(r,y,a)) = \sum_{i=1}^{r_j}\tilde{F}^i_{jj}\psi_i(r,\theta,a).$$

This proves the assertion. □

By Lemma 4.16, each $\tilde{Q}_j(r,\theta,y,a) - (\lambda_j + b_j)I_{s_j}$ verifies the hypothesis of Theorem 4.5 with $l = r_j < m$ and hence there are positive definite $K_j(r,\theta,y,a)$ which are real analytic near the origin such that

$$\tilde{Q}_j(r,\theta,y,a)K_j(r,\theta,y,a) = K_j(r,\theta,y,a)\tilde{Q}^*_j(r,\theta,y,a),$$
$$K^*_j(r,\theta,y,a) = K_j(r,\theta,y,a)$$

with $K_j(0) = I_{s_j}$. Let us define $K(r, \theta, y, a)$ as

$$K(r, \theta, y, a) = \oplus_{j=1}^{p} K_j(r, \theta, y, a)^{1/2}$$

so that

$$K(r, \theta, y, a)^{-1}(\oplus_{j=1}^{p} \tilde{Q}_j(r, \theta, y, a))K(r, \theta, y, a) = \text{Hermitian (resp. symmetric).}$$

With $V = T(r, \theta, y, a)K(r, \theta, y, a)$, this shows that

$$V^{-1}Q(r, \theta, y, a)V = \text{Hermitian (resp. symmetric).}$$

Setting $U = ST_0TK$ we conclude that $U^{-1}P(r(\omega + \theta), y)U$ becomes Hermitian (resp. symmetric) and hence

$$P(r(\omega + \theta), y)UU^* = UU^*P^*(r(\omega + \theta), y).$$

Since $UU^* = ST_0T(KK^*)T^*T_0^*S$, noting that

$$KK^* = \oplus K_j = I + O(|(r, \theta, y, a)|), \quad T_0TT^*T_0^* = I + O(|(r, \theta, y, a)|)$$

we see that

$$UU^* = S(a)(I + K')S(a)$$

where $K' = O(|(r, \theta, y, a)|)$. Hence every diagonal entry of UU^* takes the form

$$\epsilon^2 + 2\epsilon a_i + a_i^2 + O(\epsilon^2|(r, \theta, y, a)|) + O(|(r, \theta, y, a)|^2).$$

Now taking $\epsilon > 0$ small enough, by the implicit function theorem one can solve $a(r, \theta, y) = (a_1(r, \theta, y), \dots, a_m(r, \theta, y))$ so that $a_i(0) = 0$ and

$$\text{every diagonal entry of } UU^* = \epsilon^2$$

where $a(r, \theta, y)$ is real analytic near the origin. With

$$H(r, \phi, y) = \epsilon^{-2}U(r, \phi - \omega, y, a(r, \phi - \omega, y))U^*(r, \phi - \omega, a(r, \phi - \omega, y))$$

which is real analytic near $(0, \omega, 0)$ we conclude that

$$P(r\phi, y)H(r, \phi, y) = H(r, \phi, y)P^*(r\phi, y)$$

where all diagonal entries of $H(r, \phi, y)$ are 1. Since $\omega \neq 0$ is arbitrary the proof of Proposition 4.7 is completed.

Second step: We prove Proposition 4.8. We begin with proving Lemma 4.14. Recall that

$$r^{-1}P(r\omega, y) = L(\omega) + \sum_{j+|\alpha|\geq 1} r^j y^\alpha R_{j\alpha}(\omega), \quad R_{j\alpha}(\omega) = \sum_{|\beta|=j+1} R_{j\alpha\beta}\omega^\beta$$

with constant $m \times m$ matrices $R_{j\alpha\beta}$ so that $R_{j\alpha}(\omega)$ is a homogeneous polynomial in ω of degree $j+1$.

Lemma 4.17. *Let $H_i(r, \omega, y)$, $i = 1, 2$ be real analytic Hermitian (resp. symmetric) $m \times m$ matrix with diagonal entries 1 defined in open neighborhoods $\mathscr{U}_i = I \times U_i \times J$ of $(0, \omega_i, 0)$ such that*

$$P(r\omega, y)H_i(r, \omega, y) = H_i(r, \omega, y)P^*(r\omega, y) \quad \text{in } \mathscr{U}_i. \tag{4.81}$$

Then we have $H_1(r, \omega, y) = H_2(r, \omega, y)$ in $\mathscr{U}_1 \cap \mathscr{U}_2$.

Proof. We expand $H_i(r, \omega, y)$ around $(r, y) = (0, 0)$

$$H_i(r, \omega, y) = \sum_{j,\alpha} r^j y^\alpha H_{ij\alpha}(\omega), \quad H_{ij\alpha}(\omega) \in \mathscr{A}(U_i).$$

Then (4.81) yields

$$\sum_{j+k=p,\alpha+\beta=\gamma} R_{j\alpha}(\omega)H_{ik\beta}(\omega) = \sum_{j+k=p,\alpha+\beta=\gamma} H_{ik\beta}(\omega)R^*_{j\alpha}(\omega)$$

where $R_{00}(\omega) = L(\omega) = R^*_{00}(\omega)$. Hence we get

$$\begin{aligned}&[L(\omega), H_{ip\gamma}(\omega)]\\ &\quad = \sum_{j+k=p,\alpha+\beta=\gamma, j+|\alpha|\geq 1} H_{ik\beta}(\omega)R^*_{j\alpha}(\omega) - R_{j\alpha}(\omega)H_{ik\beta}(\omega).\end{aligned} \tag{4.82}$$

Note that the right-hand side of (4.82) is anti-Hermitian (resp. anti-symmetric).

For the time being we stop to continue the proof and we make more detailed look on (4.82) than needed here, which will give a key of the proof of Proposition 4.8.

Let $L \in M_m(\mathbb{C})$. We consider the mapping from $H \in M_m^h(\mathbb{C})$ with the zero diagonal entries to the space consisting of off diagonal entries of $m \times m$ matrices defined by

$$H \mapsto \text{off diagonal entries of } [L, H].$$

This is a linear mapping from the real $m(m-1)$ dimensional linear space V consisting of H to the linear space W of real dimension $m(m-1)$ consisting of off diagonal entries of $m \times m$ matrices. These vector spaces admit complex structures and we are naturally identifying $\mathbb{C}^{m(m-1)/2}$ to $\mathbb{R}^{m(m-1)}$. We denote by S the representation matrix with respect to fixed bases of V and W.

Lemma 4.18. *Let* $L = (l_{ij})$. *Then there is a real polynomial* $f \in \mathbb{R}[\operatorname{Re} l_{ij}, \operatorname{Im} l_{ij}]$ *such that* $\det S = f^2$.

Proof. [1] In the proof we are regarding components l_{ij} and $l_{ji} = \overline{l_{ij}}$ are independent variables. We write $H = (h_{ij})$ where $h_{ii} = 0$ and $h_{ji} = \overline{h_{ij}}$. We identify $(h_{ij}) \in V$ with the complex vector $(h_{12}, h_{13}, \ldots, h_{m-1,m}, \overline{h_{12}}, \overline{h_{13}}, \ldots, \overline{h_{m-1,m}})$ and $(c_{ij}) \in W$ with $(c_{12}, c_{13}, \ldots, c_{m-1,m}, -\overline{c_{12}}, -\overline{c_{13}}, \ldots, -\overline{c_{m-1,m}})$. Sometimes we write $z = (h_{12}, \ldots, h_{m-1,m}) \in \mathbb{C}^{m(m-1)/2}$ and $V = \{(z, \bar{z}) \mid z \in \mathbb{C}^{m(m-1)/2}\}$ and also write $Z = (c_{12}, \ldots, c_{m-1,m})$ and $W = \{(Z, -\bar{Z}) \mid Z \in \mathbb{C}^{m(m-1)/2}\}$. We represent S with respect to these bases and write

$$S \begin{bmatrix} z \\ \bar{z} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \end{bmatrix} = \begin{bmatrix} Z \\ -\bar{Z} \end{bmatrix}$$

where $S_{ij} \in M_{m(m-1)/2}(\mathbb{C})$. Since we have

$$-\left(\overline{S_{11} z + S_{12} \bar{z}}\right) = S_{21} z + S_{22} \bar{z}$$

for any $z \in \mathbb{C}^{m(m-1)/2}$ we have $S_{22} = -\overline{S_{11}}$ and $S_{21} = -\overline{S_{12}}$. We now show that S is a Hermitian matrix. This is checked by direct calculation. Let $L = (a_{ij})_{1 \le i,j \le m}$. (Here we use the letter a since the letter l seems confusing.) We may write $S = (s_{(i,j)}, s_{(k,l)})$ $(1 \le i \ne j \le m,\ 1 \le k \ne l \le m)$ since components of V and W are indexed by (i, j) $(1 \le i \ne j \le m)$. We compare $s_{(i,j),(k,l)}$ and $s_{(k,l),(i,j)}$ and show that

$$\overline{s_{(k,l),(i,j)}} = s_{(i,j),(k,l)}.$$

We determine $s_{(i,j),(k,l)}$. Since

$$c_{ij} = \sum_{p=1}^{m} a_{ip} h_{pj} - \sum_{p=1}^{m} h_{ip} a_{pj}$$

then $s_{(i,j),(k,l)}$ is the coefficient of h_{kl} of c_{ij}.

(i) If $(i, j) = (k, l)$, then we have

$$s_{(i,j),(i,j)} = a_{ii} - a_{jj}$$

which is a real number.

[1] We owe the proof of this lemma to T. Ibukiyama.

(ii) If $i = k$ and $j \neq l$, then

$$s_{(i,j),(k,l)} = -a_{lj}$$

so that $s_{(k,l),(i,j)} = -a_{jl} = -\overline{a_{lj}} = \overline{s_{(i,j),(k,l)}}$.

(iii) If $j = l$ and $i \neq k$, then

$$s_{(i,j),(k,l)} = a_{ik}$$

so that $s_{(k,l),(i,j)} = a_{ki} = \overline{a_{ik}} = \overline{s_{(i,j),(k,l)}}$.

(iv) If $i \neq k$ and $j \neq l$, then we have

$$s_{(i,j),(k,l)} = s_{(k,l),(i,j)} = 0.$$

These proves $S^* = S$. We summarize what we have checked.

(1) S is Hermitian and moreover

$$S = \begin{bmatrix} S_{11} & S_{12} \\ -\overline{S_{12}} & -\overline{S_{11}} \end{bmatrix}.$$

(2) If we write $S_{11} = A_1 + iB_1$ and $S_{12} = A_2 + iB_2$ with $A_j, B_j \in M_{m(m-1)/2}(\mathbb{R})$ then A_1 is symmetric and A_2, B_1, B_2 are anti-symmetric, that is ${}^tA_1 = A_1$, ${}^tA_2 = -A_2$, ${}^tB_j = -B_j$ for $j = 1, 2$.

Indeed the relation ${}^t\overline{S_{11}} = {}^tA_1 - i\,{}^tB_1 = S_{11} = A_1 + iB_1$ shows that ${}^tA_1 = A_1$ and ${}^tB_1 = -B_1$. Since S is Hermitian and hence $-\overline{S_{12}} = {}^t\overline{S_{12}}$ it follows that ${}^tA_2 = -A_2$ and ${}^tB_2 = -B_2$.

We now prove that a representation matrix of S can be taken to be an anti-symmetric matrix by a suitable change of basis. We write down matrices with respect to the real coordinates. Recall

$$\begin{bmatrix} Z \\ -\bar{Z} \end{bmatrix} = S \begin{bmatrix} z \\ \bar{z} \end{bmatrix}.$$

So writing $z = x + iy$ and $Z = X + iY$ for real vectors x, y, X, Y we have

$$\begin{bmatrix} E_m & iE_m \\ -E_m & iE_m \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = S \begin{bmatrix} E_m & iE_m \\ E_m & -iE_m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We put

$$T = \begin{bmatrix} E_m & iE_m \\ -E_m & iE_m \end{bmatrix}^{-1} S \begin{bmatrix} E_m & iE_m \\ E_m & -iE_m \end{bmatrix}.$$

Then we have

$$T = \frac{1}{2}\begin{bmatrix} S_{11} + \overline{S_{11}} + S_{12} + \overline{S_{12}} & i(S_{11} - \overline{S_{11}} + \overline{S_{12}} - S_{12}) \\ -i(S_{11} - \overline{S_{11}} + S_{12} - \overline{S_{12}}) & S_{11} + \overline{S_{11}} - S_{12} - \overline{S_{12}} \end{bmatrix}$$
$$= \begin{bmatrix} A_1 + A_2 & -B_1 + B_2 \\ B_1 + B_2 & A_1 - A_2 \end{bmatrix}.$$

Then the matrix

$$T_1 = \begin{bmatrix} O & -E_m \\ E_m & O \end{bmatrix} T = \begin{bmatrix} -B_1 - B_2 & -A_1 + A_2 \\ A_1 + A_2 & -B_1 + B_2 \end{bmatrix}$$

is an anti-symmetric matrix, in fact since B_i are anti-symmetric and ${}^t(-A_1 + A_2) = -{}^tA_1 + {}^tA_2 = -A_1 - A_2 = -(A_1 + A_2)$. So $\det T_1$ is a square of the Pfaffian, that is $\det T_1 = f^2$ where f is a real polynomial in components of T_1, that is a real polynomial in components of A_i and B_i and hence a real polynomial in $(\operatorname{Re} l_{ij}, \operatorname{Im} l_{ij})$. Thus $\det S$ is also a square of a real polynomial in $(\operatorname{Re} l_{ij}, \operatorname{Im} l_{ij})$. □

We now check

Lemma 4.19. *Let f be in Lemma 4.18. Then f is irreducible in $\mathbb{R}[\operatorname{Re} l_{ij}, \operatorname{Im} l_{ij}]$ and $\{f = 0\}$ contains a regular point.*

We postpone the proof until stating the next lemma. We now consider the real symmetric case, that is $L \in M_m^s(\mathbb{R})$ and study the mapping from $H \in M_m^s(\mathbb{R})$ with the zero diagonal entries to the space consisting of off diagonal entries of $m \times m$ real matrices defined by

$$H \mapsto \text{off diagonal entries of } [L, H].$$

This is a linear mapping from the real $m(m-1)/2$ dimensional linear space V consisting of such H to the linear space W of real dimension $m(m-1)/2$ consisting of off diagonal entries of $m \times m$ real matrices. We denote by S again the representation matrix with respect to fixed bases of V and W.

Lemma 4.20. *Let us write $L = (l_{ij})$. Then $\det S$ is irreducible in $\mathbb{R}[l_{ij}]$ and $\{\det S = 0\}$ contains a regular point.*

Proof. [2] We write $H = (h_{ij})$ where $h_{ii} = 0$ and $h_{ji} = h_{ij}$. We identify $(h_{ij}) \in V$ with $(h_{12}, h_{13}, \ldots, h_{m-1,m})$ and $(c_{ij}) \in W$ with $(c_{12}, c_{13}, \ldots, c_{m-1,m})$. We represent S with respect to these bases. In the proof of Lemma 4.18, putting $B_j = O$ and $\operatorname{Im} h_{ij} = 0$, we easily see that

$$S = A_1 + A_2.$$

[2] Another proof is found in [53].

Let us write $A_1 + A_2 = (X_{ij})$ then it is clear that $X_{ii} = l_{ii} - l_{jj}$ and for (i, j), $i \neq j$ we have either $X_{ij} = 0$ or $X_{ij} = l_{pq}$ with some (p, q) with $p \neq q$. Suppose that $\det S$ is reducible so that $\det S = fg$ where f, g are homogeneous polynomials in l_{ij} of degree greater than or equal to one. Assume that f contains X_{ii} and assume that the i-th row of S consists of $\{l_{pq}\}_{(p,q)\in J}$ and 0. Note that the i-th column consists of the same $\{l_{pq}\}_{(p,q)\in J}$ and 0 because ${}^tA_1 = A_1$ and ${}^tA_2 = -A_2$. Replace these l_{pq}, $(p, q) \in J$, by λl_{pq} with $\lambda > 0$. Then, in the $\det S$, the coefficient of X_{ii} is multiplied by λ. This proves that g is independent of these l_{pq}, $(p, q) \in J$. Renumbering if necessary we may assume that f contains $X_{11},\ldots,X_{rr}$ and g contains $X_{r+1,r+1},\ldots,X_{NN}$ with $N = m(m-1)/2$. From the above arguments it follows that f is a polynomial in $(X_{ij})_{1\le i,j\le r}$ and g is a polynomial in $(X_{ij})_{r+1\le i,j\le N}$ so that $\det S$ is independent of X_{ij} with $1 \le i \le r$, $r+1 \le j \le N$. This is a contradiction. Indeed it is easy to check that there is (i^*, j^*) with $1 \le i^* \le r$, $r+1 \le j^* \le N$ such that $X_{i^*j^*} = l_{pq}$ and this shows that $\det S$ contains the term

$$l_{pq}^2 \prod_{i\neq i^*,j^*} X_{ii}$$

up to the sign. Thus we have proved that $\det S$ is irreducible in $\mathbb{R}[l_{ij}]$. Let us set

$$S'(l_{ij}) = S(l_{ij})\big|_{l_{ij}=0, i\neq j}$$

then it is obvious that $\det S' = \prod_{i<j}(l_{ii} - l_{jj})$ which clearly shows $\{\det S' = 0\}$ contains a regular point. This proves that $\{\det S = 0\}$ contains a regular point clearly. □

Proof of Lemma 4.19. Let us put $\operatorname{Im} l_{ij} = 0$. Then it follows that

$$\det S = (\det(A_1 + A_2))^2$$

up to the sign. This shows that $f(\operatorname{Im} l_{ij} = 0) = \det(A_1 + A_2)$ up to the sign. Thus the assertion follows from Lemma 4.20. □

Completion of the Proof of Lemma 4.17. Assume that

$$[L(\omega), H(\omega)] = C(\omega).$$

Introduce a new system of coordinates $\theta = \big((\operatorname{Re} l_{ij}(\omega))_{i\le j}, (\operatorname{Im} l_{ij}(\omega))_{i<j}\big)$. From Lemmas 4.18 and 4.19 it follows that the above equation can be written with

$$\check{H}(\theta) = (h_{12}, h_{13}, \ldots, h_{m-1,m}, \overline{h_{12}}, \overline{h_{13}}, \ldots, \overline{h_{m-1,m}}),$$
$$\check{C}(\theta) = (c_{12}, c_{13}, \ldots, c_{m-1,m}, -\overline{c_{12}}, -\overline{c_{13}}, \ldots, -\overline{c_{m-1,m}})$$

so that

$$S_L(\theta)\check{H}(\theta) = \check{C}(\theta)$$

where $\det S_L(\theta) = f(\theta)^2$ with an irreducible f. We turn to (4.82). Let $H(r, \omega, y) = H_1(r, \omega, y) - H_2(r, \omega, y)$. Then with

$$H(r, \theta, y) = \sum_{j+|\alpha|\geq 0} r^j y^\alpha H_{j\alpha}(\theta), \tag{4.83}$$

$$C_{p\gamma}(\theta) = \sum_{j+k=p, \alpha+\beta=\gamma, j+|\alpha|\geq 0} H_{k\beta}(\theta) R^*_{j\alpha}(\theta) - R_{j\alpha}(\theta) H_{k\beta}(\theta)$$

(4.82) can be written as

$$S_L(\theta)\check{H}_{00}(\theta) = O, \quad S_L(\theta)\check{H}_{p\gamma}(\theta) = \check{C}_{p\gamma}(\theta), \quad p + |\gamma| \geq 1. \tag{4.84}$$

Note that $H_{j\alpha}(\theta)$ are Hermitian and the diagonal entries of $H_{j\alpha}(\theta)$ are 0 by the assumption. Since $\det S_L(\theta) \neq 0$ on a dense subset we conclude that $\check{H}_{00}(\theta, \phi) = O$ and hence $H_{00}(\theta) = O$. Then $C_{p\gamma}(\theta) = O$ for $p + |\gamma| = 1$ from (4.83). By induction on $p + |\gamma|$ it follows that $H_{p\gamma}(\theta) = O$ for all $p + |\gamma| \geq 0$. □

Completion of the Proof of Lemma 4.14. Suppose that at every $(0, \bar{\omega}, 0)$, $\bar{\omega} \neq 0$ there is an positive definite real analytic symmetrizers $H(r, \omega, y)$ with all diagonal entries 1 defined near $(0, \bar{\omega}, 0)$. By Lemma 4.17 these symmetrizers are continued analytically and yields $H(r, \omega, y)$ which is positive definite with all diagonal entries 1 and real analytic in a neighborhood of $\{0\} \times S^{k-1} \times \{0\}$. □

Remark. Note that $H_{j\alpha}(\theta)$ are Hermitian (resp. symmetric) and the diagonal entries of $H_{00}(\theta)$ and $H_{j\alpha}(\theta)$, $j + |\alpha| \geq 1$ are 1 and 0 respectively and $\check{H}_{j\alpha}(\theta)$ verifies (4.84). Since $S_L(\theta)$ is linear in θ and $R_{j\alpha}(\theta)$ are homogeneous of degree $j + 1$ in θ, then by the homogeneity, $\check{H}_{j\alpha}(\theta)$ extends uniquely to a homogeneous function in $\mathbb{R}^k \setminus \{0\}$ of degree j with respect to θ. Then $H_{j\alpha}(\theta)$ extends there as a homogeneous function of degree j in θ.

Proof of Proposition 4.8. We prove the case that $L(\omega)$ is Hermitian since the real case is similar. Let $H(r, \omega, y)$ be positive definite and satisfy (4.76). With the coordinates (r, θ, y), we again expand $H(r, \theta, y)$ around $(r, y) = (0, 0)$

$$H(r, \theta, y) = \sum_{j,\alpha} r^j y^\alpha H_{j\alpha}(\theta), \quad H_{j\alpha}(\theta) \in \mathscr{A}(S^{k-1})$$

where $H_{j\alpha}(\theta)$ are Hermitian and all diagonal entries of $H_{00}(\theta)$ and $H_{j\alpha}(\theta)$, $j + |\alpha| \geq 1$ are 1 and 0 respectively. As before $H_{p\gamma}(\theta)$ verifies

$$[L(\theta), H_{p\gamma}(\theta)] = C_{p\gamma}(\theta)$$

where $C_{p\gamma}(\theta)$ is given by (4.83). The same argument as in the proof of Lemma 4.17 gives that

$$H_{00}(\theta) = I.$$

Then it follows that $C_{p\gamma}(\theta) = R^*_{p\gamma}(\theta) - R_{p\gamma}(\theta)$ for $p + |\gamma| = 1$ which is a homogeneous polynomial of θ of degree 2. Recall that there is an $m(m-1) \times m(m-1)$ matrix $S_L(\theta)$ whose entries are linear functions of θ such that $H_{p\gamma}(\theta)$ satisfies

$$S_L(\theta)\check{H}_{p\gamma}(\theta) = \check{C}_{p\gamma}(\theta).$$

Moreover $\det S_L(\theta) = f(\theta)^2$ where $f(\theta)$ is irreducible in $\mathbb{R}[\theta]$ and $\{f(\theta) = 0\}$ contains a regular point. Let us denote by ${}^{co}S_L(\theta)$ the cofactor matrix of $S_L(\theta)$. It is clear that $\check{H}_{p\gamma}(\theta)$, $p + |\gamma| = 1$ verifies

$$f(\theta)\check{H}_{p\gamma}(\theta) = {}^{co}S_L(\theta)\check{C}_{p\gamma}(\theta)/f(\theta) = (f_{ij}(\theta)/f(\theta)).$$

Recalling that $C_{p\gamma}(\theta) = R^*_{p\gamma}(\theta) - R_{p\gamma}(\theta)$ for $p + |\gamma| = 1$ we see that $f_{ij}(\theta)$ are homogeneous polynomials of degree $m(m-1) + 1$ in θ. Since $\check{H}_{p\gamma}(\theta)$ is real analytic in $\mathbb{R}^k \setminus \{0\}$ as remarked after the proof of Lemma 4.17 it follows that $f_{ij}(\theta)$ vanishes on $\{f(\theta) = 0\}$. Since $f(\theta)$ is irreducible and $\{f(\theta) = 0\}$ contains a regular point from Lemma 4.19 we can apply Lemma 2.5 in [41] (for example) to conclude that $f_{ij}(\theta)/f(\theta)$ are homogeneous polynomials of degree $m(m-1)/2+1$ in θ. Thus $f(\theta)\check{H}_{p\gamma}(\theta)$ is a homogeneous polynomial in θ of degree $m(m-1)+1$. Repeating the same arguments we conclude that $\check{H}(\theta)$ is a homogeneous polynomial in θ of degree 1 for $p + |\gamma| = 1$ and so is $H_{p\gamma}(\theta)$ because $H_{p\gamma}(\theta)$ is Hermitian and whose diagonal entries are 0. By (4.83), $C_{p\gamma}(\theta)$, $p + |\gamma| = 2$ becomes a homogeneous polynomial in θ of degree 3. By induction on $j + |\alpha|$ we prove that $H_{j\alpha}(\theta)$ is a homogeneous polynomial of degree j in θ. In the coordinates ω, $H_{j\alpha}$ is a homogeneous polynomial in ω of degree j. Then one can write

$$r^j H_{j\alpha}(\omega) = G_{j\alpha}(r\omega).$$

where $G_{j\alpha}(x)$ is a homogeneous polynomial of degree j in x. Let us define

$$G(x, y) = \sum_{j,\alpha} y^\alpha G_{j\alpha}(x).$$

Since the convergence follows from that of $\sum_{j,\alpha} r^j y^\alpha H_{j\alpha}(\omega)$ then $G(x, y)$ becomes real analytic near $(0, 0)$ and the proof is complete. □

Remark. The arguments proving that $H_{p\gamma}(\theta)$ is a homogeneous polynomial in θ can be applied under less restrictive hypotheses. Let $f(\theta)$, $g(\theta)$ be homogeneous polynomials in θ of degree n, m respectively where $n \geq m$. Let

$$g(\theta) = \prod_{j=1}^{s} g_j(\theta)^{r_j}$$

be the irreducible factorization of $g(\theta)$ in $\mathbb{R}[\theta]$. We assume that $f(\theta)/g(\theta)$ is C^∞ apart from the origin and $V_j = \{\theta | g_j(\theta) = 0\}$, $1 \le j \le s$ contains a regular point. Then applying Lemma 2.5 in [41] again, we conclude that $f(\theta)$ is a homogeneous polynomial in θ of degree $n - m$.

4.6 Well Posed Cauchy Problem

Let us study a differential operator of order q

$$P(x, D) = \sum_{|\alpha| \le q} A_\alpha(x) D^\alpha, \quad D_j = \frac{1}{i} \frac{\partial}{\partial x_j} \tag{4.85}$$

where $A_\alpha(x)$ are $m \times m$ matrix valued smooth functions defined in a neighborhood Ω of the origin of $\mathbb{R}^n$. We assume that $x_1 = const.$ are non characteristic and without restrictions we may assume that

$$A_{(q,0,\ldots,0)}(x) = I. \tag{4.86}$$

We are concerned with the following Cauchy problem

$$\begin{cases} P(x, D)u = f, \ \ \operatorname{supp} f \subset \{x_1 \ge 0\}, \\ \operatorname{supp} u \subset \{x_1 \ge 0\}. \end{cases} \tag{4.87}$$

Let $P_q(x, \xi)$ be the principal symbol of $P(x, D)$

$$P_q(x, \xi) = \sum_{|\alpha| = q} A_\alpha(x) \xi^\alpha$$

and we assume that

$$\det P_q(x, \xi) = 0 \Longrightarrow \xi_1 \text{ is real } \ \forall x \in \Omega, \forall \xi' = (\xi_2, \ldots, \xi_n) \in \mathbb{R}^{n-1}. \tag{4.88}$$

We first study the case that P_q is of constant coefficients.

Theorem 4.6 ([22, 26]). *Let $P(\xi)$ be a homogeneous polynomial of degree q in $\xi \in \mathbb{R}^n$ with real $m \times m$ matrix values such that* $\det P(\xi)$ *satisfies (4.88) and every multiple characteristic of* $\det P(\xi)$ *is at most double and nondegenerate. Then the Cauchy problem for $P(D) + R(x, D)$ is C^∞ well posed for every R of order $q - 1$ with C^∞ $m \times m$ matrix coefficients.*

In [22, 26] their proof is based on the estimate of $P(\xi + i\tau N)^{-1}$ such that

$$|\tau|(|\xi| + |\tau|)^{m-1}|P(\xi + i\tau N)^{-1}| \le C, \quad \text{if } 0 \neq (\tau, \xi) \in \mathbb{R}^{n+1} \tag{4.89}$$

which is derived from the assumption of the non degeneracy of double characteristics. For $f \in C_0^\infty$ with supp $f \subset \{x_1 \ge 0\}$ we look for a solution

$$(P(D) + R(x, D))u = f$$

such that supp $u \subset \{x_1 \ge 0\}$. We set $u = u_\tau e^{\tau N x_1}$ and $f = f_\tau e^{\tau N x_1}$ and obtain the equivalent equation

$$(P(D - i\tau N) + R(x, D - i\tau N))u_\tau = f_\tau. \tag{4.90}$$

Let E_τ be the inverse Fourier transform of $P(\xi - i\tau N)^{-1}$ and set $u_\tau = E_\tau * v$. Then (4.90) becomes

$$v + R(x, D - i\tau N)E_\tau * v = f_\tau. \tag{4.91}$$

On the other hand assuming (4.89) we have

$$\tau\|E_\tau * w\|_{s+m-1,\tau} \le C\|w\|_{s,\tau}, \quad \|w\|^2_{s,\tau} = (2\pi)^{-n}\int |\hat{u}(\xi)|^2(|\xi|^2 + \tau^2)^s d\xi.$$

Thus from this estimate it follows that

$$\|R(x, D - i\tau N)E_\tau * v\|_{s,\tau} \le C'\tau^{-1}\|f_\tau\|_{s,\tau}.$$

Choosing $\tau > 2C'$ we conclude that (4.91) has a unique solution $v \in H^s$.

We prove the following result which extends Theorem 4.6. Our proof is completely different from that in [22, 26] and based on the smooth symmetrizability of corresponding first order system (Proposition 4.1) and hence can be applicable to differential operators with variable coefficients.

Theorem 4.7. *Assume that every characteristic $(0, \xi_1, \xi')$, $|\xi'| = 1$ of $P_q(x, \xi)$ is at most double and nondegenerate. Then the Cauchy problem for $P(x, D)$ is C^∞ well posed near the origin for arbitrary lower order term. Moreover if $\tilde{P}(x, D)$ is another system of the form (4.85) verifying (4.88) with the principal symbol $\tilde{P}_q(x, \xi) = \sum_{|\alpha|=q} \tilde{A}_\alpha(x)\xi^\alpha$ of which $\tilde{A}_\alpha$ are sufficiently close to $A_\alpha(x)$ in $C^2(\Omega)$ then the Cauchy problem for $\tilde{P}(x, D)$ is C^∞ well posed near the origin for any lower order term.*

Assuming the analyticity of the coefficients we have

Theorem 4.8. *Assume that $A_\alpha(x)$, $|\alpha| = q$ are real analytic in Ω and every characteristic $(0, \xi_0, \xi')$, $|\xi'| = 1$ of $P_q(x, \xi)$ is nondegenerate. Then the Cauchy* *problem for $P(x, D)$ is C^∞ well posed near the origin for arbitrary lower order term.*

The proof is very simple. We reduce the Cauchy problem for $P(x, D)$ to that for a first order system $\mathscr{P}(x, D)$. Taking the invariance of non degeneracy of characteristics proved in Proposition 4.1.1, to prove Theorems 4.7 and 4.8, it suffices to apply Proposition 4.2 and Theorem 4.4 respectively which asserts the existence of a smooth symmetrizer $\mathscr{S}(x, D')$ for $\mathscr{P}(x, D)$ defined near the origin.

Let us write

$$P(x, D)u = D_1^q u + \sum_{j=1}^{q} A_j(x, D') D_1^{q-j} u = f. \tag{4.92}$$

Put

$$u^{(k)} = \langle D' \rangle^{q-k} D_1^{k-1} u, \quad k = 1, \dots, q$$

where $\langle D' \rangle^2 = 1 + \sum_{j=2}^{n} D_j^2$. Then (4.92) is reduced to

$$D_1 U + \begin{bmatrix} 0 & -I & & & \\ 0 & 0 & -I & & \\ & & & \ddots & \\ 0 & & & & -I \\ A_q^{\#}(x, D') & & \cdots & & A_1^{\#}(x, D') \end{bmatrix} \langle D' \rangle U = F$$

where $U = {}^t(u^{(1)}, \dots, u^{(q)})$, $F = {}^t(0, \dots, 0, f)$ and

$$A_j^{\#}(x, D') = A_j(x, D') \langle D' \rangle^{-j}.$$

Let us denote by $A_j^0(x, \xi')$ the principal symbol of $A_j^{\#}(x, \xi')$ and set

$$\mathscr{A}(x, \xi') = \begin{bmatrix} 0 & -I & & & \\ 0 & 0 & -I & & \\ & & & \ddots & \\ 0 & & & & -I \\ A_q^0(x, \xi') & & \cdots & & A_1^0(x, \xi') \end{bmatrix}. \tag{4.93}$$

Fix $(0, \bar{\xi}')$, $|\bar{\xi}'| = 1$. Let $(0, \lambda_i, \bar{\xi}')$, $i = 1, \dots, p$, be characteristics of $\xi_1 + \mathscr{A}(x, \xi')$ where $(0, \lambda_i, \bar{\xi}')$ are nondegenerate and λ_i are different from each other. Then there exists a smooth $\mathscr{T}(x, \xi')$ defined near $(0, \bar{\xi}')$, homogeneous of degree 0, such that

$$\mathscr{T}(x, \xi')^{-1} \mathscr{A}(x, \xi') \mathscr{T}(x, \xi') = \mathscr{A}_1(x, \xi') \oplus \cdots \oplus \mathscr{A}_p(x, \xi')$$

where $(0, \lambda_i, \bar{\xi}')$ is a nondegenerate characteristic of $\mathscr{P}_i(x,\xi) = \xi_1 + \mathscr{A}_i(x,\xi')$. Then one can apply Proposition 4.2 or Theorem 4.4 to get a smooth symmetrizer $\mathscr{S}_i(x,\xi')$ of $\mathscr{A}_i(x,\xi')$ defined near $(0,\bar{\xi}')$, homogeneous of degree 0 such that

$$\mathscr{S}_i(x,\xi')^{-1}\mathscr{A}_i(x,\xi')\mathscr{S}_i(x,\xi')$$

is Hermitian. This proves that $\mathscr{A}(x,\xi')$ is smoothly symmetrizable near $(0,\bar{\xi}')$ by $\mathscr{S}_1(x,\xi') \oplus \cdots \oplus \mathscr{S}_p(x,\xi')$. By the usual argument of partition of unity one can prove that there is a smooth $\mathscr{S}(x,\xi')$ which symmetrizes $\mathscr{A}(x,\xi')$. Thus the Cauchy problem for $\mathscr{P}(x,D)$ is C^∞ well posed for arbitrary lower order term and hence so is for $P(x,D)$.

We turn to prove the second assertion of Theorem 4.7. Recall that $(0,\lambda_i,\bar{\xi}')$, $i = 1,\ldots,p$ are characteristics of $\xi_1 + \mathscr{A}(x,\xi')$ where $\{\lambda_i\}$ are different from each other. By assumption each $(0,\lambda_i,\bar{\xi}')$ is either simple characteristic or double nondegenerate characteristic. Let $\xi_1 + \tilde{\mathscr{A}}(x,\xi')$ be the symbol of first order system associated to $\tilde{P}(x,D)$. Let $(0,\lambda_i,\bar{\xi}')$ be a double nondegenerate characteristic of $\xi_1 + \mathscr{A}(x,\xi')$. Since $\tilde{\mathscr{A}}(x,\xi')$ is enough close to $\mathscr{A}(x,\xi')$, as for characteristics of $\xi_1 + \tilde{\mathscr{A}}(0,\bar{\xi}')$ enough close to $(0,\lambda_i,\bar{\xi}')$, we have either two simple characteristics $(0,\tilde{\lambda}_{ik},\bar{\xi}')$ or a double characteristic $(0,\tilde{\lambda}_i,\bar{\xi}')$. From Proposition 4.3 it follows that the double characteristic $(0,\tilde{\lambda}_i,\bar{\xi}')$ is nondegenerate. Thus we conclude that every characteristic of $\xi_1 + \tilde{\mathscr{A}}(0,\bar{\xi}')$ is nondegenerate and then repeating the same arguments as above we get the assertion.

Example 4.9. Consider the second order differential operator $P(x,D) = (p_{ik}(x,D))$ with 3×3 matrix coefficients

$$p_{ik}(x,\tau,\xi) = (\tau^2 - \sigma_i(x)|\xi|^2)\delta_{ik} - (1-\sigma_i(x))\xi_i\xi_k$$

in Example 4.6. Then from Theorem 4.7 it follows that the Cauchy problem for $P(x,D) + R(x,D)$ is C^∞ well posed for every R of first order with C^∞ 3×3 matrix coefficients. Let $A(x,\xi)$ be in Example 4.7. Then the Cauchy problem for $D_0^2 - A(x,D) + R(x,D)$ is C^∞ well posed for every R of first order with C^∞ 3×3 matrix coefficients.

Example 4.10. Let $P(\xi) = \xi_1 I + \sum_{j=2}^d F_j\xi_j$ be the symbol in Example 4.3. Consider

$$P(x,\xi) = \xi_1 I + \sum_{j=2}^d A_j(x)\xi_j$$

where $A_j(x)$ are real valued real analytic $m\times m$ matrices which are enough close to F_j in C^2 and $P(x,\xi)$ has only real eigenvalues for any x and any ξ. Then Theorem 4.8 shows that the Cauchy problem for $P(x,D) + B(x)$ is C^∞ well posed for every smooth $m\times m$ matrix $B(x)$.

4.7 Nondegenerate Characteristics of Symmetric Systems

Let P be a first order system with constant coefficients

$$P(x) = x_1 + \sum_{j=2}^{n} A_j x_j \tag{4.94}$$

where A_j are real $m \times m$ constant matrices. We always assume that $P(x)$ is hyperbolic with respect to $\theta = (1, 0, \dots, 0)$. Then from [34] (see also [13]) $P(x)$ can not be strictly hyperbolic if $n > 3$ and $m \equiv 2$ modulo 4, that is $P(x)$ has necessarily multiple characteristics $x \neq 0$. We want to check whether these multiple characteristics are nondegenerate.

For symmetric systems with constant coefficients the description of non degeneracy of characteristics becomes simple. Consider

$$\mathscr{L}(x) = \sum_{j=1}^{n} A_j x_j$$

where $A_j \in M_m^s(\mathbb{R})$. In this and the following sections we identify a symmetric system $\mathscr{L}(x)$ with the image of $\mathscr{L}(x)$ when x varies in $\mathbb{R}^n$

$$\mathscr{L} = \{\mathscr{L}(x) \mid x \in \mathbb{R}^n\}$$

which is a linear subspace in $M_m^s(\mathbb{R})$. Indeed if $\mathscr{L}$ is a linear subspace of dimension q in $M_m^s(\mathbb{R})$ which contains the identity then choosing a basis $\{I, A_2, \dots, A_q\}$, $A_j \in M_m^s(\mathbb{R})$ for $\mathscr{L}$ we have a symmetric system

$$x_1 I + \sum_{j=2}^{q} A_j x_j$$

and vice versa.

We denote by $M_m^s(k; \mathbb{R})$ the set of all $A \in M_m^s(\mathbb{R})$ with rank $m - k$. Then we have

Lemma 4.21. *In order that $\bar{x}$ is a nondegenerate characteristic of $\mathscr{L}(x)$ of order k if and only if the image $\mathscr{L}$ intersects with $M_m^s(k; \mathbb{R})$ at $\mathscr{L}(\bar{x})$ transversally.*

Proof. Since $\mathscr{L}(\bar{x})$ and $\mathscr{L}_{\bar{x}}(x)$ are symmetric, the conditions (4.15) and (4.17) in Definition 4.5 are automatically satisfied. Without restrictions we may assume that $\bar{x} = (0, \dots, 0, 1)$. Then A_n is of rank $m - k$. We can make an orthogonal transformation of the matrices so that with a block matrix notation we have

$$A_n = \begin{bmatrix} O & O \\ O & G \end{bmatrix}$$

where G is a $(m-k)\times(m-k)$ non singular matrix. The tangent space of $M_m^s(k;\mathbb{R})$ at A_n consists of matrices of the form

$$\begin{bmatrix} O & * \\ * & * \end{bmatrix} \tag{4.95}$$

with the corresponding block decomposition. On the other hand, with the same block decomposition of $\mathscr{L}(x)$

$$\mathscr{L}(x) = \begin{bmatrix} L_{11}(x) & L_{12}(x) \\ L_{21}(x) & L_{22}(x) \end{bmatrix}$$

it is clear that $\mathscr{L}_{\bar{x}}(x) = L_{11}(x)$. Thus the transversality of intersection means that $\dim L_{11} = d_k$ that is, $\dim \mathscr{L}_{\bar{x}} = d_k$ and hence $\bar{x}$ is nondegenerate. The converse follows in the same way. $\square$

We start with the special case that $\dim \mathscr{L} = d_m - 1$. Since $\mathscr{L}$ has codimension one in $M_m^s(\mathbb{R})$ then $\mathscr{L}$ is defined by

$$\mathscr{L} = \{X = (x_{ij}), x_{ij} = x_{ji} \mid \mathrm{Tr}\,(AX) = 0\} \tag{4.96}$$

with some $A \in M_m^s(\mathbb{R})$. Note that $\mathrm{Tr}\,A = 0$ because $\mathscr{L}$ contains the identity. Now we have

Proposition 4.9. *Assume that $\mathscr{L}$ is given by (4.96) with $O \neq A \in M_m^s(\mathbb{R})$ and that the rank of A is greater than k. Then every characteristic of order k of $\mathscr{L}(x)$ is nondegenerate.*

Proof. Let $\bar{x}$ be a characteristic of order k of $\mathscr{L}(x)$ and hence $H = \mathscr{L}(\bar{x}) \in \mathscr{L} \cap M_m^s(k;\mathbb{R})$. Here we note that $\dim T_H(M_m^s(k;\mathbb{R})) = d_m - d_k$ which is seen by the proof of Lemma 4.21. To show $\bar{x}$ is nondegenerate it suffices to prove that

$$\dim\,(\mathscr{L} \cap T_H(M_m^s(k;\mathbb{R})) = d_m - d_k - 1 \tag{4.97}$$

by Lemma 4.21. As in the proof of Lemma 4.21, considering $T^{-1}\mathscr{L}T$ with a suitable $T \in O(m)$ we may assume that

$$H = \begin{bmatrix} O & O \\ O & G \end{bmatrix} \tag{4.98}$$

where G is a $(m-k)\times(m-k)$ non singular matrix. Set $x_{ij} = 0$ for $1 \le i \le j \le k$. Then $\mathrm{Tr}\,(AX) = 0$, $X = (x_{ij})$ implies that

$$\sum_{k+1 \le i \le j \le m} a_{ij}x_{ij} = 0$$

where $A = (a_{ij})$. Recalling that the tangent space $T_H(M_m^s(k;\mathbb{R}))$ spanned by matrices of the form (4.95) we see that $\mathscr{L} \cap T_H(M_m^s(k;\mathbb{R}))$ consists of the matrices of the form

$$X = \begin{bmatrix} O & x_{ij} \\ x_{ij} & x_{ij} \end{bmatrix}, \quad \mathrm{Tr}\,(AX) = \sum_{k+1 \le j, i \le j} (2 - \delta_{ij}) a_{ij} x_{ij} = 0$$

where δ_{ij} is the Kronecker's delta. Since A is symmetric and the rank of A is greater than k by assumption then it follows that $(a_{ij})_{k+1 \le j, i \le j} \ne O$. This proves (4.97) and hence the assertion. □

We turn to the case that $1 \le \dim \mathscr{L} \le d_m - 1$. We first give a parametrization of the Grassmannian of l dimensional subspaces of $M_m^s(\mathbb{R})$ containing the identity.

Take a map

$$\sigma : \{1, \ldots, \nu\} \to \{(i, j) | 1 \le i \le j \le m, (i, j) \ne (m, m)\}$$

which is injective. Denote by U_σ the set of all ν-tuple of $m \times m$ symmetric matrices $A = (A_1, \ldots, A_\nu)$ such that $\mathrm{Tr}\, A_j = 0$ and the $\sigma(k)$-th entry of A_j is zero unless $k = j$ and the $\sigma(j)$-th entry of A_j is 1. It is clear that U_σ can be identified with $\mathbb{R}^{\nu(d_m - \nu - 1)}$. Taking all such injective σ, U_σ and the inverse of the map

$$\phi_\sigma : U_\sigma \ni A \mapsto \mathscr{L}, \ \ \mathscr{L} = \{X \in M_m^s(\mathbb{R}) | \mathrm{Tr}\,(A_j X) = 0, 1 \le j \le \nu\}$$

then $\{(\phi_\sigma^{-1}, \Omega_\sigma = \phi_\sigma(U_\sigma))\}$ give charts of the Grassmannian of $l = d_m - \nu$ dimensional subspaces of $M_m^s(\mathbb{R})$ containing I, which we denote by $G^l_{d_m, I}$.

Proposition 4.10. *In the Grassmannian $G^l_{d_m, I}$ consisting of l dimensional subspaces of $M_m^s(\mathbb{R})$ containing the identity I, the subset for which every characteristic of order less than m is nondegenerate is an open and dense subset.*

Let $\mathbf{P}^N(\mathbb{R})$ be the N dimensional real projective space and let $X \subset \mathbf{P}^N(\mathbb{R})$ be a non-singular algebraic manifold of dimension r and assume that $x_0 \notin T_x X$ for all $x \in X$. Let us denote

$$\tilde{G}^s_{N, x_0} = \{W \subset \mathbf{P}^N(\mathbb{R}) | W \text{ is a linear space}, \dim W = s, x_0 \in W\}$$

and set $s' = N - s$. Then we have

Lemma 4.22. *A generic $W \in \tilde{G}^s_{N, x_0}$ intersects X transversally.*

Proof.[3] Let $Y = \{(x, W) \in X \times \tilde{G}^s_{N, x_0} | x \in W\}$ and denote by p_1, p_2 the projections onto X and $\tilde{G}^s_{N, x_0}$ respectively. Note that $\dim Y = s's - s' + r$ and

[3]The author owes this simple proof to A. Gyoja.

$\dim \tilde{G}^s_{N,x_0} = s's$. Then if $r < s'$ a generic $W \in \tilde{G}^s_{N,x_0}$ does not intersect X and hence the result. Thus it is enough to study the case $r \geq s'$. Let us set

$$Z = \{(x, W) \in Y | \dim (T_x X + W) \leq N - 1\}.$$

It is not difficult to see that

$$\dim (p_1|Z)^{-1}(x) = ss' - r - 1, \quad x \in X$$

so that $\dim Z = ss' - 1 = \dim \tilde{G}^s_{N,x_0} - 1$. Thus for every W belonging to the open dense subset $\tilde{G}^s_{N,x_0} \setminus \overline{p_2(Z)}$, W intersects X transversally. This proves the assertion. □

Proof of Proposition 4.10. Take X and $\tilde{G}^s_{N,x_0}$ as the projective spaces $M^s_m(k;\mathbb{R})^{pr}$ and $(G^{s+1}_{d_m,I})^{pr}$ based on $M^s_m(k;\mathbb{R})$ and $G^{s+1}_{d_m,I}$ respectively. Applying Lemma 4.22 with $N = d_m - 1$, $r = N - d_k$, $x_0 = I$ we get the desired result. □

4.8 Hyperbolic Perturbations of Symmetric Systems

In this section, we discuss hyperbolic perturbations, of which definition is given below, of symmetric systems with constant coefficients near multiple characteristics which are not necessarily nondegenerate. To motivate our study in this section let us consider

$$L(x, D) = \sum_{j=0}^{n} A_j(x) D_j, \quad A_0(x) = I$$

where $A_j(x)$ are real $m \times m$ real analytic matrices and let ρ be a multiple characteristic of order m with involutive $\Lambda(\rho)$. If $L(x, D)$ is strongly hyperbolic near the origin we have $\dim \mathrm{Ker} L(\rho) = m$ by Theorem 2.2 which implies $L(\rho) = O$. We can assume $\rho = (0, e_n)$ so that $A_n(0) = O$ then one can write

$$\begin{aligned} L(x,\xi) &= \xi_0 I + \sum_{j=1}^{n-1} A_j(x)\xi_j + \sum_{j=0}^{n} A_{nj}(x) x_j \xi_n \\ &= \xi_n \Big\{(\xi_0/\xi_n) I + \sum_{j=1}^{n-1} A_j(x)(\xi_j/\xi_n) + \sum_{j=0}^{n} A_{nj}(x) x_j\Big\} \end{aligned}$$

and note that

$$L_\rho(x, \xi') = \xi_0 I + \sum_{j=1}^{n-1} A_j(0)\xi_j + \sum_{j=0}^{n} A_{nj}(0) x_j.$$

From Lemma 4.2 $L_\rho(x,\xi')$ is hyperbolic with respect to $\theta = (0,\dots,0,1,0,\dots,0)$. Assume that $L_\rho(x,\xi')$ is diagonalizable for every (x,ξ'). If $\dim_{\mathbb{R}} L_\rho = d_m$ so that ρ is nondegenerate then by Theorem 4.4 we see that $L(x,\xi)$ is symmetrizable near ρ. Moreover under the assumption

$$\dim_{\mathbb{R}} L_\rho \ge d_m - 1$$

it follows from Lemma 4.9 that there exists $T \in M_m(\mathbb{R})$ such that $T^{-1}L_\rho(x,\xi')T$ is symmetric for every (x,ξ'). Considering $T^{-1}L(x,\xi)T$ from the beginning we can assume that $L_\rho(x,\xi')$ is symmetric. Thus we can write

$$L(x,\xi) = \xi_n\big\{L_\rho(x,\xi'/\xi_n) + R(x,\xi'/\xi_n)\big\}, \quad R(x,\xi'/\xi_n) = O(|x|^2 + |\xi'/\xi_n|^2)$$

where $L_\rho(x,\xi') + R(x,\xi')$ is hyperbolic with respect to $(0,\dots,0,1,0,\dots,0)$.

Let us consider symmetric systems with constant coefficients

$$\mathscr{L}(x) = x_1 I + \sum_{j=2}^{q} F^j x_j = x_1 I + L(x') \tag{4.99}$$

where $F^j \in M_m^s(\mathbb{R})$ and $\{I, F^j\}$ are linearly independent. Note that if $q \le d_m - 1$ then $x = 0$ is a degenerate characteristic of $\mathscr{L}(x)$.

We perturb $\mathscr{L}(x)$ near $x = 0$ by adding $R(x) = O(|x|^2)$ as $x \to 0$. We start with

Definition 4.6. We say that $M_m(\mathbb{R})$ valued real analytic $R(x) = O(|x|^2)$, $x \to 0$ is a hyperbolic perturbation to $\mathscr{L}(x)$ near $x = 0$ if the perturbed system

$$\mathscr{P}(x) = \mathscr{L}(x) + R(x)$$

remains to be hyperbolic near $x = 0$, that is

$$\text{all eigenvalues of } \mathscr{P}(x + \lambda\Theta) \text{ are real near } x = 0 \tag{4.100}$$

where $\Theta = (1,0,\dots,0)$ and

$$R(x) = O \quad \text{if} \quad \mathscr{L}(x) = O. \tag{4.101}$$

Example 4.11. Let $\mathscr{L}(x)$ be as in (4.99) and let $T(x)$ be real analytic $m \times m$ matrix defined near $x = 0$ with $T(0) = I$. Then it is clear that

$$T^{-1}(x)\mathscr{L}(x)T(x) = x_1 I + \sum_{j=2}^{q} T^{-1}(x)F_j T(x)x_j = \mathscr{L}(x) + R(x)$$

is a hyperbolic perturbation, while it is never trivial to find $T(x)$ starting from $\mathscr{L}(x) + R(x)$.

As before, we define $S_{\mathscr{L}}(x)$ as the representation matrix of the linear map sending $M_m^s(\mathbb{R}) \ni H$ with zero diagonal entries to the anti-symmetric matrix $[\mathscr{L}(x), H]$. Note that

$$S_{\mathscr{L}}(x) = S_{\tilde{\mathscr{L}}}(x) \tag{4.102}$$

if $\tilde{\mathscr{L}}(x) - \mathscr{L}(x)$ is a scalar matrix. Let

$$g(x) = \prod_{j=1}^{s} g_j(x)^{r_j}$$

be the irreducible factorization of $\det S_{\mathscr{L}}(x)$ in $\mathbb{R}[x]$. We assume that

$$\{x | g_j(x) = 0\} \text{ contains a regular point} \tag{4.103}$$

for $1 \le j \le s$. Then we have

Theorem 4.9. *Assume that every characteristic of $\mathscr{L}(x)$ of order less than m is nondegenerate. Suppose that* $\det S_{\mathscr{L}}(x)$ *satisfies (4.103). Then for every perturbed $\mathscr{P}(x) = \mathscr{L}(x) + R(x)$ with a hyperbolic perturbation $R(x)$ we can find real analytic $A(x)$, $B(x)$ defined near the origin with $A(0) = B(0) = I$ such that*

$$A(x)\mathscr{P}(x)B(x)$$

becomes symmetric.

Proof. By a preparation theorem proved in [11], generalizing the Weierstrass preparation theorem to matrix valued functions, one can write

$$\mathscr{P}(x + \lambda\Theta) = C(x, \lambda)(\lambda I + Q(x)) \tag{4.104}$$

where $C(x, \lambda)$ is real analytic near $(0, 0)$ with $\det C(0, 0) \neq 0$ and $Q(x)$, $Q(0) = O$ is real analytic with values in $M_m(\mathbb{R})$. Comparing the first order term in the Taylor expansion at $(x, \lambda) = (0, 0)$ of both sides we see that $C(0, 0) = I$ and $Q(x) = \mathscr{L}(x) + \tilde{R}(x)$ where $\tilde{R}(x) = O(|x|^2)$. Since $\mathscr{L}(0, \dots, 0, x_{q+1}, \dots, x_n) = O$ taking $\lambda = -x_1$, $x_j = 0$, $2 \le j \le q$ in (4.104) it follows from (4.101) that $O = C(x_1, 0, \dots, 0, x_{q+1}, \dots, x_n, -x_1)\tilde{R}(x_1, 0, \dots, 0, x_{q+1}, \dots, x_n)$ and hence

$$\tilde{R}(x_1, 0, \dots, 0, x_{q+1}, \dots, x_n) = O.$$

Since $C(x, 0)^{-1}\mathscr{P}(x) = \mathscr{L}(x) + \tilde{R}(x)$ it is enough to study a perturbation term $R(x)$ which verifies $R(x_1, 0, \dots, 0, x_{q+1}, \dots, x_n) = O$. Changing notations we set $x = (x_2, \dots, x_q)$, $y = (x_1, x_{q+1}, \dots, x_n)$ and

$$P(x,y) = L(x) + R(x,y), \quad L(x) = \sum_{j=2}^{q} F^j x_j$$

where $S_L(x)$ verifies the assumptions because of (4.102). As in Sect. 4.5 we set

$$\tilde{P}(r,\omega,y,a) = r^{-1} S(a)^{-1} P(r\omega, y) S(a).$$

Since $\tilde{P}(0,\omega,0,0) = L(\omega)$ and $\{I, F^j\}$ are linearly independent the multiplicity of eigenvalues of $\tilde{P}(0,\omega,0,0)$ are less than m if $\omega \neq 0$. We then fix $\omega \neq 0$ and proceed exactly as the same way in Sect. 4.5. Take an orthogonal T_0 so that $T_0^{-1} L(\omega) T_0 = \oplus_{i=1}^{p} \lambda_i I_{s_i}$. Then we have

$$\begin{aligned} Q(r,\theta,y,a) &= r^{-1} T_0^{-1} S(a)^{-1} P(r(\omega+\theta), y) S(a) T_0 \\ &= \tilde{L}(\omega) + \tilde{L}(\theta) + \tilde{P}(\omega; r, y, a) + O(|(r,\theta,y,a)|^2) \end{aligned}$$

where $\tilde{L}(\omega) = \oplus \lambda_i I_{s_i}$ and $\tilde{L}(\theta) = T_0^{-1} L(\theta) T_0 = (\tilde{L}_{ij}(\theta))_{1\le i,j\le p}$. Let

$$\tilde{L}_{ii}(\theta) = \sum_{j=2}^{q} \tilde{F}_{ii}^{j} \theta_j$$

then we get

Lemma 4.23. $\{I_{s_i}, \tilde{F}_{ii}^j\}$ *span* $M_{s_i}^s(\mathbb{R})$.

Proof. Let $\tilde{\mathscr{L}}(x) = T_0^{-1} \mathscr{L}(x) T_0$, $x = (x_1, x_2, \ldots, x_q)$. Since $(x_1, x_2, \ldots, x_q) = (-\lambda_i, \omega)$ is a characteristic of $\tilde{\mathscr{L}}(x)$ of order less than m and hence nondegenerate by assumption. It is clear that the localization of $\tilde{\mathscr{L}}(x)$ at $(-\lambda_i, \omega)$ is

$$\tilde{\mathscr{L}}_{(-\lambda_i,\omega)}(x) = x_1 I_{s_i} + \sum_{j=2}^{q} \tilde{F}_{ii}^{j} x_j$$

because $\tilde{\mathscr{L}}(-\lambda_i, \omega)$ is diagonal. Noting that the non degeneracy of characteristics is invariant under changes of basis for $\mathbb{C}^m$, the matrices $\{I_{s_i}, \tilde{F}_{ii}^j\}$ span a subspace of dimension $s_i(s_i+1)/2$. Since $\tilde{F}_{ii}^j$ are symmetric this proves the assertion. □

Completion of the Proof of Theorem 4.9. In view of Remark at the end of Sect. 4.5, the rest of the proof of Theorem 4.9 goes exactly as the same way in Sect. 4.5. □

Taking into account the invariance of non degeneracy of characteristics under change of basis we have

Corollary 4.2. *Assume that every characteristic of $\mathscr{L}(x)$ of order less than m is nondegenerate and there is an orthogonal matrix $T \in O(m)$ such that* $\det S_{T^{-1}\mathscr{L}T}(x)$ *verifies (4.103). Then the same conclusion as in Theorem 4.9 holds.*

Remark. The condition (4.103) is not invariant under orthogonal changes of basis for $\mathbb{C}^m$. Indeed let

$$\mathscr{L}(x) = x_1 I_2 + \begin{bmatrix} 0 & x_2 \\ x_2 & 0 \end{bmatrix}$$

then it is obvious that $S_{\mathscr{L}}(x) = O$. On the other hand it is easy to see that there is an orthogonal $T \in O(2)$ such that $S_{T^{-1}\mathscr{L}T}(x)$ verifies (4.103).

Example 4.12. Let us take

$$L_1(x) = \begin{bmatrix} x_2 + x_5 & x_5 & x_5 \\ x_5 & x_3 + x_5 & x_5 \\ x_5 & x_5 & x_4 + x_5 \end{bmatrix}, \quad L_2(x) = \begin{bmatrix} x_2 & x_4 & x_5 \\ x_4 & x_3 & -x_5 \\ x_5 & -x_5 & x_4 \end{bmatrix}$$

for which constant hyperbolic perturbation must be trivial (see Definition 4.7 in the next section and Theorems 3.5 and 3.6 in [22]). Applying Theorem 4.9 we show that not only constant hyperbolic perturbations but also more general hyperbolic perturbation is trivial.

Note that it is easy to see that

$$\begin{aligned} \det S_{L_1}(x) &= x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_2 - x_2 x_3^2 - x_3 x_4^2 - x_4 x_2^2 \\ &= -(x_2 - x_3)(x_3 - x_4)(x_4 - x_2), \\ \det S_{L_2}(x) &= x_2^2 x_3 + x_3^2 x_4 + x_5^2 x_2 - x_2 x_3^2 - x_4 x_2^2 - x_3 x_5^2 \\ &= (x_2 - x_3)(x_2 x_3 - x_2 x_4 - x_3 x_4 + x_5^2). \end{aligned}$$

Let $\Theta_1 = (1,1,1,0)$ and $\Theta_2 = (2,2,1,0)$. It is obvious that $L_i(\Theta_i)$ is positive definite. Let us set

$$\tilde{L}_i(x) = L_i(\Theta_i)^{-1/2} L_i(x) L_i(\Theta_i)^{-1/2}.$$

It follows from Theorems 3.5, 3.6 in [22] and Lemma 4.3 that

Lemma 4.24. *Every characteristic of $\tilde{L}_i(x)$, $i = 1,2$ of order less than 3 is nondegenerate.*

To apply Theorem 4.9 to $\tilde{L}_i(x)$ we examine that

Lemma 4.25. $\det S_{\tilde{L}_i}(x)$, *$i = 1,2$ verifies (4.103).*

Proof. We first note that $\det S_{L_i}(x)$ verifies (4.103). The assertion for $S_{\tilde{L}_1}(x)$ is clear because $L_1(\Theta_1) = I$. To prove the assertion for $S_{\tilde{L}_2}(x)$ we note that

$$C = L_2(\Theta_2)^{-1/2} = \begin{bmatrix} \alpha & \beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{L}_2(x) = C L_2(x) C$$

with $\alpha > 0, \beta > 0$ and $\gamma = \alpha^2 - \beta^2 > 0$. Let x be so that $\det S_{L_2}(x) = 0$. Then there is a $H \in M_3^s(\mathbb{R})$, $H \neq O$ with zero diagonal entries such that $[L_2(x), H] = O$. Setting $\tilde{H} = C^{-1}HC$ it follows that

$$\tilde{L}_2(x)\tilde{H} - \tilde{H}\tilde{L}_2(x) = O.$$

Hence we have $[\tilde{L}_2(x), \tilde{H}^s] = O$ where $\tilde{H}^s$ is the symmetric part of $\tilde{H}$. It is easy to check that the diagonal entries of $\tilde{H}$ and hence those of $\tilde{H}^s$ are zero. Thus we conclude that $\det S_{\tilde{L}_2}(x) = 0$. Since $\det S_{L_2}(x)$ verifies (4.103) by Remark at the end of Sect. 4.5 we get $\det S_{\tilde{L}_2}(x) = c \det S_{L_2}(x)$ with a constant $c \neq 0$ and hence the assertion. □

4.9 Stability of Symmetric Systems Under Hyperbolic Perturbations

We start with

Definition 4.7. Let $R(x)$ be a hyperbolic perturbation to $\mathscr{L}(x)$ near $x = 0$. We say that the perturbation is trivial if there exist real analytic $A(x)$, $B(x)$ defined near the origin with $A(0)B(0) = I$ such that $A(x)\mathscr{P}(x)B(x)$ becomes symmetric.

In this section we prove that *generically* every hyperbolic perturbation of symmetric system $\mathscr{L}$

$$\mathscr{L}(x) = x_1 I + \sum_{j=2}^{n} F^j x_j, \quad F^j \in M_m^s(\mathbb{R})$$

is trivial if $\dim \mathscr{L}$ is enough large. As in Sect. 4.7 we identify $\mathscr{L}(x)$ with the subspace $\mathscr{L} = \{\mathscr{L}(x) \mid x \in \mathbb{R}^n\}$.

Theorem 4.10. *Assume $d_m - m + 3 \leq l \leq d_m$. Then in the $(d_m - l)(l - 1)$ dimensional Grassmannian of l dimensional subspaces of $M_m^s(\mathbb{R})$ containing the identity, the subset for which every hyperbolic perturbation is trivial is an open and dense subset.*

As in Sect. 4.5 we study $S_{\mathscr{L}}(x)$ for symmetric $\mathscr{L}(x)$ when $\dim \mathscr{L} = d_m - \nu$ where $1 \leq \nu \leq m - 3$. We first examine the representation matrix $S_{\mathscr{L}}(x)$. Let

$$V_m = \{H = (h_{ij}) \in M_m^s(\mathbb{R}) | h_{ii} = 0\}$$

and recall that $S_{\mathscr{L}}(x)$ is defined as the linear map between two d_{m-1} dimensional linear subspaces V_m and $W_m = M_m^{as}(\mathbb{R})$

$$V_m \ni H \mapsto [\mathscr{L}(x), H] = K \in W_m$$

where $M_m^{as}(\mathbb{R})$ denotes the set of all real $m \times m$ anti-symmetric matrices. Let us write

$$\mathscr{L}(x) = (\phi_j^i(x))_{1 \le i,j \le m}, \quad \phi_j^i(x) = \phi_i^j(x). \tag{4.105}$$

For $H \in V_m$ we write $\check{H} = {}^t(h_{12}, h_{13}, h_{23}, h_{14}, h_{24}, h_{34}, \ldots, h_{m-1m}) \in \mathbb{R}^{d_{m-1}}$. Then the equation $[\mathscr{L}(x), H] = K$ can be written as

$$S_{\mathscr{L}}(x)\check{H} = \check{K}$$

where $S_{\mathscr{L}}(x)$ is a $d_{m-1} \times d_{m-1}$ matrix. For instance when $m = 3$ we have

$$S_{\mathscr{L}}(x) = \begin{bmatrix} \phi_1^1(x) - \phi_2^2(x) & -\phi_3^2(x') & \phi_3^1(x') \\ -\phi_3^2(x') & \phi_1^1(x) - \phi_3^3(x) & \phi_2^1(x') \\ -\phi_3^1(x') & \phi_2^1(x') & \phi_2^2(x) - \phi_3^3(x) \end{bmatrix}. \tag{4.106}$$

We turn to the case $\mathscr{L}(x)$ is a $m \times m$ matrix. Let

$$\mathscr{L}(x) = \begin{bmatrix} L(x) & l(x') \\ {}^t l(x') & \phi_m^m(x) \end{bmatrix}$$

where $l(x') = {}^t(\phi_m^1(x'), \ldots, \phi_m^{m-1}(x'))$ and $L(x)$ stands for $\mathscr{L}(x)$ in (4.105) with $m - 1$. For $H \in V_m$ and $K \in W_m$ we write

$$H = \begin{bmatrix} H_1 & h \\ {}^t h & 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & k \\ {}^t k & 0 \end{bmatrix}$$

with $H_1 \in V_{m-1}$, $K_1 \in W_{m-1}$ and $h = {}^t(h_{1m}, \ldots, h_{m-1m})$. Then it is easy to see that the equation $[\mathscr{L}(x), H] = K$ is written as

$$\begin{bmatrix} S_L(x) & c(l) \\ c'(l) & L(x) - \phi_m^m I \end{bmatrix} \begin{bmatrix} \check{H}_1 \\ h \end{bmatrix} = \begin{bmatrix} \check{K}_1 \\ k \end{bmatrix} = \check{K}$$

and hence we get

$$S_{\mathscr{L}}(x) = \begin{bmatrix} S_L(x) & c(l) \\ c'(l) & L(x) - \phi_m^m I \end{bmatrix}. \tag{4.107}$$

Our aim in this section is to prove

Proposition 4.11. *Assume that $1 \le \nu \le m - 3$. Then in the Grassmannian $G_{d_m, I}^{d_m - \nu}$, the subset of $\mathscr{L}$ for which the condition (4.103) is fulfilled for $T^{-1}\mathscr{L}T$ with some $T \in O(m)$ is an open and dense subset.*

Here we use a parametrization of the Grassmannian $G^l_{d_m,I}$ used in Sect. 4.7. We set $\triangle = \{(i,i) | 1 \le i \le m\}$ and let $1 \le k \le m-1$. We first remark that

Lemma 4.26. *Assume that* $1 \le k \le m-1$. *Then one can find finitely many* $S_1, \dots, S_N \in O(m)$ *such that for any* $\mathscr{L} \in G^{d_m-k}_{d_m,I}$ *there is* $S_i \in \{S_1, \dots, S_N\}$ *so that* $S_i^{-1}\mathscr{L}S_i \in \Omega_\sigma$ *with some* σ *verifying* $\sigma(\{1,\dots,k\}) \cap \triangle = \emptyset$.

Proof. In this proof we denote $|C| = \max_{i,j} |c_{ij}|$ for a matrix $C = (c_{ij})$. Let $T_{pq}(\epsilon)$ be the orthogonal matrix obtained replacing p-th and q-th, $p < q$, rows of the identity matrix by

$$(0,\dots,0,f(\epsilon),0,\dots,0,\epsilon,0,\dots,0), \quad (0,\dots,0,-\epsilon,0,\dots,0,f(\epsilon),0,\dots,0)$$

where $\epsilon^2 + f(\epsilon)^2 = 1$. We show that it is enough to take $\{S_i\}$ as the set of all

$$K_1K_2\cdots K_m$$

where

$$K_j \in \{I, T_{pq}(\epsilon_i) \mid \epsilon_i = (C_i m^{2^{i-1}})^{-1}, i = 1,\dots,m, 1 \le p < q \le m\}$$

and $C_1 < C_2 < \cdots < C_m$ will be chosen suitably. Let $\mathscr{L} \in G^{d_m-k}_{d_m,I}$ and let $A_1,\dots,A_k$ define $\mathscr{L}$ so that $\mathscr{L}$ consists of all $X \in M^s_m(\mathbb{R})$ such that $\mathrm{Tr}\,(A_jX) = 0$, $1 \le j \le k$ where A_j are linearly independent and $\mathrm{Tr}\,A_j = 0$. We first note that we may assume

$(H)_\mu$: there is an injective $\tau : \{1,\dots,\mu\} \to \{(i,j) | 1 \le i < j \le m\}$ such that $\tau(i)$-th entry of A_j is zero unless $i = j$ and $\tau(j)$-th entry of A_j is 1, $|A_j| \le a_\mu m^{2^{\mu-1}}$ for $1 \le j \le \mu$ where $a_1 = 1$, $a_{\mu+1} = Ba_\mu C_\mu$ with a fixed large B and $A_{\mu+1}, \dots, A_k$ are diagonal matrices.

In fact if some A_j has a non-zero off diagonal entry we may assume that the off diagonal $\tau(1)$-th entry of A_1 is 1 and $|A_1| \le 1$. Replacing A_j by $A_j - \alpha_jA_1$, $j \neq 1$, with suitable α_j one can assume that $\tau(1)$-th entry of A_j is zero if $j \neq 1$. A repetition of this argument gives the assertion. If $\mu = k$ then $\tau(\{1,\dots,k\}) \cap \triangle = \emptyset$ and there is nothing to prove. Then we may assume that $\mu \le k-1$. Let $A_{\mu+1} = \mathrm{diag}\,(\lambda_1,\dots,\lambda_m)$. Since $\mathrm{Tr}\,A_{\mu+1} = 0$ it is easy to see that there are at least $m-1$ pairs (i,j), $i < j$ such that

$$3|\lambda_i - \lambda_j| \ge |\lambda_r|, \quad r = 1,\dots,m.$$

Since $\mu \le m-2$ there exists such a (p,q) with $(p,q) \notin \tau(\{1,\dots,\mu\})$. Let us set

$$A_j(\epsilon_\mu) = T_{pq}(\epsilon_\mu)^{-1}A_jT_{pq}(\epsilon_\mu), \quad 1 \le j \le k$$

and note that $|A_j(\epsilon_\mu) - A_j| \le B_1a_\mu C_\mu^{-1}$, $1 \le j \le \mu$. Choose C_μ so that $a_\mu C_\mu^{-1}$ is small enough then taking $\tilde{A}_j(\epsilon_\mu) = \sum_{i=1}^{\mu} c_{ji}A_i(\epsilon_\mu)$, $1 \le j \le \mu$, with a non singular $C = (c_{ji})$ we may suppose that $\tau(i)$-th entry of $\tilde{A}_j(\epsilon_\mu)$ is zero unless $i = j$ and $\tau(j)$-th entry of $\tilde{A}_j(\epsilon_\mu)$ is 1 and $|\tilde{A}_j(\epsilon_\mu)| \le 2|A_j|$. Note that the off diagonal entries of $A_{\mu+1}(\epsilon_\mu)$ are zero except for (p, q), (q, p)-th entries which are $\epsilon_\mu f(\epsilon_\mu)(\lambda_q - \lambda_p)$. Set

$$\tilde{A}_{\mu+1}(\epsilon_\mu) = \{\epsilon_\mu f(\epsilon_\mu)(\lambda_q - \lambda_p)\}^{-1} A_{\mu+1}(\epsilon_\mu)$$

and hence $|\tilde{A}_{\mu+1}(\epsilon_\mu)| \le B_2 C_\mu m^{2^{\mu-1}}$. Replacing $\tilde{A}_j(\epsilon_\mu)$ by $\tilde{A}_j(\epsilon_\mu) - \alpha_j \tilde{A}_{\mu+1}(\epsilon_\mu)$ with suitable α_j we can conclude that $\tau(\mu + 1) = (p, q)$-th entry of $\tilde{A}_j(\epsilon_\mu)$ is zero for $1 \le j \le \mu$ and $|\tilde{A}_j(\epsilon_\mu)| \le a_{\mu+1} m^{2^\mu}$, $1 \le j \le \mu + 1$. By subtraction again we may suppose that $A_j(\epsilon_\mu)$, $j \ge \mu + 2$ are diagonal matrices and then we get to $(H)_{\mu+1}$. The rest of the proof is clear. □

Proof of Proposition 4.11. We first assume that $\mathscr{L} \in \Omega_\tau$ with $\tau(\{1, \dots, \nu\}) \cap \triangle = \emptyset$ and let $A = (A_1, \dots, A_\nu) \in U_\tau$ be the coordinate of $\mathscr{L}$. Let us denote

$$\mathscr{L}(x) = \sum_{j=1}^{n} K_j x_j = (\phi^i_j(x))$$

where $\{K_j\}$, $1 \le j \le n = d_m - \nu$ is a basis for $\mathscr{L}$ and set $g(x) = \det S_{\mathscr{L}}(x)$. Let $J_\tau = \{(i, j) | 1 \le i \le j \le m\} \setminus \tau(\{1, \dots, \nu\})$ and note that $\phi^i_j(x)$, $(i, j) \in J_\tau$ are linearly independent and $\triangle \subset J_\tau$. With $A_k = (a^{(k)}_{ij})$ it is clear that the equations $\phi^i_j(x) = 0$, $(i, j) \in J_\tau \setminus \triangle$ and $\mathrm{Tr}\,(A_k \mathscr{L}(x)) = 0$ define a plane

$$\sum_{j=1}^{m} a^{(k)}_{jj} \phi^j_j(x) = \sum_{j=1}^{m-1} a^{(k)}_{jj} (\phi^j_j(x) - \phi^m_m(x)) = 0, \quad 1 \le k \le \nu \tag{4.108}$$

and $S_{\mathscr{L}}(x)$ is diagonal matrix on the plane with the determinant

$$g(x) = \prod_{1 \le i < j \le m} (\phi^i_i(x) - \phi^j_j(x)). \tag{4.109}$$

We show that there is a polynomial $\pi(A)$ in $a^{(k)}_{jj}$, $1 \le k \le \nu$, $1 \le j \le m - 1$ such that if $\pi(A) \ne 0$ then no two $\phi^i_i(x) - \phi^j_j(x)$, $i < j$ are proportional on the plane (4.108). To simplify notations we write y_i for $\phi^i_i(x) - \phi^m_m(x)$ so that

$$g(y) = \prod_{1 \le i < j \le m-1} (y_i - y_j) y_1 \cdots y_{m-1}$$

provided that $y\tilde{A} = 0$ where $y = (y_1, \dots, y_{m-1})$ and $\tilde{A} = (a^{(k)}_{jj})$ which is a $(m - 1) \times \nu$ matrix. Suppose that some two $y_i - y_j$ are proportional on the plane $y\tilde{A} = 0$ and hence $yb = 0$ with some $b \in \mathbb{R}^{m-1}$ for every y with $y\tilde{A} = 0$.

Then it is clear that rank $(\tilde{A}, b) = \text{rank}\,\tilde{A}$. Note that at most two components of b are the constant of the proportionality c and the other components are either 0 or 1 (at most two 1 appear). Take a $(\nu + 1) \times (\nu + 1)$ submatrix of $(\tilde{A}, b)$ and expand the determinant with respect to the last column. Equating the determinant to zero we get a linear relation of ν-minors of $\tilde{A}$ with coefficients which are either 1 or the proportional constant c. Since $\nu + 1 \leq m - 2$ we have at least $m - 1$ such linear relations. Elimination of c gives a quadratic equation in ν-minors of $\tilde{A}$. Denote this equation by $\pi(A) = 0$. Then we conclude that the rank of the matrix $(\tilde{A}, b)$ is $\nu + 1$ if $\pi(A) \neq 0$. This shows that no two $y_i - y_j$ are proportional if $\pi(A) \neq 0$.

Let $g(x) = \prod g_j(x)^{r_j}$ be the irreducible factorization in $\mathbb{R}[x]$. Without restrictions we may assume that the plane $y\tilde{A} = 0$ is given by $y_b = f(y_a)$, after a linear change of coordinates y if necessary, where $y = (y_a, y_b)$ is a partition of the coordinates y. Then we have

$$\prod g_j(y_a, f(y_a))^{r_j} = \prod p_i(y_a)$$

where $p_i(y_a)$ are linear in y_a and no two $p_i(y_a)$ are proportional if $\pi(A) \neq 0$. Then it follows that $r_j = 1$ and $g_j(y_a, f(y_a))$ is a product of some $p_i(y_a)$'s;

$$g_j(y_a, f(y_a)) = \prod_{i \in I_j} p_i(y_a).$$

From this it is obvious that $\{g_j(y_a, f(y_a)) = 0\}$ contains a regular point. Then it follows that $\{g_j(x) = 0\}$ contains a regular point. This shows that, in U_τ, the set of A such that $S_{\mathscr{L}}(x)$ does not verify (4.103) is contained in an algebraic set. We now study $\mathscr{L} \in \Omega_\sigma$ with $\sigma(\{1, \ldots, \nu\}) \cap \triangle \neq \emptyset$. By Lemma 4.26 there is $S_i \in O(m)$ such that $S_i^{-1}\mathscr{L}S_i \in \Omega_\tau$ with some τ verifying $\tau(\{1, \ldots, \nu\}) \cap \triangle = \emptyset$. Since $\{S_i\}$ is a finite set the proof is clear. □

Proof of Theorem 4.10. Let $d_m - m + 3 \leq l \leq d_m$. Then Theorem 4.10 follows immediately from Propositions 4.10, 4.11 and Corollary 4.2. □

4.10 Some Special Cases

In the case $m = 3$ one can improve Theorem 4.10.

Theorem 4.11. *Assume that $m = 3$ and $4 \leq l \leq 6 = d_3$. Then in the $(6-l)(l-1)$ dimensional Grassmannian of l dimensional subspaces of $M_3^s(\mathbb{R})$ containing the identity, the subset for which every hyperbolic perturbation is trivial is an open and dense subset.*

We assume $m = 3$ throughout the section. Let $\mathscr{L} \in G_{6,I}^l$ for $l = 4$ or 5. Taking a basis $\{K_j\}$ for $\mathscr{L}$, $\mathscr{L}$ is the image of

$$\mathscr{L}(x) = \sum_{j=1}^{n} K_j x_j.$$

We first study the case $l = 5$.

Lemma 4.27. *In the Grassmannian $G^5_{6,I}$, the subset of $\mathscr{L}$ for which the condition (4.103) is fulfilled for $T^{-1}\mathscr{L}T$ with some $T \in O(m)$ is an open and dense subset.*

Proof. Let $A = A_1 \in U_\sigma$ be the coordinate of $\mathscr{L}$ and assume that $\sigma(1) \cap \triangle = \emptyset$ so that the diagonal entries of $\mathscr{L}(x)$ are linearly independent. Considering $T^{-1}\mathscr{L}(x)T$ with suitable permutation matrix T, if necessary, we may assume that $\sigma(1) = (1, 2)$ so that with $\mathscr{L}(x) = (\phi^i_j(x))$ we have from $\mathrm{Tr}\,(A\mathscr{L}(x)) = 0$ that

$$-2\phi^1_2(x) = a_{11}(\phi^1_1 - \phi^3_3) + a_{22}(\phi^2_2 - \phi^3_3) + 2a_{13}\phi^1_3 + 2a_{23}\phi^2_3.$$

From (4.106), simplifying notations, it is enough to study

$$S(x, y) = \begin{bmatrix} x_1 - x_2 & -y_1 & y_2 \\ -y_1 & x_1 & \phi(x, y) \\ -y_2 & \phi(x, y) & x_2 \end{bmatrix}$$

where $\phi(x, y) = a_1x_1 + a_2x_2 + b_1y_1 + b_2y_2$. We show that if $a_1 + a_2 \neq 1$ and $4a_1a_2 - 1 \neq 0$ then the condition (4.103) is fulfilled. We first assume that $x_1x_2 - \phi(x, 0)^2$ is irreducible. Note that $g(x, y) = \det S(x, y)$ is then irreducible. Indeed if $g(x, y)$ were reducible so that $g(x, y) = h(x, y)k(x, y)$ then from $g(x, 0) = (x_1 - x_2)\psi(x)$ with $\psi(x) = x_1x_2 - \phi(x, 0)^2$ we may suppose that

$$h(x, y) = \psi(x) + p(x, y), \quad k(x, y) = x_1 - x_2 + q(y)$$

where $p(x, 0) = 0$, $q(y) = \alpha y_1 + \beta y_2$. Equating the coefficients of y_j in both sides of $g(x, y) = h(x, y)k(x, y)$ we see that $\alpha\psi(x)$, $\beta\psi(x)$ have a factor $x_1 - x_2$ which implies that $q = 0$. This gives $g(x, y) = h(x, y)(x_1 - x_2)$ which is a contradiction. Thus g is irreducible. It is clear that $\{g(x, 0) = 0\}$ has a regular point and hence so does $\{g(x, y) = 0\}$. This proves the assertion.

Assume now that $\psi(x) = x_1x_2 - \phi(x, 0)^2$ is reducible. From the assumption $4a_1a_2 - 1 \neq 0$ it follows that $\psi(x)$ has no multiple factor. Note that $a_1 + a_2 \neq \pm 1$ implies that $\psi(x)$ and $x_1 - x_2$ are relatively prime. The rest of the proof is a repetition of the last part of the proof of Proposition 4.11. □

We turn to the case $l = 4$. We show that

Lemma 4.28. *Assume that $l = 4$ and every double characteristic of $\mathscr{L}(x)$ is nondegenerate. Then the condition (4.103) is fulfilled for $T^{-1}\mathscr{L}(x)T$ with a suitable $T \in O(3)$.*

Proof. Following the proof of Theorems 3.5 and 3.6 in [22] we choose a specific basis for $\tilde{\mathscr{L}} = T^{-1}\mathscr{L}T$ with suitably chosen $T \in O(3)$ and show that (4.103) is fulfilled for $\tilde{\mathscr{L}}$ using this basis. From the proof of Theorem 3.3 in [22], if every double characteristic of $\mathscr{L}$ is nondegenerate, then only two cases occur, that is $\mathscr{L}$ has either four nondegenerate double characteristics or two nondegenerate double characteristics.

We first treat the case that $\mathscr{L}$ has four nondegenerate characteristics. Choosing a suitable $T \in O(3)$ we see from [22] that $A^{\pm} = \alpha_{\pm} \otimes \alpha_{\pm}$ and $B^{\pm} = \beta_{\pm} \otimes \beta_{\pm}$ is a basis for $\tilde{\mathscr{L}} = T^{-1}\mathscr{L}T$ where $\alpha_{\pm} = (a, \pm a, 1)$, $\beta_{\pm} = (b, \pm b, 1)$ and $a \neq b$, $ab \neq 0$. Now we can write

$$\tilde{\mathscr{L}}(x) = A^{+}x_1 + A^{-}x_2 + B^{+}x_3 + B^{-}x_4.$$

With $X = x_1 + x_2$, $Y = x_1 - x_2$, $Z = x_3 + x_4$, $W = x_3 - x_4$ we have

$$\tilde{\mathscr{L}} = \begin{bmatrix} a^2X + b^2Z & a^2Y + b^2W & aX + bZ \\ a^2Y + b^2W & a^2X + b^2Z & aY + bW \\ aX + bZ & aY + bW & X + Z \end{bmatrix}. \tag{4.110}$$

Therefore it follows from (4.106) and (4.110) that

$$S_{\tilde{\mathscr{L}}} = \begin{bmatrix} 0 & -aY - bW & aX + bZ \\ -aY - bW & cX + dZ & a^2Y + b^2W \\ -aX - bZ & a^2Y + b^2W & cX + dZ \end{bmatrix}$$

where $c = a^2 - 1$, $d = b^2 - 1$. Let $\tilde{g} = \det S_{\tilde{\mathscr{L}}}$. On the plane $a^2Y + b^2W = 0$, that is, if $W = -a^2Y/b^2 = eY$ we get

$$\tilde{g} = (cX + dZ)(aX + bZ + (a + be)Y)(aX + bZ - (a + be)Y).$$

Note that $a + be \neq 0$ because $a \neq b$ and no two factors in the right-hand side are proportional. Now, as the end of the proof of Proposition 4.11, it is easy to conclude that $\tilde{g}$ satisfies (4.103).

We next study the case $\mathscr{L}$ has two nondegenerate double characteristics. With a suitable $T \in O(3)$ we see that $\tilde{\mathscr{L}} = T^{-1}\mathscr{L}T$ contains $K^{\pm} = \alpha_{\pm} \otimes \alpha_{\pm}$ with $\alpha_{\pm} = (a, \pm a, 1)$, $a \neq 0$, which are intersections with $M_3^s(2; \mathbb{R})$. Since $\tilde{\mathscr{L}}$ contains the identity, as a member of basis for $\tilde{\mathscr{L}}$, one can take K_3

$$K_3 = \begin{bmatrix} 0 & 0 & -2a \\ 0 & 0 & 0 \\ -2a & 0 & 2(a^2 - 1) \end{bmatrix}$$

because $K^+ + K^- + K_3 = 2a^2 I$. The last member of basis for $\tilde{\mathscr{L}}$ can then be chosen of the form

$$K_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & \mu \\ 0 & \mu & \nu \end{bmatrix}.$$

Thus with $X = x_1 + x_2$, $Y = x_1 - x_2$, $Z = x_3$, $W = x_4$ and $c = a^2 - 1$ the matrix $K^+ x_1 + K^- x_2 + K_3 x_3 + K_4 x_4$ can be written

$$\tilde{\mathscr{L}} = \begin{bmatrix} a^2 X & a^2 Y & aX - 2aZ \\ a^2 Y & a^2 X + \lambda W & aY + \mu W \\ aX - 2aZ & aY + \mu W & X + 2cZ + \nu W \end{bmatrix}. \tag{4.111}$$

We examine if there are other double characteristics, that is, if $\tilde{\mathscr{L}}$ is of rank 1 for some (X, Y, Z, W) with $Z^2 + W^2 \neq 0$. It is not difficult to see that six 2-minors of (4.111) vanish for such (X, Y, Z, W) if and only if the equation

$$4a^2 Z^2 + 2(a^2 + 1)\lambda Z W + (\lambda\nu - \mu^2) W^2 = 0$$

has a real solution $(Z, W) \neq (0, 0)$. Thus in order that $\tilde{\mathscr{L}}$ has two nondegenerate double characteristics it is necessary and sufficient that

$$4a^2 \lambda\nu > 4a^2 \mu^2 + (a^2 + 1)^2 \lambda^2. \tag{4.112}$$

In particular λ and ν have the same sign. From (4.111) and (4.106) it follows that

$$S_{\tilde{\mathscr{L}}} = \begin{bmatrix} -\lambda W & -aY - \mu W & aX - 2aZ \\ -aY - \mu W & cX - 2cZ - \nu W & a^2 Y \\ -aX + 2aZ & a^2 Y & cX - 2cZ + (\lambda - \nu) W \end{bmatrix}.$$

If $c \neq 0$ then we consider $\tilde{g} = \det S_{\tilde{\mathscr{L}}}$ on $W = 0$ so that

$$\tilde{g} = (cX - 2cZ)(aX - 2aZ + aY)(aX - 2aZ - aY).$$

The same argument as before proves that (4.103) is verified for $\tilde{g}$. If $c = 0$ and hence $a^2 = 1$ then

$$\begin{aligned} \tilde{g} &= W(-\nu(aX - 2aZ)^2 + \lambda(\nu^2 - \mu^2)\alpha^{-1} Y^2 + (\lambda - \nu)\alpha(W - a\mu\alpha^{-1} Y)^2) \\ &= W h(X, Y, Z, W) \end{aligned}$$

where $\alpha = \lambda\nu - \mu^2$. From (4.112) it follows that $\alpha > 0$ and $\nu^2 - \mu^2 > 0$ because $\nu^2 + \lambda^2 \geq \lambda\nu > \mu^2 + \lambda^2$. Then the quadratic form h is indefinite and hence $\{h = 0\}$ contains a regular point. This proves the assertion. □

Proof of Theorem 4.11. If $l = 6$ then the assertion follows from Theorem 4.2 in [53]. If $l = 5$, combining Proposition 4.10 and Lemma 4.27 we get the result by Corollary 4.2. Let $l = 4$. Then by virtue of Proposition 4.10 and Lemma 4.28 one can apply Corollary 4.2 to get the assertion. □

4.11 Concluding Remarks

In [25], F. John discovered mysterious phenomena on the characteristics of hyperbolic systems. He considered the system P of 3 second order equations in 4 independent variables, which is the system discussed in Example 4.4. He showed that any system $\tilde{P}$ near P is hyperbolic if and only if $\tilde{P}$ has 4 double characteristics near the double characteristics of P. In [26] he showed that P is strongly hyperbolic. In [22], L. Hörmander studied hyperbolic systems with nondegenerate double characteristics. In particular, it was proved there that nondegenerate double characteristics are stable, that is we can not remove nondegenerate double characteristics by hyperbolic perturbations which shows a complexity of hyperbolic systems compared with the scalar case (see [58]).

For first order systems the notion of nondegenerate characteristics of any order is introduced in [53, 54]. We adapt this definition for higher order systems through the associated first order system in [57]. According to this definition, simple characteristics are nondegenerate characteristics of order 1 and nondegenerate double characteristics coincide with those studied in [4, 17, 22, 25, 26, 48].

Theorem 4.3 (in the real case) was proved for analytic first order systems in [53] and for systems with nondegenerate double characteristics in [22]. The results about hyperbolic perturbations of symmetric systems with constant coefficients are found in [54].

Problem. Generalize Theorem 4.4 to C^∞ $m \times m$ matrix valued $\mathscr{P}(x)$.

Problem. Determine the minimal l such that Theorem 4.10 holds.

Problem. Determine the minimal $\dim_{\mathbb{R}} \{L(x) \mid x \in \mathbb{R}^n\}$ such that Lemma 4.9 holds. In the real valued case it is known that 5 is optimal when $m = 3$ (see [59]).

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