Lyapunov Exponent Sign Reversal: Stability and Instability by the First Approximation

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1 Introduction

This chapter is a concise and updated version of authors' survey *Time-Varying Linearization and the Perron effects* [52], devoted to the rigorous mathematical justification of the use of Lyapunov exponents to investigate the stability, instability, and chaos. In his thesis A.M. Lyapunov [57] proved that if the first approximation system is *regular* and its largest Lyapunov exponent is negative, then the solution of the original system is asymptotically stable. Then it was stated by O. Perron [62] that the requirement of regularity is substantial: he constructed an example of second-order system such that a solution of the first approximation system has negative largest Lyapunov exponent while the solution of the original system with the same initial data has positive largest Lyapunov exponent. The effect of Lyapunov exponent sign reversal of solutions of the first approximation system and of the original system under the same initial data, we shall call the Perron effect.

Later, [14, 58, 60, 63] there were obtained sufficient conditions of stability by the first approximation for nonregular linearizations generalizing the Lyapunov theorem. At the same time, according to [58]: "… The counterexample of Perron shows that the negativeness of Lyapunov exponents is not a sufficient condition of stability by the first approximation. In the general case necessary and sufficient conditions of stability by the first approximation are not obtained."

Recently, it was also shown [47, 52] that, in general, the positiveness of the largest Lyapunov exponent is not a sufficient condition of instability by the first approximation and chaos.

In the 1940s N.G. Chetaev [15] published the criterion of instability by the first approximation for regular linearizations. However, in the proof of these criteria a

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flaw was discovered [48, 52] and, at present, a complete proof of Chetaev theorems is given for a more weak condition in comparison with that for instability in the sense of Lyapunov, namely, for instability in the sense of Krasovsky.

The discovery of strange attractors and chaos in the investigation of complex nonlinear dynamical systems led to the use and study of instability by the first approximation. At present, many specialists in chaotic dynamics use various numerical methods for computation of Lyapunov exponent (see, e.g., [3, 8, 10, 12, 13, 16, 17, 24, 26–28, 30, 55, 56, 64, 65, 68, 70–72, 75–77, 79], and others) and believe that the positiveness of the largest Lyapunov exponent of linear first approximation system implies the instability of solutions of the original system.

As a rule, the authors ignore the justification of linearization procedure and use numerical values of exponents so obtained to construct various numerical characteristics of attractors of the original nonlinear systems (Lyapunov dimensions, metric entropies, and so on). Sometimes, computer experiments serve as arguments for the partial justification of the linearization procedure. For example, some computer experiments [61, 67] show the coincidence of the Lyapunov and Hausdorff dimensions of the attractors of Henon, Kaplan–Yorke, and Zaslavskii. But for B-attractors of Henon and Lorenz, such a coincidence does not hold [46, 48].

So, the approach, based on linearizations along the nonstationary trajectories on the strange attractors, requires justification. This motivates the development of nonstationary theory of instability by the first approximation.

In this work for the discrete and continuous systems the results of stability by the first approximation for regular and nonregular linearizations are given, the Perron effects are considered, the criteria of stability and instability of flow and cascade of solutions, and the criteria of instability in the sense of Lyapunov and Krasovsky are demonstrated. Some recent consideration of Lyapunov exponents, stability, and chaos can be found, e.g., [7, 18, 22, 31–33, 39, 53, 54, 59, 66, 69, 78]

2 Classical Definitions of Stability

Consider a continuous system

$$\begin{aligned} &\frac{dx}{dt} = F(x,t), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R} \\ &F(\cdot,\cdot)\colon\ \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \end{aligned} \tag{1}$$

and its discrete analog

$$\begin{aligned} &x(t+1) = F\big(x(t),t\big), \qquad x \in \mathbb{R}^n,\ t \in \mathbb{Z}, \\ &F(\cdot,\cdot)\colon\ \mathbb{R}^n \times \mathbb{Z} \to \mathbb{R}^n. \end{aligned} \tag{1$'$}$$

Consider the solution $x(t)$ of system (1) or (1$'$), given on the interval $a < t < +\infty$.

Definition 1. The solution $x(t)$ is said to be stable in the sense of Lyapunov (Lyapunov stable) if for any $\varepsilon > 0$ and $t_0 > a$ there exists a number $\delta = \delta(\varepsilon, t_0)$ such that

1. all solutions $y(t)$, satisfying the condition
$$|y(t_0) - x(t_0)| < \delta,$$
are defined in the interval $t_0 \le t < +\infty$;
2. for these solutions the inequality
$$|x(t) - y(t)| < \varepsilon, \quad \forall t \ge t_0$$

is valid. If $\delta(\varepsilon, t_0)$ is independent of t_0, then Lyapunov stability is called uniform.

Definition 2. The solution $x(t)$ is said to be asymptotically Lyapunov stable if it is Lyapunov stable and for any $t_0 > a$ there exists a positive number $\Delta = \Delta(t_0)$ such that all solutions $y(t)$, defined in the interval $t_0 \le t < +\infty$ and satisfying the condition
$$|y(t_0) - x(t_0)| < \Delta,$$
have the following property:
$$\lim_{t \to +\infty} |y(t) - x(t)| = 0.$$

In other words, for any $\varepsilon' > 0$ there exists a positive number $T = T(\varepsilon', y(t_0), t_0)$ such that the inequality $|x(t) - y(t)| < \varepsilon'$, $\forall t \ge t_0 + T$ is valid. If $x(t)$ is uniformly stable and $\Delta(t_0)$ and $T(\varepsilon', y(t_0), t_0)$ is independent of t_0, then Lyapunov asymptotic stability is called uniform.

Definition 3. The solution $x(t)$ is said to be exponentially stable if for any $t_0 > a$ there exist positive numbers $\delta = \delta(t_0)$, $R = R(t_0)$, and $\alpha = \alpha(t_0)$ such that

1. all solutions $y(t)$, satisfying the condition
$$|y(t_0) - x(t_0)| < \delta,$$
are defined in the interval $t_0 \le t < +\infty$;
2. the inequality
$$|y(t) - x(t)| \le R \exp\big(-\alpha\,(t - t_0)\big)|y(t_0) - x(t_0)|, \quad \forall t \ge t_0$$

is satisfied. If δ, R, and α are independent of t_0, then exponential stability is called uniform.

Assuming $\alpha = 0$, one obtains the following definition.

Definition 4. The solution $x(t)$ is said to be stable in the sense of Krasovsky (Krasovsky stable) if for any $t_0 > a$ there exist positive numbers $\delta = \delta(t_0)$ and $R = R(t_0)$ such that

1. all solutions $y(t)$, satisfying the condition
$$|y(t_0) - x(t_0)| < \delta,$$
are defined in the interval $t_0 \le t < +\infty$;
2. the following inequality
$$|x(t) - y(t)| \le R|y(t_0) - x(t_0)|, \quad \forall t \ge t_0$$

is valid. If δ and R are independent of t_0, then stability in the sense of Krasovsky is called uniform.

Hence, it follows that the stability of solution in the sense of Krasovsky yields its stability in the sense of Lyapunov. Relations with uniform stability can be found in [Willems, 1970].

Further without loss of generality, consider solutions with $t_0 = 0$. Denote by $x(t, x_0)$ a solution of either system (1) or system (1′) with the initial data $x(0, x_0) = x_0$, and suppose that all solutions $x(t, x_0)$ of continuous system are defined on the interval $[0, +\infty)$ and the solutions of discrete system are defined on the set $\mathbb{N}_0 = 0, 1, 2 \ldots.$

3 Characteristic Exponents, Regular Systems, Lyapunov Exponents

The problem of the investigation of the solution $x(t, x_0)$ can be reduced to the problem of the stability of the trivial solution $y(t) \equiv 0$ by transformation $x = y + x(t, x_0)$. Then one can consider systems (1) and (1′) with a marked linear part. In the continuous case one has

$$\frac{dx}{dt} = A(t)x + f(t, x), \; x \in \mathbb{R}^n, \quad t \in [0, +\infty), \tag{2}$$

where $A(t)$ is a continuous $(n \times n)$-matrix, $f(\cdot, \cdot) : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector-function.

In the discrete case, one has

$$x(t + 1) = A(t)x(t) + f\big(t, x(t)\big), \; x(t) \in \mathbb{R}^n, \quad t \in \mathbb{N}_0, \tag{2′}$$

where $A(t)$ is an $(n \times n)$-matrix, $f(\cdot, \cdot)\colon \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^n$.

Suppose, in a certain neighborhood $\Omega(0)$ of the point $x = 0$ the nonlinear parts of systems (2) and (2′) satisfy the following condition

$$|f(t,x)| \leq \kappa |x|^{\nu} \quad \forall t \geq 0, \quad \forall x \in \Omega(0), \quad \kappa > 0, \nu > 1. \tag{3}$$

We shall say that the first approximation system for (2) *is the following linear system*

$$\frac{dx}{dt} = A(t)x \tag{4}$$

and that for discrete system (2′) *is the linear system*

$$x(t+1) = A(t)x(t). \tag{4′}$$

Consider a fundamental matrix $X(t) = \big(x_1(t), \ldots, x_n(t)\big)$, consisting of the linear-independent solutions $\{x_i(t)\}_1^n$ of the first approximation system. For the determinant of the fundamental matrix one has the Ostrogradsky–Liouville formula, which in the continuous case is as follows

$$\det X(t) = \det X(0) \exp\left(\int_0^t \mathrm{Tr}A(\tau)d\tau\right), \tag{5}$$

and in the discrete one takes the form

$$\det X(t) = \det X(0) \prod_{j=0}^{t-1} \det A(j). \tag{5′}$$

The fundamental matrices are often considered to satisfy the following condition

$$X(0) = I_n,$$

where I_n is a unit $(n \times n)$-matrix.

The following definitions and results are valid for continuous system as well as for the discrete one. The proofs will be given, if necessary, for each situation separately.

Consider the vector-function $f(t)$ such that $\lim\limits_{t\to+\infty} \sup |f(t)| \neq 0$.

Definition 5. The value (or the symbol $+\infty, or -\infty$), defined by formula

$$\mathscr{X}[f(t)] = \lim_{t\to+\infty} \sup \frac{1}{t} \ln |f(t)|,$$

is called a characteristic exponent (or upper characteristic exponent) of the vector-function $f(t)$. The value

$$\lim_{t\to+\infty} \inf \frac{1}{t} \ln |f(t)|,$$

is called lower characteristic exponent.

The characteristic exponent is equal to that taken with inverse sign characteristic number, introduced by [57].

Definition 6. The characteristic exponent of the vector-function $f(t)$ is said to be exact if the finite limit

$$\lim_{t\to+\infty} \frac{1}{t} \ln |f(t)|$$

exists.

Consider the characteristic exponents of solutions of linear system (4) or (4′).

Definition 7 ([19]). A set of distinctive characteristic exponents of all solutions (except a zero solution), being different from $\pm\infty$, of linear system is called its spectrum.

Note that the number of different characteristic exponents is bounded by the dimension of the considered space of system states. Imposing conditions on $A(t)$ one can get boundedness of characteristic exponents (see, e.g., [19, 20, 43]).

3.1 Regular Systems

Consider the normal fundamental systems of solutions [57].

Definition 8. A fundamental matrix is said to be normal if the sum of characteristic exponents of its columns is minimal in comparison with other fundamental matrices.

For continuous systems [19] and discrete [20] the following result is well known.

Lemma 1. *In all normal fundamental systems of solutions, the number of solutions with equal characteristic exponents is the same. Each normal fundamental system realizes a spectrum of linear system.*

Thus, one can introduce the following definition.

Definition 9 ([19]). The set of characteristic exponents

$$\lambda_1, \ldots, \lambda_n$$

of a certain normal fundamental system of solutions is called a complete spectrum and the number $\sigma = \sum_1^n \lambda_i$ is a sum of characteristic exponents of linear system.

Note that any fundamental system of solutions has a solution with the largest characteristic exponent $\max\limits_{1 \le j \le n} \lambda_j$.

Consider a class of regular systems, introduced by Lyapunov.

Definition 10. A linear system is said to be regular if for the sum of its characteristic exponents σ the following relation holds

$$\sigma = \lim_{t \to +\infty} \inf \frac{1}{t} \ln |\det X(t)|.$$

Taking into account formula (5), in the continuous case one obtains a classical definition [1, 19] of the regularity of system

$$\sigma = \lim_{t \to +\infty} \inf \frac{1}{t} \int\limits_0^t \mathrm{Tr} A(\tau)\, \mathrm{d}\tau.$$

Similarly, formula (5′) gives a definition of regularity [20] in the discrete case

$$\sigma = \lim_{t \to +\infty} \inf \frac{1}{t} \ln \prod_{j=0}^{t-1} |\det A(j)|.$$

Definition 11. The number

$$\Gamma = \sigma - \lim_{t \to +\infty} \inf \frac{1}{t} \ln |\det X(t)|$$

is called an irregularity coefficient of linear system.

As was shown in [19], the systems with constant and periodic coefficients are regular.

For continuous [19] and discrete systems [20, 25] the following is well known

Lemma 2 (Lyapunov inequality). *Let all characteristic exponents of solutions of linear system be $< +\infty$ (or all characteristic exponents be $> -\infty$.) Then, for any fundamental system of solutions $X(t)$ the following inequality*

$$\lim_{t \to +\infty} \sup \frac{1}{t} \ln |\det X(t)| \le \sigma_X, \tag{6}$$

where σ_X is a sum of characteristic exponents of the system of solutions $X(t)$, is satisfied.

Thus, for regular systems there exists the limit $\lim\limits_{t \to +\infty} \frac{1}{t} \ln |\det X(t)|$.

Note that also from the condition of regularity of linear system it follows [19] that for its solutions $x(t) \neq 0$ there exist the limits

$$\lim_{t \to +\infty} \frac{1}{t} \ln |x(t)|.$$

Example 1 (Nonregular system with exact characteristic exponents). As was shown in [11], the opposite, generally speaking, is not valid. Consider an example of nonregular system, all characteristic exponents of which are exact [11]. Consider system (4) with the matrix

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & (\cos \ln t - \sin \ln t - 1) \end{pmatrix}, \quad t \geq 1 \tag{7}$$

and its fundamental matrix $X(t)$

$$X(t) = \big(x_1(t), x_2(t)\big) = \begin{pmatrix} 1 & \int\limits_1^t e^{\gamma(\tau)}\, \mathrm{d}\tau \\ 0 & e^{\gamma(t)} \end{pmatrix},$$

where $\gamma(t) = t(\cos \ln t - 1)$. In this case for the determinant of fundamental matrix the following relation

$$\lim_{t \to +\infty} \inf \frac{1}{t} \ln |\det X(t)| = -2 \tag{8}$$

is satisfied. Consider characteristic exponents of solutions. For $x_1(t)$ one has

$$\lim_{t \to +\infty} \sup \frac{1}{t} \ln |x_1(t)| = \lim_{t \to +\infty} \inf \frac{1}{t} \ln |x_1(t)| = 0. \tag{9}$$

Since $e^{\gamma(t)} \leq 1$ for $t \geq 1$, one concludes that the characteristic exponent $x_2(t)$ is less than or equal to zero

$$\lim_{t \to +\infty} \sup \frac{1}{t} \ln |x_2(t)| \leq 0.$$

On the other hand, since the integral of $e^{\gamma(\tau)}$ is divergent, namely

$$\int\limits_1^{+\infty} e^{\gamma(\tau)}\, \mathrm{d}\tau = +\infty, \tag{10}$$

for $x_2(t)$ one has the following estimate

$$\lim_{t \to +\infty} \inf \frac{1}{t} \ln |x_2(t)| \geq 0.$$

This implies that

$$\lim_{t\to+\infty}\sup\frac{1}{t}\ln|x_2(t)| = \lim_{t\to+\infty}\inf\frac{1}{t}\ln|x_2(t)| = 0. \tag{11}$$

Thus, by (8), (9), and (11) the linear system with matrix (7) has exact characteristic exponents but it is nonregular:

$$\Gamma = 2.$$

Let us prove that the integral of $e^{\gamma(\tau)}$ is divergent.

Suppose, $t^u(k) = e^{2k\pi+\delta(k)}$ and $t^l(k) = e^{2k\pi-\delta(k)}$, where $\delta(k) = e^{-k\pi}$, $k = 1, 2\ldots$. From the definition of $t^l(k)$ and $t^u(k)$ one obtains

$$t^u(k) - t^l(k) \geq e^{2k\pi-\delta(k)}(e^{2\delta(k)} - 1) \geq e^{2k\pi-\delta(k)}2\delta(k) \geq 2e^{k\pi-1}. \tag{12}$$

In the case $\tau \in \left[t^l(k), t^u(k)\right]$ for $\gamma(\tau)$ the estimate

$$-\gamma(\tau) \leq \tau\big(1-\cos(\delta(k))\big) \leq t^u(k)\frac{\delta^2(k)}{2} \leq \frac{1}{2}e^{2k\pi+\delta(k)}e^{-2\pi k} \leq \frac{e^{\delta(k)}}{2} \leq \frac{e}{2} \tag{13}$$

is valid. Then one has

$$\int_1^{t^u(k)} e^{\gamma(\tau)}\mathrm{d}\tau \geq \big(t^u(k) - t^l(k)\big)\,e^{-e/2} \geq 2e^{k\pi-1-e/2} \to +\infty$$

as $k \to +\infty$. ■

3.2 Lyapunov Exponents and Singular Values

Consider singular values (see, e.g., [9]) of the matrix $X(t)$.

Definition 12. The singular values $\{\alpha_j\big(X(t)\big)\}_1^n$ of the matrix $X(t)$ are the square roots of the eigenvalues of the matrix $X(t)^*X(t)$.

The following geometric interpretation of singular values is known: the numbers $\alpha_j\big(X(t)\big)$ coincide with a principal semiaxis of the ellipsoid $X(t)B$, where B is a ball of unit radius.

Definition 13 ([73]). The Lyapunov exponent μ_j is as follows

$$\mu_j = \lim_{t\to+\infty}\sup\frac{1}{t}\ln\alpha_j\big(X(t)\big). \tag{14}$$

In the case (14) the terms *upper singular exponent* are also used [5].

Let μ_1 and λ_1 be the largest Lyapunov exponent and the largest characteristic exponent, respectively.

Lemma 3. *For the linear systems the largest characteristic exponent is equal to the largest Lyapunov exponent.*

Proof. Recall that a geometric interpretation of singular values implies the relation $|X(t)| = \alpha_1\big(X(t)\big)$. Here $|X|$ is a norm of the matrix X, defined by formula $|X| = \max\limits_{|x|=1} |Xx|, \quad x \in \mathbb{R}^n$. Then the relation $\lim\limits_{t\to+\infty} \sup \frac{1}{t} \ln |X(t)| = \lambda_1$ yields the relation $\lambda_1 = \mu_1$. ■

Example 2 (Characteristic exponents do not coincide with Lyapunov exponents). Consider [51] system (4) with the matrix

$$A(t) = \begin{pmatrix} 0 & \sin(\ln t) + \cos(\ln t) \\ \sin(\ln t) + \cos(\ln t) & 0 \end{pmatrix} \quad t > 1$$

and with the fundamental normal matrix

$$X(t) = \begin{pmatrix} e^{\gamma(t)} & e^{-\gamma(t)} \\ e^{\gamma(t)} & -e^{-\gamma(t)} \end{pmatrix},$$

where $\gamma(t) = t\sin(\ln t)$. It is obvious that $\lambda_1 = \lambda_2 = 1$ and

$$\alpha_1(X(t)) = \sqrt{2}\max(e^{\gamma(t)}, e^{-\gamma(t)}), \quad \alpha_2(X(t)) = \sqrt{2}\min(e^{\gamma(t)}, e^{-\gamma(t)}).$$

This implies the following relations $\mu_1 = 1$, $\mu_2 = 0$. Thus, one has $\lambda_2 \neq \mu_2$. ■

Example 3 (Nemytskii–Vinograd counterexample). Consider [11] a continuous system

$$\frac{dx}{dt} = A(t)x$$

with the matrix

$$A(t) = \begin{pmatrix} 1 - 4(\cos 2t)^2 & 2 + 2\sin 4t \\ -2 + 2\sin 4t & 1 - 4(\sin 2t)^2 \end{pmatrix}.$$

In this case, its solution is the vector-function

$$x(t) = \begin{pmatrix} e^t \sin 2t \\ e^t \cos 2t \end{pmatrix}. \tag{15}$$

It follows that

$$\det(A(t) - pI_n) = p^2 + 2p + 1.$$

Therefore for the eigenvalues $\nu_1(t)$ and $\nu_2(t)$ of the matrix $A(t)$ one has

$$\nu_1(t) = \nu_2(t) = -1.$$

On the other hand, the characteristic exponent λ of solution (15) is equal to 1. ■

This counterexample shows that all eigenvalues of the matrix $A(t)$ can have negative real parts even if the corresponding linear system has positive characteristic exponents.

It also shows that the formulas, obtained in the book [2], namely

$$\lambda_j = \limsup_{t\to+\infty} \frac{1}{t}\int_0^t \mathrm{Re}\,\nu_j(\tau)\,d\tau$$

are untrue.

4 The Perron Effects

In 1930, O. Perron [62] showed that the negativeness of the largest Lyapunov exponent of the first approximation system does not always result in the stability of zero solution of the original system. Furthermore, in an arbitrary small neighborhood of zero, the solutions of the original system with positive Lyapunov exponent can be found.

We now present the outstanding result of Perron [1930] and its discrete analog [25, 41] (see, also, [4, 6, 21, 29, 34–38]).

Consider the following system

$$\begin{aligned} \frac{dx_1}{dt} &= -ax_1 \\ \frac{dx_2}{dt} &= \big(\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a\big)x_2 + x_1^2 \end{aligned} \tag{16}$$

and its discrete analog

$$\begin{aligned} x_1(t+1) &= \exp(-a)x_1(t) \\ x_2(t+1) &= \frac{\exp\big((t+2)\sin\ln(t+2) - 2a(t+1)\big)}{\exp\big((t+1)\sin\ln(t+1) - 2at\big)}x_2(t) + x_1(t)^2. \end{aligned} \tag{16'}$$

Here a is a number satisfying the following inequalities

$$1 < 2a < 1 + \frac{1}{2}\exp(-\pi). \tag{17}$$

The solution of the first approximation system for systems (16) and (16′) takes the form

$$\begin{aligned} x_1(t) &= \exp(-at)x_1(0) \\ x_2(t) &= \exp\big((t+1)\sin(\ln(t+1)) - 2at\big)x_2(0). \end{aligned}$$

It is obvious that by condition (17) for the solution of the first approximation system for $x_1(0) \neq 0, x_2(0) \neq 0$ one has

$$\mathscr{X}[x_1(t)] = -a, \ \mathscr{X}[x_2(t)] = 1 - 2a < 0.$$

This implies that a zero solution of linear system of the first approximation is Lyapunov stable.

Consider the solution of system (16)

$$\begin{aligned} & x_1(t) = \exp(-at)x_1(0), \\ & x_2(t) = \exp\big((t+1)\sin(\ln(t+1)) - 2at\big)\times \\ & \times\left(x_2(0) + x_1(0)^2\int\limits_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)d\tau\right). \end{aligned} \tag{18}$$

Assuming $t = t_k = \exp\left(\big(2k + \frac{1}{2}\big)\pi\right) - 1$, where k is an integer, one obtains

$$\exp\big((t+1)\sin(\ln(t+1)) - 2at\big) = \exp\big((1-2a)t + 1\big), \quad (1+t)e^{-\pi} - 1 > 0,$$

$$\int\limits_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)d\tau >$$

$$> \int\limits_{f(k)}^{g(k)} \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)\,d\tau >$$

$$> \int\limits_{f(k)}^{g(k)} \exp\left(\frac{1}{2}(\tau+1)\right)\,d\tau > \int\limits_{f(k)}^{g(k)} \exp\left(\frac{1}{2}(\tau+1)\exp(-\pi)\right)\,d\tau =$$

$$= \exp\left(\frac{1}{2}(t+1)\exp(-\pi)\right)(t+1)\left(\exp\big(-\frac{2\pi}{3}\big) - \exp(-\pi)\right),$$

where

$$f(k) = (1+t)\exp(-\pi) - 1,$$

$$g(k) = (1+t)\exp\left(-\frac{2\pi}{3}\right) - 1.$$

Hence one has the following estimate

$$\begin{aligned}
&\exp\big((t+1)\sin(\ln(t+1)) - 2at\big)\int_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)d\tau > \\
&> \exp\left(\frac{1}{2}(2+\exp(-\pi))\right)\left(\exp\big(-\frac{2\pi}{3}\big) - \exp(-\pi)\right)(t+1)\times \\
&\times \exp\left((1-2a+\frac{1}{2}\exp(-\pi))t\right).
\end{aligned} \tag{19}$$

From the last inequality and condition (17) it follows that for $x_1(0) \neq 0$ one of the characteristic exponents of solutions of system (16) is positive:

$$\mathscr{X}[x_1(t)] = -a,\ \mathscr{X}[x_2(t)] \geq 1 - 2a + e^{-\pi}/2 > 0. \tag{20}$$

Thus, one obtains that all characteristic exponents of the first approximation system are negative but almost all solutions of the original system (16) tend exponentially to infinity as $t_k \to +\infty$.

Consider now the solution of discrete system ($16'$)

$$\begin{aligned}
x_1(t) &= x_1(0)e^{-at} \\
x_2(t) &= \exp\big((t+1)\sin\ln(t+1) - 2at\big)\times \\
&\times\left(x_2(0) + x_1(0)^2\sum_{k=0}^{t-1}\exp\big(-(k+2)\sin\ln(k+2) + 2a\big)\right),
\end{aligned} \tag{21}$$

and show that for this system inequalities (20) are also satisfied. For this purpose one obtains the estimate similar to estimate (19) in the discrete case.

Obviously, for any $N > 0$ and $\delta > 0$ there exists a natural number ($t' = t'(N,\delta),\ t' > N$) such that

$$\sin\ln(t'+1) > 1 - \delta.$$

Then

$$\exp\big((t'+1)\sin\ln(t'+1) - 2at'\big) \geq \exp\big((1-\delta-2a)t' + 1 - \delta\big). \tag{22}$$

Estimate from below the second multiplier in the expression for $x_2(t)$. For sufficiently large t' there exists a natural number m

$$m \in \left(\frac{t'+1}{e^{\pi}} - 2, t'\right)$$

such that

$$\sin \ln(m+2) \leq -\frac{1}{2}.$$

Then one has

$$-(m+2)\sin\ln(m+2) + 2a \geq \frac{t'+1}{2e^{\pi}}.$$

This implies the following estimate

$$\sum_{k=0}^{t'-1} \exp\big(-(k+2)\sin\ln(k+2) + 2a\big) \geq \exp\left((t'+1)\frac{1}{2}e^{-\pi}\right). \tag{23}$$

From (22), (23), and condition (17) it follows that for $x_1(0) \neq 0$ one of characteristic exponents of solutions (21) of system (16′) is positive and inequalities (20) are satisfied.

Consider an example, which shows the possibility of the sign reversal of characteristic exponents "on the contrary," namely the solution of the first approximation system has a positive characteristic exponent while the solution of the original system with the same initial data has a negative exponent [47].

Consider the following continuous system [49]

$$\begin{aligned}
\dot{x}_1 &= -ax_1\\
\dot{x}_2 &= -2ax_2\\
\dot{x}_3 &= \big(\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a\big)x_3 + x_2 - x_1^2
\end{aligned} \tag{24}$$

and its discrete analog

$$\begin{aligned}
x_1(t+1) &= e^{-a}x_1(t)\\
x_2(t+1) &= e^{-2a}x_2(t)
\end{aligned}$$

$$x_3(t+1) = \frac{\exp\big((t+2)\sin\ln(t+2) - 2a(t+1)\big)}{\exp\big((t+1)\sin\ln(t+1) - 2at\big)}x_3(t) + x_2(t) - x_1(t)^2 \tag{24′}$$

on the invariant manifold

$$M = \{x_3 \in \mathbb{R}^1,\ x_2 = x_1^2\}.$$

Here the value a satisfies condition (17).

The solutions of (24) and (24′) on the manifold M take the form

$$\begin{aligned} x_1(t) &= \exp\big(-at\big)x_1(0) \\ x_2(t) &= \exp\big(-2at\big)x_2(0) \\ x_3(t) &= \exp\big((t+1)\sin(\ln(t+1)) - 2at\big)x_3(0), \\ x_1(0)^2 &= x_2(0). \end{aligned} \tag{25}$$

Obviously, these solutions have negative characteristic exponents.

For system (24) in the neighborhood of its zero solution, consider the first approximation system

$$\begin{aligned} \dot{x}_1 &= -ax_1 \\ \dot{x}_2 &= -2ax_2 \\ \dot{x}_3 &= \big(\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a\big)x_3 + x_2. \end{aligned} \tag{26}$$

The solutions of this system are the following

$$\begin{aligned} x_1(t) &= \exp\big(-at\big)x_1(0) \\ x_2(t) &= \exp\big(-2at\big)x_2(0) \\ x_3(t) &= \exp\big((t+1)\sin(\ln(t+1)) - 2at\big)\times \\ &\qquad \times\Big(x_3(0) + x_2(0)\int_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)d\tau\Big). \end{aligned} \tag{27}$$

For system (24′) in the neighborhood of its zero solution, the first approximation system is as follows

$$\begin{aligned} x_1(t+1) &= \exp(-a)x_1(t) \\ x_2(t+1) &= \exp(-2a)x_2(t) \\ x_3(t+1) &= \frac{\exp\big((t+2)\sin\ln(t+2) - 2a(t+1)\big)}{\exp\big((t+1)\sin\ln(t+1) - 2at\big)}x_3(t) + x_2(t). \end{aligned} \tag{26′}$$

Then the solutions of system (26′) take the form

$$\begin{aligned} x_1(t) &= \exp\big(-at\big)x_1(0) \\ x_2(t) &= \exp\big(-2at\big)x_2(0) \\ x_3(t) &= \exp\big((t+1)\sin\ln(t+1) - 2at\big)\times \\ &\qquad \times\Big(x_3(0) + x_2(0)^2\sum_{k=0}^{t-1}\exp\big(-(k+2)\sin\ln(k+2) + 2a\big)\Big). \end{aligned} \tag{27′}$$

By estimates (19) and (23) for solutions (27) and (27′) for $x_2(0) \neq 0$ one obtains

$$\mathscr{X}[x_3(t)] > 0.$$

It is easily shown that for the solutions of systems (24) and (26) the following relations

$$\big(x_1(t)^2 - x_2(t)\big)^{\bullet} = -2a\big(x_1(t)^2 - x_2(t)\big)$$

are valid. Similarly, for system (26′) one has

$$x_1(t+1)^2 - x_2(t+1) = \exp(-2a)\big(x_1(t)^2 - x_2(t)\big).$$

Then

$$x_1(t)^2 - x_2(t) = \exp\big(-2at\big)\big(x_1(0)^2 - x_2(0)\big).$$

It follows that the manifold M is an invariant exponentially attractive manifold for solutions of continuous systems (24) and (26), and for solutions of discrete systems (24′) and (26′).

This means that the relation $x_1(0)^2 = x_2(0)$ yields the relation $x_1(t)^2 = x_2(t)$ for all $t \in \mathbb{R}^1$ and for any initial data one has

$$\big|x_1(t)^2 - x_2(t)\big| \leq \exp\big(-2at\big)\big|x_1(0)^2 - x_2(0)\big|.$$

Thus, systems (24) and (26) have the same invariant exponentially attractive manifold M on which almost all solutions of the first approximation system (26) have a positive characteristic exponent and all solutions of the original system (24) have negative characteristic exponents. The same result can be obtained for discrete systems (24′) and (26′).

The Perron effect occurs here on the whole manifold

$$\{x_3 \in \mathbb{R}^1,\ x_2 = x_1^2 \neq 0\}.$$

To construct exponentially stable system, the first approximation of which has a positive characteristic exponent we represent system (24) in the following way

$$\begin{aligned}
\dot{x}_1 &= F(x_1, x_2)\\
\dot{x}_2 &= G(x_1, x_2)\\
\dot{x}_3 &= \big(\sin\ln(t+1) + \cos\ln(t+1) - 2a\big)x_3 + x_2 - x_1^2.
\end{aligned} \tag{28}$$

Here the functions $F(x_1, x_2)$ and $G(x_1, x_2)$ have the form

$$F(x_1, x_2) = \pm 2x_2 - ax_1, \quad G(x_1, x_2) = \mp x_1 - \varphi(x_1, x_2),$$

in which case the upper sign is taken for $x_1 > 0$, $x_2 > x_1^2$ and for $x_1 < 0$, $x_2 < x_1^2$, the lower one for $x_1 > 0$, $x_2 < x_1^2$ and for $x_1 < 0$, $x_2 > x_1^2$.

The function $\varphi(x_1, x_2)$ is defined as

$$\varphi(x_1, x_2) = \begin{cases} 4ax_2 \text{ for } |x_2| > 2x_1^2 \\ 2ax_2 \text{ for } |x_2| < 2x_1^2. \end{cases}$$

The solutions of system (28) are regarded in the sense of Filippov [23]. By definition of $\varphi(x_1, x_2)$ the following system

$$\begin{aligned} \dot{x}_1 &= F(x_1, x_2) \\ \dot{x}_2 &= G(x_1, x_2) \end{aligned} \tag{29}$$

on the lines of discontinuity $\{x_1 = 0\}$ and $\{x_2 = x_1^2\}$ has sliding solutions, which are given by the equations

$$x_1(t) \equiv 0, \quad \dot{x}_2(t) = -4ax_2(t)$$

and

$$\dot{x}_1(t) = -ax_1(t), \quad \dot{x}_2(t) = -2ax_2(t), \quad x_2(t) \equiv x_1(t)^2.$$

In this case the solutions of system (29) with the initial data $x_1(0) \neq 0$, $x_2(0) \in \mathbb{R}^1$ attain the curve $\{x_2 = x_1^2\}$ in a finite time, which is less than or equal to 2π.

This implies that for the solutions of system (28) with the initial data $x_1(0) \neq 0$, $x_2(0) \in \mathbb{R}^1$, $x_3(0) \in \mathbb{R}^1$, for $t \geq 2\pi$ one obtains the relations $F(x_1(t), x_2(t)) = -ax_1(t)$, $G(x_1(t), x_2(t)) = -2ax_2(t)$. Therefore, based on these solutions for $t \geq 2\pi$ system (26) is a system of the first approximation for system (28).

System (26), as was shown above, has a positive characteristic exponent. At the same time, all solutions of system (28) tend exponentially to zero. ∎

The considered technique permits us to construct the different classes of nonlinear continuous and discrete systems for which the Perron effects occur.

5 Stability Criteria by the First Approximation

Consider a normal fundamental matrix $X(t)$ of the linear part of the system, and let

$$\Lambda = \max_j \lambda_j, \quad \lambda = \min_j \lambda_j.$$

Here $\{\lambda_j\}_1^n$ is a complete spectrum of linear system. We shall say that $X(t)X(\tau)^{-1}$ is a Cauchy matrix. Represent the solutions of systems (2) and (2′) in the Cauchy form. In the continuous case one has

$$x(t) = X(t)x(0) + \int_0^t X(t)X(\tau)^{-1} f\big(\tau, x(\tau)\big)\, d\tau, \tag{30}$$

and in the discrete one

$$x(t) = X(t)x(0) + \sum_{\tau=0}^{t-1} X(t)X(\tau+1)^{-1} f\big(\tau, x(\tau)\big),\ t = 1, 2 \dots . \tag{31}$$

The following result is well known and is often used.

Theorem 1. *For any number $\varepsilon > 0$ there exists a number $C > 0$ such that the following inequalities*

$$|X(t)X(\tau)^{-1}| \le C \exp\big((\Lambda + \varepsilon)(t - \tau) + (\Gamma + \varepsilon)\tau\big), \quad \forall\, t \ge \tau \ge 0 \tag{32}$$

$$|X(t)X(\tau)^{-1}| \le C \exp\big(\lambda(t - \tau) + (\Gamma + \varepsilon)\tau\big), \quad \forall\, \tau \ge t \ge 0, \tag{33}$$

where Γ is the irregularity coefficient, are satisfied.

Recall that by condition (3) the nonlinear part $f(t, x)$ of systems (2) and (2′) in a certain neighborhood $\Omega(0)$ of the point $x = 0$ satisfies the following condition

$$|f(t, x)| \le \kappa |x|^\nu\ \forall t \ge 0, \quad \forall x \in \Omega(0), \quad \kappa > 0, \nu > 1.$$

Let us describe the most famous stability criteria by the first approximation.

Consider the continuous case. Assume that there exists a number $C > 0$ and a piecewise continuous function $p(t)$ such that for the Cauchy matrix $X(t)X(\tau)^{-1}$ the estimate

$$|X(t)X(\tau)^{-1}| \le C \exp \int_\tau^t p(s)\, ds, \quad \forall t \ge \tau \ge 0 \tag{34}$$

is valid.

Theorem 2 ([51]). *If condition (3) with $\nu = 1$ and the inequality*

$$\limsup_{t \to +\infty} \frac{1}{t} \int_0^t p(s)\, ds + C\kappa < 0$$

are satisfied, then the solution $x(t) \equiv 0$ of system (2) is asymptotically Lyapunov stable.

Proof. From (30) and the hypotheses of theorem one has

$$|x(t)| \leq C \exp\left(\int_0^t p(s)\,ds\right)|x(0)| + C\int_0^t \exp\left(\int_\tau^t p(s)\,ds\right)\kappa|x(\tau)|\,d\tau.$$

This estimate can be rewritten as

$$\exp\left(-\int_0^t p(s)\,ds\right)|x(t)| \leq C|x(0)| + C\kappa\int_0^t \exp\left(-\int_0^\tau p(s)\,ds\right)|x(\tau)|)\,d\tau.$$

By Bellman–Gronwall Lemma the following estimate

$$|x(t)| \leq C|x(0)|\exp\left(\int_0^t p(s)\,ds + C\kappa t\right), \quad \forall t \geq 0$$

is satisfied. This completes the proof of theorem. ■

Consider a discrete analog of this theorem. In the discrete case it is assumed that in place of inequality (34) one has

$$|X(t)X(\tau)^{-1}| \leq C\prod_{s=\tau}^{t-1} p(s), \quad \forall t > \tau \geq 0, \tag{35}$$

where $p(s)$ is a positive function.

In the discrete case one has a similar theorem

Theorem 3 ([44, 52]). *If condition* (3) *with* $\nu = 1$ *and the inequality*

$$\lim_{t\to+\infty}\sup\ \frac{1}{t}\ln\prod_{s=0}^{t-1}\big(p(s) + C\kappa\big) < 0 \tag{36}$$

are satisfied, then the solution $x(t) \equiv 0$ *of system* ($2'$) *is asymptotically Lyapunov stable.*

Corollary 1. *For the first-order system* (2) *or* ($2'$) *the negativeness of characteristic exponent implies the asymptotic stability of its zero solution.*

Assume that for the Cauchy matrix $X(t)X(\tau)^{-1}$ the following estimate

$$|X(t)X(\tau)^{-1}| \leq C\exp\big(-\alpha(t-\tau) + \gamma\tau\big), \quad \forall t \geq \tau \geq 0, \tag{37}$$

where $\alpha > 0$, $\gamma \geq 0$, is satisfied.

Theorem 4 ([14, 58, 60]). *Let condition* (3) *with sufficiently small* κ *and condition* (37) *be valid. Then if the inequality*

$$(\nu - 1)\alpha - \gamma > 0 \tag{38}$$

holds, then the solution $x(t) \equiv 0$ *is asymptotically Lyapunov stable.*

Theorem 4 strengthens the well-known Lyapunov theorem [57] on stability by the first approximation for regular systems.

5.1 Stability Criteria for the Flow and Cascade of Solutions

Consider system (1) or (1′) where $F(\cdot,\cdot)$ is a twice continuously differentiable vector-function. Consider the linearizations of these systems along solutions with the initial data $y = x(0, y)$ from the open set Ω, which is bounded in $\mathbb{R}^n$

$$\frac{dz}{dt} = A_y(t)z, \tag{39}$$

$$z(t+1) = A_y(t)z(t). \tag{39′}$$

Here the matrix

$$A_y(t) = \frac{\partial F(x,t)}{\partial x}\Big|_{x=x(t,y)}$$

is Jacobian matrix of the vector-function $F(x,t)$ on the solution $x(t, y)$. Let $X(t, y)$ be a fundamental matrix of linear system and $X(0, y) = I_n$.

Assume that for the largest singular value $\alpha_1(t, y)$ of systems (39) and (39′) for all t the following estimate

$$\alpha_1(t, y) < \alpha(t), \quad \forall y \in \Omega, \tag{40}$$

where $\alpha(t)$ is a scalar function, is valid.

Theorem 5 ([43, 50]). *Suppose the function* $\alpha(t)$ *is bounded on the interval* $(0, +\infty)$. *Then the flow (cascade) of solutions* $x(t, y)$, $y \in \Omega$, *of systems* (1) *and* (1′) *is Lyapunov stable. If, in addition,*

$$\lim_{t\to+\infty} \alpha(t) = 0,$$

then the flow (cascade) of solutions $x(t, y)$, $y \in \Omega$, *is asymptotically Lyapunov stable.*

Proof. It is well known that

$$\frac{\partial x(t,y)}{\partial y} = X(t,y), \quad \forall t \geq 0.$$

It is also known that for any vectors y, z, and a number $t \geq 0$ there exists a vector w such that the relations

$$|w - y| \leq |y - z|,$$

$$|x(t,y) - x(t,z)| \leq \left|\frac{\partial x(t,w)}{\partial w}\right| |y - z|$$

are satisfied. Therefore for any vector z from the ball centered at y and placed entirely in Ω the following estimate

$$|x(t,y) - x(t,z)| \leq |y - z| \sup \alpha_1(t,w) \leq \alpha(t)|y - z|, \quad \forall t \geq 0 \tag{41}$$

is valid. Here the supremum is taken over all w from the ball $\{w : |w-y| \leq |y-z|\}$.

Estimate (41) gives at once the assertions of theorem. ∎

Corollary 2. *The Perron effects are possible on the boundary of the stable by the first approximation solutions flow (cascade) only.*

Consider the flow of solutions of system (16) with the initial data in a neighborhood of the point $x_1 = x_2 = 0$: $x_1(0, x_{10}, x_{20}) = x_{10}$, $x_2(0, x_{10}, x_{20}) = x_{20}$.

Hence it follows easily that

$$x_1(t, x_{10}, x_{20}) = \exp\big(-at\big)x_{10}.$$

Therefore for continuous system the matrix $A(t)$ of linear system takes the form

$$A(t) = \begin{pmatrix} -a & 0 \\ 2\exp\big(-at\big)x_{10} & r(t) \end{pmatrix}, \tag{42}$$

where

$$r(t) = \sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a.$$

For the discrete system one has

$$A(t) = \begin{pmatrix} e^{-a} & 0 \\ 2\exp\big(-at\big)x_{10} & r(t) \end{pmatrix}, \tag{42$'$}$$

$$r(t) = \frac{\exp\big((t+2)\sin\ln(t+2) - 2a(t+1)\big)}{\exp\big((t+1)\sin\ln(t+1) - 2at\big)}.$$

The solutions of system (39) and (39′) with matrices (42) and (42′), respectively, are the following

$$
\begin{aligned}
z_1(t) &= \exp\big(-at\big)z_1(0),\\
z_2(t) &= p(t)(z_2(0) + 2x_{10}z_1(0))q(t)).
\end{aligned}
\tag{43}
$$

Here

$$
p(t) = \exp\big((t+1)\sin(\ln(t+1)) - 2at\big),
$$

$$
q(t) = \int_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)\,d\tau
$$

in the continuous case and

$$
p(t) = \exp\big((t+1)\sin(\ln(t+1)) - 2at\big),
$$

$$
q(t) = \sum_{k=0}^{t-1} \exp\big(-(k+2)\sin\ln(k+2) + 2a\big)
$$

in the discrete case.

As was shown above (20), if relations (17) are satisfied and

$$
z_1(0)x_{10} \neq 0,
$$

then the characteristic exponent of $z_2(t)$ is positive.

Hence in an arbitrary small neighborhood of the trivial solution $x_1(t) \equiv x_2(t) \equiv 0$ there exist the initial data x_{10}, x_{20} such that for $x_1(t, x_{10}, x_{20})$, $x_2(t, x_{10}, x_{20})$ the first approximation system has the positive largest characteristic exponent (and Lyapunov exponent μ_1).

Therefore in this case there does not exist a neighborhood Ω of the point $x_1 = x_2 = 0$ such that uniform estimates (40) are satisfied. Thus, for systems (16), (16′) the Perron effect occurs.

6 Instability Criteria by the First Approximation

6.1 The Perron–Vinograd Triangulation Method

One of the basic procedures for analysis of instability is a reduction of the linear part of the system to the triangular form. In this case the Perron–Vinograd triangulation method for a linear system [11, 19] turns out to be most effective.

Let $Z(t) = \big(z_1(t), \ldots, z_n(t)\big)$ be a fundamental system of solutions of linear continuous system (4) or discrete system (4′). Apply the Schmidt orthogonalization procedure to the solutions $z_j(t)$.

$$\begin{aligned}
&v_1(t) = z_1(t)\\
&v_2(t) = z_2(t) - v_1(t)^* z_2(t)\frac{v_1(t)}{|v_1(t)|^2}\\
&\ldots\ldots\ldots\ldots\ldots\\
&v_n(t) = z_n(t) - v_1(t)^* z_n(t)\frac{v_1(t)}{|v_1(t)|^2} - \ldots - v_{n-1}(t)^* z_n(t)\frac{v_{n-1}(t)}{|v_{n-1}(t)|^2}.
\end{aligned} \tag{44}$$

Relations (44) yield the following relations

$$v_i(t)^* v_j(t) = 0, \quad \forall j \neq i, \tag{45}$$

$$|v_j(t)|^2 = v_j(t)^* z_j(t). \tag{46}$$

If for the fundamental matrix $Z(t)$ the relation $Z(0) = I_n$ holds, one concludes that $V(0) = \big(v_1(0), .., v_n(0)\big) = I_n$.

Proceed now to the description of *the triangulation procedure of Perron–Vinograd.*

Consider the unitary matrix

$$U(t) = \left(\frac{v_1(t)}{|v_1(t)|}, \cdots, \frac{v_n(t)}{|v_n(t)|}\right),$$

and make the change of variable: $z = U(t)w$ in the linear system. In the continuous case one obtains the system

$$\frac{dw}{dt} = B(t)w, \tag{47}$$

where

$$B(t) = U(t)^{-1}A(t)U(t) - U(t)^{-1}\dot{U}(t), \tag{48}$$

and in the discrete case the system

$$w(t+1) = B(t)w(t), \tag{47'}$$

where

$$B(t) = U(t+1)^{-1}A(t)U(t). \tag{48'}$$

The unitarity of the matrix $U(t)$ implies that for the columns $w(t)$ of the fundamental matrix

$$W(t) = \big(w_1(t), \dots, w_n(t)\big) = U(t)^* Z(t), \tag{49}$$

the relations $|w_j(t)| = |z_j(t)|$ are satisfied.

By (44)–(46) one obtains that the matrix $W(t)$ has the upper triangular form with the diagonal elements $|v_1(t)|$,..,$|v_n(t)|$, namely

$$W(t) = \begin{pmatrix} |v_1(t)| & \cdots & \\ & \ddots & \vdots \\ 0 & & |v_n(t)| \end{pmatrix}. \tag{50}$$

From the fact that $W(t)$ is an upper triangular matrix it follows that $W(t)^{-1}$, $\dot{W}(t)$ are also upper triangular matrices. Hence $B(t)$ is an upper triangular matrix with the diagonal elements $b_1(t)$,..,$b_n(t)$:

$$B(t) = \begin{pmatrix} b_1(t) & \cdots & \\ & \ddots & \vdots \\ 0 & & b_n(t) \end{pmatrix}, \tag{51}$$

where in the continuous case $b_i(t) = (\ln|v_i(t)|)^\bullet$ and in the discrete one

$$b_i(t) = \frac{|v_i(t+1)|}{|v_i(t)|}.$$

Thus, it is proved the following

Theorem 6 (Perron triangulation). *By means of the unitary transformation $z = U(t)w$ the linear system can be reduced to the linear system with the upper triangular matrix $B(t)$.*

Note that if $|A(t)|$ is bounded for $t \geq 0$, then $|B(t)|$, $|U(t)|$, and $|\dot{U}(t)|$ are also bounded for $t \geq 0$. If in the discrete case, in addition, $|A(t)^{-1}|$ is bounded for $t \geq 0$, then $|B(t)^{-1}|$ is also bounded for $t \geq 0$.

Lemma 4. *The following estimate*

$$\frac{|v_n(t)|}{|v_n(\tau)|} \geq \frac{|\det Z(t)|}{|\det Z(\tau)|} \prod_{j=1}^{n-1} \frac{|v_j(\tau)|}{|z_j(t)|}, \quad \forall t, \tau \geq 0 \tag{52}$$

is valid.

Define the vector $z_i' = z_i - v_i$. Then the vector z_i' is orthogonal to the vector v_i, where $i \geq 2$. Consider the angle included between the vectors z_i and z_i'. Note that from definition of the angle included between the vectors one has $\angle(z_i, z_i') \leq \pi$. In this case the following relation

$$|v_i| = |z_i| \sin\big(\angle(z_i, z_i')\big) \quad i \geq 2 \tag{53}$$

is valid.

By (53) from (49) and (50) one has

$$|\det Z(t)| = |\det U(t)| \prod_{i=1}^{n} |v_i| = \prod_{i=1}^{n} |z_i| \prod_{k=2}^{n} |\sin\big(\angle(z_k, z_k')\big)|.$$

With the help of this relation in [74] the following criterion of system regularity was obtained.

Theorem 7 ([74]). *Consider a linear system with bounded coefficients and its certain fundamental system of solutions* $Z(t) = \big(z_1(t), \ldots, z_n(t)\big)$. *Let there exist the exact characteristic exponents of* $|z_i(t)|$

$$\lim_{t \to +\infty} \frac{1}{t} \ln |z_i(t)| \quad i = 1, \ldots, n \tag{54}$$

and let there exist and be equal to zero the exact characteristic exponents of sines of the angles $\angle(z_i, z_i')$

$$\lim_{t \to +\infty} \frac{1}{t} \ln \big| \sin\big(\angle(z_i, z_i')\big)\big| = 0 \quad i = 2, \ldots, n. \tag{55}$$

Then the linear system is regular and $Z(t)$ *is a normal system of solutions.*

Conversely, if the linear system is regular and $Z(t)$ *is a normal system of solutions, then (54) and (55) are satisfied.*

6.2 Instability Criterion by Krasovsky

Consider instability in the sense of Krasovsky for the solution $x(t) \equiv 0$ of continuous system (2) and of discrete system (2′).

Theorem 8 ([42,45,49]). *If the relation*

$$\sup_{1\le k\le n}\ \lim_{t\to+\infty}\ \inf\left[\frac{1}{t}\left(\ln\left|\det Z(t)\right| - \sum_{j\neq k}\ln\left|z_j(t)\right|\right)\right] > 1 \tag{56}$$

is satisfied, then the solution $x(t)\equiv 0$ is unstable in the sense of Krasovsky.[1]

Proof. One can assume, without loss of generality, that in (56) the supremum, taken over k, is attained for $k = n$. Then by Lemma 4 with $\tau = 0$ from condition (56) one obtains that there exists a number $\mu > 1$ such that for sufficiently large t the following estimate

$$\ln|v_n(t)| \ge \mu t, \quad \mu > 1 \tag{57}$$

holds. Suppose now that the solution $x(t)\equiv 0$ is stable by Krasovsky. This means that in a certain neighborhood of the point $x = 0$ there exists a number $R > 0$ such that the estimate

$$|x(t, x_0)| \le R|x_0|, \quad \forall t \ge 0 \tag{58}$$

is valid. Make use of the Perron–Vinograd change of variable

$$x = U(t)y \tag{59}$$

to obtain a system with the upper triangular matrix $B(t)$ of the type (51).

1. Consider the continuous case. Using (59), from continuous system (2) one obtains

$$\frac{dy}{dt} = B(t)y + g(t,y), \quad g(t,y) = U(t)^{-1} f\big(t, U(t)y\big). \tag{60}$$

Thus, the last equation of system (60) takes the form

$$\frac{dy_n}{dt} = \big(\ln|v_n(t)|\big)^{\bullet} y_n + g_n(t,y). \tag{61}$$

Here y_n and g_n are the nth components of the vectors y and g, respectively. Conditions (3) and (58) yield the estimate

$$\big|g\big(t, y(t)\big)\big| \le \varkappa R^{\nu}\big|y(0)\big|^{\nu}. \tag{62}$$

Note that the solution $y_n(t)$ of (61) can be represented in the form

[1]Condition (56) can be weakened [40,42,52].

$$y_n(t) = \frac{|v_n(t)|}{|v_n(0)|}\left(y_n(0) + \int_0^t \frac{|v_n(0)|}{|v_n(s)|} g\big(s, y(s)\big) ds\right). \tag{63}$$

Estimate (57) implies that there exists a number $\rho > 0$ such that the following inequalities

$$\int_0^t \frac{|v_n(0)|}{|v_n(s)|} ds \le \rho, \quad \forall t \ge 0 \tag{64}$$

are valid. Take the initial condition $x_0 = U(0)y(0)$ in such a way that $y_n(0) = |y(0)| = \delta$, where the number δ satisfies the inequality

$$\delta > \rho \varkappa R^\nu \delta^\nu. \tag{65}$$

Then from (62)–(64) for sufficiently large $t \ge 0$ one obtains the following estimate

$$y_n(t) \ge t^\mu(\delta - \rho \varkappa R^\nu \delta^\nu), \quad \mu > 1.$$

By (65)

$$\lim_{t\to+\infty} \inf y_n(t) = +\infty.$$

The latter contradicts the assumption on stability in the sense of Krasovsky of a trivial solution of system (2).

2. Let us prove the theorem in the discrete case. By (59), from discrete system (2′) one has

$$y(t+1) = B(t)y(t) + g\big(t, y(t)\big), \tag{66}$$

where

$$g\big(t, y(t)\big) = U(t+1)^{-1} f\big(t, U(t)y(t)\big).$$

Then the last equation of system (66) takes the form

$$y_n(t+1) = \frac{|v_n(t+1)|}{|v_n(t)|} y_n(t) + g_n\big(t, y(t)\big), \tag{67}$$

where y_n and g_n are the nth components of the vectors y and g, respectively. Conditions (3) and (58) give the following estimate

$$\big|g\big(t, y(t)\big)\big| \le \varkappa R^\nu \big|y(0)\big|^\nu. \tag{68}$$

Note that the solution $y_n(t)$ of (67) can be represented as

$$y_n(t) = \frac{|v_n(t)|}{|v_n(0)|}\left(\sum_{j=0}^{t-1} \frac{|v_n(0)|}{|v_n(j+1)|} g_n\big(j, y(j)\big) + y_n(0)\right). \tag{69}$$

Estimate (57) implies that there exists a number $\rho > 0$ such that the following inequality

$$\sum_{j=0}^{t-1} \frac{|v_n(0)|}{|v_n(j+1)|} < \rho, \quad t \geq 1 \tag{70}$$

is satisfied. Taking the same initial data as in the continuous case (65), one obtains

$$\lim_{t \to +\infty} \inf y_n(t) = +\infty.$$

The latter contradicts the assumption on stability in the sense of Krasovsky of a trivial solution of system (2′).

This proves the theorem. ■

Remark. Concerning the method for the proof of theorem.

Assuming that the zero solution of the considered system is stable in the sense of Lyapunov and using the same reasoning as in the case of stability in the sense of Krasovsky, one need to prove in the continuous case the following inequality

$$y_n(0) + \int_0^{+\infty} \frac{|v_n(0)|}{|v_n(s)|} g\big(s, y(s)\big)\, ds \neq 0. \tag{71}$$

While the above inequality is easily proved in the case of stability in the sense of Krasovsky, this becomes an intractable problem in the case of stability in the sense of Lyapunov.

A scheme similar to that, considered above for reducing the problem to one scalar equation of the type (61), was used by N.G. Chetaev [1990; 1948] to obtain instability criteria. In the scheme, suggested by N.G. Chetaev for proving inequality (71), a similar difficulty occurs. Therefore, at present, Chetaev's technique permits us to obtain the criteria of instability in the sense of Krasovsky only.

The method to obtain the criteria of instability in the sense of Lyapunov invites further development. Such development under certain additional restrictions will be presented in Theorem 10.

Corollary 3. *Condition (56) of Theorem 8 is satisfied if the following inequality*

$$\Lambda - \Gamma > 0 \tag{72}$$

is valid, where Λ is the largest Lyapunov exponent, Γ is the irregularity coefficient.

Note 1. An open problem is to prove Chetayev theorem on Lyapunov instability or refute the assertion, i.e. to build an example of a regular (non-regular) system with a positive Lyapunov exponent, which is stable in the sence of Lyapunov but unstable in the sense of Krasovsky.

Recall here stability condition (38) of Theorem 4, which by Theorem 1 can be represented as

$$(\nu - 1)\Lambda + \Gamma < 0. \tag{73}$$

Since Theorems 2–4 give, at the same time, the criteria of stability in the sense of Krasovsky, one can formulate the following

Theorem 9 ([49]). *If*

$$\Lambda < \frac{-\Gamma}{(\nu - 1)},$$

then the solution $x(t) \equiv 0$ is stable in the sense of Krasovsky and if

$$\Lambda > \Gamma,$$

then the solution $x(t) \equiv 0$ is unstable in the sense of Krasovsky.

For regular systems (the case $\Gamma = 0$), Theorem 9 gives a complete solution of the problem of stability in the sense of Krasovsky in the noncritical case ($\Lambda \neq 0$).

Note that for system (26) the relation $\Gamma = \Lambda + 2a + 1$ holds. Therefore for system (26) condition (72) is untrue.

Consider now Lyapunov instability of the solution $x(t) \equiv 0$ of multidimensional continuous system (2) and of discrete system (2′).

Theorem 10 ([42, 49, 52]). *Let for certain values $C > 0$, $\beta > 0$, $\alpha_1, \ldots, \alpha_{n-1}$ ($\alpha_j < \beta$ for $j = 1, \ldots, n-1$) the following conditions hold:*

1.

$$\begin{gathered} |z_j(t)| \leq C \exp(\alpha_j (t - \tau))|z_j(\tau)|, \\ \forall\, t \geq \tau \geq 0, \quad j = 1, \ldots, n-1, \end{gathered} \tag{74}$$

2.

$$\frac{1}{(t - \tau)} \ln|\det Z(t)| > \beta + \sum_{j=1}^{n-1} \alpha_j, \quad \forall\, t \geq \tau \geq 0, \tag{75}$$

and, if $n > 2$,

3.

$$\prod_{j=1}^{n} |z_j(t)| \leq C\,|\det Z(t)|, \quad \forall t \geq 0. \tag{76}$$

Then the zero solution of the system considered is Lyapunov unstable.

Corollary 4. *For the first-order system* (2) *or* (2′) *with bounded coefficients the positiveness of lower characteristic exponent of the first approximation system results in exponential instability of zero solution of the original system.*

The problem arises naturally as to the weakening of instability conditions, which are due to Theorems 8 and 10. However the Perron effects impose restrictions on such weakening.

Consider continuous and discrete systems (1) and (1′), respectively.

Suppose, for a certain vector-function $\xi(t)$ the following relations

$$|\xi(t)| = 1, \quad \inf_{y\in\Omega} \left|X(t,y)\xi(t)\right| \geq \alpha(t), \quad \forall t \geq t_0 \tag{77}$$

hold.

Theorem 11 ([42, 50]). *Let for the function $\alpha(t)$ the following condition*

$$\lim_{t\to+\infty} \sup \alpha(t) = +\infty \tag{78}$$

be satisfied.

Then the flow (cascade) of solutions $x(t,y)$, $y \in \Omega$ is Lyapunov unstable.

Proof. Holding a certain pair $x_0 \in \Omega$ and $t \geq t_0$ fixed, choose the vector y_0 in any δ-neighborhood of the point x_0 in such a way that

$$x_0 - y_0 = \delta\xi(t). \tag{79}$$

Let δ be so small that the ball of radius δ centered at x_0 is entirely placed in Ω.

For any fixed values t, j and for the vectors x_0, y_0 there exists a vector $w_j \in \mathbb{R}^n$ such that

$$|x_0 - w_j| \leq |x_0 - y_0|,$$

$$x_j(t,x_0) - x_j(t,y_0) = X_j(t,w_j)(x_0 - y_0). \tag{80}$$

Here $x_j(t,x_0)$ is the jth component of the vector-function $x(t,x_0)$, $X_j(t,w)$ is the jth row of the matrix $X(t,w)$.

By (80) one has

$$|x(t,x_0)-x(t,y_0)| = \sqrt{\sum_j |X_j(t,w_j)(x_0-y_0)|^2} \geq$$

$$\geq \delta \max\{|X_1(t,w_1)\xi(t)|,\dots,|X_n(t,w_n)\xi(t)|\} \geq$$

$$\geq \delta \max_j \inf_\Omega |X_j(t,x_0)\xi(t)| = \delta \inf_\Omega \max_j |X_j(t,x_0)\xi(t)| \geq$$

$$\geq \frac{\delta}{\sqrt{n}} \inf_\Omega |X(t,x_0)\xi(t)| \geq \frac{\alpha(t)\delta}{\sqrt{n}}.$$

This estimate and conditions (78) imply that for any positive numbers ε and δ there exist a number $t \geq t_0$ and a vector y_0 such that

$$|x_0-y_0| = \delta, \quad |x(t,x_0)-x(t,y_0| > \varepsilon.$$

The latter means that the solution $x(t,x_0)$ is Lyapunov unstable. ■

Consider the hypotheses of Theorem 11.

The hypotheses of Theorem 11 is, in essence, the requirement that, at least, one Lyapunov exponent of the linearizations of the flow of solutions with the initial data from Ω is positive under the condition that the "unstable directions $\xi(t)$" (or unstable manifolds) of these solutions depend continuously on the initial data x_0. Actually, if this property holds, then, regarding (if necessary) the domain Ω as the union of the domains Ω_i, of arbitrary small diameter, on which conditions (77) and (78) are valid, one obtains Lyapunov instability of the whole flow of solutions with the initial data from Ω.

Apply Theorem 11 to systems (24) and (24′).

For the solutions $x(t,t_0,x_0)$ with the initial data $t_0=0$,

$$x_1(0,x_{10},x_{20},x_{30}) = x_{10},$$

$$x_2(0,x_{10},x_{20},x_{30}) = x_{20},$$

$$x_3(0,x_{10},x_{20},x_{30}) = x_{30}$$

in the continuous case one has the following relations

$$x_1(t,x_{10},x_{20},x_{30}) = \exp(-at)x_{10},$$

$$\frac{\partial F(x,t)}{\partial x}\Big|_{x=x(t,0,x_0)} = \begin{pmatrix} -a & 0 & 0 \\ 0 & -2a & 0 \\ -2\exp(-at)x_{10} & 1 & r(t) \end{pmatrix}, \tag{81}$$

where

$$r(t) = \sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a.$$

For discrete system one obtains

$$\frac{\partial F(x,t)}{\partial x}\Big|_{x=x(t,0,x_0)} = \begin{pmatrix} \exp(-a) & 0 & 0 \\ 0 & \exp(-2a) & 0 \\ -2\exp(-at)x_{10} & 1 & r(t) \end{pmatrix}, \qquad (81')$$

where

$$r(t) = \frac{\exp\big((t+2)\sin\ln(t+2) - 2a(t+1)\big)}{\exp\big((t+1)\sin\ln(t+1) - 2at\big)}.$$

Solutions (39) and (39′) with matrices (81) and (81′), respectively, have the form

$$\begin{aligned} z_1(t) &= \exp(-at)z_1(0), \\ z_2(t) &= \exp(-2at)z_2(0), \\ z_3(t) &= p(t)\big(z_3(0) + (z_2(0) - 2x_{10}z_1(0))\,q(t)\big). \end{aligned} \qquad (82)$$

Here in the continuous case one has

$$p(t) = \exp\big((t+1)\sin(\ln(t+1)) - 2at\big),$$

$$q(t) = \int_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)\,d\tau.$$

and in the discrete case

$$p(t) = \exp\big((t+1)\sin(\ln(t+1)) - 2at\big),$$

$$q(t) = \sum_{k=0}^{t-1} \exp(-(k+2)\sin\ln(k+2) + 2a).$$

Relations (82) give

$$X(t,0,x_0) = \begin{pmatrix} \exp(-at) & 0 & 0 \\ 0 & \exp(-2at) & 0 \\ -2x_{10}p(t)q(t) & p(t)q(t) & p(t) \end{pmatrix}.$$

If it is assumed that

$$\xi(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

then for $\Omega = \mathbb{R}^n$ and

$$\alpha(t) = \sqrt{\exp(-4at) + (p(t)q(t))^2}$$

relations (77) and (78) are satisfied (see estimate (19)).

Thus, by Theorem 11 any solution of system (24) is Lyapunov unstable.

Restrict ourselves to the consideration of the manifold

$$M = \{x_3 \in \mathbb{R}^1, \quad x_2 = x_1^2\}.$$

In this case the initial data of the unperturbed solution x_0 and the perturbed solution y_0 belong to the manifold M:

$$x_0 \in M, \quad y_0 \in M. \tag{83}$$

The analysis of the proof of Theorem 11 (see (79)) implies that the vector-function $\xi(t)$ satisfies the following additional condition: if (79) and (83) hold, then the inequality $\xi_2(t) \neq 0$ yields the relation $\xi_1(t) \neq 0$.

In this case (77) and (78) are not valid since for either $2x_{10}\xi_1(t) = \xi_2(t) \neq 0$ or $\xi_2(t) = 0$ the value

$$|X(t, x_0)|$$

is bounded on $[0, +\infty)$.

Thus, since in conditions (77) and (78) the uniformity with respect to x_0 is violated, for system (24) on the set M the Perron effects are possible under certain additional restrictions on the vector-function $\xi(t)$. ■

7 Conclusion

We summarize the investigations of stability by the first approximation.

Theorems 5 and 11 give a complete solution for the problem on the flows and cascade of solutions in the noncritical case when for small variations of the initial data of the original system, the first approximation system preserves its stability (or instability in the certain "direction" $\xi(t)$). Thus, the classical problem of stability by the first approximation of nonstationary motions is completely proved in the general case [58].

The Perron effects of largest Lyapunov exponent sign reversal are possible only on the boundaries of the flows that are either stable or unstable by the first approximation. Thus, the difficulties, arising in studying the individual solutions, are connected to the fact that these solutions can be situated on the boundaries of the flows that are stable (or unstable) by the first approximation. In this case a special situation occurs which requires the development of more complicated tools for investigation. Such methods of investigation of the individual solutions are given in the present study.

It is shown that Perron effects may occur on the boundaries of a flow of solutions that is stable by the first approximation. Inside a flow, stability is completely determined by the negativeness of the characteristic exponents of linearized systems.

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