Chapter 2
Network Identification via Node Knockout

In this chapter, we examine the problem of identifying the interaction geometry among a known number of agents, adopting a consensus-type algorithm for their coordination. The proposed identification process is facilitated by introducing "ports" for stimulating a subset of network vertices via an appropriately defined interface and observing the network's response at another set of vertices.

2.1 Problem Formulation

Consider the weighted consensus protocol described in (2.1) as

$$\dot{x}(t) = A(\mathcal{G})x(t) + Bu(t), \quad y(t) = Cx(t), \tag{2.1}$$

where $A(\mathcal{G}) = -L_w(\mathcal{G}) \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times r_{\mathcal{I}}}$, and $C \in \mathbf{R}^{r_{\mathcal{O}}\times n}$.

Example 2.1.1. *Consider the network in Fig. 2.1; set $\mathcal{I} = \{1, 2\}$, $\mathcal{O} = \{1, 4\}$, and the corresponding weights equal to one on every edge. Then,*

$$A(\mathcal{G}) = -\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{2.2}$$

Even though in general, sets $\mathcal{I}$ and $\mathcal{O}$ can be distinct and contain more than one element, for the convenience of our presentation, we will assume that they are identical- and at times, we will assume that the resulting input-output system is in fact SISO. The extension of the presented results to the case when $\mathcal{I}$ and $\mathcal{O}$ are distinct will be discussed after introducing the basic setup and approach.

M. Nabi-Abdolyousefi, *Controllability, Identification, and Randomness in Distributed Systems*, Springer Theses, DOI: 10.1007/978-3-319-02429-5_2,

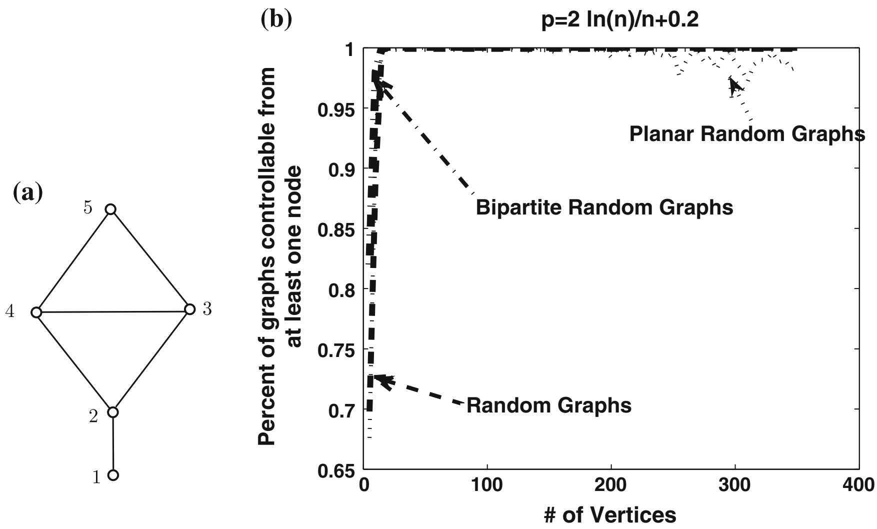

Fig. 2.1 **a** The network in Example 2.1.1, and **b** the percentage of random planar graphs that are controllable from at least one node

We now pose the *inverse problem* of graph-based coordination algorithms, namely, the feasibility of identifying the spectral and structural properties of the underlying network $\mathcal{G}$ via data facilitated by the input-output ports $\mathcal{I}$ and $\mathcal{O}$. In order to implement this program we will assume that: (1) the identification procedure has knowledge of the number of agents in the network, (2) if the removal of one or two nodes disconnects the underlying graph, the input/output sets $\mathcal{I}$ and $\mathcal{O}$ have non-trivial intersections with each of the resulting connected components,[1] and (3) the input/output sets $\mathcal{I}$ and $\mathcal{O}$ have been chosen such that the system described in (2.1) is controllable and observable. Although the first assumption is reasonable and the second an artifact of our approach,[2] the last assumption requires more justification which we now provide. In the trivial case when $\mathcal{I} = \mathcal{V}$ and B is equal to the identity matrix, the input-output weighted consensus (2.1) is clearly controllable, and by duality, observable. However, more generally, the controllability/observability of the network from a subset of its boundary nodes, is less trivial, and more to the point, not guaranteed for general graphs [1]. In the meantime, since we will need controllability and observability of the network for its identifiability, we will rely on an intriguing topical conjecture in algebraic graph theory, to the effect that for large values of n, the ratio of graphs with n nodes that are not controllable from any single node to the total number of graphs on n nodes approaches zero as $n \to \infty$ [2]. This phenomena is depicted in Fig. 2.1b;

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[1] Thus, for example, when the graph is 3-connected, the input/output sets can be chosen arbitrary to satisfy this connection.

[2] The procedure, knowing the number of nodes in the network, can identify when in fact the graph is disconnected.

for every node n, the percent of controllable networks from one node is calculated from 400 sample random graphs. *In the present research, we take the controllability and the observability of the underlying graph from the input and output nodes as our working assumption.* In the meantime, it is always convenient to know when the network is uncontrollable from a given node.

Lemma 2.1.2. *Let $P(s) = C(sI - A)^{-1}B$ be the input-output realization of* (2.1). *The uncontrollable/unobservable eigenvalues of* (2.1) *will not appear in the corresponding entry of $P(s)$. Specifically, $P(s)$ will be order $n - i$ polynomial for the SISO case with n agents and i uncontrollable/unobservable eigenvalues.*

Proof. Since the underlining graph is undirected, the matrix $A(\mathcal{G})$ in (2.1) is symmetric, and there exists a unitary matrix U and a real nonnegative diagonal matrix $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$ such that $A(\mathcal{G}) = U\Lambda U^T$. In this case, the columns of U are an orthonormal set of eigenvectors for $A(\mathcal{G})$ and the corresponding diagonal entries of Λ are its eigenvalues. Therefore,

$$P(s) = C(sI - A(\mathcal{G}))^{-1}B = C(sI - U\Lambda U^T)^{-1}B = CU(sI - \Lambda)^{-1}U^T B. \tag{2.3}$$

From the PBH test, if the system (2.1) is not controllable, there is an eigenvector that is orthogonal to B. Therefore, for an arbitrary uncontrollable eigenvalue λ_i, the i-th row of U^T is orthogonal to B, and λ_i will not appear in $(sI - \Lambda)^{-1}U^T B$. An analogous argument works for the unobservable case as well. □

2.2 System Identification

We now consider various standard system identification procedures in the context of identifying the spectra of the underlying graph Laplacian, and subsequently, gaining insights into the interconnection structure that underscores the agents' coordinated behavior.

System identification methods are implemented via sampling of the system (2.1) at discrete time instances[3] $\delta, 2\delta, \ldots, k\delta, \ldots$, with $\delta > 0$, so that the assumes the form

$$z(k+1) = A_d z(k) + B_d v(k), \quad w(k) = C_d z(k), \tag{2.4}$$

where $z(k) = x(k\delta)$, $v(k) = u(k\delta)$, $w(k) = y(k\delta)$, $A_d = e^{\delta A}$, $B_d = \left(\int_0^\delta e^{At} dt\right) B$, and $C_d = C$.[4] In fact, the system identification process leads to a realization of the model

[3] The system identification methods work based on data sampling from the system. Since we aimed to identify the interaction geometry of the network, we originally considered a continuous system. Therefore, we need to discretize the system (2.1).

[4] The notation e^A for a square matrix A refers to its matrix exponential.

$$\widetilde{z}(k+1) = \widetilde{A}_d\widetilde{z}(k) + \widetilde{B}_d v(k), \quad \widetilde{w}(k) = \widetilde{C}_d\widetilde{z}(k), \tag{2.5}$$

where $(\widetilde{A}_d, \widetilde{B}_d, \widetilde{C}_d)$ is the realization of (A_d, B_d, C_d) in (2.4). The identified system (2.5), on the other hand, corresponds to the continuous-time system

$$\dot{\widetilde{x}}(t) = \widetilde{A}\widetilde{x}(t) + \widetilde{B}u(k), \quad y(t) = \widetilde{C}\widetilde{x}(t), \tag{2.6}$$

with $\widetilde{A}_d = e^{\delta\widetilde{A}}$, $\widetilde{B}_d = \left(\int_0^\delta e^{\widetilde{A}t}dt\right)\widetilde{B}$, and $\widetilde{C}_d = \widetilde{C}$; in this case, $\widetilde{A} = (1/\delta)\log_M \widetilde{A}_d$ where $\log_M$ denotes the matrix logarithm. Since the system (2.5) is a realization of the system (2.4), it follows that the estimated triplet $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is a realization of (A, B, C) in (2.1). As a result, there exists a similarity transformation induced by the matrix T, such that $\widetilde{A} = TAT^{-1}$, $\widetilde{B} = TB$, and $\widetilde{C} = CT^{-1}$. In fact, in the controllable/observable case, the eigenvalues of $\widetilde{A}_d$ are precisely matched with the eigenvalues of A_d. Obtaining a zero as eigenvalue of $\widetilde{A}_d$, which is equivalent of obtaining $-\infty$ as the eigenvalue of $\widetilde{A}$, is a sign of an uncontrollable and/or unobservable mode in (2.1).[5] For example in the identification procedure called *Iterative Prediction-Error Minimization Method*, the model (2.4) for every input v_i and output w_j can be represented as $\mathbf{A}(q)w_j(k) = \mathbf{B}(q)v_i(k)$, where $\mathbf{A}(q) = 1 + a_1q^{-1} + \cdots + a_nq^{-n}$ and $\mathbf{B}(q) = b_1q^{-1} + \cdots + b_{r_\mathcal{I}}q^{-r_\mathcal{I}}$. The unknown model parameters $\theta = [a_1, \ldots, a_n, b_1, \ldots, b_{r_\mathcal{I}}]$ can then be estimated by comparing the actual output $w_j(k)$ with the predicted output $\widetilde{w}_{ji}(k|k-1)$ using the mean-square minimization. In this case, the output predictor is constructed as $\widetilde{w}_{ji}(k|k-1) = [-w_j(k-n), \ldots, -w_j(1), v_i(k-r_\mathcal{I}), \ldots, v_i(1)]$. In yet another candidate system identification procedure, namely the *Subspace Identification Method*, the system (2.4) is approximated by another system in the form (2.5) using a state trajectory of the dynamic system that has been determined from input-output observations. The Hankel matrix, which can be constructed from the gathered input-output data, plays an important role in this method. By constructing the Hankel matrix, the discrete time system matrices $\widetilde{A}_d$, $\widetilde{B}_d$, and $\widetilde{C}_d$ can then be determined. Subsequently, the continuous-time estimated matrices $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ can be identified; see [3] for an extensive treatment of system identification methods.

In summary, an identification procedure such as the above two methods, implemented on a controllable and observable steered-and-observed coordination protocol (2.1), leads to a system realization whose state matrix is similar to the underlying graph Laplacian and in particular shares the same spectra and characteristic polynomial. However, a distinct and fundamental issue in our setup is that having found a matrix that is "similar" to the Laplacian of a network is far from having exact knowledge of the network structure itself [4]. This observation motivates the following question: *to what extent does the knowledge of the spectra of the graph, combined with the knowledge of the input-output matrices, reduce the search space for the underlying interaction geometry?* In this chapter, we explore this question using techniques based on integer partitioning and degree-based graph reconstruction.

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[5] This follows from Lemma 2.1.2 since $-\infty$ will appear as zero in the corresponding entries of $P(s)$.

Inspired by how biologists use gene knockouts for experimentally identifying genetic interaction networks in cellular organisms, we propose a node-knockout procedure for the complete characterization of the interaction geometry in consensus-type networks. In our context, the node knockout is essentially a grounding procedure–where the node broadcasts a zero state to its neighbors without being removed from the network. The proposed identification process is also facilitated by introducing "ports" for stimulating a subset of network vertices via an appropriately defined interface and observing the network's response at another set of vertices. We then provide an example for the utility of such a network identification process in the context of fault detection for networked systems.

2.3 Characterization of the Network Topology via Node Knockout

We now explore means by which a system identification procedure as discussed in Sect. 2.2 can be used for the exact characterization of the interaction topology for consensus-type networks. Consider again the input-output LTI model (2.1). For notational simplicity, we consider unweighted Laplacian matrices; the extension of this work to the weighted Laplacian is straightforward from our analysis. In this section, we explore a method for characterizing the network by resorting to "grounding" the graph at a vertex. First, recall that via one of the system identification methods discussed previously, the characteristic equation of the system (2.1) can be found to be

$$\phi_{\mathcal{G}}(s) = \mathbf{det}(sI - A(\mathcal{G})) = \mathbf{det}(sI + L(\mathcal{G})). \tag{2.7}$$

Definition 2.3.1. *The grounded consensus at node v evolves according the dynamics*

$$\dot{x}(t) = -L_v(\mathcal{G})x(t),$$

where $L_v(\mathcal{G}) = L(\mathcal{G}\backslash v) + \Delta_v$, $\mathcal{G}\backslash v$ is the graph obtained after removing v from $\mathcal{G}$, and the diagonal matrix Δ_v is such that $[\Delta_v]_{ii} = 1$ (or $[\Delta_v]_{ii} = w_{iv}$ in the weighted consensus protocol) when $\{v, i\} \in \mathcal{E}$ and 0 otherwise. We refer to the grounding operation as node knockout.

In this section, we make the standing assumption that each vertex in $\mathcal{G}$ can be instructed to "ground" itself upon request. Now, define the polynomial

$$\phi^v_{\mathcal{G}}(s) := \mathbf{det}(sI + (L(\mathcal{G}\backslash v) + \Delta_v)) \tag{2.8}$$

which is the characteristic polynomial of the grounded consensus at node v. Following our presentation in Sect. 2.2, it then follows that the system identification procedure can be applied for the cases when one or two nodes in the graph are grounded. And in fact, provided that the grounded consensus protocol is controllable and observable, the characteristic polynomials $\phi^v_{\mathcal{G}}(s)$ and $\phi^{uv}_{\mathcal{G}}(s)$ for situations where either vertex v has been grounded or when both vertices v and u have been grounded can be obtained. But before we proceed, let us address the controllability/observability of the influenced/observed grounded consensus.

Proposition 2.3.2. *Consider the controllable and observable steered-and-observed system* (2.1) *on the n-node graph $\mathcal{G}$ with $n \geq 2$. Then as long as none of the input-output vertices are identical to the grounded node(s), the grounded consensus on $L_v(\mathcal{G})$ remains controllable and observable with the corresponding reduced input and observation matrices if and only if the graph $\mathcal{G}\backslash v$ stays connected.*

Proof. Let the grounded node v be different from the input-output nodes. Without loss of generality, assume that v is the last indexed node in $L(\mathcal{G})$ and also that the input set contains just one node. In this case, since the input set does not include v, we can rewrite B as $B = [\,\hat{B}^T, 0\,]^T$. From the definition of controllability, if the grounded consensus is controllable with the pair $(L_v(G), \hat{B})$, the graph $\mathcal{G}\backslash v$ is connected. The next step is to prove that if the graph $\mathcal{G}\backslash v$ is connected, the grounded consensus is controllable, or equivalently, that if the grounded consensus is uncontrollable, the graph $\mathcal{G}\backslash v$ has to be disconnected. This is proven as follows. Since the original graph is controllable, from the PBH test there does not exist a nonzero z and λ such that $L(\mathcal{G})z = \lambda z$ and $z^T B = 0$. Thus after partitioning the matrix $L(\mathcal{G})$, one has

$$\nexists\, z \neq 0,\ \lambda \quad \text{s.t.} \quad \begin{bmatrix} L(\mathcal{G}\backslash v) + \Delta_v - \lambda I & \delta_v \\ \delta_v^T & \mathbf{deg}\ v - \lambda \\ \hat{B}^T & 0 \end{bmatrix} z = 0, \tag{2.9}$$

where I is the identity matrix with proper dimensions, and δ_v is the vector formed from the diagonal of Δ_v. Since the grounded consensus is uncontrollable, there exists a nonzero $s \in \mathbf{R}^{n-1}$ and $\hat{\lambda} \in \mathbf{R}$ such that $(L(\mathcal{G}\backslash v) + \Delta_v)s = \hat{\lambda}s$ and $s^T \hat{B} = 0$ with $\hat{\lambda} \neq 0$ since otherwise the proof is done. Therefore,

$$\begin{bmatrix} L(\mathcal{G}\backslash v) + \Delta_v - \hat{\lambda} I \\ \hat{B}^T \end{bmatrix} s = 0, \tag{2.10}$$

where I is the identity matrix with proper dimensions. Note that in order to show that $\mathcal{G}\backslash v$ is disconnected, it suffices to show that there exists a vector $r \notin \mathbf{span}\{\mathbf{1}\} \subseteq \mathbf{R}^{n-1}$ such that $L(\mathcal{G}\backslash v)r = 0$. The matrix on the left hand-side of (2.9) is full rank for all values of λ. Thus for any choice of $q \in \mathbf{R}^n$, there exists $p \in \mathbf{R}^{n+1}$ such that

$$\begin{bmatrix} L(\mathcal{G}\backslash v) + \Delta_v - \lambda I & \delta_v & \hat{B} \\ \delta_v^T & \mathbf{deg}\ v - \lambda & 0 \end{bmatrix} p = q. \tag{2.11}$$

By partitioning $p = [p_1^T,\ p_2^T,\ p_3^T]^T$ and $q = [q_1^T,\ q_2^T]^T$, we obtain

$$(L(\mathcal{G}\backslash v) + \Delta_v - \lambda I)p_1 + \delta_v p_2 + \hat{B}^T p_3 = q_1, \quad \text{and}$$
$$\delta_v^T p_1 + (\mathbf{deg}\ v - \lambda)p_2 = q_2. \qquad (2.12)$$

Choosing $\lambda = \hat{\lambda},\ q_1 = \delta_v$, and multiplying both sides of the first identity in (2.12) by s, we obtain $p_2 = 1$. Let us choose $q_2 = \mathbf{deg}\ v - \hat{\lambda} + \alpha_1$ in (2.12) where $\alpha_1 \neq 0$. Therefore, $\delta_v^T p_1 = \alpha_1$ or $\sum_{i\in\mathcal{N}(v)} p_1(i) = \alpha_1$. In view of (2.12), we conclude that $L(\mathcal{G}\backslash v)p_1 = (\hat{\lambda} I - \Delta_v)p_1 - \hat{B}p_3$. If $(\hat{\lambda} I - \Delta_v)p_1 - \hat{B}p_3 = 0$, choosing $r = p_1$ will prove the claim if $p_1 \notin \mathbf{span}\{\mathbf{1}\}$. By choosing $p_1 = [0,\ 0,\ \ldots, \alpha_1]^T$, it can be verified that $(\hat{\lambda} I - \Delta_v)p_1 - \hat{B}p_3 = 0$ when $p_3 = \frac{\alpha_1(\hat{\lambda}-1)}{\sum_{i\in\mathcal{N}(v)} \hat{B}(i)}$. $\square$

When the candidate grounding vertex belongs to the input set $\mathcal{I}$ in the original LTI system (2.1), the controllability and observability of the grounded consensus should be preserved by switching the control/observe channel to another vertex in the grounded consensus. The same argument holds valid when the graph $\mathcal{G}\backslash v$ is disconnected, in which case the grounded consensus looses its controllability and observability. We also note that the controllability/observability of the grounded network at two vertices follows from the same argument in the Proposition 2.3.2 and is thereby omitted. We are ready to state the main result of this section.

Theorem 2.3.3. *System identification on the steered-and-observed consensus* (2.1)*, while allowing the grounding operation (Definition 2.3.1) at all nodes and all pairs of nodes, allows for a complete characterization of the underlying network.*

We will prove this theorem via a number of observations—and most importantly—with the help of a powerful construct in combinatorics, namely that of generating functions [5].

Definition 2.3.4. *Let $a_0, a_1, \ldots$ be a finite or infinite sequence of real numbers. Then the ordinary generating function $\chi(s)$ of the sequence is the power series*

$$\chi(s) = a_0 + a_1 s + a_2 s^2 + \cdots = \sum_{k=0}^{\infty} a_k s^k. \qquad (2.13)$$

On one level, generating functions can be regarded as algebraic objects whose formal manipulation allows one to address combinatorial problems by means of algebra [7]. Yet on another level, generating functions can be considered as power series expansions of infinitely differentiable functions. Generating functions can conveniently be extended to matrices and in particular to graph Laplacians. In this venue, define the generating function $\chi_{\mathcal{G}}(s)$ with respect to the sequence of powers of the graph Laplacian as

$$\chi_{\mathcal{G}}(s) : \mathbb{C} \to \mathbb{C}^{n\times n}, \quad \chi_{\mathcal{G}}(s) := \sum_{k=0}^{\infty} s^k(-L(\mathcal{G}))^k = (I + sL(\mathcal{G}))^{-1}. \quad (2.14)$$

Generating functions based on the adjacency matrix of a graph have been studied extensively; see for example [6] (Chap. 4). In view of the resemblance of (2.14) to the transfer matrix of (2.1), our aim will be on clarifying the role of generating functions for the characterization of steered and observed consensus-type networks. In doing so, we are then able to devise a procedure for network identification for (2.1) by identifying the corresponding generating functions for the grounded and ungrounded characteristic polynomials. The ingredients of such a program are clarified by the following observations.

In order to determine the entries of the generating function of $L(\mathcal{G})$, define the matrix $\Psi_{\mathcal{G}}(s)$ as the adjugate of $sI + L(\mathcal{G})$, i.e., the complex conjugate transpose of the matrix of its cofactors. From the definition of the matrix inverse, we then have

$$\Psi_{\mathcal{G}}(s)(sI + L(\mathcal{G})) = \det(sI + L(\mathcal{G}))I = \phi_{\mathcal{G}}(s)I. \tag{2.15}$$

Lemma 2.3.5. *Let v be a vertex in the graph $\mathcal{G}$. Then*

$$s^{-1}[\chi_{\mathcal{G}}(s^{-1})]_{vv} = \frac{\phi_{\mathcal{G}}^{v}(s)}{\phi_{\mathcal{G}}(s)}, \tag{2.16}$$

where $[\chi_{\mathcal{G}}(s)]_{vv}$ is the v-th diagonal entry of the generating function $\chi_{\mathcal{G}}(s)$.

Proof. From (2.14), it follows that

$$s^{-1}\chi_{\mathcal{G}}(s^{-1}) = s^{-1}(I + s^{-1}L(\mathcal{G}))^{-1} = (sI + L(\mathcal{G}))^{-1} = \frac{\Psi_{\mathcal{G}}(s)}{\phi_{\mathcal{G}}(s)}; \tag{2.17}$$

the last equality in (2.17) is Cramer's rule. Then (2.16) follows immediately from (2.17); we note that $\phi_{\mathcal{G}}^{v}(s)$, as the v-th diagonal entry of $\Psi_{\mathcal{G}}(s)$, is the characteristic polynomials of $L(\mathcal{G}\backslash v) + \Delta_v$. On the other hand, as defined in (2.8), $\phi_{\mathcal{G}}^{v}(s)$ is the characteristic polynomial of the grounded consensus system matrix at node v. $\square$

Before we can state our next result, we introduce a new notation. For $D \subseteq \mathcal{V}$, $[\chi_{\mathcal{G}}(s)]_D$ denotes the submatrix of $\chi_{\mathcal{G}}(s)$ with rows and columns indexed by the vertices in D.

Theorem 2.3.6. *(Jacobi)* [7] *Let D be a subset of d vertices of the graph $\mathcal{G}$. Then*

$$s^{-d}\,\mathbf{det}[\chi_{\mathcal{G}}(s^{-1})]_D = \frac{\phi_{\mathcal{G}}^{D}(s)}{\phi_{\mathcal{G}}(s)}; \tag{2.18}$$

note that this is an extension of (2.16).

Proof. Without loss of generality, we may assume that D consists of the first d vertices of $\mathcal{G}$. Let $\mathcal{K}$ be the matrix obtained by replacing the first d columns of the $n \times n$ identity matrix with the corresponding columns of $\Psi_{\mathcal{G}}(s)$. Consider the product

$(sI + L(\mathcal{G}))\mathcal{K}$. We have

$$(sI + L(\mathcal{G}))\mathcal{K} = \begin{bmatrix} \phi_{\mathcal{G}}(s)I_d & X \\ 0 & sI + L(\mathcal{G}\backslash D) + \Delta_D \end{bmatrix},$$

where $\Delta_D = \sum_{v \in D} \Delta_v$, and the exact form of the matrix X is inconsequential. Taking the determinant of both sides of this equation yields

$$\phi_{\mathcal{G}}(s)\ \mathbf{det}\mathcal{K} = \phi_{\mathcal{G}}(s)^d\ \mathbf{det}(sI + L(\mathcal{G}\backslash D) + \Delta_D), \tag{2.19}$$

where $\mathcal{G}\backslash D$ is the graph obtained from removing the node set D from $\mathcal{G}$ (as well as all edges incident on the nodes in D). Note that $\mathbf{det}(sI + L(\mathcal{G}\backslash D) + \Delta_D) = \phi_{\mathcal{G}}^D(s)$. From (2.17) and the definition of $\mathcal{K}$, one has

$$s^{-d}\ \mathbf{det}[\chi_{\mathcal{G}}(s^{-1})]_D = \phi_{\mathcal{G}}(s)^{-d}\ \mathbf{det}\,\mathcal{K},$$

and in combination with (2.19), this yields the statement of the theorem. □

If D consists of a pair of vertices u and v, then we obtain

$$\mathbf{det}\ [\chi_{\mathcal{G}}(s^{-1})]_{D=\{u,v\}} = [\chi_{\mathcal{G}}(s^{-1})]_{uu}[\chi_{\mathcal{G}}(s^{-1})]_{vv} - [\chi_{\mathcal{G}}(s^{-1})]_{uv}[\chi_{\mathcal{G}}(s^{-1})]_{vu}. \tag{2.20}$$

Since $\mathcal{G}$ is undirected, $[\chi_{\mathcal{G}}(s^{-1})]_{uv} = [\chi_{\mathcal{G}}(s^{-1})]_{vu}$. Hence, we can determine the off-diagonal entries of the generating function of $L(\mathcal{G})$.

Lemma 2.3.7. *Let $\mathcal{G}$ be a graph and $u, v \in \mathcal{G}$. Then*

$$s^{-1}[\chi_{\mathcal{G}}(s^{-1})]_{uv} = \frac{[\Psi_{\mathcal{G}}(s)]_{uv}}{\phi_{\mathcal{G}}(s)}, \tag{2.21}$$

where $[\chi_{\mathcal{G}}(s^{-1})]_{uv}$ is the uv-entry of the generating function $\chi_{\mathcal{G}}(s)$, and

$$[\Psi_{\mathcal{G}}(s)]_{uv} = (\phi_{\mathcal{G}}^u(s)\phi_{\mathcal{G}}^v(s) - \phi_{\mathcal{G}}(s)\phi_{\mathcal{G}}^{uv}(s))^{1/2}.$$

Proof. From Theorem 2.3.6, set $D = \{u, v\}$. Thus,

$$s^{-2}\mathbf{det}[\chi_{\mathcal{G}}(s^{-1})]_{\{u,v\}} = \frac{\phi_{\mathcal{G}}^{uv}(s)}{\phi_{\mathcal{G}}(s)}. \tag{2.22}$$

From (2.20) and (2.22), it follows that

$$s^{-2}([\chi_{\mathcal{G}}(s^{-1})]_{uu}[\chi_{\mathcal{G}}(s^{-1})]_{vv} - [\chi_{\mathcal{G}}(s^{-1})]_{uv}^2) = \frac{\phi_{\mathcal{G}}^{uv}(s)}{\phi_{\mathcal{G}}(s)}.$$

From (2.16), on the other hand, it follows that

$$s^{-2}(s^2\frac{\phi^u_{\mathcal{G}}(s)}{\phi_{\mathcal{G}}(s)}\frac{\phi^v_{\mathcal{G}}(s)}{\phi_{\mathcal{G}}(s)} - [\chi_{\mathcal{G}}(s^{-1})]^2_{uv}) = \frac{\phi^{uv}_{\mathcal{G}}(s)}{\phi_{\mathcal{G}}(s)}.$$

Hence,

$$s^{-2}[\chi_{\mathcal{G}}(s^{-1})]^2_{uv} = \frac{\phi^u_{\mathcal{G}}(s)\phi^v_{\mathcal{G}}(s) - \phi_{\mathcal{G}}(s)\phi^{uv}_{\mathcal{G}}(s)}{\phi_{\mathcal{G}}(s)^2},$$

and

$$s^{-1}[\chi_{\mathcal{G}}(s^{-1})]_{uv} = \frac{(\phi^u_{\mathcal{G}}(s)\phi^v_{\mathcal{G}}(s) - \phi_{\mathcal{G}}(s)\phi^{uv}_{\mathcal{G}}(s))^{1/2}}{\phi_{\mathcal{G}}(s)}.$$

□

Since $s^{-1}[\chi_{\mathcal{G}}(s^{-1})]_{uv}$ is an entry of $s^{-1}(I + s^{-1}L(\mathcal{G}))^{-1} = (sI + L)^{-1}$, it is a rational function with $\phi_{\mathcal{G}}(s)$ as its denominator and the numerator is the $uv-$entry of $\Psi_{\mathcal{G}}(s)$.

In order to draw a direct connection between the generating function (2.14) and the steered-and-observed consensus protocol (2.1), note that $(sI + L(\mathcal{G}))^{-1}$ is defined as $s^{-1}\chi_{\mathcal{G}}(s^{-1})$. And rather conveniently, according to Lemmas 2.3.5 and 2.3.7, all entries of $s^{-1}\chi_{\mathcal{G}}(s^{-1})$ can be determined by the characteristic polynomials of the ungrounded and grounded consensus through the system identification procedure. In fact, the impulse response of $(sI + L(\mathcal{G}))^{-1}$ is the state transition matrix of the consensus system and can be calculated from $s^{-1}\chi_{\mathcal{G}}(s^{-1})$ as,[6]

$$\mathcal{L}^{-1}\{(sI + L(\mathcal{G}))^{-1}\} = e^{-L(\mathcal{G})t} = \mathcal{L}^{-1}\{s^{-1}\chi_{\mathcal{G}}(s^{-1})\}, \tag{2.23}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. Hence, the graph Laplacian $L(\mathcal{G})$ can be uniquely identified by running the system identification on the ungrounded and grounded systems corresponding to (2.1).

The next example demonstrates the proposed procedure for the network in Fig. 2.2a. Notice that if we run system identification with the input and output set to $\mathcal{I} = \mathcal{O} = \{1\}$, we obtain the characteristic polynomial $\phi_{\mathcal{G}}(s) = s^4+6s^3+10s^2+4s$. In the meantime, by running the system identification on the grounded consensus at node 1 while choosing $\mathcal{I} = \mathcal{O} = \{4\}$, we obtain the characteristic polynomial $\phi^{\{1\}}_{\mathcal{G}}(s) = s^3 + 5s^2 + 6s + 1$. Notice that in this case

$$A(\mathcal{G}) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

[6] Incidentally we should mention that the impulse response of the system with the transfer matrix function as (2.14), is the state transition matrix for the Markov chain associated with the consensus protocol [1] (Chap. 3).

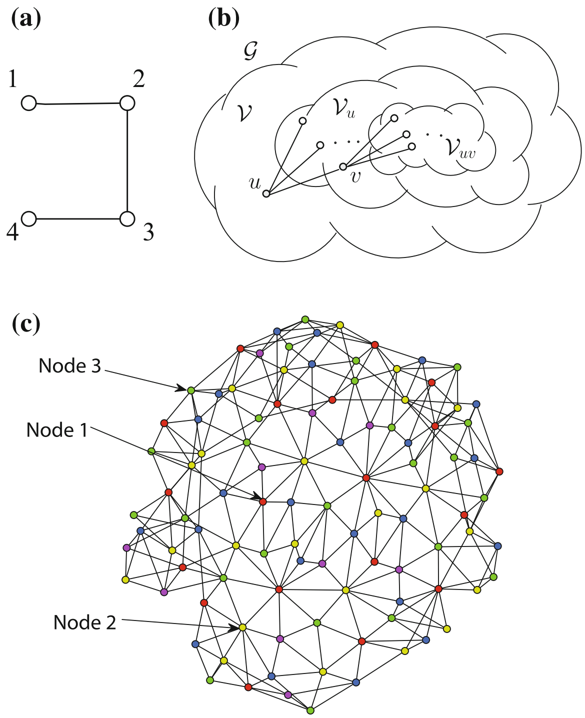

Fig. 2.2 **a** Graph $\mathcal{G}$, **b** grounding the consensus protocol, and **c** the graph considered for the example

In the same manner $\phi_{\mathcal{G}}^{\{1\}}(s) = \phi_{\mathcal{G}}^{\{2\}}(s)$. As we ground node 2, two disconnected components will be generated, and the input and output nodes must be chosen from these distinct components of the graph. We therefore determine $\phi_{\mathcal{G}}^{\{2\}}(s)$ and $\phi_{\mathcal{G}}^{\{3\}}(s)$ as $\phi_{\mathcal{G}}^{\{2\}}(s) = s^3 + 4s^2 + 4s + 1$ and $\phi_{\mathcal{G}}^{\{3\}}(s) = \phi_{\mathcal{G}}^{\{2\}}(s)$. In the next step, in order to calculate the off-diagonal entries of $s^{-1}\chi_{\mathcal{G}}(s^{-1})$, one needs to ground a pair of nodes simultaneously. For example, if we run the system identification procedure on the consensus system grounded at nodes $\{1, 2\}$ and choose $\mathcal{I} = \mathcal{O} = \{3\}$, we obtain $\phi_{\mathcal{G}}^{\{1,2\}}(s) = s^2 + 3s + 1$. Notice that the system matrix in this case is

$$A(\mathcal{G}) = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Considering the symmetry in the generating matrix $\chi_{\mathcal{G}}(s)$, by running multiple sessions of the system identification procedure, we obtain all entries of $s^{-1}\chi_{\mathcal{G}}(s^{-1})$. Thus, the Laplacian matrix, $L(\mathcal{G})$, can be explicitly found from (2.23).

2.4 Edge Faults in the Network

In this section, we explore the utility of the network identification procedure in the context of fault detection for networked systems. Consider a remote administrator of a network, running a protocol similar to (2.1),[7] monitoring the network's behavior by occasionally sending signals through certain boundary nodes and observing its reflection. As we discussed in previous sections, assuming the observability/controllability of the underlying network, the identification process provides us with the characteristic equation for the system and the number of edges in the network. Thus, if there is an edge failure, the characteristic equation of the modified network will reflect this failure. We would like to explore now the possibility of *identifying the broken link* from running the identification procedure on the network when the grounding operation is permissible. It is assumed that the nodes can be instructed to ground themselves, i.e., sending a zero state value to their neighbors, while not being able to independently and locally determine whether an edge in the network has been broken. In fact it is assumed that the administrator can only detect faults indirectly by observing and monitoring a few selected signals from the network. In this venue, let $E = 2|\mathcal{E}(\mathcal{G})|$ be twice the number of edges in the fault-free network $\mathcal{G}$, which is equal to the sum of the roots of the characteristic polynomial $\phi_{\mathcal{G}}(s)$. In the same vein, let us denote the sum of the roots of the characteristic equations of the grounded consensus matrices $L(\mathcal{G}_u)$ and $L(\mathcal{G}_{uv})$ as E_u and E_{uv}, respectively.

Proposition 2.4.1. *Consider the consensus protocol* (2.1) *over the graph* $\mathcal{G}$. *Let* E, E_u, E_v, *and* E_{uv} *denote, respectively, the sum of the roots of the characteristic polynomials* $\phi_{\mathcal{G}}(s)$, $\phi^u_{\mathcal{G}}(s)$, $\phi^v_{\mathcal{G}}(s)$, *and* $\phi^{uv}_{\mathcal{G}}(s)$. *Then* $E - E_u - E_v + E_{uv} = 0$ *if there is no edge between nodes* u *and* v, *while* $E - E_u - E_v + E_{uv} = 2$ *indicates that there is an edge between* u *and* v.

Proof. Let $\mathcal{V}_u$ denote the subset of $\mathcal{V}$ with node u excluded; see Fig. 2.2. By running the system identification on $\mathcal{G}$, we obtain the sum of the roots of the characteristic polynomial $\phi_{\mathcal{G}}(s)$ which is also equal to the sum of the degrees of all nodes in $\mathcal{V}$. In the meantime, by running the system identification on the grounded consensus at u, we obtain $E_u = E - \textbf{deg}\ u$. Analogously, one obtains $E_v = E - \textbf{deg}\ v$ when node v is grounded. Thus by grounding the pair of nodes u and v in the consensus protocol, the sum of the roots of the characteristic polynomial $\phi^{uv}_{\mathcal{G}}(s)$ is $E_{uv} = E - \textbf{deg}\ u - \textbf{deg}\ v + 2I_{\{u,v\}\in\mathcal{E}}$, where $I_{\{u,v\}\in\mathcal{E}}$ is equal to one if nodes u and v are incident and zero otherwise. Thereby, $E_u + E_v - E_{uv} = E - 2I_{\{u,v\}\in\mathcal{E}}$ and the statement of the proposition now follows. □

As an example, consider the graph in Fig. 2.2 with $|\mathcal{V}| = 100$ nodes and $|\mathcal{E}| = 284$, running the consensus protocol. Our goal is to resolve whether there is an edge between nodes 1 and 2 via the proposed network identification procedure. We note that if we run the system identification with $r_{\mathcal{I}} = r_{\mathcal{O}} = 50$, we obtain the

[7] Including any class of consensus-type protocols for formation control and distributed estimation [1].

number of edges in the network, E, as the sum of the roots of the characteristic polynomial $\phi_{\mathcal{G}}(s)$; in this case there are 568 edges in the network. Applying the system identification procedure on the grounded consensus at node 1 leads the value of $E_1 = 562$, where E_1 is the sum of the roots of the characteristic polynomial $\phi^1_{\mathcal{G}}(s)$. Analogously, the values of E_2 and $E_{1,2}$ are 560 and 554, respectively. Since $E = E_1 + E_2 - E_{1,2}$, from Proposition 2.4.1, it follows that there is no edge between nodes 1 and 2.

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http://www.springer.com/978-3-319-02428-8

Controllability, Identification, and Randomness in Distributed Systems

Nabi, M.

2014, XVII, 151 p. 39 illus., 30 illus. in color., Hardcover
ISBN: 978-3-319-02428-8