Chapter 1
Preliminaries on (Almost-)Complex Manifolds

Abstract In this preliminary chapter, we summarize some basic notions and some classical results in (almost-)complex and symplectic geometry. In particular, we start by setting some definitions and notation concerning (almost-)complex structures, Sect. 1.1, symplectic structures, Sect. 1.2, and generalized complex structures, Sect. 1.3; then we recall the main results in the Hodge theory for Kähler manifolds, Sect. 1.4, and in the Kodaira, Spencer, Nirenberg, and Kuranishi theory of deformations of complex structures, Sect. 1.5; furthermore, we summarize some basic definitions and some useful facts about currents and de Rham homology, Sect. 1.6, and about nilmanifolds and solvmanifolds, Sect. 1.7, in order to set the notation for the following chapters. (As a matter of notation, unless otherwise stated, by "manifold" we mean "connected differentiable manifold", and by "compact manifold" we mean "closed manifold".)

1.1 Almost-Complex Geometry and Complex Geometry

The tangent bundle of a complex manifold X is naturally endowed with an endomorphism $J \in \mathrm{End}(TX)$ such that $J^2 = -\mathrm{id}_{TX}$, satisfying a further integrability property. It is hence natural to study differentiable manifolds endowed with such an endomorphism, the so-called *almost-complex* manifolds. It turns out that the vanishing of the *Nijenhuis tensor* Nij_J characterizes the almost-complex structures J on X naturally induced by a structure of complex manifold [NN57, Theorem 1.1].

In this section, we recall the notions of almost-complex structure, complex manifold, and Dolbeault cohomology, and some of their properties.

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1.1.1 Almost-Complex Structures

Let X be a (differentiable) manifold. We start by recalling the notion of almost-complex structure.

Definition 1.1 ([Ehr49]). Let X be a differentiable manifold. An *almost-complex structure* J on X is an endomorphism $J \in \mathrm{End}(TX)$ such that $J^2 = -\,\mathrm{id}_{TX}$.

Let J be an almost-complex structure on X. Extending J by $\mathbb{C}$-linearity to $TX \otimes \mathbb{C}$, we get the decomposition

$$TX \otimes \mathbb{C} \;=\; T^{1,0}X \oplus T^{0,1}X \;,$$

where $T^{1,0}X$ (respectively, $T^{0,1}X$) is the sub-bundle of $TX\otimes\mathbb{C}$ given by the i-eigen-spaces (respectively, the $(-\,\mathrm{i})$-eigen-spaces) of $J \in \mathrm{End}\,(TX \otimes \mathbb{C})$: that is, for every $x \in X$,

$$\begin{aligned}\left(T^{1,0}X\right)_x &= \{v_x - \mathrm{i}\; J_x v_x \;:\; v_x \in T_x X\}\,,\\ \left(T^{0,1}X\right)_x &= \{v_x + \mathrm{i}\; J_x v_x \;:\; v_x \in T_x X\}\;.\end{aligned}$$

Considering the dual of J, again denoted by $J \in \mathrm{End}\,(T^*X)$, we get analogously a decomposition at the level of the cotangent bundle:

$$T^*X \otimes \mathbb{C} \;=\; \left(T^{1,0}X\right)^* \oplus \left(T^{0,1}X\right)^* \;,$$

where $\left(T^{1,0}X\right)^*$ (respectively, $\left(T^{0,1}X\right)^*$) is the sub-bundle of $T^*X \otimes \mathbb{C}$ given by the i-eigen-spaces (respectively, the $(-\,\mathrm{i})$-eigen-spaces) of the $\mathbb{C}$-linear extension $J \in \mathrm{End}\,(T^*X \otimes \mathbb{C})$. Extending the endomorphism J to the bundle $\wedge^\bullet\,(T^*X)\otimes\mathbb{C}$ of complex-valued differential forms, we get, for every $k \in \mathbb{N}$, the bundle decomposition

$$\wedge^k\left(T^*X\right)\otimes\mathbb{C} \;=\; \bigoplus_{p+q=k} \wedge^p\left(T^{1,0}X\right)^* \otimes \wedge^q\left(T^{0,1}X\right)^* \;.$$

As a matter of notation, we will denote by $\mathcal{C}^\infty\,(X;F)$ the space of smooth sections of a vector bundle F over X, and, for every $k \in \mathbb{N}$ and $(p,q) \in \mathbb{N}^2$, we will denote by $\wedge^k X \;:=\; \mathcal{C}^\infty\left(X;\,\wedge^k\,(T^*X)\right)$ the space of smooth sections of $\wedge^k\,(T^*X)$ over X and by $\wedge^{p,q}X := \wedge^{p,q}_J X := \mathcal{C}^\infty\left(X;\,\wedge^p\left(T^{1,0}X\right)^* \otimes \wedge^q\left(T^{0,1}X\right)^*\right)$ the space of smooth sections of $\wedge^p\left(T^{1,0}X\right)^* \otimes \wedge^q\left(T^{0,1}X\right)^*$ over X.

Since $\mathrm{d}\left(\wedge^0 X \otimes_{\mathbb{R}} \mathbb{C}\right) \subseteq \wedge^{1,0}X \oplus \wedge^{0,1}X$ and $\mathrm{d}\left(\wedge^1 X \otimes_{\mathbb{R}} \mathbb{C}\right) \subseteq \wedge^{2,0}X \oplus \wedge^{1,1}X \oplus \wedge^{0,2}X$, since every differential form is locally a finite sum of decomposable differential forms, and by the Leibniz rule, the $\mathbb{C}$-linear extension of the exterior differential, $\mathrm{d}\colon \wedge^\bullet X \otimes \mathbb{C} \to \wedge^{\bullet+1}X \otimes \mathbb{C}$, splits into four components:

$$\mathrm{d} \;=\; A + \partial + \overline{\partial} + \bar{A}$$

where

$$A\colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet+2,\bullet-1} X \;, \qquad \partial\colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet+1,\bullet} X \;, \qquad \overline{\partial}\colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet,\bullet+1} X \;,$$
$$\bar{A}\colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet-1,\bullet+2} X \;;$$

in terms of these components, the condition $\mathrm{d}^2 = 0$ is written as

$$\begin{cases} A^2 = 0 \\ A\,\partial + \partial\,A = 0 \\ A\,\overline{\partial} + \partial^2 + \overline{\partial}\,A = 0 \\ A\,\bar{A} + \partial\,\overline{\partial} + \overline{\partial}\,\partial + A\,\bar{A} = 0 \\ \partial\,\bar{A} + \overline{\partial}^2 + \bar{A}\,\partial = 0 \\ \bar{A}\,\overline{\partial} + \overline{\partial}\,\bar{A} = 0 \\ \bar{A}^2 = 0 \end{cases} .$$

Remark 1.1. More in general, given a $2n$-dimensional manifold X, an *almost-c.p.s. structure*, [Vai07, Sect. 1], is the datum of an endomorphism $K \in \mathrm{End}(TX)$ such that $K^2 = \lambda\ \mathrm{id}_{TX}$ where $\lambda \in \{-1,\, 0,\, 1\}$: more precisely, the case $\lambda = -1$ corresponds to *almost-complex structures*, the case $\lambda = 0$ corresponds to *almost-subtangent structures*, and the case $\lambda = 1$ corresponds to *almost-product structures*.

Among almost-product structures, **D**-complex structures deserve special interest. By definition, an *almost-***D***-complex structure* (also called *almost-para-complex structure*) on a manifold X is an endomorphism $K \in \mathrm{End}(TX)$ such that $K^2 = \mathrm{id}_{TX}$ and $\mathrm{rk}\, T^+X = \mathrm{rk}\, T^-X = \frac{1}{2} \dim X$, where, for $\pm \in \{+, -\}$, the sub-bundle $T^{\pm}X$ of TX is the (± 1)-eigen-bundle of $K \in \mathrm{End}(TX)$. Recall also that a **D***-holomorphic map* between two almost-**D**-complex manifolds $(X_1,\, K_1)$ and $(X_2,\, K_2)$ is a smooth map $f\colon X_1 \to X_2$ such that $\mathrm{d}\, f \circ K_1 = K_2 \circ \mathrm{d}\, f$. Recall that an almost-**D**-complex structure is said to be *integrable* (and hence it is called a **D***-complex structure*, or also a *para-complex structure*) if $\left[T^+X,\, T^+X\right] \subseteq T^+X$ and $[T^-X,\, T^-X] \subseteq T^-X$. Equivalently, the integrability condition can be straightforwardly expressed as the vanishing of the *Nijenhuis tensor* of K, defined as $\mathrm{Nij}_K(\cdot,\, \cdot\cdot) := [\cdot,\, \cdot\cdot] + [K\cdot,\, K\cdot\cdot] - K\,[K\cdot,\, \cdot\cdot] - K\,[\cdot,\, K\cdot\cdot]$. Furthermore, for an almost-**D**-complex structure on an n-dimensional manifold X, the integrability is also equivalent to being naturally associated to a structure on X defined in terms of local homeomorphisms with open sets in $\mathbf{D}^n$ and **D**-holomorphic changes of coordinates, see, e.g., [CMMS04, Proposition 3], where $\mathbf{D}^n := \mathbb{R}^n + \tau\,\mathbb{R}^n$, with $\tau^2 = 1$, is the algebra of *double numbers*.

Finally, we recall that, given a $2n$-dimensional manifold endowed with an almost-**D**-complex structure K, a **D**-*Hermitian metric* on X is a pseudo-Riemannian metric of signature (n, n) such that $g(K\cdot, K\cdot\cdot) = -g(\cdot, \cdot\cdot)$. A **D**-*Kähler structure* on a manifold X is the datum of an integrable **D**-complex structure K and a **D**-Hermitian metric g such that its associated K-anti-invariant form $\omega := g(K\cdot, \cdot\cdot)$ is d-closed, equivalently, the datum of a K-*compatible* (that is, a K-anti-invariant) *symplectic form* on X, see, e.g., [AMT09, Sect. 5.1], [CMMS04, Theorem 1].

We refer, e.g., to [HL83, AMT09, CMMS04, CMMS05, CM09, CFAG96, KMW10, ABDMO05, AS05, Kra10, Ros12a, Ros12b, AR12, Ros13] and the references therein for more notions and results in **D**-complex geometry and for motivations for its study.

1.1.2 Complex Structures, and Dolbeault Cohomology

If X is a complex manifold, then there is a natural almost-complex structure on X: locally, in a holomorphic coordinate chart $\left(U,\ \left\{z^\alpha =: x^{2\alpha-1} + \mathrm{i}\ x^{2\alpha}\right\}_{\alpha\in\{1,\ldots,\dim_{\mathbb{C}} X\}}\right)$, with $\left(U,\ \{x^\alpha\}_{\alpha\in\{1,\ldots,2\dim_{\mathbb{C}} X\}}\right)$ a (differential) coordinate chart, one defines, for every $\alpha \in \{1, \ldots, \dim_{\mathbb{C}} X\}$,

$$J\left(\frac{\partial}{\partial x^{2\alpha-1}}\right) \stackrel{\text{loc}}{:=} \frac{\partial}{\partial x^{2\alpha}}\,, \qquad J\left(\frac{\partial}{\partial x^{2\alpha}}\right) \stackrel{\text{loc}}{:=} -\frac{\partial}{\partial x^{2\alpha-1}}\,;$$

note that this local definition does not depend on the coordinate chart, by the Cauchy and Riemann equations.

Conversely, we recall that an almost-complex structure J on a differentiable manifold X is called *integrable* if it is the natural almost-complex structure induced by a structure of complex manifold on X.

The following theorem by A. Newlander and L. Nirenberg characterizes the integrable almost-complex structures on a manifold X in terms of the *Nijenhuis tensor* Nij_J, defined as

$$\mathrm{Nij}_J(\cdot, \cdot\cdot) \ :=\ [\cdot, \cdot\cdot] + J\,[J\,\cdot, \cdot\cdot] + J\,[\cdot, J\,\cdot\cdot] - [J\cdot, J\,\cdot\cdot]\ .$$

(See also [KN96, Appendix 8] for the integrability of real-analytic almost-complex structures.)

Theorem 1.1 ([NN57, Theorem 1.1]). *Let X be a differentiable manifold. An almost-complex structure J on X is integrable if and only if* $\mathrm{Nij}_J = 0$.

By a straightforward computation, the integrability of an almost-complex structure J turns out to be equivalent to the vanishing of the components A and $\bar{A}$ of the exterior differential, equivalently, to $\left(\wedge^{\bullet,\bullet}X,\ \partial,\ \overline{\partial}\right)$ being a double complex of $\mathcal{C}^\infty(X;\mathbb{C})$-modules (see, e.g., [Wel08, Sect. 2.6], [Mor07, Proposition 8.2]).

Therefore, for a complex manifold X, one can consider, for every $p \in \mathbb{N}$, the differential complex $\left(\wedge^{p,\bullet} X,\, \overline{\partial}\right)$ and its cohomology, defining the *Dolbeault cohomology*, as the bi-graded $\mathbb{C}$-vector space

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \;:=\; \frac{\ker \overline{\partial}}{\operatorname{im} \overline{\partial}} \,.$$

For every $(p, q) \in \mathbb{N}^2$, denote by $\mathcal{A}^{p,q}_X$ the (fine) sheaf of germs of (p, q)-forms on X. For every $p \in \mathbb{N}$, denote by Ω^p_X the sheaf of germs of *holomorphic p-forms* on X, that is, the kernel sheaf of the map $\overline{\partial}\colon \mathcal{A}^{p,0}_X \to \mathcal{A}^{p,1}_X$. By the Dolbeault and Grothendieck Lemma, see, e.g., [Dem12, I.3.29], one has that

$$0 \to \Omega^p_X \to \mathcal{A}^{p,\bullet}_X$$

is a fine resolution of Ω^p_X; hence, one gets the following result.

Theorem 1.2 (Dolbeault Theorem, [Dol53]). *Let X be a complex manifold. For every p, $q \in \mathbb{N}$,*

$$H^{p,q}_{\overline{\partial}}(X) \;\simeq\; \check{H}^q\left(X; \Omega^p\right) \,.$$

This gives a sheaf-theoretic interpretation of the Dolbeault cohomology. On the other hand, also an analytic interpretation can be provided.

Suppose X is a compact complex manifold of complex dimension n, and fix g a Hermitian metric on X and vol the induced volume form on X (recall that every complex manifold is orientable, see, e.g., [GH94, pp. 17–18]); denote by $\omega := g(J\cdot, \cdot\cdot) \in \wedge^{1,1} X \cap \wedge^2 X$ the associated $(1, 1)$-form to g. Recall that g induces a Hermitian inner product $\langle \cdot, \cdot\cdot \rangle$ on the space $\wedge^{\bullet,\bullet} X$ of global differential forms on X, and that the *Hodge-$*$-operator* associated to g is the $\mathbb{C}$-linear map

$$* \lfloor_{\wedge^{p,q} X} \colon \wedge^{p,q} X \to \wedge^{n-q,n-p} X$$

defined requiring that, for every $\alpha, \beta \in \wedge^{p,q} X$,

$$\alpha \wedge *\bar{\beta} \;=\; \langle \alpha,\, \beta \rangle \;\mathrm{vol} \,.$$

Define

$$\overline{\partial}^* \;:=\; -*\,\partial\,*\colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet,\bullet-1} X \,;$$

the operator $\overline{\partial}^*\colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet,\bullet-1} X$ is the adjoint[1] of $\overline{\partial}\colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet,\bullet+1} X$ with respect to $\langle \cdot, \cdot\cdot \rangle$. Define

[1] This fact can be generalized for a *bi-differential bi-graded algebra of* PD*-type*, [Kas12b, Sect. 2], see also [AK12, Sect. 2.1].

$$\overline{\square} := \left[\overline{\partial}, \overline{\partial}^*\right] := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}\colon \wedge^{\bullet,\bullet} X \to \wedge^{\bullet,\bullet} X \;;$$

$\overline{\square}$ being a 2nd order self-adjoint elliptic differential operator (see, e.g., [Kod05, Theorem 3.16]), one gets the following result.

Theorem 1.3 (Hodge Theorem, [Hod89]). *Let X be a compact complex manifold endowed with a Hermitian metric. There is an orthogonal decomposition*

$$\wedge^{\bullet,\bullet} X \;=\; \ker \overline{\square} \overset{\perp}{\oplus} \overline{\partial} \wedge^{\bullet,\bullet-1} X \overset{\perp}{\oplus} \overline{\partial}^* \wedge^{\bullet,\bullet+1} X \;,$$

and hence an isomorphism

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \;\simeq\; \ker \overline{\square} \;.$$

In particular, $\dim_{\mathbb{C}} H^{\bullet,\bullet}_{\overline{\partial}}(X) < +\infty$.

Note that, for any $(p,q) \in \mathbb{N}^2$, the Hodge-$*$-operator $*\colon \wedge^{p,q} X \to \wedge^{n-q,n-p} X$ sends a $\overline{\square}$-harmonic (p,q)-form ψ (that is, $\psi \in \wedge^{p,q} X$ is such that $\overline{\square}\psi = 0$) to a $\square$-harmonic $(n-q, n-p)$-form $*\psi$, where $\square := [\partial, \partial^*] := \partial\partial^* + \partial^*\partial \in \operatorname{End}(\wedge^{\bullet,\bullet} X)$ is the conjugate operator to $\overline{\square}$, and hence, by conjugating, one gets a $\overline{\square}$-harmonic $(n-p, n-q)$-form $\overline{*\psi}$. Hence, one gets the following result.

Theorem 1.4 (Serre Duality, [Ser55, Théorème 4]). *Let X be a compact complex manifold of complex dimension n, endowed with a Hermitian metric. For every p, $q \in \mathbb{N}$, the Hodge-$*$-operator induces an isomorphism*

$$*\colon H^{p,q}_{\overline{\partial}}(X) \xrightarrow{\simeq} \overline{H^{n-p,n-q}_{\overline{\partial}}(X)} \;.$$

Since a $\overline{\partial}$-closed form is not necessarily d-closed, Dolbeault cohomology classes do not define, in general, de Rham cohomology classes, that is, in general, on a compact complex manifold, there is no natural map between the Dolbeault cohomology and the de Rham cohomology (as we will see, in the special case of compact Kähler manifolds, or more in general of compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma, the de Rham cohomology actually can be decomposed by means of the Dolbeault cohomology groups, [Wei58, Théorème IV.3], [DGMS75, Lemma 5.15, Remark 5.16, 5.21]). Nevertheless, the Frölicher inequality provides a relation between the dimension of the Dolbeault cohomology and the dimension of the de Rham cohomology; it follows by considering the Hodge and Frölicher spectral sequence, which we recall here.

The structure of double complex of $\left(\wedge^{\bullet,\bullet} X, \partial, \overline{\partial}\right)$ gives rise to two natural filtrations of $\wedge^{\bullet} X \otimes \mathbb{C}$, namely (for $(p,q) \in \mathbb{N}^2$ and for $k \in \mathbb{N}$,)

$${}'F^p\left(\wedge^k X \otimes \mathbb{C}\right) := \bigoplus_{\substack{r+s=k\\ r\geq p}} \wedge^{r,s} X \quad \text{and} \quad {}''F^q\left(\wedge^k X \otimes \mathbb{C}\right) := \bigoplus_{\substack{r+s=k\\ s\geq q}} \wedge^{r,s} X \;;$$

these filtrations induce two spectral sequences (see, e.g., [McC01, Sect. 2.4], [GH94, Sect. 3.5]),

$$\left\{\left(E_r^{\bullet,\bullet},\, \mathrm{d}_r\right) = \left({}'E_r^{\bullet,\bullet},\, {}'\mathrm{d}_r\right)\right\}_{r\in\mathbb{N}} \qquad \text{and, respectively,} \qquad \left\{\left({}''E_r^{\bullet,\bullet},\, {}''\mathrm{d}_r\right)\right\}_{r\in\mathbb{N}} \;,$$

called *Hodge and Frölicher spectral sequences* (or *Hodge to de Rham spectral sequences*): one has

$${}'E_1^{\bullet,\bullet} \simeq H^{\bullet,\bullet}_{\overline{\partial}}(X) \;\Rightarrow\; H^{\bullet}_{dR}(X;\mathbb{C}) \quad \text{and} \quad {}''E_1^{\bullet,\bullet} \;\simeq\; H^{\bullet,\bullet}_{\partial}(X) \;\Rightarrow\; H^{\bullet}_{dR}(X;\mathbb{C}) \;.$$

An explicit description of $\{(E_r,\, \mathrm{d}_r)\}_{r\in\mathbb{N}}$ is given in [CFUG97]: for any $p,\, q \in \mathbb{N}$ and $r \in \mathbb{N}$, its terms are

$$E_r^{p,q} \;\simeq\; \frac{\mathcal{X}_r^{p,q}}{\mathcal{Y}_r^{p,q}} \;,$$

where, for $r = 1$,

$$\mathcal{X}_1^{p,q} \;:=\; \left\{\alpha \in \wedge^{p,q}X \;:\; \overline{\partial}\alpha = 0\right\} \;, \qquad \mathcal{Y}_1^{p,q} \;:=\; \overline{\partial} \wedge^{p,q-1} X \;,$$

and, for $r \geq 2$,

$$\begin{aligned}\mathcal{X}_r^{p,q} := \Big\{&\alpha^{p,q} \in \wedge^{p,q}X \;:\; \overline{\partial}\alpha^{p,q} = 0 \text{ and, for any } i \in \{1,\ldots,r-1\}\;, \\ &\text{there exists } \alpha^{p+i,q-i} \in \wedge^{p+i,q-i}X \\ &\text{such that } \partial\alpha^{p+i-1,q-i+1} + \overline{\partial}\alpha^{p+i,q-i} = 0\Big\} \;,\end{aligned}$$

$$\begin{aligned}\mathcal{Y}_r^{p,q} := \Big\{&\partial\beta^{p-1,q} + \overline{\partial}\beta^{p,q-1} \in \wedge^{p,q}X \;:\; \text{for any } i \in \{2,\ldots,r-1\}\;, \\ &\text{there exists } \beta^{p-i,q+i-1} \in \wedge^{p-i,q+i-1}X \\ &\text{such that } \partial\beta^{p-i,q+i-1} + \overline{\partial}\beta^{p-i+1,q+i-2} = 0 \text{ and } \overline{\partial}\beta^{p-r+1,q+r-2} = 0\Big\} \;,\end{aligned}$$

see [CFUG97, Theorem 1], and, for any $r \geq 1$, the map $\mathrm{d}_r \colon E_r^{\bullet,\bullet} \to E_r^{\bullet+r,\bullet-r+1}$ is given by

$$\mathrm{d}_r \colon \{[\alpha^{p,q}] \in E_r^{p,q}\}_{(p,q)\in\mathbb{N}^2} \mapsto \left\{\left[\partial\alpha^{p+r-1,q-r+1}\right] \in E_r^{p+r,q-r+1}\right\}_{(p,q)\in\mathbb{N}^2} \;,$$

see [CFUG97, Theorem 3].

As a consequence of ${}'E_1^{\bullet,\bullet} \simeq H^{\bullet,\bullet}_{\overline{\partial}}(X) \;\Rightarrow\; H^{\bullet}_{dR}(X;\mathbb{C})$, one gets the following inequality by A. Frölicher.

Theorem 1.5 (Frölicher Inequality, [Frö55, Theorem 2]). *Let X be a compact complex manifold. Then, for every $k \in \mathbb{N}$,*

$$\dim_{\mathbb{C}} H^k_{dR}(X;\mathbb{C}) \;\leq\; \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X)\,.$$

As a matter of notation, for $k \in \mathbb{N}$ and $(p,q) \in \mathbb{N}^2$, we will denote by $b_k := \dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R})$, respectively $h^{p,q}_{\overline{\partial}} := \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X)$, the *$k$th Betti number*, respectively the $(p,q)^{th}$ *Hodge number* of X.

Remark 1.2. Other than the Dolbeault cohomology, other cohomologies can be defined for a complex manifold X; more precisely, since, for every $(p,q) \in \mathbb{N}^2$,

$$\wedge^{p-1,q-1}X \stackrel{\partial\overline{\partial}}{\to} \wedge^{p,q}X \stackrel{\partial+\overline{\partial}}{\to} \wedge^{p+1,q}X \oplus \wedge^{p,q+1}X \qquad \text{and}$$

$$\wedge^{p-1,q}X \oplus \wedge^{p,q-1}X \stackrel{(\partial,\overline{\partial})}{\to} \wedge^{p,q}X \stackrel{\partial\overline{\partial}}{\to} \wedge^{p+1,q+1}X$$

are complexes, one can define the *Bott-Chern cohomology* $H^{\bullet,\bullet}_{BC}(X)$ and the *Aeppli cohomology* $H^{\bullet,\bullet}_{A}(X)$ of X as

$$H^{\bullet,\bullet}_{BC}(X) \;:=\; \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}(X) \;:=\; \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial + \operatorname{im}\overline{\partial}}\;;$$

we refer to Sect. 2.1 for further details.

In the next chapter, we will provide a Frölicher-type inequality also for the Bott-Chern cohomology, Theorem 2.13, showing that it allows to characterize the compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma just in terms of the dimensions of the Bott-Chern cohomology and of the de Rham cohomology, Theorem 2.14.

(We refer also to [AT13a], where Dolbeault, Bott-Chern, and Aeppli cohomologies are considered in a more general context.)

Remark 1.3. In Complex Analysis, properties concerning the existence of exhaustion functions with convexity properties may have consequences on the vanishing of the cohomology. In fact, the Hörmander theorem, [Hör65, Theorems 2.2.4 and 2.2.5], [Hör90, Theorem 4.2.2, Corollary 4.2.6], states that the Dolbeault cohomology groups $H^{p,q}_{\overline{\partial}}(D)$ of a *strictly pseudo-convex* domain D in $\mathbb{C}^n$ (that is, a domain admitting a smooth proper strictly pluri-sub-harmonic exhaustion function) vanish for $q \geq 1$, for any $p \in \mathbb{N}$. More in general, A. Andreotti and H. Grauert proved in [AG62, Proposition 27] that the Dolbeault cohomology groups $H^{r,s}_{\overline{\partial}}(D)$ of a *q-complete* domain in $\mathbb{C}^n$ (that is, a domain in $\mathbb{C}^n$ admitting a smooth proper exhaustion function whose Levi form has at least $n-q+1$ positive eigen-values, [AG62, Rot55]) vanish for $s \geq q$, for any $r \in \mathbb{N}$, see also [AV65a, AV65b, Theorem 5], see also [Ves67]. (Furthermore, given a domain D of a Stein manifold with boundary of class $\mathcal{C}^2$, if $H^{r,s}_{\overline{\partial}}(D) = \{0\}$ for $s \geq q$ and for any $r \in \mathbb{N}$, then D is q-complete, [EVS80, Theorem 3.8].)

See [AC12, AC13] for other results connecting convexity properties (see [HL12, HL11]) and vanishing of the cohomology, see also [Sha86, Theorem 1], [Wu87, Theorem 1], see also [HL11, Proposition 5.7].

1.2 Symplectic Geometry

In this section, we recall some definitions and results concerning symplectic manifolds, that is, differentiable manifolds endowed with a non-degenerate d-closed 2-form. An interesting class of examples of symplectic manifolds is provided by the Kähler manifolds. Moreover, given a differentiable manifold X, its cotangent bundle T^*X is endowed with a natural symplectic structure (see, e.g., [CdS01, Sect. 2]): in fact, symplectic geometry has applications and motivations in the study of Hamiltonian Mechanics, see, e.g., [CdS01, Part VII].

1.2.1 Symplectic Structures

Let X be a compact $2n$-dimensional manifold. We start by recalling the notion of symplectic structure.

Definition 1.2 ([Wey97, Sect. VI]). Let X be a differentiable manifold. A *symplectic form* on X is a non-degenerate d-closed 2-form $\omega \in \wedge^2 X$.

The main difference between symplectic geometry and Riemannian geometry is provided by G. Darboux's theorem.

Theorem 1.6 (Darboux Theorem, [Dar82]). *Let X be a $2n$-dimensional manifold endowed with a symplectic form ω. Then, for every $x \in X$, there exists a coordinate chart $\left(U,\, \left\{x^j\right\}_{j\in\{1,\dots,2n\}}\right)$, with $x \in U$, such that*

$$\omega \stackrel{loc}{=} \sum_{j=1}^{n} \mathrm{d}\,x^{2j-1} \wedge \mathrm{d}\,x^{2j} \;.$$

As for (almost-)complex manifolds, on a symplectic manifold X endowed with a symplectic form ω, one has a decomposition of differential forms in symplectic-type components, the so-called Lefschetz decomposition; it is a consequence of an $\mathfrak{sl}(2;\mathbb{R})$-representation on $\wedge^2 X$ by means of operators related to the symplectic structure.

More precisely, define the operators $L,\ \Lambda,\ H \in \mathrm{End}^\bullet\left(\wedge^\bullet X\right)$ as

$$L\colon \wedge^\bullet X \to \wedge^{\bullet+2} X\ , \qquad \alpha \mapsto \omega \wedge \alpha\ ,$$

$$\Lambda\colon \wedge^\bullet X \to \wedge^{\bullet-2} X\ , \qquad \alpha \mapsto -\iota_\Pi \alpha\ ,$$

$$H\colon \wedge^\bullet X \to \wedge^\bullet X\ , \qquad \alpha \mapsto \sum_k (n-k)\ \pi_{\wedge^k X}\alpha$$

(where $\iota_\xi\colon \wedge^\bullet X \to \wedge^{\bullet-2} X$ denotes the interior product with $\xi \in \wedge^2(TX)$, and, for $k \in \mathbb{N}$, the map $\pi_{\wedge^k X}\colon \wedge^\bullet X \to \wedge^k X$ denotes the natural projection onto $\wedge^k X$). Note that, using the symplectic-$\star$-operator $\star_\omega$, one can write, [Yan96, Lemma 1.5],

$$\Lambda \;=\; -\star_\omega\ L\ \star_\omega\ .$$

The following result holds.[2]

Theorem 1.7 ([Yan96, Corollary 1.6]). *Let X be a manifold endowed with a symplectic structure. Then*

$$[L,\ H] \;=\; 2\,L\ , \qquad [\Lambda,\ H] \;=\; -2\,\Lambda\ , \qquad [L,\ \Lambda] \;=\; H\ ,$$

and hence

$$\mathfrak{sl}(2;\mathbb{R}) \;\simeq\; \langle L,\ \Lambda,\ H\rangle \to \mathrm{End}^\bullet\left(\wedge^\bullet X\right)$$

gives an $\mathfrak{sl}(2;\mathbb{R})$-representation on $\wedge^\bullet X$.

(See also [Huy05, Proposition 1.2.2] for a proof.)

The above $\mathfrak{sl}(2;\mathbb{R})$-representation, having finite H-spectrum,[3] induces a decomposition of the space of the differential forms.

Theorem 1.8 ([Yan96, Corollary 2.6]). *Let X be a manifold endowed with a symplectic structure. Then one has the* Lefschetz decomposition *on differential forms,*

$$\wedge^\bullet X \;=\; \bigoplus_{r\in\mathbb{N}} L^r\,\mathrm{P}\wedge^{\bullet-2r} X\ ,$$

[2] See, e.g., [Hum78, Sect. 7] for general results concerning $\mathfrak{sl}(2;\mathbb{K})$-representations.

[3] Recall that an $\mathfrak{sl}(2;\mathbb{R})$-representation on a (possibly non-finite dimensional) $\mathbb{R}$-vector space V is called of *finite H-spectrum* if V can be decomposed into the direct sum of eigen-spaces of H and H has only finitely-many distinct eigen-values, [Yan96, Definition 2.2].

where

$$\mathrm{P}\wedge^{\bullet} X \;:=\; \ker \Lambda$$

is the space of primitive forms.

Remark 1.4. Note that, for every $k \in \mathbb{N}$,

$$\mathrm{P}\wedge^{k} X \;=\; \ker L^{n-k+1}\lfloor_{\wedge^k X} \;,$$

see, e.g., [Huy05, Proposition 1.2.30(v)].

In general, see, e.g., [TY12b, p. 422], the Lefschetz decomposition of $A^{(k)} \in \wedge^k X$ reads as

$$A^{(k)} \;=\; \sum_{r \geq \max\{k-n,\, 0\}} \frac{1}{r!}\, L^r\, B^{(k-2r)}$$

where, for $r \geq \max\{k-n,\, 0\}$,

$$B^{(k-2r)} \;:=\; \left(\sum_{\ell \in \mathbb{N}} a_{r,\ell,(n,k)}\, \frac{1}{\ell!}\, L^{\ell}\, \Lambda^{r+\ell} \right) A^{(k)} \;\in\; \mathrm{P}\wedge^{k-2r} X$$

and, for $r \geq \max\{k-n,\, 0\}$ and $\ell \in \mathbb{N}$,

$$a_{r,\ell,(n,k)} := (-1)^{\ell} \cdot (n-k+2r+1)^2 \cdot \prod_{i=0}^{r} \frac{1}{n-k+2r+1-i} \cdot \prod_{j=0}^{\ell} \frac{1}{n-k+2r+1+j} \in \mathbb{Q} \,.$$

We recall that

$$L\lfloor_{\bigoplus_{k=-1}^{n-2} \wedge^{n-k-2} X} \colon \bigoplus_{k=-1}^{n-2} \wedge^{n-k-2} X \to \wedge^{n-k} X$$

is injective, [Yan96, Corollary 2.8], and that, for every $k \in \mathbb{N}$,

$$L^k \colon \wedge^{n-k} X \to \wedge^{n+k} X$$

is an isomorphism, [Yan96, Corollary 2.7].

1.2.1.1 Primitive Currents

Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Denote by $\mathcal{D}_\bullet X := \mathcal{D}^{2n-\bullet} X$ the space of currents, and consider the de Rham homology $H_\bullet^{dR}(X;\mathbb{R}) := H^\bullet(\mathcal{D}_\bullet X, \mathrm{d})$. (See Sect. 1.6 for definitions and results concerning currents and de Rham homology.)

Following [Lin13, Definition 4.1], set, by duality,

$$\begin{aligned} L\colon \mathcal{D}_\bullet X &\to \mathcal{D}_{\bullet-2} X\;, \qquad S \mapsto S\,(L\,\cdot)\;, \\ \Lambda\colon \mathcal{D}_\bullet X &\to \mathcal{D}_{\bullet+2} X\;, \qquad S \mapsto S\,(\Lambda\,\cdot)\;, \\ H\colon \mathcal{D}_\bullet X &\to \mathcal{D}_\bullet X\;, \qquad S \mapsto S\,(-H\,\cdot)\;; \end{aligned}$$

note that

$$[L,\, H] \;=\; 2\,L\;, \qquad [\Lambda,\, H] \;=\; -2\,\Lambda\;, \qquad [L,\, \Lambda] \;=\; H\;.$$

A current $S \in \mathcal{D}^k X$ is said *primitive* if $\Lambda S = 0$, equivalently, if $L^{n-k+1} S = 0$, see, e.g., [Lin13, Proposition 4.3]; denote by $\mathrm{P}\mathcal{D}^\bullet X := \mathrm{P}\mathcal{D}_{2n-\bullet} X$ the space of primitive currents on X.

In [Lin13], Y. Lin proved the following result.

Theorem 1.9 ([Lin13, Lemma 4.2, Proposition 4.3]). *Let X be a compact manifold endowed with a symplectic structure ω. Then $\langle L,\, \Lambda,\, H\rangle$ gives an $\mathfrak{sl}(2;\mathbb{R})$-module structure on $\mathcal{D}^\bullet X$. In particular, one has the Lefschetz decomposition on the space of currents,*

$$\mathcal{D}^\bullet X \;=\; \bigoplus_{r\in\mathbb{N}} L^r\, \mathrm{P}\mathcal{D}^{\bullet-2r} X \;=\; \bigoplus_{r\in\mathbb{N}} L^r\, \mathrm{P}\mathcal{D}_{2n-\bullet+2r} X\;.$$

Finally, if $j\colon Y \hookrightarrow X$ is a compact oriented submanifold of X of codimension k (possibly with non-empty boundary), then the *dual current* $[Y] \in \mathcal{D}_k X$ associated with Y is defined, by setting, for every $\varphi \in \wedge^k X$,

$$[Y]\,(\varphi) \;:=\; \int_Y j^*(\varphi)\;.$$

If Y is a closed oriented submanifold, then the dual current $[Y]$ is d-closed. According to [TY12a, Lemma 4.1], the dual current $[Y]$ is primitive if and only if Y is co-isotropic.

1.2.2 Cohomological Aspects of Symplectic Geometry

Cohomological properties of symplectic manifolds have been studied starting from the works by J.-L. Koszul, [Kos85], and by J.-L. Brylinski, [Bry88]. Drawing a parallel between the symplectic and the Riemannian cases, J.-L. Brylinski proposed in [Bry88] a Hodge theory for compact symplectic manifolds $(X,\ \omega)$, introducing a symplectic Hodge-$\star$-operator $\star_\omega$ and the notion of ω-symplectically-harmonic form (i.e., a form being both d-closed and d^Λ-closed, where the symplectic co-differential is defined as $\mathrm{d}^\Lambda \lfloor_{\wedge^k X} := (-1)^{k+1} \star_\omega \mathrm{d} \star_\omega$ for every $k \in \mathbb{N}$): in this context, O. Mathieu in [Mat95], and D. Yan in [Yan96], proved that any de Rham cohomology class admits an ω-symplectically-harmonic representative if and only if the Hard Lefschetz Condition is satisfied. In [TY12a, TY12b], see also [TY11], L.-S. Tseng and S.-T. Yau introduced new cohomologies for symplectic manifolds $(X,\ \omega)$: among them, in particular, they defined and studied

$$H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) := \frac{\ker\left(\mathrm{d}+\mathrm{d}^\Lambda\right)}{\operatorname{im}\mathrm{d}\,\mathrm{d}^\Lambda},$$

developing a Hodge theory for this cohomology; furthermore, they studied the dual currents of Lagrangian and co-isotropic submanifolds, and they defined a homology theory on co-isotropic chains, which turns out to be naturally dual to a primitive cohomology. In the context of generalized geometry, [Gua04a, Gua11, Cav05, Cav07], the cohomology $H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R})$ can be interpreted as the symplectic counterpart of the Bott-Chern cohomology of a complex manifold, see [TY11]. Inspired also by their works, Y. Lin developed in [Lin13] a geometric measure theoretic approach to symplectic Hodge theory, proving in particular that, on any compact symplectic manifold, every primitive cohomology class of positive degree admits a symplectically-harmonic representative not supported on the entire manifold, [Lin13, Theorems 1.1 and 5.3].

In this section, we recall some notions and results concerning Hodge theory for compact symplectic manifolds; we refer to [Bry88, Mat95, Yan96, Cav05, TY12a, TY12b, Lin13] for further details.

1.2.2.1 Symplectic Hodge Theory

Let X be a compact $2n$-dimensional manifold endowed with a symplectic form ω. By exploiting the parallelism with Riemannian geometry, one can try to develop a Hodge theory also for compact symplectic manifolds, [Bry88]. The first tool that can be introduced is an analogue of the Hodge-$*$-operator; hence one can define a symplectic counterpart of the co-differential operator, and define the notion of symplectically-harmonic form, investigating the existence of symplectically-harmonic representatives in any de Rham cohomology class.

Note that every symplectic manifold is orientable, $\frac{\omega^n}{n!}$ giving a canonical orientation.

Denote by I the natural isomorphism of vector bundles induced by ω, namely, for every $x \in X$,

$$T_xX \ni v \mapsto I(v)(\cdot) := \omega(v,\cdot) \in \operatorname{Hom}(T_xX;\mathbb{R}) .$$

Then, for every $k \in \mathbb{N}$, the form ω gives rise to a bi-$\mathcal{C}^\infty(X;\mathbb{R})$-linear form on $\wedge^k X$ denoted by $\left(\omega^{-1}\right)^k$, which is skew-symmetric, respectively symmetric, according that k is odd, respectively even, and defined on the simple elements $\alpha^1 \wedge \ldots \wedge \alpha^k \in \wedge^k X$ and $\beta^1 \wedge \ldots \wedge \beta^k \in \wedge^k X$ as

$$\left(\omega^{-1}\right)^k \left(\alpha^1 \wedge \ldots \wedge \alpha^k, \beta^1 \wedge \ldots \wedge \beta^k\right) := \det\left(\omega^{-1}\left(\alpha^\ell, \beta^m\right)\right)_{\ell,m\in\{1,\ldots,k\}} ,$$

where

$$\omega^{-1}\left(\alpha^\ell, \beta^m\right) := \omega\left(I^{-1}\left(\alpha^\ell\right), I^{-1}\left(\beta^m\right)\right)$$

for every $\ell, m \in \{1, \ldots, k\}$. In a Darboux coordinate chart $\left(U, \left\{x^j\right\}_{j\in\{1,\ldots,2n\}}\right)$, the canonical *Poisson bi-vector* $\Pi := \omega^{-1} \in \wedge^2 TX$ associated to ω is written as $\omega^{-1} \stackrel{\text{loc}}{=} \sum_{j=1}^{n} \frac{\partial}{\partial x^{2j-1}} \wedge \frac{\partial}{\partial x^{2j}}$.

The *symplectic-⋆-operator*

$$\star_\omega \colon \wedge^\bullet X \to \wedge^{2n-\bullet} X ,$$

introduced by J.-L. Brylinski, [Bry88, Sect. 2], is defined requiring that, for every $k \in \mathbb{N}$, and for every $\alpha,\ \beta \in \wedge^k X$,

$$\alpha \wedge \star_\omega \beta = \left(\omega^{-1}\right)^k (\alpha, \beta) \frac{\omega^n}{n!} .$$

By continuing in the parallelism between Riemannian geometry and symplectic geometry, one can introduce the d^Λ operator with respect to a symplectic structure ω as

$$\mathrm{d}^\Lambda \lfloor_{\wedge^k X} := (-1)^{k+1} \star_\omega \mathrm{d} \star_\omega$$

for any $k \in \mathbb{N}$, and interpret it as the symplectic counterpart of the Riemannian d^* operator with respect to a Riemannian metric. In light of this, J.-L. Brylinski proposed in [Bry88] a Hodge theory for compact symplectic manifolds, conjecturing that, on a compact manifold endowed with a symplectic structure ω, every de Rham cohomology class admits a (possibly non-unique) *ω-symplectically-harmonic representative*, namely, a d-closed d^Λ-closed representative, [Bry88, Conjecture 2.2.7]. (Note that $\mathrm{d}\,\mathrm{d}^\Lambda + \mathrm{d}^\Lambda\,\mathrm{d} = 0$, [Bry88, Theorem 1.3.1], [Kos85, p. 265], provides a strong difference in the parallelism between symplectic geometry and Riemannian geometry; in particular, it follows that a ω-symplectically-harmonic representative, whenever it exists, is not unique.)

Remark 1.5. For an almost-Kähler structure $(J,\ \omega,\ g)$ on a compact manifold X (that is, $\omega \in \wedge^2 X$ is a symplectic form on X, and $J \in \mathrm{End}(TX)$ is an almost-complex structure on X, and g is a J-Hermitian metric on X such that ω is the associated $(1,1)$-form to g), the symplectic-$\star$-operator $\star_\omega$ and the Hodge-$*$-operator $*_g$ are related by

$$\star_\omega \;=\; J *_g \;,$$

and hence

$$\mathrm{d}^\Lambda \;=\; -\left(\mathrm{d}^c\right)^{*_g}$$

where $\mathrm{d}^c \;:=\; J^{-1}\,\mathrm{d}\,J$ and $\left(\mathrm{d}^c\right)^{*_g}\lfloor_{\wedge^k X} := \; (-1)^{k+1} *_g \mathrm{d} *_g$ for every $k \in \mathbb{N}$ (note that, when J is integrable, then $\mathrm{d}^c = -\mathrm{i}\left(\partial - \overline{\partial}\right)$). Moreover, on a compact manifold X endowed with a Kähler structure $(J,\,\omega,\,g)$, by the Hodge decomposition theorem, [Wei58, Théorème IV.3], the pure-type components with respect to J of the harmonic representatives of the de Rham cohomology classes are themselves harmonic. Hence, it follows that Brylinski's conjecture holds true for compact Kähler manifolds, [Bry88, Corollary 2.4.3].

Remark 1.6. In the framework of generalized complex geometry, see Sect. 1.3, the d^Λ operator associated to a symplectic structure should be interpreted as the symplectic counterpart of the operator $\mathrm{d}^c := -\mathrm{i}\left(\partial - \overline{\partial}\right)$ associated to a complex structure, [Cav05].

Consider now, for $k \in \mathbb{N}$, the map $L^k \colon \wedge^{n-k} X \to \wedge^{n+k} X$. Since $[L,\,\mathrm{d}] = 0$, it induces a map $L^k \colon H^{n-k}_{dR}(X;\mathbb{R}) \to H^{n+k}_{dR}(X;\mathbb{R})$ in cohomology. One says that X satisfies the *Hard Lefschetz Condition*, shortly HLC, if

$$\text{for every } k \in \mathbb{N}\;, \qquad L^k \colon H^{n-k}_{dR}(X;\mathbb{R}) \stackrel{\simeq}{\to} H^{n+k}_{dR}(X;\mathbb{R})\;. \tag{HLC}$$

O. Mathieu in [Mat95], and D. Yan in [Yan96], provided counterexamples to Brylinski's conjecture, characterizing the compact symplectic manifolds satisfying Brylinski's conjecture in terms of the validity of the Hard Lefschetz Condition. Furthermore, S.A. Merkulov in [Mer98], see also [Cav05], and V. Guillemin in [Gui01], proved that the Hard Lefschetz Condition on compact symplectic manifolds is equivalent to satisfying the $\mathrm{d}\,\mathrm{d}^\Lambda$-Lemma, namely, to every d-exact d^Λ-closed form being $\mathrm{d}\,\mathrm{d}^\Lambda$-exact. See Theorem 1.15 for a summary of several equivalent statements to Hard Lefschetz Condition.

Note that, by the Lefschetz decomposition theorem, [Wei58, Théorème IV.5] (see Sect. 1.4), compact Kähler manifolds satisfy the Hard Lefschetz Condition.

1.2.2.2 Symplectic Cohomologies

Let X be a compact $2n$-dimensional manifold endowed with a symplectic structure ω.

In [TY12a], L.-S. Tseng and S.-T. Yau introduced also the $\left(\mathrm{d}+\mathrm{d}^{\Lambda}\right)$-*cohomology*, [TY12a, Sect. 3.2],

$$H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \;:=\; \frac{\ker\left(\mathrm{d}+\mathrm{d}^{\Lambda}\right)}{\operatorname{im}\mathrm{d}\,\mathrm{d}^{\Lambda}} \;,$$

and the $\left(\mathrm{d}\,\mathrm{d}^{\Lambda}\right)$-*cohomology*, [TY12a, Sect. 3.3],

$$H^{\bullet}_{\mathrm{d}\,\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \;:=\; \frac{\ker\mathrm{d}\,\mathrm{d}^{\Lambda}}{\operatorname{im}\mathrm{d}+\operatorname{im}\mathrm{d}^{\Lambda}} \;;$$

such cohomologies are, in a sense, the symplectic counterpart of the Bott-Chern and Aeppli cohomologies of complex manifolds, see [TY12a, Sect. 5] and [TY11] for further discussions.

Furthermore, they provided a Hodge theory for such cohomologies, proving the following result.

Theorem 1.10 ([TY12a, Theorem 3.5, Corollary 3.6]). *Let X be a compact manifold endowed with a symplectic structure ω. Let $(J,\ \omega,\ g)$ be an almost-Kähler structure on X. For a fixed $\lambda > 0$, the 4^{th} order self-adjoint differential operator*

$$D_{\mathrm{d}+\mathrm{d}^{\Lambda}} := \left(\mathrm{d}\mathrm{d}^{\Lambda}\right)\left(\mathrm{d}\mathrm{d}^{\Lambda}\right)^{*} + \left(\mathrm{d}\mathrm{d}^{\Lambda}\right)^{*}\left(\mathrm{d}\mathrm{d}^{\Lambda}\right) + \left(\mathrm{d}^{*}\mathrm{d}^{\Lambda}\right)\left(\mathrm{d}^{*}\mathrm{d}^{\Lambda}\right)^{*} + \left(\mathrm{d}^{*}\mathrm{d}^{\Lambda}\right)^{*}\left(\mathrm{d}^{*}\mathrm{d}^{\Lambda}\right)$$
$$+\lambda\left(\mathrm{d}^{*}\mathrm{d}+\left(\mathrm{d}^{\Lambda}\right)^{*}\mathrm{d}^{\Lambda}\right) \;.$$

is elliptic, with $\ker D_{\mathrm{d}+\mathrm{d}^{\Lambda}} = \ker\mathrm{d}\cap\ker\mathrm{d}^{\Lambda}\cap\ker\left(\mathrm{d}\,\mathrm{d}^{\Lambda}\right)^{*}$.

Furthermore, there exist an orthogonal decomposition

$$\wedge^{\bullet}X \;=\; \ker D_{\mathrm{d}+\mathrm{d}^{\Lambda}} \oplus \mathrm{d}\,\mathrm{d}^{\Lambda}\wedge^{\bullet}X \oplus \left(\mathrm{d}^{*}\wedge^{\bullet+1}X + \left(\mathrm{d}^{\Lambda}\right)^{*}\wedge^{\bullet-1}X\right)$$

and an isomorphism

$$H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \;\simeq\; \ker D_{\mathrm{d}+\mathrm{d}^{\Lambda}} \;.$$

In particular, $\dim_{\mathbb{R}} H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) < +\infty$.

An analogous statement holds for the $\left(\mathrm{d}\,\mathrm{d}^{\Lambda}\right)$-cohomology.

Theorem 1.11 ([TY12a, Theorem 3.16, Corollary 3.17]). *Let X be a compact manifold endowed with a symplectic structure ω. Let $(J,\ \omega,\ g)$ be an almost-Kähler structure on X. For a fixed $\lambda > 0$, the 4^{th} order self-adjoint differential operator*

$$D_{\mathrm{d}\,\mathrm{d}^{\Lambda}} := \left(\mathrm{d}\,\mathrm{d}^{\Lambda}\right)\left(\mathrm{d}\,\mathrm{d}^{\Lambda}\right)^{*} + \left(\mathrm{d}\,\mathrm{d}^{\Lambda}\right)^{*}\left(\mathrm{d}\,\mathrm{d}^{\Lambda}\right) + \left(\mathrm{d}\left(\mathrm{d}^{\Lambda}\right)^{*}\right)\left(\mathrm{d}\left(\mathrm{d}^{\Lambda}\right)^{*}\right)^{*}$$
$$+\left(\mathrm{d}\left(\mathrm{d}^{\Lambda}\right)^{*}\right)^{*}\left(\mathrm{d}\left(\mathrm{d}^{\Lambda}\right)^{*}\right) + \lambda\left(\mathrm{d}\,\mathrm{d}^{*}+\mathrm{d}^{\Lambda}\left(\mathrm{d}^{\Lambda}\right)^{*}\right) \;.$$

is elliptic, with $\ker D_{\mathrm{d}\,\mathrm{d}^{\Lambda}} = \ker\left(\mathrm{d}\,\mathrm{d}^{\Lambda}\right)\cap\ker\mathrm{d}^{*}\cap\ker\left(\mathrm{d}^{\Lambda}\right)^{*}$.

Furthermore, there exist an orthogonal decomposition

$$\wedge^\bullet X \;=\; \ker D_{\mathrm{d}\,\mathrm{d}^\Lambda} \oplus \left(\mathrm{d} \wedge^{\bullet-1} X + \mathrm{d}^\Lambda \wedge^{\bullet+1} X\right) \oplus \left(\mathrm{d}\,\mathrm{d}^\Lambda\right)^* \wedge^\bullet X$$

and an isomorphism

$$H^\bullet_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R}) \;\simeq\; \ker D_{\mathrm{d}\,\mathrm{d}^\Lambda} \;.$$

In particular, $\dim_{\mathbb{R}} H^\bullet_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R}) < +\infty$.

As for the Bott-Chern and the Aeppli cohomologies, the $\left(\mathrm{d}+\mathrm{d}^\Lambda\right)$-cohomology and the $\left(\mathrm{d}\,\mathrm{d}^\Lambda\right)$-cohomology groups turn out to be isomorphic by means of the Hodge-$*$-operator associated to any Riemannian metric being compatible with ω.

Theorem 1.12 ([TY12a, Lemma 3.23, Proposition 3.24, Corollary 3.25]). *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Let $(J,\, \omega,\, g)$ be an almost-Kähler structure on X. The operators $D_{\mathrm{d}+\mathrm{d}^\Lambda}$ and $D_{\mathrm{d}\,\mathrm{d}^\Lambda}$ satisfy*

$$*_g\, D_{\mathrm{d}+\mathrm{d}^\Lambda} \;=\; D_{\mathrm{d}\,\mathrm{d}^\Lambda}\, *_g \;,$$

and hence $*_g$ induces an isomorphism

$$*_g \colon H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) \xrightarrow{\simeq} H^{2n-\bullet}_{\mathrm{d}\,\mathrm{d}^\Lambda}(X;\mathbb{R}) \;.$$

Moreover, the cohomology $H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R})$ is invariant under symplectomorphisms and Hamiltonian isotopies, [TY12a, Proposition 2.8].

One has the following commutation relations between the differential operators d, d^Λ, and $\mathrm{d}\,\mathrm{d}^\Lambda$, and the elements L, Λ, and H of the $\mathfrak{sl}(2;\mathbb{R})$-triple, see, e.g., [TY12a, Lemma 2.3]:

$$\begin{aligned}
[\mathrm{d},\, L] &= 0 \;, & \left[\mathrm{d}^\Lambda,\, L\right] &= -\,\mathrm{d} \;, & \left[\mathrm{d}\,\mathrm{d}^\Lambda,\, L\right] &= 0 \;,\\
[\mathrm{d},\, \Lambda] &= \mathrm{d}^\Lambda \;, & \left[\mathrm{d}^\Lambda,\, \Lambda\right] &= 0 \;, & \left[\mathrm{d}\,\mathrm{d}^\Lambda,\, \Lambda\right] &= 0 \;,\\
[\mathrm{d},\, H] &= \mathrm{d} \;, & \left[\mathrm{d}^\Lambda,\, H\right] &= -\,\mathrm{d}^\Lambda \;, & \left[\mathrm{d}\,\mathrm{d}^\Lambda,\, H\right] &= 0 \;.
\end{aligned}$$

Hence, by defining the *primitive* $\left(\mathrm{d}+\mathrm{d}^\Lambda\right)$*-cohomology* as

$$\mathrm{P}H^\bullet_{\mathrm{d}+\mathrm{d}^\Lambda}(X;\mathbb{R}) \;:=\; \frac{\ker \mathrm{d} \cap \ker \mathrm{d}^\Lambda \cap \mathrm{P}\wedge^\bullet X}{\operatorname{im} \mathrm{d}\,\mathrm{d}^\Lambda \cap \mathrm{P}\wedge^\bullet X} \;=\; \frac{\ker \mathrm{d} \cap \mathrm{P}\wedge^\bullet X}{\operatorname{im} \mathrm{d}\,\mathrm{d}^\Lambda \lfloor_{\mathrm{P}\wedge^\bullet X}}$$

(where the second equality follows from [TY12a, Lemma 3.9]), one gets the following result.

Theorem 1.13 ([TY12a, Theorem 3.11]). *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Then there exist a decomposition*

$$H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} L^{r}\,\mathrm{P}H^{\bullet-2r}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R})$$

and, for every $k\in\mathbb{N}$, an isomorphism

$$L^{k}\colon H^{n-k}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \stackrel{\simeq}{\to} H^{n+k}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R})\;,$$

Analogously, by defining the *primitive* $\left(\mathrm{d}\,\mathrm{d}^{\Lambda}\right)$*-cohomology* as

$$\begin{aligned}\mathrm{P}H^{\bullet}_{\mathrm{d}\,\mathrm{d}^{\Lambda}}(X;\mathbb{R}) &:= \frac{\ker \mathrm{d}\,\mathrm{d}^{\Lambda}\cap \mathrm{P}\wedge^{\bullet}X}{\left(\mathrm{im}\,\mathrm{d}+\mathrm{im}\,\mathrm{d}^{\Lambda}\right)\cap \mathrm{P}\wedge^{\bullet}X}\\ &= \frac{\ker \mathrm{d}\,\mathrm{d}^{\Lambda}\cap \mathrm{P}\wedge^{\bullet}X}{\mathrm{im}\,\left(\mathrm{d}+LH^{-1}\,\mathrm{d}^{\Lambda}\right)\lfloor_{\mathrm{P}\wedge^{\bullet-1}X}+\mathrm{im}\,\mathrm{d}^{\Lambda}\lfloor_{\mathrm{P}\wedge^{\bullet+1}X}}\end{aligned}$$

(where the second equality follows from [TY12a, Lemma 3.20]), one gets the following result.

Theorem 1.14 ([TY12a, Theorem 3.21]). *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Then there exist a decomposition*

$$H^{\bullet}_{\mathrm{d}\,\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} L^{r}\,\mathrm{P}H^{\bullet-2r}_{\mathrm{d}\,\mathrm{d}^{\Lambda}}(X;\mathbb{R})$$

and, for every $k\in\mathbb{N}$, an isomorphism

$$L^{k}\colon H^{n-k}_{\mathrm{d}\,\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \stackrel{\simeq}{\to} H^{n+k}_{\mathrm{d}\,\mathrm{d}^{\Lambda}}(X;\mathbb{R})\;,$$

The identity map induces the following natural maps in cohomology:

$$\begin{tikzcd}[column sep=small] & H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \arrow[dl] \arrow[dr] & \\ H^{\bullet}_{dR}(X;\mathbb{R}) \arrow[dr] & & H^{\bullet}_{\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \arrow[dl] \\ & H^{\bullet}_{\mathrm{d}\,\mathrm{d}^{\Lambda}}(X;\mathbb{R}) & \end{tikzcd}$$

Recall that a symplectic manifold is said to satisfy the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-*Lemma* if every d-exact d^{Λ}-closed form is $\mathrm{d}\,\mathrm{d}^{\Lambda}$-exact, [DGMS75], namely, if $H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \to H^{\bullet}_{dR}(X;\mathbb{R})$ is injective.

Remark 1.7. (In view of [DGMS75, Lemma 5.15] and [AT13a, Lemma 1.4]) note that

$$\ker \mathrm{d}^{\Lambda} \cap \operatorname{im} \mathrm{d} \;=\; \operatorname{im} \mathrm{d}\,\mathrm{d}^{\Lambda} \qquad \text{if and only if} \qquad \ker \mathrm{d} \cap \operatorname{im} \mathrm{d}^{\Lambda} \;=\; \operatorname{im} \mathrm{d}\,\mathrm{d}^{\Lambda} \;.$$

Indeed, since $\star_{\omega}^{2} = \mathrm{id}_{\wedge^{\bullet} X}$, [Bry88, Lemma 2.1.2], and $\mathrm{d}\,\mathrm{d}^{\Lambda} + \mathrm{d}^{\Lambda}\,\mathrm{d} = 0$, [Bry88, Theorem 1.3.1], one has

$$\star_{\omega} \ker \mathrm{d} \;=\; \ker \mathrm{d}^{\Lambda} \;, \qquad \star_{\omega} \operatorname{im} \mathrm{d} \;=\; \operatorname{im} \mathrm{d}^{\Lambda} \;, \qquad \star_{\omega} \operatorname{im} \mathrm{d}\,\mathrm{d}^{\Lambda} \;=\; \operatorname{im} \mathrm{d}\,\mathrm{d}^{\Lambda} \;.$$

Recalling that a $2n$-dimensional compact manifold X endowed with a symplectic form ω is said to satisfy the Hard Lefschetz Condition if and only if

$$\text{for every } k \in \mathbb{N} \;, \qquad L^{k} \colon H^{n-k}_{dR}(X;\mathbb{R}) \stackrel{\simeq}{\to} H^{n+k}_{dR}(X;\mathbb{R}) \;, \tag{HLC}$$

we summarize in the following result the connection between the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-Lemma, the Hard Lefschetz Condition, and the existence of ω-symplectically harmonic representatives in any de Rham cohomology class.

Theorem 1.15 ([Mat95, Corollary 2], [Yan96, Theorem 0.1], [Mer98, Proposition 1.4], [Gui01], [DGMS75, Lemma 5.15, Remarks 5.16, 5.21], [TY12a, Proposition 3.13], [Cav05, Theorem 5.4], [AT13a, Theorem 4.4], [AT12b, Remark 2.3]). *Let X be a compact manifold endowed with a symplectic structure ω. The following conditions are equivalent:*

(i) *every de Rham cohomology class admits a representative being both* d*-closed and* d^{Λ}*-closed (i.e., Brylinski's conjecture [Bry88, Conjecture 2.2.7] holds on X);*
(ii) *the Hard Lefschetz Condition holds on X;*
(iii) *the natural map $H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \to H^{\bullet}_{dR}(X;\mathbb{R})$ induced by the identity is injective;*
(iv) *the natural map $H^{\bullet}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) \to H^{\bullet}_{dR}(X;\mathbb{R})$ induced by the identity is actually an isomorphism;*
(v) *X satisfies the* $\mathrm{d}\,\mathrm{d}^{\Lambda}$*-Lemma;*
(vi) *for every $k \in \mathbb{Z}$, it holds* $\dim_{\mathbb{R}} H^{k}_{\mathrm{d}+\mathrm{d}^{\Lambda}}(X;\mathbb{R}) + \dim_{\mathbb{R}} H^{k}_{\mathrm{d}\mathrm{d}^{\Lambda}}(X;\mathbb{R}) = 2 \dim_{\mathbb{R}} H^{k}_{dR}(X;\mathbb{R})$*;*
(vii) *the Lefschetz decomposition $\wedge^{\bullet} X = \bigoplus_{r\in\mathbb{N}} L^{r}\, \mathrm{P}\wedge^{\bullet-2r} X$ on forms induces a decomposition in cohomology, namely, $H^{\bullet}_{dR}(X;\mathbb{R}) = \bigoplus_{r\in\mathbb{N}} L^{r}\, H^{(0,\bullet-2r)}_{\omega}(X;\mathbb{R})$, where $H^{(r,s)}_{\omega}(X;\mathbb{R}) := \left\{\left[L^{r}\,\beta^{(s)}\right] \in H^{2r+s}_{dR}(X;\mathbb{R}) \;:\; \beta^{(s)} \in \mathrm{P}\wedge^{s} X\right\}$.*

Note that, by the Lefschetz decomposition theorem, the compact Kähler manifolds satisfy the Hard Lefschetz Condition; in other terms, note that, given a Kähler structure (J, ω, g) on a compact manifold X, one has $\star_\omega = J *_g$, [Bry88, Theorem 2.4.1], and hence every de Rham cohomology class admits an ω-symplectically-harmonic representative.

1.3 Generalized Geometry

Generalized complex geometry, introduced by N.J. Hitchin in [Hit03] and developed, among others, by M. Gualtieri, [Gua04a, Gua11], and G.R. Cavalcanti, [Cav05], see also [Hit10,Cav07], allows to frame symplectic structures and complex structures in the same context. (In a sense, this add more significance to the term "symplectic", which was invented by H. Weyl, [Wey97, Sect. VI], substituting the Greek root in the term "complex" with the corresponding Latin root.)

We recall here the notion of generalized complex structure, referring to [Cav07, Gua04a, Gua11, Cav05] for more details.

1.3.1 Generalized Complex Structures

Let X be a compact differentiable manifold of dimension $2n$.

Note that a complex structure J on X can be seen as an isomorphism $J\colon TX \stackrel{\simeq}{\to} TX$ satisfying the algebraic condition $J^2 = -\mathrm{id}_{TX}$ and the integrability condition $\mathrm{Nij}_J = 0$. Analogously, a symplectic structure ω on X can be seen as an isomorphism $\omega(\cdot)(\cdot\cdot) := \omega(\cdot, \cdot\cdot)\colon TX \stackrel{\simeq}{\to} T^*X$ satisfying the algebraic condition $\omega(\cdot)(\cdot\cdot) = -\omega(\cdot\cdot)(\cdot)$ and the integrability condition $\mathrm{d}\,\omega = 0$.

Therefore, consider the bundle $TX \oplus T^*X$. It can be endowed with the natural symmetric pairing

$$\langle\cdot|\cdot\cdot\rangle : \left(TX \oplus T^*X\right) \times \left(TX \oplus T^*X\right) \to \mathbb{R}, \quad \langle X + \xi | Y + \eta\rangle := \frac{1}{2}\left(\xi(Y) + \eta(X)\right).$$

Fixed a d-closed 3-form H on X (possibly, the zero form $H = 0$), define a suitable bracket for an integrability condition on $TX \oplus T^*X$ as follows. On the space $\mathcal{C}^\infty(X; TX \oplus T^*X)$ of smooth sections of $TX \oplus T^*X$ over X, define the *H-twisted Courant bracket* as

$$[\cdot, \cdot\cdot]_H : \mathcal{C}^\infty\left(X; TX \oplus T^*X\right) \times \mathcal{C}^\infty\left(X; TX \oplus T^*X\right) \to \mathcal{C}^\infty\left(X; TX \oplus T^*X\right),$$

$$[X + \xi, Y + \eta]_H := [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}\,\mathrm{d}\left(\iota_X\eta - \iota_Y\xi\right) + \iota_Y\iota_X H$$

(where $\iota_X \in \mathrm{End}^{-1}\left(\wedge^\bullet X\right)$ denotes the interior product with $X \in \mathcal{C}^\infty(X; TX)$ and $\mathcal{L}_X := [\iota_X, \mathrm{d}] \in \mathrm{End}^0\left(\wedge^\bullet X\right)$ denotes the Lie derivative along $X \in \mathcal{C}^\infty(X; TX)$);

the H-twisted Courant bracket can be seen also as a derived bracket induced by the H-twisted differential $\mathrm{d}_H := \mathrm{d} + H \wedge \cdot$, see [Gua04a, Sect. 3.2], [Gua11, Sect. 2].

Consider also the *Clifford action*[4] of $TX \oplus T^*X$ on the space of differential forms with respect to the natural pairing $\langle \cdot \mid \cdot\cdot \rangle$,

$$\mathrm{Cliff}\left(TX \oplus T^*X,\, \langle \cdot \mid \cdot\cdot \rangle\right) \times \wedge^\bullet X \to \wedge^{\bullet-1} X \oplus \wedge^{\bullet+1} X\ , \quad (X+\xi)\cdot\varphi = \iota_X\varphi + \xi\wedge\varphi\ ,$$

and its bi-$\mathbb{C}$-linear extension $\mathrm{Cliff}\left((TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}\right) \times \left(\wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C}\right) \to \left(\wedge^{\bullet-1} X \otimes_{\mathbb{R}} \mathbb{C}\right) \oplus \left(\wedge^{\bullet+1} X \otimes_{\mathbb{R}} \mathbb{C}\right)$.

We give now several equivalent definitions of generalized complex structure.

Definition 1.3 ([Gua04a, Definitions 4.14, 4.18], [Gua11, Definition 3.1]). Let X be a differentiable manifold and H be a d-closed 3-form on X. An *H-twisted generalized complex structure* is an endomorphism $\mathcal{J} \in \mathrm{End}\,(TX \oplus T^*X)$ such that

- $\mathcal{J}^2 = -\mathrm{id}_{TX\oplus T^*X}$, and
- $\mathcal{J}$ is orthogonal with respect to $\langle \cdot \mid \cdot\cdot \rangle$, and
- the Nijenhuis tensor

$$\begin{aligned}\mathrm{Nij}_{\mathcal{J},H} &:= -[\mathcal{J}\,\cdot,\, \mathcal{J}\,\cdot\cdot]_H + \mathcal{J}\,[\mathcal{J}\,\cdot,\, \cdot\cdot]_H + \mathcal{J}\,[\cdot,\, \mathcal{J}\,\cdot\cdot]_H + \mathcal{J}\,[\cdot,\, \cdot\cdot]_H \\ &\in \left(TX \oplus T^*X\right) \otimes_{\mathbb{R}} \left(TX \oplus T^*X\right) \otimes_{\mathbb{R}} \left(TX \oplus T^*X\right)^*\end{aligned}$$

  of $\mathcal{J}$ with respect to the H-twisted Courant bracket vanishes identically.

Equivalently, by setting $L := L_{\mathcal{J}}$ the i-eigen-bundle of the $\mathbb{C}$-linear extension of $\mathcal{J}$ to $(TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$, one gets the following definition.

Proposition 1.1 ([Gua04a, Proposition 4.3]). *Let X be a differentiable manifold and H be a* d*-closed* 3*-form on X. A generalized complex structure on X is identified by a sub-bundle L of $(TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$ such that*

- *L is maximal isotropic with respect to $\langle \cdot \mid \cdot\cdot \rangle$, and*
- *L is involutive with respect to the H-twisted Courant bracket, and*
- *$L \cap \bar{L} = \{0\}$.*

Equivalently, by choosing a complex form ρ whose Clifford annihilator

$$L_\rho \ :=\ \left\{v \in \left(TX \oplus T^*X\right) \otimes_{\mathbb{R}} \mathbb{C} \ :\ v \cdot \rho = 0\right\}$$

is the i-eigen-bundle $L_{\mathcal{J}}$ of the $\mathbb{C}$-linear extension of $\mathcal{J}$ to $(TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C})$, one gets the following definition.

[4] We recall that the Clifford algebra associated to $TX \oplus T^*X$ and $\langle \cdot \mid \cdot\cdot \rangle$ is

$$\mathrm{Cliff}\,(TX \oplus T^*X,\, \langle \cdot \mid \cdot\cdot \rangle) = \left(\bigoplus_{k\in\mathbb{Z}} \bigotimes_{j=1}^{k} (TX \oplus T^*X)\right) \Big/ \{v \otimes_{\mathbb{R}} v - \langle v \mid v \rangle : v \in TX \oplus T^*X\}\ .$$

Proposition 1.2 ([Gua04a, Theorem 4.8]). *Let X be a differentiable manifold and H be a* d*-closed* 3*-form on X. A generalized complex structure on X is identified by a sub-bundle $U := U_{\mathcal{J}}$ of complex rank* 1 *of $\wedge^{\bullet} X \otimes_{\mathbb{R}} \mathbb{C}$ being locally generated by a form $\rho = \exp(B + \mathrm{i}\,\omega) \wedge \Omega$, where $B \in \wedge^2 X$, and $\omega \in \wedge^2 X$, and $\Omega = \theta^1 \wedge \cdots \wedge \theta^k \in \wedge^k X \otimes_{\mathbb{R}} \mathbb{C}$ with $\left\{\theta^1, \ldots, \theta^k\right\}$ a set of linearly independent complex* 1*-forms, such that*

- $\Omega \wedge \bar{\Omega} \wedge \omega^{n-k} \neq 0$*, and*
- *there exists $v \in (TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$ such that $\mathrm{d}_H\, \rho = v \cdot \rho$, where $\mathrm{d}_H := \mathrm{d} + H \wedge \cdot$.*

According to [Gua04a, Sect. 4.1], [Gua11, Definition 3.7], the bundle $U_{\mathcal{J}}$ as in Proposition 1.2 is called the *canonical bundle* of the generalized complex structure $\mathcal{J}$.

By definition, the *type* of a generalized complex structure $\mathcal{J}$ on X, [Gua04a, Sect. 4.3], [Gua11, Definition 3.5], is the upper-semi-continuous function

$$\mathrm{type}\,(\mathcal{J}) \;:=\; \frac{1}{2}\,\dim_{\mathbb{R}}\left(T^*X \cap \mathcal{J}T^*X\right)$$

on X, equivalently, [Gua11, Definition 1.1], the degree of the form Ω.

A point x of a generalized complex manifold is called a *regular point* if the type of the generalized complex structure is locally constant at x.

(See Remark 1.10 for the notion of H-twisted generalized Kähler structure.)

A generalized complex structure $\mathcal{J}$ on X induces a $\mathbb{Z}$-graduation on the space of complex differential forms on X, [Gua04a, Sect. 4.4], [Gua11, Proposition 3.8]. Namely, define, for $k \in \mathbb{Z}$,

$$U^k \;:=\; U^k_{\mathcal{J}} \;:=\; \wedge^{n-k} \bar{L}_{\mathcal{J}} \cdot U_{\mathcal{J}} \;\subseteq\; \wedge^{\bullet} X \otimes_{\mathbb{R}} \mathbb{C}\,,$$

where $L_{\mathcal{J}}$ is the i-eigenspace of the $\mathbb{C}$-linear extension of $\mathcal{J}$ to $(TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$ and $U^n_{\mathcal{J}} := U_{\mathcal{J}}$ is the canonical bundle of $\mathcal{J}$.

For a $\langle \cdot \mid \cdot\cdot \rangle$-orthogonal endomorphism $\mathcal{J} \in \mathrm{End}\,(TX \oplus T^*X)$ satisfying $\mathcal{J}^2 = -\,\mathrm{id}_{TX \oplus T^*X}$, the $\mathbb{Z}$-graduation $U^{\bullet}_{\mathcal{J}}$ still makes sense, and the condition $\mathrm{Nij}_{\mathcal{J},H} = 0$ turns out to be equivalent, [Gua04a, Theorem 4.3], [Gua11, Theorem 3.14], to

$$\mathrm{d}_H \colon U^{\bullet}_{\mathcal{J}} \to U^{\bullet+1}_{\mathcal{J}} \oplus U^{\bullet-1}_{\mathcal{J}}\;.$$

Therefore, on a compact differentiable manifold endowed with a generalized complex structure $\mathcal{J}$, one has, [Gua04a, Sect. 4.4], [Gua11, Sect. 3],

$$\mathrm{d}_H = \partial_{\mathcal{J},H} + \overline{\partial}_{\mathcal{J},H} \quad \text{where} \quad \partial_{\mathcal{J},H} \colon U^{\bullet}_{\mathcal{J}} \to U^{\bullet+1}_{\mathcal{J}} \quad \text{and} \quad \overline{\partial}_{\mathcal{J},H} \colon U^{\bullet}_{\mathcal{J}} \to U^{\bullet-1}_{\mathcal{J}}\;.$$

Define also, [Gua04a, p. 52], [Gua11, Remark at page 97],

$$\mathrm{d}^{\mathcal{J}}_H \;:=\; -\mathrm{i}\left(\partial_{\mathcal{J},H} - \overline{\partial}_{\mathcal{J},H}\right) \colon U^{\bullet}_{\mathcal{J}} \to U^{\bullet+1}_{\mathcal{J}} \oplus U^{\bullet-1}_{\mathcal{J}}\;.$$

1.3.2 Cohomological Aspects of Generalized Complex Geometry

Let X be a compact complex manifold endowed with an H-twisted generalized complex structure.

By considering the decomposition $\wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{k\in\mathbb{Z}} U^k_{\mathcal{J}}$ and the operators $\mathrm{d}_H \colon U^\bullet_{\mathcal{J}} \to U^{\bullet+1}_{\mathcal{J}} \oplus U^{\bullet-1}_{\mathcal{J}}$, and $\partial_{\mathcal{J},H} \colon U^\bullet_{\mathcal{J}} \to U^{\bullet+1}_{\mathcal{J}}$ and $\overline{\partial}_{\mathcal{J},H} \colon U^\bullet_{\mathcal{J}} \to U^{\bullet-1}_{\mathcal{J}}$, one can study the following cohomologies:

$$GH_{dR_H}(X) := \frac{\ker \mathrm{d}_H}{\mathrm{im}\, \mathrm{d}_H} ,$$

and

$$GH^\bullet_{\overline{\partial}_{\mathcal{J},H}}(X) := \frac{\ker \overline{\partial}_{\mathcal{J},H}}{\mathrm{im}\, \overline{\partial}_{\mathcal{J},H}} \qquad \text{and} \qquad GH^\bullet_{\partial_{\mathcal{J},H}}(X) := \frac{\partial_{\mathcal{J},H}}{\partial_{\mathcal{J},H}} ,$$

and

$$GH^\bullet_{BC_{\mathcal{J},H}}(X) := \frac{\ker \partial_{\mathcal{J},H} \cap \ker \overline{\partial}_{\mathcal{J},H}}{\mathrm{im}\, \partial_{\mathcal{J},H}\overline{\partial}_{\mathcal{J},H}} \qquad \text{and}$$

$$GH^\bullet_{A_{\mathcal{J},H}}(X) := \frac{\ker \partial_{\mathcal{J},H}\overline{\partial}_{\mathcal{J},H}}{\mathrm{im}\, \partial_{\mathcal{J},H} + \mathrm{im}\, \overline{\partial}_{\mathcal{J},H}} .$$

Note that, for $H = 0$, one has $GH_{dR_0}(X) = \bigoplus_{k\in\mathbb{Z}} H^k_{dR}(X;\mathbb{C})$.

By [Gua04a, Proposition 5.1], [Gua11, Proposition 3.15], it follows that $\dim_{\mathbb{C}} GH^\bullet_{\partial_{\mathcal{J},H}}(X) < +\infty$ and $\dim_{\mathbb{C}} GH^\bullet_{\overline{\partial}_{\mathcal{J},H}}(X) < +\infty$.

By abuse of notation, one says that X *satisfies the* $\partial_{\mathcal{J},H}\overline{\partial}_{\mathcal{J},H}$*-Lemma* if $\left(U^\bullet_{\mathcal{J}},\, \partial_{\mathcal{J},H},\, \overline{\partial}_{\mathcal{J},H}\right)$ satisfies the $\partial_{\mathcal{J},H}\overline{\partial}_{\mathcal{J},H}$-Lemma, and one says that X *satisfies the* $\mathrm{d}_H\, \mathrm{d}^{\mathcal{J}}_H$*-Lemma* if $\left(U^\bullet_{\mathcal{J}},\, \mathrm{d}_H,\, \mathrm{d}^{\mathcal{J}}_H\right)$ satisfies the $\mathrm{d}_H\, \mathrm{d}^{\mathcal{J}}_H$-Lemma. Actually, it turns out that X satisfies the $\mathrm{d}_H\, \mathrm{d}^{\mathcal{J}}_H$-Lemma if and only if X satisfies the $\partial_{\mathcal{J},H}\overline{\partial}_{\mathcal{J},H}$-Lemma, [Cav06, Remark at page 129]: indeed, note that $\ker \partial_{\mathcal{J},H}\overline{\partial}_{\mathcal{J},H} = \ker \mathrm{d}_H\, \mathrm{d}^{\mathcal{J}}_H$, and $\ker \partial_{\mathcal{J},H} \cap \ker \overline{\partial}_{\mathcal{J},H} = \ker \mathrm{d}_H \cap \ker \mathrm{d}^{\mathcal{J}}_H$, and $\mathrm{im}\, \partial_{\mathcal{J},H} + \mathrm{im}\, \overline{\partial}_{\mathcal{J},H} = \mathrm{im}\, \mathrm{d}_H + \mathrm{im}\, \mathrm{d}^{\mathcal{J}}_H$.

Moreover, the following result by G.R. Cavalcanti holds.

Theorem 1.16 ([Cav05, Theorem 4.2], [Cav06, Theorem 4.1, Corollary 2]). *A manifold X endowed with an H-twisted generalized complex structure $\mathcal{J}$ satisfies the $\mathrm{d}_H\, \mathrm{d}^{\mathcal{J}}_H$-Lemma if and only if $\left(\ker \mathrm{d}^{\mathcal{J}}_H,\, \mathrm{d}_H\right) \hookrightarrow \left(U^\bullet_{\mathcal{J}},\, \mathrm{d}_H\right)$ is a quasi-isomorphism of differential $\mathbb{Z}$-graded $\mathbb{C}$-vector spaces. In this case, it follows that the splitting $\wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{k\in\mathbb{Z}} U^k_{\mathcal{J}}$ gives rise to a decomposition in cohomology.*

An application of [DGMS75, Propositions 5.17, 5.21] gives the following result.

Theorem 1.17 ([Cav05, Theorem 4.4], [Cav06, Theorem 5.1]). *A manifold X endowed with an H-twisted generalized complex structure $\mathcal{J}$ satisfies the $\mathrm{d}_H\,\mathrm{d}_H^{\mathcal{J}}$-lemma if and only if the canonical spectral sequence*[5] *degenerates at the first level and the decomposition of complex forms into sub-bundles $U^k_{\mathcal{J}}$, varying $k \in \mathbb{Z}$, induces a decomposition in cohomology.*

(See Remark 1.10 for $\mathrm{d}_H\,\mathrm{d}_H^{\mathcal{J}}$-Lemma on H-twisted generalized Kähler manifolds.)

1.3.3 Complex Structures and Symplectic Structures in Generalized Complex Geometry

In view of the following generalized Darboux theorem by M. Gualtieri, symplectic structures and complex structures provide the basic examples of generalized complex structures.

Theorem 1.18 ([Gua04a, Theorem 4.35], [Gua11, Theorem 3.6]). *For any regular point of a $2n$-dimensional generalized complex manifold with type equal to k, there is an open neighbourhood endowed with a set of local coordinates such that the generalized complex structure is a B-field transform of the standard generalized complex structure of $\mathbb{C}^k \times \mathbb{R}^{2n-2k}$.*

Hence, in the following examples, we recall the standard generalized complex structure of constant type n (that is, locally equivalent to the standard complex structure of $\mathbb{C}^n$), the generalized complex structure of constant type 0 (that is, locally equivalent to the standard symplectic structure of $\mathbb{R}^{2n}$), and the B-field transform of a generalized complex structure, on a $2n$-dimensional manifold. See also [Gua04a, Example 4.12].

Example 1.1 (Generalized Complex Structures of Type n, [Gua04a, Examples 4.11 and 4.25]). Let X be a compact $2n$-dimensional manifold endowed with a complex structure $J \in \mathrm{End}(TX)$. Consider the (0-twisted) generalized complex structure

$$\mathcal{J}_J \;:=\; \left(\begin{array}{c|c} -J & 0 \\ \hline 0 & J^* \end{array}\right) \;\in\; \mathrm{End}\left(TX \oplus T^*X\right) \;,$$

where $J^* \in \mathrm{End}(T^*X)$ denotes the dual endomorphism of $J \in \mathrm{End}(TX)$.

[5] We recall that, given a manifold X endowed with a generalized complex structure $\mathcal{J}$, the *canonical spectral sequence* is the spectral sequence being naturally associated to the double complex obtained from $\left(U^\bullet_{\mathcal{J}},\, \partial_{\mathcal{J},H},\, \overline{\partial}_{\mathcal{J},H}\right)$: more precisely, to the double complex $\left(U^{\bullet_1-\bullet_2}_{\mathcal{J}_J} \otimes_{\mathbb{C}} \mathbb{C}\beta^{\bullet_2},\, \partial_{\mathcal{J}_J} \otimes_{\mathbb{R}} \mathrm{id},\, \overline{\partial}_{\mathcal{J}_J} \otimes_{\mathbb{R}} \beta\right)$, where $\{\beta^m \;:\; m \in \mathbb{Z}\}$ is an infinite cyclic multiplicative group generated by some β, [Cav05, Sect. 4.2], see [Bry88, Sect. 1.3], [Goo85, Sect. II.2], [Con85, Sect. II]; (compare also [AT13a, Sect. 1]).

The i-eigenspace of the $\mathbb{C}$-linear extension of $\mathcal{J}_J$ to $(TX \oplus T^*X) \otimes_{\mathbb{C}} \mathbb{C}$ is

$$L_{\mathcal{J}_J} \;=\; T_J^{0,1}X \oplus \left(T_J^{1,0}X\right)^* \;.$$

The canonical bundle is

$$U^n_{\mathcal{J}_J} \;=\; \wedge_J^{n,0} X \;.$$

Hence, one gets that, [Gua04a, Example 4.25],

$$U^\bullet_{\mathcal{J}_J} \;=\; \bigoplus_{p-q=\bullet} \wedge_J^{p,q} X \;, \qquad \text{and } \partial_{\mathcal{J}_J} \;=\; \partial_J \quad \text{and} \quad \overline{\partial}_{\mathcal{J}_J} \;=\; \overline{\partial}_J \;;$$

note that $\mathrm{d}^{\mathcal{J}_J}$ is the operator $\mathrm{d}^c_J := -\mathrm{i}(\partial - \overline{\partial})$, [Gua04a, Remark 4.26].

Take an infinite cyclic multiplicative group $\{\beta^m \;:\; m \in \mathbb{Z}\}$ generated by some β. The Hodge and Frölicher spectral sequence associated to the canonical double complex $\left(U^{\bullet_1 - \bullet_2}_{\mathcal{J}_J} \otimes_{\mathbb{C}} \mathbb{C}\beta^{\bullet_2},\, \partial_{\mathcal{J}_J} \otimes_{\mathbb{R}} \mathrm{id},\, \overline{\partial}_{\mathcal{J}_J} \otimes_{\mathbb{R}} \beta\right)$ degenerates at the first level if and only if the Hodge and Frölicher spectral sequence associated to the double complex $\left(\wedge_J^{\bullet,\bullet} X,\, \partial_J,\, \overline{\partial}_J\right)$ does, [Cav05, Remark at page 76]. X satisfies the $\mathrm{d}\,\mathrm{d}^{\mathcal{J}_J}$-Lemma if and only if X satisfies the $\mathrm{d}\,\mathrm{d}^c_J$-Lemma.

For $\sharp \in \left\{\overline{\partial},\, \partial,\, BC,\, A\right\}$,

$$GH^\bullet_{\sharp_{\mathcal{J}_J}}(X) \;=\; \mathrm{Tot}^\bullet\, H^{\bullet,-\bullet}_{\sharp_J}(X) \;=\; \bigoplus_{p-q=\bullet} H^{p,q}_{\sharp_J}(X) \;.$$

Example 1.2 (Generalized Complex Structures of Type 0*, [Gua04a, Example 4.10]).* Let X be a compact $2n$-dimensional manifold endowed with a symplectic structure $\omega \;\in\; \wedge^2 X \;\simeq\; \mathrm{Hom}\,(TX; T^*X)$. Consider the (0-twisted) generalized complex structure

$$\mathcal{J}_\omega \;:=\; \left(\begin{array}{c|c} 0 & -\omega^{-1} \\ \hline \omega & 0 \end{array}\right) \;,$$

where $\omega^{-1} \in \mathrm{Hom}\,(T^*X; TX)$ denotes the inverse of $\omega \in \mathrm{Hom}\,(TX; T^*X)$.

The i-eigenspace of the $\mathbb{C}$-linear extension of $\mathcal{J}_\omega$ to $(TX \otimes_{\mathbb{R}} \mathbb{C}) \oplus (T^*X \otimes_{\mathbb{R}} \mathbb{C})$ is

$$L_{\mathcal{J}_\omega} \;=\; \{X - \mathrm{i}\,\omega\,(X, \cdot) \;:\; X \in TX \otimes_{\mathbb{R}} \mathbb{C}\} \;,$$

which has Clifford annihilator $\exp(\mathrm{i}\;\omega)$. The canonical bundle is

$$U^n_{\mathcal{J}_\omega} \;=\; \mathbb{C}\,\langle \exp\,(\mathrm{i}\;\omega)\rangle \;.$$

In particular, one gets that, [Cav06, Theorem 2.2],

$$U^{n-\bullet}_{\mathcal{J}_\omega} \;=\; \exp(\mathrm{i}\,\omega)\,\left(\exp\left(\frac{\Lambda}{2\,\mathrm{i}}\right)\left(\wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C}\right)\right) \;.$$

Note that, [Cav06, Sect. 2.2],

$$\mathrm{d}^{\mathcal{J}_\omega} \;=\; \mathrm{d}^{\Lambda} \;.$$

By considering the natural isomorphism

$$\varphi\colon \bigoplus_{k\in\mathbb{Z}} \wedge^k X \otimes_{\mathbb{R}} \mathbb{C} \ni \alpha \mapsto \exp(\mathrm{i}\,\omega)\,\left(\exp\left(\frac{\Lambda}{2\,\mathrm{i}}\right)\alpha\right) \in \bigoplus_{k\in\mathbb{Z}} \wedge^k X \otimes_{\mathbb{R}} \mathbb{C} \,,$$

one gets that, [Cav06, Corollary 1],

$$\varphi\left(\wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C}\right) \simeq U^{n-\bullet} \,,$$

and

$$\varphi\,\mathrm{d} \;=\; \overline{\partial}_{\mathcal{J}_\omega}\varphi \qquad \text{and} \qquad \varphi\,\mathrm{d}^{\mathcal{J}_\omega} \;=\; -2\,\mathrm{i}\,\partial_{\mathcal{J}_\omega}\varphi \;;$$

in particular,

$$GH^\bullet_{\overline{\partial}_{\mathcal{J}_\omega}}(X) \;\simeq\; H^{n-\bullet}_{dR}(X;\mathbb{C}) \;.$$

Example 1.3 (B-Transform, [Gua04a, Sect. 3.3]). Let X be a compact $2n$-dimensional manifold endowed with an H-twisted generalized complex structure $\mathcal{J}$, and let B be a d-closed 2-form. Consider the H-twisted generalized complex structure

$$\mathcal{J}^B \;:=\; \exp(-B)\;\mathcal{J}\;\exp B \qquad \text{where} \qquad \exp B \;=\; \left(\begin{array}{c|c} \mathrm{id}_{TX} & 0 \\ \hline B & \mathrm{id}_{T^*X} \end{array}\right) \;.$$

The i-eigenspace of the $\mathbb{C}$-linear extension of $\mathcal{J}$ to $(TX \oplus T^*X)\otimes_{\mathbb{R}}\mathbb{C}$ is, [Cav05, Example 2.3],

$$L_{\mathcal{J}^B} \;=\; \left\{X + \xi - \iota_X B \;:\; X + \xi \in L_{\mathcal{J}}\right\} \;,$$

and the canonical bundle is, [Cav05, Example 2.6],

$$U^n_{\mathcal{J}^B} \;=\; \exp B \wedge U^n_{\mathcal{J}} \;.$$

Hence one gets that, [Cav06, Sect. 2.3],

$$U^{\bullet}_{\mathcal{J}^B} = \exp B \wedge U^{\bullet}_{\mathcal{J}} \,, \qquad \text{and that} \qquad \partial_{\mathcal{J}^B} \;=\; \exp(-B)\; \partial_{\mathcal{J}} \; \exp B \quad \text{and}$$

$$\overline{\partial}_{\mathcal{J}^B} = \exp(-B)\; \overline{\partial}_{\mathcal{J}} \; \exp B \;.$$

In particular, $\mathcal{J}$ satisfies the $\partial_{\mathcal{J}}\overline{\partial}_{\mathcal{J}}$-Lemma if and only if $\mathcal{J}^B$ satisfies the $\partial_{\mathcal{J}^B}\overline{\partial}_{\mathcal{J}^B}$-Lemma.

1.4 Kähler Geometry

Note that, given a manifold X endowed with a symplectic form ω, there is always a (possibly non-integrable) almost-complex structure J on X such that $g := \omega(\cdot,\, J\cdot\cdot)$ is a Hermitian metric on X with ω as the associated $(1,1)$-form, see, e.g., [CdS01, Corollary 12.7] (in fact, the set of such almost-complex structures is contractible, see, e.g., [AL94, Corollary II.1.1.7], [CdS01, Proposition 13.1]; see also [Gro85, Corollary 2.3.C$'_2$], which proves that the space of almost-complex structures on X tamed by a given 2-form on X is contractible). Instead, the datum of an integrable almost-complex structure with the above property yields a Kähler structure on X. The notion of Kähler manifold has been studied for the first time by J.A. Schouten and D. van Dantzig [SvD30], see also [Sch29], and by E. Kähler [Käh33], and the terminology has been fixed by A. Weil [Wei58].

Kähler structures can be defined in different ways, according to the point of view which is stressed, Sect. 1.4.1. The presence of three different structures (complex, symplectic, and Riemannian) allows to make use of the tools available for any of them; in addition, the relations between such structures make available further tools, which yield many interesting results on Hodge theory, Sect. 1.4.2. Finally, we will study a cohomological property of compact Kähler manifolds, namely, the $\partial\overline{\partial}$-Lemma, Sect. 1.4.3: other than being a very useful tool in Kähler geometry (compare, e.g., its role in S.-T. Yau's proof [Yau77, Yau78] of E. Calabi's conjecture [Cal57]), it provides obstructions to the existence of Kähler structures on differentiable manifolds, by means of the notion of formality introduced by D.P. Sullivan, [Sul77, Sect. 12].

1.4.1 Kähler Metrics

Let X be a compact complex manifold of complex dimension n, and denote by J its natural integrable almost-complex structure.

Definition 1.4 ([SvD30, Sch29, Käh33, Wei58]). Let X be a compact complex manifold of complex dimension n and J be its natural integrable almost-complex structure. A *Kähler metric* on X is a Hermitian metric g such that the associated $(1,1)$-form $\omega := g(J\cdot, \cdot\cdot)$ is d-closed (that is, ω is a symplectic form on X).

Remark 1.8. Let X be a complex manifold endowed with a Kähler metric g, and denote the associated $(1,1)$-form to g by ω. By the Poincaré lemma, see, e.g., [Dem12, I.1.22, Theorem I.2.24], and the Dolbeault and Grothendieck lemma, see, e.g., [Dem12, I.3.29], the property that $\mathrm{d}\omega = 0$ is equivalent to ask that, for every $x \in X$, there exist an open neighbourhood U in X with $x \in U$ and a smooth function $u \in \mathcal{C}^\infty(U;\mathbb{R})$ such that $\omega \stackrel{\text{loc}}{=} \mathrm{i}\,\partial\overline{\partial}u$ in U, that is, the metric has a local potential, [Käh33] (see, e.g., [Mor07, Proposition 8.8]).

Remark 1.9. For every $n \in \mathbb{N}$, the complex projective space $\mathbb{CP}^n$ admits a Kähler metric, the so-called *Fubini and Study metric*, [Fub04, Stu05], which is induced by the fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n$; more precisely, by using the homogeneous coordinates $[z_0 : \cdots : z_n]$, one has that the associated $(1,1)$-form ω_{FS} to the Fubini and Study metric is

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\,\partial\overline{\partial}\log\left(\sum_{\ell=0}^{n}\left|z^\ell\right|^2\right).$$

It follows that complex projective manifolds provide examples of Kähler manifolds. Conversely, by the Kodaira embedding theorem [Kod54, Theorem 4], if X is a compact complex manifold endowed with a Kähler metric ω such that $[\omega] \in H^2_{dR}(X;\mathbb{R}) \cap \mathrm{im}\left(H^2(X;\mathbb{Z}) \to H^2_{dR}(X;\mathbb{R})\right)$, then there exists a complex-analytic embedding of X into a complex projective space $\mathbb{CP}^N$ for some $N \in \mathbb{N}$. In a sense, this suggest that projective manifolds are to Kähler manifolds as $\mathbb{Q}$ is to $\mathbb{R}$. Hence, it is natural to ask if every compact Kähler manifold is a deformation of a projective manifold (which is known as the *Kodaira problem*). Since Riemann surfaces are projective, this is trivially true in complex dimension 1. Furthermore, K. Kodaira proved in [Kod63, Theorem 16.1] that every compact Kähler surface is a deformation of an algebraic surface, as conjectured by W. Hodge; another proof, which does not make use of the classification of elliptic surfaces, has been given by N. Buchdahl, [Buc08, Theorem]. In higher dimension, a negative answer to the Kodaira problem has been given by C. Voisin, who constructed examples of compact Kähler manifolds, of any complex dimension greater than or equal to 4, which do not have the homotopy type of a complex projective manifold, [Voi04, Theorem 2] (indeed, recall that, by Ehresmann's theorem, if two compact complex manifolds can be obtained by deformation, then they are homeomorphic, and hence they have the same homotopy type). The examples in [Voi04] being, by construction, bimeromorphic to manifolds that can be deformed to projective manifolds, one could ask (as done by N. Buchdahl, F. Campana, S.-T. Yau) whether, in higher dimension, a birational version of the Kodaira problem may hold true; in [Voi06, Theorem 3], C. Voisin provided a negative answer to the birational version of the Kodaira problem, proving that, in any even complex dimension greater that or equal to 10, there exist compact Kähler manifolds X such that, for any compact

Kähler manifold X' bimeromorphic to X, X' does not have the homotopy type of a projective complex manifold. Positive results under an additional semi-positivity or semi-negativity condition on the canonical bundle have been provided by J. Cao in [Cao12].

In the definition of a Kähler manifold, three different structures are involved: a complex structure, a symplectic structure, and a metric structure. Therefore, changing the point of view allows to give several equivalent definitions of Kähler structure (see, e.g., [Bal06, Theorem 4.17]): we review here two of these characterizations.

Firstly, it is straightforward to prove that a Hermitian metric g on a compact complex manifold X is a Kähler metric if and only if, for every point $x \in X$, there exists a holomorphic coordinate chart $\left(U,\ \left\{z^j\right\}_{j\in\{1,\ldots,n\}}\right)$, with $x \in U$, such that

$$g \;=\; \sum_{\alpha,\beta=1}^{n} \left(\delta_{\alpha\beta} + \mathrm{o}\left(|z|\right)\right) \mathrm{d}\, z^{\alpha} \odot \mathrm{d}\, \bar{z}^{\beta} \qquad \text{at } x \;,$$

that is, g *osculates to order* 2 the standard Hermitian metric of $\mathbb{C}^n$ (see, e.g., [GH94, pp. 107–108], [Huy05, Proposition 1.3.12], [Mor07, Theorem 11.6]).

As regards the second characterization, we recall that, on a compact complex manifold X endowed with a Hermitian metric g, there is a unique connection ∇^C such that

(i) $\nabla^C g = 0$,
(ii) $\nabla^C J = 0$, and
(iii) $\pi_{\wedge^{0,1}X} \nabla^C \lfloor_{\mathcal{C}^\infty(X;\mathbb{C})} = \overline{\partial} \lfloor_{\mathcal{C}^\infty(X;\mathbb{C})}$;

such a connection is called the *Chern connection* of X (see, e.g., [Huy05, Proposition 4.2.14], [Bal06, Theorem 3.18], [Mor07, Theorem 10.3], [Dem12, Sect. V.12]). Let g be a Hermitian metric on a compact complex manifold X, and set $\omega := g(J\cdot,\ \cdot\cdot)$ its associated $(1,1)$-form, where J is the natural integrable almost-complex structure on X; consider the Levi Civita connection ∇^{LC}. One can prove that, for every $x, y, z \in \mathcal{C}^\infty\left(X; TX\right)$,

$$\mathrm{d}\,\omega(x,y,z) \;=\; g\left(\left(\nabla^{LC}_x J\right) y,\ z\right) + g\left(\left(\nabla^{LC}_y J\right) z,\ x\right) + g\left(\left(\nabla^{LC}_z J\right) x,\ y\right) \;,$$

and

$$2\, g\left(\left(\nabla^{LC}_x J\right) y,\ z\right) \;=\; \mathrm{d}\,\omega\left(x,\ y,\ z\right) - \mathrm{d}\,\omega\left(x, Jy,\ Jz\right) - g\left(\mathrm{Nij}_J\left(y,\ Jz\right),\ x\right) \;;$$

(see, e.g., [Bal06, Theorem 4.16], [Tia00, Proposition 1.5]); in particular, it follows that g is a Kähler metric if and only if $\nabla^{LC} J = 0$ if and only if the Chern connection is the Levi Civita connection (see, e.g., [Bal06, Theorem 4.17], [Mor07, Proposition 11.8]).

Remark 1.10. We recall that, given a d-closed 3-form H on a manifold X, an H*-twisted generalized Kähler structure* on X is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of H-twisted generalized complex structures on X such that *(i)* $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, and *(ii)* the symmetric pairing $\langle \mathcal{J}_1 \cdot, \mathcal{J}_2 \cdot\cdot \rangle$ is positive definite. Generalized Kähler geometry is equivalent to a bi-Hermitian geometry with torsion, [Gua04b, Theorem 2.18].

We recall that a compact manifold X endowed with an H-twisted generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ satisfies both the $\mathrm{d}_H\, \mathrm{d}_H^{\mathcal{J}_1}$-Lemma and the $\mathrm{d}_H\, \mathrm{d}_H^{\mathcal{J}_2}$-Lemma, [Gua04b, Corollary 4.2].

Any Kähler structure provide an example of a 0-twisted generalized Kähler structure. A left-invariant non-trivial twisted generalized Kähler structure on a (non-completely solvable) solvmanifold (which is the total space of a $\mathbb{T}^2$-bundle over the Inoue surface, [FT09, Proposition 3.2]) has been constructed by A.M. Fino and A. Tomassini, [FT09, Theorem 3.5].

Furthermore, we note that A. Tomasiello proved in [Tom08, Sect. B] satisfying the $\mathrm{d}\,\mathrm{d}^{\mathcal{J}}$-Lemma is a stable property under small deformations.

1.4.2 Hodge Theory for Kähler Manifolds

The complex, symplectic, and metric structures being related on a Kähler manifold, one gets the following identities concerning the corresponding operators (see, e.g., [Huy05, Proposition 3.1.12]); see also [Hod35, Hod89]. (In [Dem86, Theorems 1.1 and 2.12], commutation relations on arbitrary Hermitian manifolds are provided; see also [Gri66], [Dem12, Sect. VI.6.2].)

Theorem 1.19 (Kähler Identities, [Wei58, Théorème II.1, II.2, Corollaire II.1]). *Let X be a compact Kähler manifold. Consider the differential operators ∂ and $\overline{\partial}$ associated to the complex structure, the symplectic operators L and Λ associated to the symplectic structure, and the Hodge-$*$-operator associated to the Hermitian metric. Then, these operators are related as follows:*

(i) $\left[\overline{\partial},\, L\right] = [\partial,\, L] = 0$ *and* $\left[\Lambda,\, \overline{\partial}^*\right] = [\Lambda,\, \partial^*] = 0$*;*

(ii) $\left[\overline{\partial}^*,\, L\right] = \mathrm{i}\,\partial$ *and* $[\partial^*,\, L] = -\mathrm{i}\,\overline{\partial}$*, and* $\left[\Lambda,\, \overline{\partial}\right] = -\mathrm{i}\,\partial^*$ *and* $[\Lambda,\, \partial] = \mathrm{i}\,\overline{\partial}^*$.

Therefore, considering the 2^{nd} order self-adjoint elliptic differential operators $\square := [\partial,\, \partial^*]$, $\overline{\square} := \left[\overline{\partial},\, \overline{\partial}^*\right]$, *and* $\Delta := [\mathrm{d},\, \mathrm{d}^*]$, *one gets that*

(iii) $\square = \overline{\square} = \frac{1}{2}\Delta$*, and Δ commutes with* $*,\, \partial,\, \overline{\partial},\, \partial^*,\, \overline{\partial}^*,\, L,\, \Lambda$.

The previous identities can be proven either using the $\mathfrak{sl}(2;\mathbb{C})$ representation $\langle L,\, \Lambda,\, H \rangle \to \mathrm{End}^\bullet\left(\wedge^\bullet X \otimes \mathbb{C}\right)$, or reducing to prove the corresponding identities on $\mathbb{C}^n$ with the standard Kähler structure (which are known as Y. Akizuki and S. Nakano's identities, [AN54, Sect. 3]) and hence using that every Kähler metric osculates to order 2 the standard Hermitian metric on $\mathbb{C}^n$.

As a consequence, one gets the following theorems, stating a decomposition of the de Rham cohomology of a Kähler manifold related to the complex, respectively symplectic, structure (see, e.g., [Huy05, Corollary 3.2.12], respectively [Huy05, Proposition 3.2.13]).

Theorem 1.20 (Hodge Decomposition Theorem, [Wei58, Théorème IV.3]). *Let X be a compact complex manifold endowed with a Kähler structure. Then there exist a decomposition*

$$H^{\bullet}_{dR}(X;\mathbb{C}) \simeq \bigoplus_{p+q=\bullet} H^{p,q}_{\overline{\partial}}(X) ,$$

and, for every $(p,q) \in \mathbb{N}^2$, an isomorphism

$$H^{p,q}_{\overline{\partial}}(X) \simeq \overline{H^{q,p}_{\overline{\partial}}(X)} .$$

Theorem 1.21 (Lefschetz Decomposition Theorem, [Wei58, Théorème IV.5]). *Let X be a compact complex manifold, of complex dimension n, endowed with a Kähler structure. Then there exist a decomposition*

$$H^{\bullet}_{dR}(X;\mathbb{C}) = \bigoplus_{r\in\mathbb{N}} L^r \left(\ker\left(\Lambda\colon H^{\bullet-2r}_{dR}(X;\mathbb{C}) \to H^{\bullet-2r-2}_{dR}(X;\mathbb{C})\right)\right) ,$$

and, for every $k \in \mathbb{N}$, an isomorphism

$$L^k\colon H^{n-k}_{dR}(X;\mathbb{C}) \stackrel{\simeq}{\to} H^{n+k}_{dR}(X;\mathbb{C}) .$$

1.4.3 $\partial\overline{\partial}$-Lemma and Formality for Compact Kähler Manifolds

The Hodge decomposition theorem and the Lefschetz decomposition theorem provide obstructions to the existence of Kähler structures on a compact complex manifold. In this section, we study another property of compact Kähler manifolds, namely, formality, which provides an obstruction to the existence of a Kähler structure on a compact (differentiable) manifold. Such a property turns out to be a consequence of the validity of the $\partial\overline{\partial}$-Lemma on compact complex manifolds.

Firstly, we need to recall some general notions regarding homotopy theory of differential algebras; we will then summarize some results concerning the homotopy type of Kähler manifolds: by the classical result by P. Deligne, Ph.A. Griffiths, J. Morgan, and D.P. Sullivan, [DGMS75, Main Theorem], the real homotopy type of a Kähler manifold X is a formal consequence of its cohomology ring $H^{\bullet}_{dR}(X;\mathbb{R})$.

We recall that a *differential graded algebra* (shortly, *dga*) over a field $\mathbb{K}$ is a graded $\mathbb{K}$-algebra $A^{\bullet}$ (where the structure of $\mathbb{K}$-algebra is induced by an inclusion $\mathbb{K} \subseteq A^0$) being graded-commutative[6] and endowed with a differential $\mathrm{d}\colon A^\bullet \to A^{\bullet+1}$ satisfying the graded-Leibniz rule.[7] A *morphism of differential graded algebras* $F\colon (A^\bullet, \mathrm{d}_{A^\bullet}) \to (B^\bullet, \mathrm{d}_{B^\bullet})$ is a morphism $A^\bullet \to B^\bullet$ of $\mathbb{K}$-algebras such that $F \circ \mathrm{d}_{A^\bullet} = \mathrm{d}_{B^\bullet} \circ F$.

Given a dga $(A^\bullet, \mathrm{d})$ over $\mathbb{K}$, the cohomology $H^\bullet(A^\bullet, \mathrm{d}) := \frac{\ker \mathrm{d}}{\operatorname{im} \mathrm{d}}$ endowed with the zero differential has a natural structure of dga over $\mathbb{K}$; furthermore, every morphism $F\colon (A^\bullet, \mathrm{d}_{A^\bullet}) \to (B^\bullet, \mathrm{d}_{B^\bullet})$ of dgas induces a morphism $F^*\colon (H^\bullet(A^\bullet, \mathrm{d}_{A^\bullet}), 0) \to (H^\bullet(B^\bullet, \mathrm{d}_{B^\bullet}), 0)$ of dgas in cohomology; a morphism $F\colon (A^\bullet, \mathrm{d}_{A^\bullet}) \to (B^\bullet, \mathrm{d}_{B^\bullet})$ of dgas is called a *quasi-isomorphism* (shortly, *qis*) if the corresponding morphism $F^*\colon (H^\bullet(A^\bullet, \mathrm{d}_{A^\bullet}), 0) \to (H^\bullet(B^\bullet, \mathrm{d}_{B^\bullet}), 0)$ is an isomorphism.

The de Rham complex $(\wedge^\bullet X, \mathrm{d})$ of a compact (differentiable) manifold X has a structure of dga over $\mathbb{R}$, whose cohomology is the dga $\left(H^\bullet_{dR}(X;\mathbb{R}), 0\right)$.

Given a dga $(A^\bullet, \mathrm{d}_{A^\bullet})$ over $\mathbb{K}$, the differential $\mathrm{d}_{A^\bullet}$ is called *decomposable* if

$$\mathrm{d}_{A^\bullet}\left(A^\bullet\right) \subseteq \left(\bigoplus_{k\in\mathbb{N}\setminus\{0\}} A^k\right) \cdot \left(\bigoplus_{k\in\mathbb{N}\setminus\{0\}} A^k\right) .$$

Given a dga $(A^\bullet, \mathrm{d}_{A^\bullet})$ over $\mathbb{K}$, an *elementary extension* of $(A^\bullet, \mathrm{d}_{A^\bullet})$ is a dga $(B^\bullet, \mathrm{d}_{B^\bullet})$ over $\mathbb{K}$ such that

(i) $B^\bullet = A^\bullet \otimes_{\mathbb{K}} \wedge^\bullet V_k$ for V_k a finite-dimensional $\mathbb{K}$-vector space and $k > 0$, where $\wedge^\bullet V_k$ is the free graded $\mathbb{K}$-algebra generated by V_k, the elements of V_k having degree k, and
(ii) $\mathrm{d}_{B^\bullet}\lfloor_{A^\bullet} = \mathrm{d}_{A^\bullet}$ and $\mathrm{d}_{B^\bullet}(V_k) \subseteq A^\bullet$.

A dga $(M^\bullet, \mathrm{d}_{M^\bullet})$ over $\mathbb{K}$ is called *minimal* if it can be written as an increasing union of sub-dgas,

$$(\mathbb{K}, 0) = \left(M^\bullet_0, \mathrm{d}_{M^\bullet_0}\right) \subset \left(M^\bullet_1, \mathrm{d}_{M^\bullet_1}\right) \subset \left(M^\bullet_2, \mathrm{d}_{M^\bullet_2}\right) \subseteq \cdots ,$$

$$\left(M^\bullet, \mathrm{d}_{M^\bullet}\right) = \bigcup_{j\in\mathbb{N}} \left(M^\bullet_j, \mathrm{d}_{M^\bullet_j}\right) ,$$

such that

(i) for any $j \in \mathbb{N}$, the dga $\left(M^\bullet_{j+1}, \mathrm{d}_{M^\bullet_{j+1}}\right)$ is an elementary extension of the dga $\left(M^\bullet_j, \mathrm{d}_{M^\bullet_j}\right)$, and
(ii) $\mathrm{d}_{M^\bullet}$ is decomposable.

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[6]We recall that a graded $\mathbb{K}$-algebra $A^\bullet$ is *graded-commutative* if, for every $x \in A^{\deg x}$ and $y \in A^{\deg y}$, it holds $x \cdot y = (-1)^{\deg x \cdot \deg y}\, y \cdot x$.

[7]We recall that a differential $\mathrm{d}\colon A^\bullet \to A^{\bullet+1}$ on a graded $\mathbb{K}$-algebra $A^\bullet$ satisfies the *graded-Leibniz rule* if, for every $x \in A^{\deg x}$ and $y \in A^{\deg y}$, it holds $\mathrm{d}(x \cdot y) = \mathrm{d}x \cdot y + (-1)^{\deg x}\, x \cdot \mathrm{d}y$.

A *minimal model* for a dga $(A^\bullet, d_{A^\bullet})$ over $\mathbb{K}$ is the datum of a minimal dga $(M^\bullet, d_{M^\bullet})$ over $\mathbb{K}$ and a quasi-isomorphism $\rho\colon (M^\bullet, d_{M^\bullet}) \stackrel{\text{qis}}{\to} (A^\bullet, d_{A^\bullet})$ of dgas.

Two dgas $(A^\bullet, d_{A^\bullet})$ and $(B^\bullet, d_{B^\bullet})$ over $\mathbb{K}$ are *equivalent* if there exist an integer $n \in \mathbb{N} \setminus \{0\}$, a family $\left\{\left(C_j^\bullet, d_{C_j^\bullet}\right)\right\}_{j \in \{0,\dots,2n\}}$ of dgas over $\mathbb{K}$ with $\left(C_0^\bullet, d_{C_0^\bullet}\right) = (A^\bullet, d_{A^\bullet})$ and $\left(C_{2n}^\bullet, d_{C_{2n}^\bullet}\right) = (B^\bullet, d_{B^\bullet})$, and a family

$$\begin{tikzcd} & \left(C_{2j+1}^\bullet, d_{C_{2j+1}^\bullet}\right) \arrow[dl, "\text{qis}"'] \arrow[dr, "\text{qis}"] & \\ \left(C_{2j}^\bullet, d_{C_{2j}^\bullet}\right) & & \left(C_{2j+2}^\bullet, d_{C_{2j+2}^\bullet}\right) \end{tikzcd}$$

of quasi-isomorphisms, varying $j \in \{0, \dots, n-1\}$. A dga $(A^\bullet, d_{A^\bullet})$ over $\mathbb{K}$ is called *formal* if it is equivalent to a dga $(B^\bullet, 0)$ over $\mathbb{K}$ with zero differential, that is, if it is equivalent to $(H^\bullet(A^\bullet, d_{A^\bullet}), 0)$.

A compact manifold X is called *formal* if its de Rham complex $(\wedge^\bullet X, d)$ is a formal dga over $\mathbb{R}$.

Let $(A^\bullet, d_{A^\bullet})$ be a dga over $\mathbb{K}$. Given

$$[\alpha_{12}] \in H^{\deg \alpha_{12}}\left(A^\bullet, d_{A^\bullet}\right), \qquad [\alpha_{23}] \in H^{\deg \alpha_{23}}\left(A^\bullet, d_{A^\bullet}\right), \qquad \text{and}$$
$$[\alpha_{34}] \in H^{\deg \alpha_{34}}\left(A^\bullet, d_{A^\bullet}\right)$$

such that

$$[\alpha_{12}] \cdot [\alpha_{23}] = 0 \qquad \text{and} \qquad [\alpha_{23}] \cdot [\alpha_{34}] = 0,$$

let $\alpha_{13} \in A^{\deg \alpha_{12} + \deg \alpha_{23} - 1}$ and $\alpha_{24} \in A^{\deg \alpha_{23} + \deg \alpha_{34} - 1}$ be such that

$$(-1)^{\deg \alpha_{12}} \alpha_{12} \cdot \alpha_{23} = d_{A^\bullet} \alpha_{13} \qquad \text{and} \qquad (-1)^{\deg \alpha_{23}} \alpha_{23} \cdot \alpha_{34} = d_{A^\bullet} \alpha_{24};$$

one can then define the *triple Massey product* $\langle [\alpha_{12}], [\alpha_{23}], [\alpha_{34}] \rangle$ as

$$\langle [\alpha_{12}], [\alpha_{23}], [\alpha_{34}] \rangle := \left[(-1)^{\deg \alpha_{12}} \alpha_{12} \cdot \alpha_{24} + (-1)^{\deg \alpha_{13}} \alpha_{13} \cdot \alpha_{34}\right]$$
$$\in \frac{H^{\deg \alpha_{12} + \deg \alpha_{23} + \deg \alpha_{34} - 1}(A^\bullet, d_{A^\bullet})}{H^{\deg \alpha_{12}}(A^\bullet, d_{A^\bullet}) \cdot H^{\deg \alpha_{23} + \deg \alpha_{34} - 1}(A^\bullet, d_{A^\bullet}) + H^{\deg \alpha_{34}}(A^\bullet, d_{A^\bullet}) \cdot H^{\deg \alpha_{12} + \deg \alpha_{23} - 1}(A^\bullet, d_{A^\bullet})}.$$

One can define the higher order Massey product by induction. Fixed $m \in \mathbb{N}$ such that $m \geq 4$, and given

$$[\alpha_{12}] \in H^{\deg \alpha_{12}}\left(A^\bullet, d_{A^\bullet}\right), \qquad \dots, \qquad [\alpha_{m,m+1}] \in H^{\deg \alpha_{m,m+1}}\left(A^\bullet, d_{A^\bullet}\right)$$

such that all the Massey products of order lower than or equal to $m-1$ vanish, let $\{\alpha_{rs}\}_{1\leq r<s\leq m+1} \subseteq A^\bullet$ be such that

$$\sum_{h<\ell<k} (-1)^{\deg \alpha_{h\ell}}\ \alpha_{h\ell}\cdot\alpha_{\ell k} \;=\; \mathrm{d}\,\alpha_{hk}\ ,$$

for any $h,k \in \{1,\ldots,m+1\}$ with $k-h<m$. Then define the m^{th} *order Massey product* as

$$\langle[\alpha_{12}]\,,\,\ldots\,,[\alpha_{m,m+1}]\rangle \;:=\; \left[\sum_{1<\ell<m+1} (-1)^{\deg \alpha_{1k}}\ \alpha_{1k}\cdot\alpha_{k,m+1}\right]$$

belonging to a quotient of $H^\bullet\left(A^\bullet,\ \mathrm{d}_{A^\bullet}\right)$.

As a direct consequence of the definitions, the Massey products (of any order) on a formal dga are zero.

Now, let X be a compact manifold endowed with a Kähler structure.

The Kähler identities allow to prove the following result, known as $\partial\overline{\partial}$*-Lemma* (see, e.g., [Huy05, Corollary 3.2.10]), which, in a sense, summarizes many of the cohomological properties of compact Kähler manifolds.

Theorem 1.22 ($\partial\overline{\partial}$-Lemma for Compact Kähler Manifolds, [DGMS75, Lemma 5.11]). *Let X be a compact Kähler manifold. Then every ∂-closed, $\overline{\partial}$-closed,* d*-exact form is also $\partial\overline{\partial}$-exact.*

Proof. We recall the idea of the proof, as can be found, e.g., in [Huy05, Corollary 3.2.10], see also [DGMS75, pp. 266–267].

Let $\alpha \in \wedge^{p,q}X$ be a ∂-closed $\overline{\partial}$-closed d-exact form on X. In particular, by the Hodge decomposition theorem for the de Rham cohomology (see, e.g., [War83, 6.8]), the formα is orthogonal to the space of Δ-harmonic forms. Note that, by the Kähler identities, the space of Δ-harmonic forms coincide with the space of $\square$-harmonic forms and with the space of $\overline{\square}$-harmonic forms. Since α is ∂-closed and orthogonal to the space of $\square$-harmonic forms, the conjugate version of the Hodge decomposition theorem for the Dolbeault cohomology, [Hod89], yields $\alpha=\partial\gamma$ for some $\gamma \in \wedge^{p-1,q}X$. By applying the Hodge decomposition theorem for the Dolbeault cohomology, [Hod89], to the form γ, one gets a $\overline{\square}$-harmonic form h_γ, a form $\beta \in \wedge^{p-1,q-1}X$, and a form $\eta \in \wedge^{p-1,q+1}X$ such that $\gamma = h_\gamma + \overline{\partial}\beta + \overline{\partial}^*\eta$. By the Kähler identities, one has $\left[\partial, \overline{\partial}^*\right] = 0$ and that h_γ is also $\square$-harmonic. Hence $\alpha = \partial\gamma = \partial\overline{\partial}\beta - \overline{\partial}^*\partial\eta$. It suffices to prove that $\overline{\partial}^*\partial\eta = 0$. Indeed, since α is $\overline{\partial}$-closed, one has $\overline{\partial}\overline{\partial}^*\partial\eta = 0$, and hence $\left\|\overline{\partial}^*\partial\eta\right\|^2 = \left\langle\overline{\partial}^*\partial\eta,\, \overline{\partial}^*\partial\eta\right\rangle = \left\langle\overline{\partial}\overline{\partial}^*\partial\eta,\, \partial\eta\right\rangle = 0$. Hence $\alpha = \partial\overline{\partial}\beta$ is $\partial\overline{\partial}$-exact. □

Using the differential operator $\mathrm{d}^c := J^{-1}\,\mathrm{d}\,J = -\mathrm{i}\left(\partial - \overline{\partial}\right)$ (where J is the integrable almost-complex structure naturally associated to the structure of complex manifold on X), and noting that $\ker \partial \cap \ker \overline{\partial} = \ker \mathrm{d} \cap \ker \mathrm{d}^c$ and $\operatorname{im} \partial\overline{\partial} = \operatorname{im} \mathrm{d}\,\mathrm{d}^c$, the following equivalent formulation can be provided.

Theorem 1.23 ($\mathrm{d}\,\mathrm{d}^c$**-Lemma for Compact Kähler Manifolds, [DGMS75, Lemma 5.11]).** *Let X be a compact Kähler manifold. Then every* d*-closed,* d^c*-closed,* d*-exact form is also* $\mathrm{d}\,\mathrm{d}^c$*-exact.*

Actually, the $\partial\overline{\partial}$-Lemma holds true for a larger class of compact complex manifolds than the compact Kähler manifolds: indeed, it holds, for examples, for any compact complex manifold that can be blown up to a Kähler manifold, [DGMS75, Theorem 5.22], e.g., for compact complex manifolds in class $\mathcal{C}$ of Fujiki, or for Moĭšezon manifolds; we refer to Sect. 2.1.3 for further results concerning the $\partial\overline{\partial}$-Lemma for compact complex manifolds.

If X is a compact Kähler manifold (or, more in general, any compact complex manifold for which the $\partial\overline{\partial}$-Lemma, equivalently the $\mathrm{d}\,\mathrm{d}^c$-Lemma, holds), then one has the following quasi-isomorphisms of dgas:

$$\begin{tikzcd}[column sep=small] & \left(\ker \mathrm{d}^c,\ \mathrm{d}\lfloor_{\ker \mathrm{d}^c}\right) \arrow[dl, "\text{qis}"'] \arrow[dr, "\text{qis}"] & \\ \left(\wedge^\bullet X,\ \mathrm{d}\right) & & \left(\frac{\ker \mathrm{d}^c}{\operatorname{im} \mathrm{d}^c},\ 0\right) \end{tikzcd} ;$$

in particular, the dga $(\wedge^\bullet X,\ \mathrm{d})$ is equivalent to a dga with zero differential, and hence it is formal. This proves the following result by P. Deligne, Ph.A. Griffiths, J. Morgan, and D.P. Sullivan.

Theorem 1.24 ([DGMS75, Main Theorem]). *Let X be a compact complex manifold for which the $\partial\overline{\partial}$-Lemma holds (e.g., a compact Kähler manifold, or a manifold in class $\mathcal{C}$ of Fujiki). Then the differentiable manifold underlying X is formal (that is, the differential graded algebra* $(\wedge^\bullet X,\ \mathrm{d})$ *is formal).*

In particular, all Massey products (of any order) on a compact complex manifold satisfying the $\partial\overline{\partial}$-Lemma are zero, [DGMS75, Corollary 1]. This provide an obstruction to the existence of Kähler structures on compact differentiable manifolds.

1.5 Deformations of Complex Structures

A natural way to construct new complex structures on a manifold is by "deforming" a given complex structure. Natural questions arise naturally from this construction, concerning, for example, what properties (e.g., the existence of some special metric) remain still valid after such a small deformation.

We recall in this section the basic notions and the classical results concerning the K. Kodaira, D.C. Spencer, L. Nirenberg, and M. Kuranishi theory of deformations of complex manifolds, [KS58, KS60, KNS58, Kur62], referring to [Huy05], see also, e.g., [Kod05, MK06].

Let B be a complex (respectively, differentiable) manifold. A family $\{X_t\}_{t\in B}$ of compact complex manifolds is said to be a *complex-analytic* (respectively, *differentiable*) *family of compact complex manifolds* if there exist a complex (respectively, differentiable) manifold $\mathcal{X}$ and a surjective holomorphic (respectively, smooth) map $\pi\colon \mathcal{X} \to B$ such that *(i)* $\pi^{-1}(t) = X_t$ for any $t \in B$, and *(ii)* π is a proper holomorphic (respectively, smooth) submersion. A compact complex manifold X is said to be a *deformation* of a compact complex manifold Y if there exist a complex-analytic family $\{X_t\}_{t\in B}$ of compact complex manifolds, and $b_0,\, b_1 \in B$ such that $X_{b_0} = X_s$ and $X_{b_1} = X_t$.

A complex-analytic (respectively, differentiable) family $\mathcal{X} \stackrel{\pi}{\to} B$ of compact complex manifolds is said to be *trivial* if $\mathcal{X}$ is bi-holomorphic (respectively, diffeomorphic) to $B \times X_b \stackrel{\pi_B}{\to} B$ for some $b \in B$ (where $\pi_B\colon B \times X_b \to B$ denotes the natural projection onto B); it is said to be *locally trivial* if, for any $b \in B$, there exists an open neighbourhood U of b in B such that $\pi^{-1}(U) \stackrel{\pi\lfloor_{\pi^{-1}(U)}}{\to} U$ is trivial. The following theorem by Ch. Ehresmann states the local triviality of a differentiable family of compact complex manifolds (see, e.g., [Kod05, Theorems 2.3 and 2.5], [MK06, Theorem 1.4.1]).

Theorem 1.25 (Ehresmann Theorem, [Ehr47]). *Let $\{X_t\}_{t\in B}$ be a differentiable family of compact complex manifolds. For any $s,\, t \in B$, the manifolds X_s and X_t are diffeomorphic.*

As a consequence of Ehresmann's theorem, a complex-analytic family $\{X_t\}_{t\in B}$ of compact complex manifolds with B contractible can be viewed as a family of complex structures on a compact differentiable manifold.

We recall some other useful definitions, see, e.g., [Huy05, Sect. 6.2]. Let $\pi\colon \mathcal{X} \to B$ be a complex-analytic family of compact complex manifolds, deformations of $X := \pi^{-1}(0)$. We recall that, given $f\colon (B',\, 0') \to (B,\, 0)$ a morphism of germs with a distinguished point, the pull-back $f^*\mathcal{X} := \mathcal{X} \times_B B'$ gives a complex-analytic family of deformations of X. The complex-analytic family $\pi\colon \mathcal{X} \to B$ of deformations of X is called *complete* if, for any complex-analytic family $\pi'\colon \mathcal{X}' \to B'$ of deformations of X, there exists a morphism $f\colon B' \to B$ of germs with a distinguished point such that $\mathcal{X}' = f^*\mathcal{X}$. The complex-analytic family $\pi\colon \mathcal{X} \to B$ of deformations of X is called *universal* if, for any complex-analytic family $\pi'\colon \mathcal{X}' \to B'$ of deformations of X, there exists a unique morphism $f\colon B' \to B$ of germs with a distinguished point such that $\mathcal{X}' = f^*\mathcal{X}$. The complex-analytic family $\pi\colon \mathcal{X} \to B$ of deformations of X is called *versal* if, for any complex-analytic family $\pi'\colon \mathcal{X}' \to B'$ of deformations of X, there exists a morphism $f\colon B' \to B$ of germs with a distinguished point such that $\mathcal{X}' = f^*\mathcal{X}$ and such that $\mathrm{d} f\colon T_{0'}B' \to T_0 S$ is uniquely determined.

The theory of complex-analytic deformations of compact complex manifolds has been introduced by K. Kodaira and D.C. Spencer, [KS58, KS60], and developed also by L. Nirenberg, [KNS58], and M. Kuranishi, [Kur62, Kur65], see also

[Kod05, MK06]. In recalling the main results of this theory, we follow the approach in [Huy05], based on the construction of a differential graded Lie algebra structure on $\mathcal{C}^\infty\left(X;\ T^{1,0}X \otimes \wedge^{0,\bullet}X\right)$, see also [Man04].

Let X be a compact manifold endowed with an integrable almost-complex structure J. Every section $s \in \mathcal{C}^\infty\left(X; T_J^{1,0}X \otimes \wedge_J^{0,1}X\right)$ near to the zero section determines an almost-complex structure J', defined in such a way that $\wedge_{J'}^{1,0}X$ is the graph of $-s\colon \wedge_J^{1,0}X \to \wedge_J^{0,1}X$; it turns out that J' is integrable if and only if the *Maurer and Cartan equation*

$$\overline{\partial}s + \frac{1}{2}\,[s,s] \;=\; 0 \tag{MC}$$

holds (see, e.g., [Huy05, Lemma 6.1.2]), where

- $[\cdot,\cdot\cdot]\colon \mathcal{C}^\infty\left(X;\ T_J^{1,0}X \otimes \wedge_J^{0,p}X\right) \times \mathcal{C}^\infty\left(X;\ T_J^{1,0}X \otimes \wedge_J^{0,q}X\right) \to \mathcal{C}^\infty\Big(X;\ T_J^{1,0}X \otimes \wedge_J^{0,p+q}X\Big)$ is defined as

  $$\left[X \otimes \bar{\alpha},\ Y \otimes \bar{\beta}\right] \;:=\; X \otimes \left(\bar{\beta} \wedge \mathcal{L}_Y\bar{\alpha}\right) + Y \otimes \left(\bar{\alpha} \wedge \mathcal{L}_X\bar{\beta}\right) + [X,\ Y] \otimes \left(\bar{\alpha} \wedge \bar{\beta}\right)\ ,$$

  where $\mathcal{L}_W\varphi := \iota_W\,\mathrm{d}\varphi + \mathrm{d}\left(\iota_W\varphi\right)$ is the Lie derivative of φ along W; locally, in a chart with holomorphic coordinates $\left\{z^j\right\}_j$, one has

  $$\begin{aligned}&\left[w \otimes \mathrm{d}\bar{z}^{\ell_1} \wedge \cdots \wedge \mathrm{d}\bar{z}^{\ell_p},\ w' \otimes \mathrm{d}\bar{z}^{m_1} \wedge \cdots \wedge \mathrm{d}\bar{z}^{m_q} \wedge\right]\\ &\overset{\text{loc}}{=} \left[w,\ w'\right] \otimes \mathrm{d}\bar{z}^{\ell_1} \wedge \cdots \wedge \mathrm{d}\bar{z}^{\ell_p} \wedge \mathrm{d}\bar{z}^{m_1} \wedge \cdots \wedge \mathrm{d}\bar{z}^{m_q}\ ;\end{aligned}$$

- $\overline{\partial}\colon \mathcal{C}^\infty\left(X;\ T_J^{1,0}X \otimes \wedge_J^{0,p}X\right) \to \mathcal{C}^\infty\left(X;\ T_J^{1,0}X \otimes \wedge_J^{0,p+1}X\right)$ is defined as

  $$\overline{\partial}\varphi\left(\bar{Z},\ \bar{W}\right) \;:=\; \left[\bar{Z},\ \varphi\left(\bar{W}\right)\right]^{1,0} - \left[\bar{W},\ \varphi\left(\bar{Z}\right)\right]^{1,0} - \varphi\left(\left[\bar{Z},\ \bar{W}\right]\right)\ ,$$

  where $X^{1,0} := X - \mathrm{i}\ J\ X$ is the $(1,0)$-component of X; locally, in a chart with holomorphic coordinates $\left\{z^j\right\}_j$, one has

  $$\overline{\partial}\left(\frac{\partial}{\partial z^\ell} \otimes \alpha\right) \overset{\text{loc}}{=} \frac{\partial}{\partial z^\ell} \otimes \overline{\partial}\alpha\ .$$

Hence, to study complex-analytic families of infinitesimal deformations of a compact complex manifold X, it suffices to study complex-analytic families $\{s(\mathbf{t})\}_{\mathbf{t}\in\Delta(0,\varepsilon)\subset\mathbb{C}^m} \subseteq \mathcal{C}^\infty\left(X;\ T^{1,0}X \otimes \wedge^{0,1}X\right)$ (where $\varepsilon > 0$ is small enough) with $s(0) = 0$. Consider the power series expansion in $\mathbf{t}$ of $s(\mathbf{t})$,

$$s(\mathbf{t}) \;=:\; \sum_{k\in\mathbb{N}} s_k(\mathbf{t})\ ,$$

where $s_k(\mathbf{t}) \in \mathcal{C}^\infty\left(X;\, T^{1,0}X \otimes \wedge^{0,1}X\right)$ is homogeneous of degree k in $\mathbf{t}$, and $s_0(\mathbf{t}) = 0$. Then the Maurer and Cartan equation (MC) can be rewritten, for every $\mathbf{t} \in \Delta(0,\varepsilon)$, as the system

$$\begin{cases} \overline{\partial} s_1(\mathbf{t}) = 0 \\ \overline{\partial} s_k(\mathbf{t}) = -\sum_{1\leq j\leq k-1}\left[s_j(\mathbf{t}),\, s_{k-j}(\mathbf{t})\right] \text{ for } k \geq 2 \end{cases};$$

in particular, $s_1(\mathbf{t})$ defines a class in $H^{0,1}(X;\, \Theta_X)$, where Θ_X denotes the sheaf of the germs of holomorphic vector fields on X; up to the action of $\mathrm{Diff}(X)$, one has that $s_1(\mathbf{t})$ is uniquely determined by its class in $H^{0,1}(X;\, \Theta_X)$ (see, e.g., [Huy05, Lemma 6.14]).

Fix now a Hermitian metric g on X. Consider the decomposition

$$T^{1,0}X \otimes \wedge^{0,1}X = \left(T^{1,0}X \otimes \ker \overline{\square}\lfloor_{\wedge^{0,1}X}\right) \oplus \left(T^{1,0}X \otimes \overline{\partial} \wedge^{0,0} X\right) \\ \oplus \left(T^{1,0}X \otimes \overline{\partial}^* \wedge^{0,2} X\right),$$

and the corresponding projections

$$H_{\overline{\partial}}\colon T^{1,0}X \otimes \wedge^{0,1}X \to T^{1,0}X \otimes \ker \overline{\square}\lfloor_{\wedge^{0,1}X}, \quad P_{\overline{\partial}}\colon T^{1,0}X \otimes \wedge^{0,1}X \to T^{1,0}X \otimes \overline{\partial} \wedge^{0,0} X.$$

In order that $s(\mathbf{t})$ satisfies (MC), for every $\mathbf{t} \in \Delta(0,\varepsilon)$, one should have

$$\overline{\partial} s_k(\mathbf{t}) = -P_{\overline{\partial}}\left(\sum_{1\leq j\leq k-1}\left[s_j(\mathbf{t}),\, s_{k-j}(\mathbf{t})\right]\right).$$

Hence, one gets

$$\overline{\partial} s(\mathbf{t}) + [s(\mathbf{t}),\, s(\mathbf{t})] = H_{\overline{\partial}}\left([s(\mathbf{t}),\, s(\mathbf{t})]\right).$$

Therefore, define the map

$$\mathrm{obs}\colon H^{0,1}(X;\, \Theta_X) \to H^{0,2}(X;\, \Theta_X)$$

as follows. Let $\left\{X_j \otimes \bar{\omega}^k\right\}_{\substack{j\in\{1,\ldots,n\}\\ k\in\{1,\ldots,m\}}}$ be a basis of $H^{0,1}(X;\, \Theta_X)$. Given $\mu = \sum_{\substack{j\in\{1,\ldots,n\}\\ k\in\{1,\ldots,m\}}} t_k^j\, X_j \otimes \bar{\omega}^k$, denote $\mathbf{t} = \left(t_k^j\right)_{\substack{j\in\{1,\ldots,n\}\\ k\in\{1,\ldots,m\}}}$, and define $s_1(\mathbf{t}) := \mu$ and $s_k(\mathbf{t})$ such that $\overline{\partial} s_k(\mathbf{t}) := -P_{\overline{\partial}}\left(\sum_{1\leq j\leq k-1}\left[s_j(\mathbf{t}),\, s_{k-j}(\mathbf{t})\right]\right)$ for $k \geq 2$; hence, define the formal power series $s(\mathbf{t}) := \sum_{k\in\mathbb{N}} s_k(\mathbf{t})$. Define

$$\mathrm{obs}(\mu) := H_{\overline{\partial}}\left([s(\mathbf{t}),\, s(\mathbf{t})]\right).$$

Hence, one has then that $\{s(\mathbf{t})\}_{\mathbf{t}\in\Delta(0,\varepsilon)\subset\mathbb{C}^m} \subseteq \mathcal{C}^\infty\left(X; T^{1,0}_J X \otimes \wedge^{0,1}_J X\right)$ (where $\varepsilon > 0$ is small enough) defines an infinitesimal family of compact complex manifolds if $\mathrm{obs}\,(s_1\,(\mathbf{t})) = 0$ for every $\mathbf{t} \in \Delta(0,\varepsilon)$ (indeed, for $\varepsilon > 0$ small enough, the formal power series converges, see, e.g., [Kod05, Sect. 5.3], [MK06, Sect. 2.3]).

One gets the following result by M. Kuranishi.

Theorem 1.26 ([Kur62, Theorem 2]). *Let X be a compact complex manifold. Then X admits a versal complex-analytic family of deformations.*

Fixed a Hermitian metric on X, such a family of deformations, which is called the *Kuranishi space* $\mathrm{Kur}(X)$ of X, is parametrized by

$$\mathrm{Kur}(X) \;=\; \left\{\mu \in H^{0,1}\left(X;\,\Theta_X\right) \;:\; \|\mu\| \ll 1,\ \mathrm{obs}(\mu) = 0\right\} .$$

Remark 1.11. A compact complex manifold X is called *non-obstructed* if $\mathrm{Kur}(X)$ is non-singular. In particular, if $H^{0,2}\left(X;\,\Theta_X\right) = \{0\}$, then X is non-obstructed. There are other interesting cases in which the Kuranishi space turns out to be non-singular: as announced by F.A. Bogomolov, [Bog78], and proven by G. Tian, [Tia87], and, independently, by A.N. Todorov, [Tod89, Theorem 1], this happens for *Calabi-Yau manifolds* (that is, compact complex manifolds X of complex dimension n endowed with a Kähler structure $(J,\,\omega,\,g)$ and with a nowhere vanishing $\epsilon \in \wedge^{n,0}X$ such that *(i)* $\nabla^{LC}\epsilon = 0$,where ∇^{LC} denotes the Levi Civita connection associated to g, and *(ii)* $\epsilon \wedge \bar{\epsilon} = (-1)^{\frac{n(n+1)}{2}}\,\mathrm{i}^n\,\frac{\omega^n}{n!}$). In [dBT12], P. de Bartolomeis and A. Tomassini introduced the notion of *quantum inner state manifold*, [dBT12, Definition 2.2], as a possible generalization of Calabi-Yau manifolds, proving that, under a suitable hypothesis, the moduli space of quantum inner state deformations of a compact Calabi-Yau manifold is unobstructed, [dBT12, Theorem 3.6]. On the other hand, in [Rol11b], S. Rollenske studied the Kuranishi space of holomorphically parallelizable nilmanifolds, proving that it is cut out by polynomial equations of degree at most equal to the step of nilpotency of the nilmanifold, [Rol11b, Theorem 4.5], and it is smooth if and only if the associated Lie algebra is a free 2-step nilpotent Lie algebra, [Rol11b, Corollary 4.9].

Remark 1.12. We refer to [Rol11b, Rol09b, Rol09a] by S. Rollenske and to [Kas12c] by H. Kasuya for results concerning deformations of nilmanifolds and, respectively, solvmanifolds.

It could be interesting to study what properties of a complex manifold are, in a sense, compatible with the construction of small deformations of the complex structure. In such a context, a property $\mathcal{P}$ concerning compact complex manifolds is called *open under (holomorphic) deformations of the complex structure* (or *stable under small deformations of the complex structure*) if, for every complex-analytic family $\{X_t\}_{t\in B}$ of compact complex manifolds, and for every $b_0 \in B$, if X_{b_0} has the property $\mathcal{P}$, then X_b has the property $\mathcal{P}$ for every b in an open neighbourhood of b_0. A property $\mathcal{P}$ is called is called *closed under (holomorphic) deformations of the complex structure* if, for every complex-analytic family $\{X_t\}_{t\in\Delta_\varepsilon(0)}$ of compact complex manifolds, where $\Delta_\varepsilon(0) := \{z \in \mathbb{C}^m \;:\; |z| < \varepsilon\}$ for some $m \in \mathbb{N} \setminus \{0\}$ and for some $\varepsilon > 0$, if X_t has the property $\mathcal{P}$ for every $t \in \Delta_\varepsilon(0) \setminus \{0\}$, then also X_0 has the property $\mathcal{P}$.

Remark 1.13 (Gunnar Þór Magnússon[8]). In a sense, the notion of closedness is considered in the Zariski sense. In fact, as suggested to us by Gunnar Þór Magnússon, by considering, for example, moduli space of complex tori of complex dimension 2, and by using [Kod63, Theorem 16.1] (compare also [Cao12, Main Theorem]), one can construct examples of complex-analytic families $\{X_t\}_{t\in B}$ of compact complex manifolds such that there exists a converging sequence $\{b_k\}_{k\in\mathbb{N}} \subset B$ with $b_\infty := \lim_{k\to+\infty} b_k \in B$ for which X_{b_k} is Moĭšezon for every $k \in \mathbb{N}$, while X_{b_∞} is not.

We recall here the following classical result by K. Kodaira and D.C. Spencer, stating that admitting a Kähler metric is a stable property under deformations of the complex structure.

Theorem 1.27 ([KS60, Theorem 15]). *Let $\{X_t\}_{t\in B}$ be a differentiable family of compact complex manifolds. If X_{t_0} admits a Kähler metric for some $t_0 \in B$, then X_s admits a Kähler metric for every s in an open neighbourhood of t_0 in B. Moreover, given any Kähler metric ω on X_{t_0}, one can choose an open neighbourhood U of t_0 in B and a Kähler metric ω_s on X_s for any $s \in U$ such that ω_s depends differentiably in s and $\omega_{t_0} = \omega$.*

Proof (sketch). We just sketch the idea of the proof, as one can found, e.g., in [Sch07, Lemme 3.3], see also [Voi02, Théorème 9.23] and [KS60, Sect. 6].

Consider a family of compact complex manifolds $\{X_t\}_t$. By the Ehresmann theorem, [Ehr47], for every $k \in \mathbb{Z}$, the function $t \mapsto \dim_\mathbb{C} H^k_{dR}(X_t;\mathbb{C})$ is locally constant. By the theory of families of elliptic differential operators (see, e.g., [Kod05, Sect. 7.1]), for every $(p,q) \in \mathbb{Z}^2$, the functions $t \mapsto \dim_\mathbb{C} H^{p,q}_{BC}(X_t)$ and $t \mapsto \dim_\mathbb{C} H^{p,q}_{A}(X_t)$ are upper-semi-continuous, see, e.g., [Sch07, Lemme 3.2]. Recall also that, by the inequality *à la* Frölicher for the Bott-Chern cohomology, [AT13b, Theorem A], it holds that $\sum_{p+q=k}\left(\dim_\mathbb{C} H^{p,q}_{BC}(X_t) + \dim_\mathbb{C} H^{p,q}_{A}(X_t)\right) \geq 2\dim_\mathbb{C} H^k_{dR}(X_t;\mathbb{C})$ for any $k \in \mathbb{Z}$ and for any t.

If we assume that X_{t_0} is Kähler for some t_0, then in particular X_{t_0} satisfies the $\partial\overline{\partial}$-Lemma. Therefore, for any $k \in \mathbb{Z}$, it holds $\sum_{p+q=k}\left(\dim_\mathbb{C} H^{p,q}_{BC}(X_{t_0}) + \dim_\mathbb{C} H^{p,q}_{A}(X_{t_0})\right) = 2\dim_\mathbb{C} H^k_{dR}(X_{t_0};\mathbb{C})$. In particular, from the facts remarked above, it follows that, for any $(p,q) \in \mathbb{Z}^2$, the functions $t \mapsto \dim_\mathbb{C} H^{p,q}_{BC}(X_t)$ and $t \mapsto \dim_\mathbb{C} H^{p,q}_{A}(X_t)$ are locally constant at t_0.

In particular, we will use that the function $t \mapsto \dim_\mathbb{C} H^{1,1}_{BC}(X_t)$ is locally constant at t_0. In fact, by the theory of families of elliptic differential operators (see, e.g., [Kod05, Sect. 7.1]), this implies that the map $h_s\colon \wedge^2 X \otimes \mathbb{C} \to \ker \tilde{\Delta}_{BC_s}\lfloor_{\wedge^{1,1}X_s}$

[8]The author would like to thank Luis Ugarte for having pointed out the subject, and Junyan Cao and Gunnar Þór Magnússon for useful discussions on the matter.

depends smoothly in s in a neighbourhood of t_0, [KS60, Theorem 7], see also [Kod05, Theorem 7.4]. (For simplicity, we identify the smooth structures on X_{t_0} and X_s, for s in a neighbourhood of t_0, by means of the diffeomorphism given by the Ehresmann theorem, [Ehr47].) Hence define, for s in a neighbourhood of t_0,

$$\omega_s \; := \; \frac{1}{2}\left(h_s\omega + \overline{h_s\omega}\right) \; .$$

For any s, the form ω_s is a real d-closed $(1,1)$-form on X_s. Furthermore, the family $\{\omega_s\}_s$ depends smoothly in s. Since $\omega_{t_0} = \omega$, it follows that the form ω_s is positive for s in a neighbourhood of X_{t_0}. In particular, ω_s is a Kähler form on X_s depending smoothly in s in a neighbourhood of t_0. □

Remark 1.14. In [Hir62], it is proven that admitting a Kähler structure is not a closed property under deformations of the complex structure: indeed, H. Hironaka provided an explicit example of a complex-analytic family of compact complex manifolds of complex dimension 3 such that *(i)* one of the complex manifold is non-Kähler (indeed, it carries a positive 1-cycle algebraically equivalent to zero), and *(ii)* the others are Kähler and, in fact, bi-regularly embedded in a projective space (and hence projective, [Moı66, Theorem 11]), [Hir62, Theorem]. (Note that, in complex dimension 2, the Kähler property is closed under deformations of the complex structure, since a compact complex surface is Kähler if and only if its 1st Betti number is even, by [Kod64, Miy74, Siu83], or [Lam99, Corollaire 5.7], or [Buc99, Theorem 11].) It is not known whether the limit of compact Kähler manifolds admits some special structure; J.-P. Demailly and M. Păun conjectured that, given a complex-analytic family $\{X_t\}_{t\in S}$ of compact complex manifolds such that one of the fibers, X_{t_0}, is endowed with a Kähler structure, then there exists a countable union $S' \subsetneq S$ of analytic subsets in the base such that X_t admits a Kähler structure for $t \in S \setminus S'$, [DP04, Conjecture 5.1]; they also guessed that a "natural expectation" is that the remaining fibres, X_t for $t \in S'$, are in class $\mathcal{C}$ of Fujiki, [DP04, p. 1272]. In [Pop13, Pop09, Pop10], D. Popovici studied limits of projective and Moĭšezon manifolds under holomorphic deformations of complex structures, proving, in particular, that the limit of projective manifolds is Moĭšezon under some additional conditions either on Hodge numbers or on some special Hermitian metrics called strongly-Gauduchon metrics, [Pop13]. C. LeBrun and Y.S. Poon [LP92], and F. Campana [Cam91] showed that being in class $\mathcal{C}$ of Fujiki is not a stable property under small deformations of the complex structures, [LP92, Theorem 1], [Cam91, Corollary 3.13], studying twistor spaces. It is conjectured that being in class $\mathcal{C}$ of Fujiki is a closed property under deformations of the complex structure, see, e.g., [Pop11, Standard Conjecture 1.17].

We refer to [Pop11] for a review on the behaviour under holomorphic deformations of properties concerning, e.g., the existence of various types of Hermitian metrics on compact complex manifolds. See also Corollary 2.2, Theorem 4.7 for some results concerning stability or instability of special properties of complex manifolds.

1.6 Currents and de Rham Homology

In this section, we recall the basic notions and results concerning currents on (differentiable) manifolds and de Rham homology: they turn out to be a useful tool to study the geometry of complex manifolds (as an example, we recall F.R. Harvey and H.B. Lawson's intrinsic characterization of Kähler manifolds by means of currents, [HL83, Proposition 12, Theorem 14], or M.L. Michelsohn's intrinsic characterization of balanced manifolds by means of currents [Mic82, Theorem 4.7], see also Theorem 4.18, or J.P. Demailly and M. Păun's characterization of compact complex manifolds in class $\mathcal{C}$ of Fujiki by means of Kähler currents[DP04, Theorem 3.4]). We refer, e.g., to [dR84, Chap. 3], [Dem12, Sect. I.2], and [Fed69] (see also [Ale98, Ale10]) for further details.

Let X be a m-dimensional oriented differentiable manifold.

For every compact set $L \subseteq X$ and for every $s \in \mathbb{N}$, define the semi-norm ρ^s_L on $\wedge^\bullet X$ as follows: chosen $\left(U, \left\{x^j\right\}_{j\in\{1,\ldots,m\}}\right)$ a coordinate chart with $U \supset L$, and given

$$\varphi \stackrel{\text{loc}}{=} \sum_{\substack{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,m\} \\ i_1<\cdots<i_k}} \varphi_I \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \in \wedge^\bullet X ,$$

set

$$\rho^s_L(\varphi) := \sup_L \sup_{\substack{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,m\} \\ i_1<\cdots<i_k}} \sup_{\substack{(\alpha_1,\ldots,\alpha_m)\in\mathbb{N}^m \\ \alpha_1+\cdots+\alpha_m\leq s}} \left| \frac{\partial^{\alpha_1+\cdots+\alpha_m}\varphi_I}{\left(\partial x^1\right)^{\alpha_1}\cdots\left(\partial x^m\right)^{\alpha_m}} \right| \in \mathbb{R} .$$

Consider $\wedge^\bullet X$ endowed with the topology induced by the family of semi-norms ρ^s_L, varying L among the compact sets in X, and $s \in \mathbb{N}$: the manifold X being second-countable, $\wedge^\bullet X$ has a structure of a Fréchet space. Let $\wedge^\bullet_c X$ be the topological subspace of $\wedge^\bullet X$ consisting of differential forms with compact support in X.

For any $k \in \mathbb{N}$, the space of *currents* of dimension k (or degree $m-k$), denoted by

$$\mathcal{D}_k X := \mathcal{D}^{m-k} X ,$$

is defined as the topological dual space of $\wedge^k_c X$; the space $\mathcal{D}_\bullet X$ is endowed with the weak-$*$ topology.

Two basic examples of currents are the following.

- If Z is a (possibly non-closed) k-dimensional oriented compact submanifold of X, then

$$[Z] := \int_Z \cdot \in \mathcal{D}_k X$$

is a current of dimension k.

- If $\varphi \in \wedge^k X$, then

$$T_\varphi := \int_X \varphi \wedge \cdot \in \mathcal{D}^k X$$

is a current of degree k.

The exterior differential $\mathrm{d}\colon \wedge^\bullet X \to \wedge^{\bullet+1} X$ induces a differential on $\mathcal{D}_\bullet X$ by duality:

$$\mathrm{d}\colon \mathcal{D}_\bullet X \to \mathcal{D}_{\bullet-1} X$$

is defined, for every $T \in \mathcal{D}^k X$, as

$$\mathrm{d}\, T := (-1)^{k+1}\, T\, (\mathrm{d}\,\cdot) \ .$$

In particular, if Z is a k-dimensional oriented closed submanifold of X, then $\mathrm{d}\,[Z] = (-1)^{m-k+1}\,[\mathrm{b}\, Z]$, where b is the boundary operator; if $\varphi \in \wedge^k X$, then $\mathrm{d}\, T_\varphi = T_{\mathrm{d}\varphi}$.

By definition, the *de Rham homology* $H_\bullet^{dR}(X; \mathbb{R})$ of X is the homology of the differential complex $(\mathcal{D}_\bullet X, \mathrm{d})$. By means of a regularization process, [dR84, Theorem 12], (see also [Dem12, Sects. 2.D.3 and 2.D.4],) one can prove, [dR84, Theorem 14], that

$$H^\bullet_{dR}(X; \mathbb{R}) \simeq H^{dR}_{2n-\bullet}(X; \mathbb{R}) \ .$$

Since, for every $k \in \mathbb{N}$, the sheaf $\mathcal{A}^k_X$ of germs of k-forms is a sheaf of $\mathcal{C}^\infty_X$-module over a paracompact space (where $\mathcal{C}^\infty_X$ denotes the sheaf of germs of smooth functions over X), and by the Poincaré lemma for forms, see, e.g., [Dem12, I.1.22], one has that

$$0 \to \underline{\mathbb{R}}_X \to \left(\mathcal{A}^\bullet_X, \mathrm{d}\right)$$

is a fine (and hence acyclic, see, e.g., [Dem12, Corollary IV.4.19]) resolution of the constant sheaf $\underline{\mathbb{R}}_X$, and hence

$$\check{H}^\bullet\,(X; \underline{\mathbb{R}}_X) \simeq \frac{\ker\left(\mathrm{d}\colon \wedge^\bullet X \to \wedge^{\bullet+1} X\right)}{\mathrm{im}\,(\mathrm{d}\colon \wedge^{\bullet-1} X \to \wedge^\bullet X)} =: H^\bullet_{dR}\,(X; \mathbb{R}) \ ,$$

see, e.g., [Dem12, IV.6.4].

Analogously, the regularization process [dR84, Theorem 12] allows to prove the analogue of the Poincaré lemma for currents, see, e.g., [Dem12, Theorem I.2.24], and hence, the sheaf $\mathcal{D}^k_X$ of germs of currents of degree k being fine for every $k \in \mathbb{N}$ since it is a sheaf of $\mathcal{C}^\infty_X$-module over a paracompact space, one has that

$$0 \to \underline{\mathbb{R}}_X \to \left(\mathcal{D}^\bullet_X, \mathrm{d}\right)$$

is a fine (and hence acyclic, see, e.g., [Dem12, Corollary IV.4.19]) resolution of the constant sheaf $\underline{\mathbb{R}}_X$ over X, and hence

$$\check{H}^\bullet\left(X;\underline{\mathbb{R}}_X\right) \simeq \frac{\ker\left(\mathrm{d}\colon \mathcal{D}^\bullet X \to \mathcal{D}^{\bullet+1} X\right)}{\operatorname{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet-1} X \to \mathcal{D}^\bullet X\right)} =: H^{dR}_{2n-\bullet}(X;\mathbb{R}) \ ,$$

see, e.g., [Dem12, IV.6.4].

If X is compact, then it follows that the map $T.\colon \wedge^\bullet X \to \mathcal{D}^\bullet X$ is injective and a quasi-isomorphism of differential complexes: indeed, fixed a Riemannian metric g on X, if α is a Δ-harmonic form (i.e., a d-closed d^*-closed form), then $T_\alpha(*\alpha) = \|\alpha\|^2$.

Suppose now that X is a $2n$-dimensional manifold endowed with an almost-complex structure $J \in \operatorname{End}(TX)$. Considering the induced endomorphisms $J \in \operatorname{End}(\wedge^\bullet X)$ and $J \in \operatorname{End}\left(\wedge^\bullet_{\mathrm{c}} X\right)$, one can define $J \in \operatorname{End}(\mathcal{D}^\bullet X)$ by duality. In the same way as $J \in \operatorname{End}(\wedge^\bullet X)$ defines a bi-graduation on $\wedge^\bullet X \otimes \mathbb{C}$, one has that $J \in \operatorname{End}(\mathcal{D}^\bullet X)$ defines the splitting

$$\mathcal{D}_\bullet X \otimes \mathbb{C} = \bigoplus_{(p,q)\in\mathbb{N}^2} \mathcal{D}_{p,q} X \ ;$$

note that $\mathcal{D}_{p,q} X := \mathcal{D}^{n-p,n-q} X$ is the topological dual of $\wedge^{p,q} X \cap \left(\wedge^\bullet_{\mathrm{c}} X \otimes_{\mathbb{R}} \mathbb{C}\right)$, for every $(p,q) \in \mathbb{N}^2$.

1.7 Solvmanifolds

Nilmanifolds and solvmanifolds provide an important class of examples in non-Kähler geometry. Indeed, on the one hand, in studying their properties, one often can reduce to study left-invariant objects on them, which is the same to study linear objects on the corresponding Lie algebra (this allows, for example, to reduce the study of the de Rham cohomology of a nilmanifold to the study of the cohomology of a complex of finite-dimensional vector spaces, [Nom54, Theorem 1]); on the other hand, they do not admit too strong structures, e.g., they do not admit any Kähler structure.

In this section, we recall the main definitions and results concerning the theory of nilmanifolds and solvmanifolds, setting also the notation for the following chapters.

1.7.1 Lie Groups and Lie Algebras

We briefly recall the notions of Lie groups and Lie algebras and the relations between them; see, e.g., [War83, Kir08, TO97, FOT08], for further results on the subject.

A *Lie group* (respectively a *Lie group of class* $\mathcal{C}^k$, for $k \in \mathbb{N} \cup \{\infty, \omega\}$, respectively a *complex Lie group*) is[9] a smooth (respectively $\mathcal{C}^k$, respectively complex) manifold endowed with a structure of group given by differentiable (respectively $\mathcal{C}^k$, respectively holomorphic) maps $\cdot \colon G \times G \to G$ and $(\cdot)^{-1} \colon G \to G$. A *homomorphism of Lie groups* (respectively, of Lie groups of class $\mathcal{C}^k$, respectively of complex Lie groups) is a smooth (respectively $\mathcal{C}^k$, respectively holomorphic) map being also a homomorphism of groups.

Remark 1.15. According to A.M. Gleason's, and D. Montgomery and L. Zippin's theorems, [Gle52, MZ52], answering to Hilbert's fifth problem, every topological Lie group (that is, a Lie group of class $\mathcal{C}^0$) admits a unique real-analytic structure with respect to which it is a Lie group of class $\mathcal{C}^\omega$.

A *Lie algebra* over a field $\mathbb{K}$ (with characteristic $\operatorname{char}\mathbb{K} \neq 2$) is a $\mathbb{K}$-vector space $\mathfrak{g}$ endowed with an operation denoted by the *bracket* $[\cdot, \cdot\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ being bi-$\mathbb{K}$-linear, skew-symmetric (i.e., for every $x, y \in \mathfrak{g}$, it holds $[x, y] + [y, x] = 0$), and satisfying the *Jacobi identity* (i.e., for every $x, y, z \in \mathfrak{g}$, it holds $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$). An *homomorphism of Lie algebras* between the Lie algebra $\mathfrak{g}$ endowed with the bracket $[\cdot, \cdot\cdot]_{\mathfrak{g}}$ and the Lie algebra $\mathfrak{h}$ endowed with the bracket $[\cdot, \cdot\cdot]_{\mathfrak{h}}$ is a $\mathbb{K}$-linear map $\varphi \colon \mathfrak{g} \to \mathfrak{h}$ preserving the brackets (namely, such that $\varphi [\cdot, \cdot\cdot]_{\mathfrak{g}} = [\varphi(\cdot), \varphi(\cdot\cdot)]_{\mathfrak{h}}$).

Given a $\mathbb{K}$-vector space V, the $\mathbb{K}$-vector space $\operatorname{End}(V)$ endowed with the bracket $[x, y] := x \circ y - y \circ x$ has a structure of Lie algebra over $\mathbb{K}$. Given a Lie algebra $\mathfrak{g}$ endowed with the bracket $[\cdot, \cdot\cdot]$, define the *adjoint representation*

$$\operatorname{ad} \colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g}) , \qquad \mathfrak{g} \ni x \stackrel{\operatorname{ad}}{\mapsto} \operatorname{ad}_x := [x, \cdot] \in \operatorname{End}(\mathfrak{g}) ;$$

in other words, the Jacobi identity states that $\operatorname{ad} \colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ is a representation of the Lie algebra $\mathfrak{g}$, that is, a homomorphism of Lie algebras.

Let G be a Lie group. For any $g \in G$, consider the *left-translation* homomorphism of Lie groups $L_g \colon G \to G$ defined by $L_g(h) := g \cdot h$, where $\cdot$ is the group operation on G. An object on G is called *G-left-invariant* if it is invariant under the actions of left-translations L_g varying $g \in G$. The Lie bracket of vector fields yields a structure of Lie algebra on the space of G-left-invariant vector fields on G, see, e.g., [War83, Proposition 3.7]: such a Lie algebra is the *Lie algebra naturally associated to the Lie group G*. By S. Lie's and E. Cartan's theorem, [Lie80, Car30], the functor that associates to every Lie group its naturally associated Lie algebra

[9] Let **C** be a category with finite products, and consider **Grp** the category of groups; a *group-object* G in **C** is an object of **C** such that $\operatorname{Hom}(\cdot, G) \colon \mathbf{C} \to \mathbf{Grp}$ is a contravariant functor.

In such a notation, a Lie group (respectively a Lie group of class $\mathcal{C}^k$, for $k \in \mathbb{N} \cup \{\infty, \omega\}$, respectively a complex Lie group) is a group-object in the category of differentiable manifolds with differentiable maps (respectively in the category of manifolds of class $\mathcal{C}^k$ with maps of class $\mathcal{C}^k$, respectively in the category of complex manifolds with holomorphic maps). A homomorphism of Lie groups between G and H is a morphism $f \colon G \to H$ of manifolds inducing, for every manifold X, a homomorphism of groups between $\operatorname{Hom}(X, G) \to \operatorname{Hom}(X, H)$.

is an equivalence between the full sub-category of connected simply-connected Lie groups and the full sub-category of finite-dimensional Lie algebras, see, e.g., [Kir08, Corollary 3.44].

Let $\mathfrak{g}$ be a Lie algebra. Define the *descending central series* $\left\{\mathfrak{g}^{[n]}\right\}_{n\in\mathbb{N}}$ of $\mathfrak{g}$ as

$$\begin{cases} \mathfrak{g}^{[0]} := \mathfrak{g} \\ \mathfrak{g}^{[n]} := \left[\mathfrak{g},\, \mathfrak{g}^{[n-1]}\right] \text{ for } n \in \mathbb{N} \setminus \{0\} \end{cases} ;$$

a Lie algebra is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $\mathfrak{g}^{[n]} = \{0\}$, and the *step of nilpotency* of $\mathfrak{g}$ is defined as $\operatorname{nilstep}(\mathfrak{g}) := \inf\left\{n \in \mathbb{N} \,:\, \mathfrak{g}^{[n]} = \{0\}\right\}$; a Lie group is called *nilpotent* if its associated Lie algebra is nilpotent.

Let $\mathfrak{g}$ be a Lie algebra. Define the *descending derived series* $\left\{\mathfrak{g}^{\{n\}}\right\}_{n\in\mathbb{N}}$ of $\mathfrak{g}$ as

$$\begin{cases} \mathfrak{g}^{\{0\}} := \mathfrak{g} \\ \mathfrak{g}^{\{n\}} := \left[\mathfrak{g}^{\{n-1\}},\, \mathfrak{g}^{\{n-1\}}\right] \text{ for } n \in \mathbb{N} \setminus \{0\} \end{cases} ;$$

a Lie algebra is called *solvable* if there exists $n \in \mathbb{N}$ such that $\mathfrak{g}^{\{n\}} = \{0\}$; a Lie group is called *solvable* if its associated Lie algebra is solvable. One has that $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, see, e.g., [Kir08, Corollary 5.32]; in particular, every nilpotent Lie algebra is also solvable.

Let $\mathfrak{g}$ be a solvable Lie algebra, and consider the adjoint representation $\operatorname{ad}\colon \mathfrak{g} \ni x \mapsto [x, \cdot] \in \operatorname{End}(\mathfrak{g})$. One says that $\mathfrak{g}$ is *completely-solvable* (or *of type (R)*) if, for any $x \in \mathfrak{g}$, all the eigen-values of the endomorphism $\operatorname{ad}_x \in \operatorname{End}(\mathfrak{g})$ are in $\mathbb{R}$. One says that $\mathfrak{g}$ is *of rigid type* (or *of type (I)*) if, for any $x \in \mathfrak{g}$, all the eigen-values of the endomorphism $\operatorname{ad}_x \in \operatorname{End}(\mathfrak{g})$ are in $\mathrm{i}\,\mathbb{R}$, [Vin94, p. 65].

1.7.2 Nilmanifolds and Solvmanifolds

A *nilmanifold*, [Mal49, p. 278], respectively *solvmanifold*, $X = \Gamma \backslash G$ is a compact quotient of a connected simply-connected nilpotent, respectively solvable, Lie group G by a co-compact discrete subgroup Γ (see, e.g., [TO97, Definitions 2.1.1 and 3.1.1]). A solvmanifold $X = \Gamma \backslash G$ is called *completely-solvable* if its naturally associated Lie algebra is completely-solvable, that is, if, for any $g \in G$, all the eigenvalues of $\operatorname{Ad} g \in \operatorname{End}(\mathfrak{g})$ are real, equivalently, if, for any $X \in \mathfrak{g}$, all the eigenvalues of $\operatorname{ad} X \in \operatorname{End}(\mathfrak{g})$ are real.

In particular, nilmanifolds and solvmanifolds are *homogeneous spaces*, that is, differentiable manifolds endowed with a transitive action of a Lie group.

Recall that, given an homogeneous manifold X together with a transitive action $\psi\colon G \to \operatorname{Hom}_{\mathbf{Diff}}(X; X)$ of a Lie group G, then X is diffeomorphic to the quotient $H_x \backslash G$ where $x \in X$ and $H_x := \{g \in G \,:\, \psi(g)(x) = x\}$ is the *isotropy group* of G at x, see, e.g., [War83, Theorem 3.62].

Remark 1.16 ([Aus61, Has06]). We notice that in the literature, different definitions of solvmanifolds are also considered: for example, L. Auslander defines a solvmanifold as a "homogeneous space of a connected solvable Lie group", [Aus61, p. 398], see also [Aus73a, Aus73b]; K. Hasegawa defines a solvmanifold X as "a compact homogeneous space of solvable Lie group, that is, a compact differentiable manifold on which a connected solvable (nilpotent) Lie group G acts transitively", [Has06, p. 132]; as noticed in [Has06, p. 132], one can assume that $X = D \backslash \tilde{G}$, where $\tilde{G}$ is a connected simply-connected Lie group (namely, the universal covering group of G) and D is a closed subgroup of $\tilde{G}$. Note that a closed subgroup of a solvable non-nilpotent group is not necessarily discrete; however, every compact solvmanifold in the Auslander sense has a solvmanifold $\Gamma \backslash \tilde{G}$ with discrete isotropy subgroup Γ as a finite covering, [Aus61].

Remark 1.17. Nilmanifolds and solvmanifolds $\Gamma \backslash G$ are *aspherical*, that is, $\pi_j(\Gamma \backslash G) = \{0\}$ for $j \geq 2$; (in fact, they are *Eilenberg and MacLane spaces of type* $K(\pi; 1)$ with $\pi = \Gamma$). Furthermore, they are uniquely determined, up to diffeomorphism, by their fundamental group, as proven by A.I. Mal'tsev for nilmanifolds, [Mal49, Corollary, p. 293], and by G.D. Mostow for solvmanifolds, [Mos54, Mos57, Theorem A], see also, e.g., [Rag72, Corollary 2, p. 34, Theorem 3.6], [Aus73a, pp. 235, 244], [GOV97, Theorems II.1.3(i) and II.2.6].

Obviously, every nilmanifold is in particular a solvmanifold; on the other hand, there exist solvmanifolds which are not diffeomorphic to any nilmanifold, since their fundamental group is not nilpotent.[10] In fact, given a solvmanifold $\Gamma \backslash G$, if $\pi_1(\Gamma \backslash G)$ is nilpotent, then $\Gamma \backslash G$ is diffeomorphic to some nilmanifold, see, e.g., [GOV97, Corollary II.2]; if $\pi_1(\Gamma \backslash G)$ is Abelian, then $\Gamma \backslash G$ is diffeomorphic to a torus, [Mos54, Mos57, Theorem 1], see, e.g., [GOV97, Corollary II.2]. More precisely, one can prove that a compact aspherical homogeneous manifold having nilpotent fundamental group is diffeomorphic to some nilmanifold, which is unique up to diffeomorphism, see, e.g., [GOV97, Theorem II.2.7].

Remark 1.18. Note that nilmanifolds and solvmanifolds are parallelizable,[11] [TO97, Theorem 2.3.11].

Given a $2n$-dimensional solvmanifold $X = \Gamma \backslash G$, consider $(\mathfrak{g}, [\cdot, \cdot\cdot])$ the Lie algebra naturally associated to the Lie group G; given a basis $\{e_1, \ldots, e_{2n}\}$ of $\mathfrak{g}$, the Lie algebra structure of $\mathfrak{g}$ is characterized by the *structure constants* $\left\{c_{\ell m}^k\right\}_{\ell, m, k \in \{1, \ldots, 2n\}} \subset \mathbb{R}$: namely, for any $k \in \{1, \ldots, 2n\}$,

$$\mathrm{d}_{\mathfrak{g}} e^k =: \sum_{\ell, m} c_{\ell m}^k \, e^\ell \wedge e^m \ ,$$

[10] We recall that the following theorem by A.I. Mal'tsev: a group Γ is isomorphic to a discrete co-compact subgroup in a simply-connected connected nilpotent Lie group if and only if it is finitely-generated, nilpotent, and torsion-free, [Mal49, Corollary 1], see, e.g., [Rag72, Theorem 2.18].

[11] Recall that a manifold is called *parallelizable* if its tangent bundle is trivial.

where $\left\{e^1, \ldots, e^{2n}\right\}$ is the dual basis of $\mathfrak{g}^*$ of $\{e_1, \ldots, e_{2n}\}$ and $\mathrm{d}_{\mathfrak{g}} \colon \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*$ is defined by

$$\mathfrak{g}^* \ni \alpha \mapsto \mathrm{d}_{\mathfrak{g}}\, \alpha(\cdot, \cdot\cdot) := -\alpha\left([\cdot, \cdot\cdot]\right) \in \wedge^2 \mathfrak{g}^* .$$

To shorten the notation, as in [Sal01], we will refer to a given solvmanifold $X = \Gamma \backslash G$ writing the structure equations of its Lie algebra: for example, writing

$$X := \left(0^4,\ 12,\ 13\right) , \qquad (\text{or } \mathfrak{g} := \left(0^4,\ 12,\ 13\right) ,)$$

we mean that $X = \Gamma \backslash G$ and there exists a basis of the Lie algebra $\mathfrak{g}$ naturally associated to G, let us say $\{e_1, \ldots, e_6\}$, whose dual will be denoted by $\left\{e^1, \ldots, e^6\right\}$, such that the structure equations with respect to such basis are

$$\begin{cases} \mathrm{d} e^1 = \mathrm{d} e^2 = \mathrm{d} e^3 = \mathrm{d} e^4 = 0 \\ \mathrm{d} e^5 = e^1 \wedge e^2 =: e^{12} \\ \mathrm{d} e^6 = e^1 \wedge e^3 =: e^{13} \end{cases} ,$$

where we also shorten $e^{AB} := e^A \wedge e^B$.

The following theorem by A.I. Mal'tsev characterizes the nilpotent Lie algebras $\mathfrak{g}$ for which the naturally associated connected simply-connected Lie group admits a co-compact discrete subgroup, and hence such that there exists a nilmanifold with $\mathfrak{g}$ as Lie algebra; (see also, e.g., [Rag72, Theorem 2.12]).

Theorem 1.28 ([Mal49, Theorem 7]). *In order that a simply-connected connected nilpotent Lie group contain a discrete co-compact Lie group it is necessary and sufficient that the Lie algebra of this group have rational constant structures with respect to an appropriate basis.*

As regards the existence of discrete co-compact subgroups in solvable Lie groups, one has the following obstruction provided by J. Milnor. (Recall that a Lie group G, with associated Lie algebra $\mathfrak{g}$, is called *unimodular* if, for all $X \in \mathfrak{g}$, it holds $\operatorname{tr} \operatorname{ad} X = 0$; see also [Mil76, Lemmas 6.1, 7.1, and 6.3] for other equivalent formulations.)

Lemma 1.1 ([Mil76, Lemma 6.2]). *Any connected Lie group that admits a discrete subgroup with compact quotient is unimodular and in particular admits a bi-invariant volume form η.*

Dealing with *G-left-invariant* objects on X, we mean objects induced by objects on G being invariant under the left-action of G on itself given by left-translations. By means of left-translations, G-left-invariant objects will be identified with objects on the Lie algebra $\mathfrak{g}$.

For example, a G-left-invariant complex structure $J \in \operatorname{End}(TX)$ on X is uniquely determined by a linear complex structure $J \in \operatorname{End}(\mathfrak{g})$ on $\mathfrak{g}$ satisfying the integrability condition $\operatorname{Nij}_J = 0$, [NN57, Theorem 1.1], where

$$\mathrm{Nij}_J(\cdot, \cdot\cdot) \;:=\; [\cdot, \cdot\cdot] + J\,[J\cdot, \cdot\cdot] + J\,[\cdot, J\cdot\cdot] - [J\cdot, J\cdot\cdot] \;\in\; \wedge^2\mathfrak{g}^* \otimes_{\mathbb{R}} \mathfrak{g}\;;$$

we will denote the set of G-left-invariant complex structures on X by

$$\mathcal{C}(\mathfrak{g}) := \left\{J \in \mathrm{End}(\mathfrak{g}) \;:\; J^2 = -\,\mathrm{id}_{\mathfrak{g}} \;\text{ and }\; \mathrm{Nij}_J = 0\right\}\;.$$

By the Leibniz rule, the map $\mathrm{d}_{\mathfrak{g}}\colon \wedge^1\mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ induces a differential operator $\mathrm{d}\colon \wedge^\bullet\mathfrak{g}^* \to \wedge^{\bullet+1}\mathfrak{g}^*$ giving a graded differential algebra $(\wedge^\bullet\mathfrak{g}^*, \mathrm{d})$, and hence a differential complex $(\wedge^\bullet\mathfrak{g}^*, \mathrm{d})$, which is usually called the *Chevalley and Eilenberg complex* of $\mathfrak{g}$, [CE48]; we will denote by $H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) := H^\bullet(\wedge^\bullet\mathfrak{g}^*, \mathrm{d})$ the cohomology of such a differential complex.

In general, on a solvmanifold, the inclusion $(\wedge^\bullet\mathfrak{g}^*, \mathrm{d}) \hookrightarrow (\wedge^\bullet X, \mathrm{d})$ induces an injective map in cohomology, $i\colon H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) \hookrightarrow H^\bullet_{dR}(X;\mathbb{R})$ (see [Rag72, Remark 7.30], [TO97, Theorem 3.2.10], and compare [CF01, Lemma 9] and Lemma 3.2 for the Dolbeault and, respectively, Bott-Chern cohomologies), which is not always an isomorphism, as the example in [dBT06, Corollary 4.2, Remark 4.3] shows. On the other hand, the following theorem by K. Nomizu says that the de Rham cohomology of a nilmanifold can be computed as the cohomology of the subcomplex of left-invariant forms (some results in this direction have been provided also by Y. Matsushima in [Mat51, Theorems 5 and 6]).

Theorem 1.29 ([Nom54, Theorem 1]). *Let $X = \Gamma\backslash G$ be a nilmanifold and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. The complex $(\wedge^\bullet\mathfrak{g}^*, \mathrm{d})$ is a minimal model for $(\wedge^\bullet X, \mathrm{d})$. In particular, the map $(\wedge^\bullet\mathfrak{g}^*, \mathrm{d}) \to (\wedge^\bullet X, \mathrm{d})$ of differential complexes is a quasi-isomorphism:*

$$i\colon H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) \xrightarrow{\simeq} H^\bullet_{dR}(X;\mathbb{R})\;.$$

The proof rests on an inductive argument, which can be performed since every nilmanifold can be seen as a principal torus-bundle over a lower dimensional nilmanifold, see [Mal49, Lemma 4], [Mat51, Theorem 3]; see also [Rag72, Corollary 7.28].

A similar result holds also in the case of completely-solvable solvmanifolds, as proven by A. Hattori, as a consequence of the Mostow structure theorem,[12] [Mos54, Mos57].

[12] We recall the following, referring, e.g., to [TO97, Theorem 1.2]; see also [Rag72, Theorem 3.3, Corollary 3.5] for a different proof.

Theorem 1.30 (Mostow Structure Theorem, [Mos54, Mos57, Theorem 2]). *Any solvmanifold can be naturally fibred over a torus with a nilmanifold as a fibre. More precisely, let $\Gamma\backslash G$ be a solvmanifold, and consider the maximal connected nilpotent subgroup N of G. Then (i) $N\Gamma$ is a closed subgroup in G, (i) $N \cap \Gamma$ is a lattice in N, and (i) $N\Gamma\backslash G$ is a torus. Hence, one gets the* Mostow bundle

$$N \cap \Gamma\backslash N \;=\; \Gamma\backslash N\Gamma \;\hookrightarrow\; \Gamma\backslash G \;\to\; N\Gamma\backslash G\;\;.$$

Theorem 1.31 ([Hat60, Corollary 4.2]). *Let $X = \Gamma \backslash G$ be a completely-solvable solvmanifold and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. The map $(\wedge^\bullet \mathfrak{g}^*, \mathrm{d}) \to (\wedge^\bullet X, \mathrm{d})$ of differential complexes is a quasi-isomorphism:*

$$i \colon H^\bullet_{dR}(\mathfrak{g}; \mathbb{R}) \xrightarrow{\simeq} H^\bullet_{dR}(X; \mathbb{R}) \; .$$

The previous result holds true also for solvmanifolds satisfying the so-called Mostow condition, [Mos54, Theorem 8.2, Corollary 8.1], see also [Rag72, Corollary 7.29]. For some results concerning the de Rham cohomology of (non-necessarily completely-solvable) solvmanifolds, see [Gua07, CF11, Kas13a, Kas12a, CFK13]. We refer also to Sect. 3.1.2 for further classical results concerning the computation of the de Rham and Dolbeault cohomologies of nilmanifolds and solvmanifolds.

In some cases, we will see that the study of (properties of) geometric structures on a solvmanifold is reduced to the study of the corresponding (properties of) geometric structures on the associated Lie algebra (see, e.g., Theorem 4.13, Proposition 4.2, Theorem 4.6). To this aim, we need the following trick by F.A. Belgun (see also [FG04, Theorem 2.1]).

Lemma 1.2 (F.A. Belgun's Symmetrization Trick, [Bel00, Theorem 7]). *Let $X = \Gamma \backslash G$ be a solvmanifold, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Let η be a G-bi-invariant volume form on G such that $\int_X \eta = 1$, whose existence follows from J. Milnor's lemma [Mil76, Lemma 6.2]. Up to identifying G-left-invariant forms on X and linear forms over $\mathfrak{g}^*$ through left-translations, define the* F. A. Belgun's symmetrization map

$$\mu \colon \wedge^\bullet X \to \wedge^\bullet \mathfrak{g}^* \;, \qquad \mu(\alpha) \;:=\; \int_X \alpha\lfloor_m \, \eta(m) \; .$$

One has that

$$\mu\lfloor_{\wedge^\bullet \mathfrak{g}^*} \;=\; \mathrm{id}\lfloor_{\wedge^\bullet \mathfrak{g}^*} \;,$$

and that

$$\mathrm{d} \circ \mu \;=\; \mu \circ \mathrm{d} \; .$$

In particular, the symmetrization map μ induces a map $\mu \colon (\wedge^\bullet X, \mathrm{d}) \to (\wedge^\bullet \mathfrak{g}^*, \mathrm{d})$ of differential complexes, and hence a map $\mu \colon H^\bullet_{dR}(X; \mathbb{R}) \to H^\bullet_{dR}(\mathfrak{g}; \mathbb{R})$ in cohomology. Since $\mu\lfloor_{\wedge^\bullet \mathfrak{g}^*} = \mathrm{id}\lfloor_{\wedge^\bullet \mathfrak{g}^*}$, if the inclusion $(\wedge^\bullet \mathfrak{g}^*, \mathrm{d}) \hookrightarrow (\wedge^\bullet X, \mathrm{d})$ is a quasi-isomorphism (for example, if X is a nilmanifold, by [Nom54, Theorem 1],

(See also [Boc09, Theorem 3.6], which gives a sufficient condition for the Mostow bundle to be a principal bundle.)

or a completely-solvable solvmanifold, by [Hat60, Corollary 4.2]), then the map $\mu\colon (\wedge^\bullet X, \mathrm{d}) \to (\wedge^\bullet \mathfrak{g}^*, \mathrm{d})$ turns out to be a quasi-isomorphism.

K. Nomizu's theorem [Nom54, Theorem 1], A. Hattori's theorem [Hat60, Corollary 4.2], and F.A. Belgun's theorem [Bel00, Theorem 7] suggest that nilmanifolds, and, more in general, solvmanifolds, may provide a very useful and interesting class of examples in non-Kähler geometry. On the other hand, another reason for this statement is given by the following result by Ch. Benson and C.S. Gordon, and by K. Hasegawa: it answers the *Weinstein and Thurston problem* for nilmanifolds, namely, it characterizes the nilmanifolds admitting a Kähler structure.

Theorem 1.32 ([BG88, Theorem A]). *Let X be a nilmanifold endowed with a symplectic structure ω such that the Hard Lefschetz Condition holds. Then X is diffeomorphic to a torus.*

Actually, one can prove that any $2n$-dimensional nilmanifold X endowed with a symplectic structure ω such that the map $[\omega]^{n-1} : H^1_{dR}(X;\mathbb{R}) \to H^{2n-1}_{dR}(X;\mathbb{R})$ is an isomorphism is diffeomorphic to a torus, [LO94], see, e.g., [FOT08, Theorem 4.98]. A minimal model proof of Ch. Benson and C.S. Gordon's theorem [BG88, Theorem A] is due to G. Lupton and J. Oprea, [LO94, Theorem 3.5].

Remark 1.19. For some results on Hard Lefschetz Condition for solvmanifolds, obtained by H. Kasuya, see [Kas13a, Kas09].

Theorem 1.33 ([Has89, Theorem 1, Corollary]). *Let X be a nilmanifold. If X is formal, then X is diffeomorphic to a torus.*

Remark 1.20. For some results on formality for solvmanifolds, obtained by H. Kasuya, see [Kas13a, Kas12d, Kas09].

In particular, since compact Kähler manifolds satisfy the Hard Lefschetz Condition, [Wei58, Théorème IV.5], and are formal, [DGMS75, Main Theorem], it follows that a nilmanifold admits a Kähler structure if and only if it is diffeomorphic to a torus (compare also [Han57, Theorem II, Footnote 1]). More in general, compact completely-solvable Kähler solvmanifolds are tori, as proven by A. Tralle and J. Kedra in [TK97, Theorem 1], solving a conjecture by Ch. Benson and C.S. Gordon, [BG90, p. 972]. In fact, the following result by K. Hasegawa gives a complete characterization of Kähler solvmanifolds.

Theorem 1.34 ([Has06, Main Theorem]). *Let X be a compact homogeneous space of solvable Lie group, that is, a compact differentiable manifold on which a connected solvable Lie group acts transitively. Then X admits a Kähler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus-bundle over a complex torus. In particular, a completely-solvable solvmanifold has a Kähler structure if and only if it is a complex torus.*

Remark 1.21. For some results on Hodge decomposition and $\partial\overline{\partial}$-Lemma for solvmanifolds, obtained by H. Kasuya, see [Kas11, Kas12b].

Remark 1.22. In this context, another class that could provide several interesting examples is given by complex orbifolds[13] of the type $\tilde{X} = X/G$, where X is a complex manifold (possibly, a nilmanifold or a solvmanifold[14]) and G is a finite group of biholomorphisms of X (compare the Bochner linearization theorem, [Boc45, Theorem 1], see also [Rai06, Theorem 1.7.2]). Orbifolds of such a global-quotient-type have been considered and studied, e.g., by D.D. Joyce in constructing examples of compact seven-dimensional manifolds with holonomy G_2, [Joy96b] and [Joy00, Chaps. 11–12], and examples of compact eight-dimensional manifolds with holonomy Spin(7), [Joy96a, Joy99] and [Joy00, Chaps. 13–14]. We refer to [Sat56, Bai56, Bai54, Ang13a] for results concerning the de Rham, Dolbeault, and Bott-Chern cohomologies of orbifolds.

Appendix: Low Dimensional Solvmanifolds and Special Structures

In this section, we summarize some results concerning low-dimensional solvable Lie algebras, especially focusing on their classification and on the geometric properties of the possibly associated solvmanifolds. As regards nilpotent Lie algebras, we recall that, up to isomorphisms, the nilpotent Lie algebras of dimension less than or equal to 6 are finitely-many; on the other hand, in dimension 7, there are infinitely-many non-isomorphic nilpotent Lie algebras.

We refer mainly to [Boc09] for the classification, and to [TO97, Sal01, Mac13] for results on the geometric properties of nilmanifolds and solvmanifolds. (As regards classification of nilpotent Lie algebras up to dimension 7, see also, e.g., [Gon98] and the references therein.)

A.1 Solvmanifolds up to Dimension 4

In Table 1.1, we list the isomorphism classes of solvable Lie algebras of dimension less or equal to 4 and whose connected simply-connected Lie group admit a lattice (at least for some parameter), [Boc09, Sect. 5, Theorem 5.3, Table 1], [Boc09, Sect. 6, Theorem 6.2, Table 2], [Boc09, Table 8]. The notation follows [Mub63c,

[13]The notion of orbifold has been introduced by I. Satake in [Sat56], with the name of *V-manifold*, and has been studied, among others, by W.L. Baily, [Bai56, Bai54]. We recall that a *complex orbifold of complex dimension n* is a singular complex space whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n/G$, for finite subgroups $G \subset \mathrm{GL}(n;\mathbb{C})$, [Sat56, Definition 2]. We refer, e.g., to [Joy07, Joy00, Sat56, Bai56, Bai54] for more results concerning complex orbifolds and their cohomology.

[14]Recall that nilmanifolds and solvmanifolds are Eilenberg and MacLane spaces of type $K(\pi;1)$; in particular, they are not simply-connected.

Table 1.1 Isomorphism classes of solvable Lie algebras of dimension less or equal to 4 and whose connected simply-connected Lie group admit a lattice (at least for some parameter), [Boc09, Sect. 5, Theorem 5.3, Table 1], [Boc09, Sect. 6, Theorem 6.2, Table 2], [Boc09, Table 8]

Name	Structure equations	Conditions	Type	b_1	Lattice	Complex?	Symplectic?	Formal?	HLC[a]?
$\mathfrak{g}_1$	(0)		Abelian	1	Torus	×	×	✓	×
$2\mathfrak{g}_1$	$(0, 0)$		Abelian	2	Torus	Torus	✓	Kähler	✓
$3\mathfrak{g}_1$	$(0, 0, 0)$		Abelian	3	Torus	×	×	✓	×
$\mathfrak{g}_{3.1}$	$(-23, 0, 0)$		Nilpotent	2	[Boc09, Theorem 5.4]	×	×	×	×
$\mathfrak{g}_{3.4}^{-1}$	$(-13, 23, 0)$		Completely-solvable	1	[Boc09, p. 25], [TO97, Theorem 1.9]	×	×	✓	×
$\mathfrak{g}_{3.5}^{0}$	$(-23, 13, 0)$		Solvable	1	[Boc09, p. 25], [TO97, Theorem 1.9]	×	×	✓	×
$4\mathfrak{g}_1$	$(0, 0, 0, 0)$		Abelian	4	Torus	Torus	✓	Kähler	✓
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$	$(-23, 0, 0, 0)$		Nilpotent	3	[Boc09, p. 34]	Primary Kodaira surface	✓	×	×
$\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_1$	$(-13, 23, 0, 0)$		Completely-solvable	2	[Boc09, p. 34]	×	✓	✓	✓
$\mathfrak{g}_{3.5}^{0} \oplus \mathfrak{g}_1$	$(-23, 13, 0, 0)$		Solvable	2	[Boc09, p. 35]	Hyper-elliptic surface	✓	Kähler	✓
$\mathfrak{g}_{4.1}$	$(-24, -34, 0, 0)$		Nilpotent	2	[Boc09, p. 35]	×	✓	×	×
$\mathfrak{g}_{4.5}^{p,-p-1}$	$\left(-14, -p^{-1}24, (p+1)^{-1}34, 0\right)$	$-\frac{1}{2} \leq p < 0$	Completely-solvable	1	[Boc09, p. 35]	×	×	✓	×

(continued)

Table 1.1 (continued)

Name	Structure equations	Conditions	Type	b_1	Lattice	Complex?	Symplectic?	Formal?	HLC[a]?
$\mathfrak{g}_{4.6}^{-2p,p}$	$\left((2p)^{-1}14, -p^{-1}24 - 34, 24 - p^{-1}34, 0\right)$	$p > 0$	Solvable	1	[Boc09, p. 35]	Inoue surface of type S^0	×	✓	×
$\mathfrak{g}_{4.8}^{-1}$	$(-23, -24, 34, 0)$		Completely-solvable	1	[Boc09, p. 35]	Inoue surface of type S^+	×	✓	×
$\mathfrak{g}_{4.9}^{0}$	$(-23, -34, 24, 0)$		Solvable	1	[Boc09, p. 36]	Secondary Kodaira surface	×	✓	×

[a] By [Boc09, Theorem 9.2], a four-dimensional symplectic solvmanifold does not satisfy the Hard Lefschetz Condition if and only if it is a non-torus nilmanifold; in particular, the Hard Lefschetz Condition does not depend on the symplectic structure for four-dimensional solvmanifolds

Boc09]. (For the classification of three-dimensional solvable Lie algebras and, respectively, of three-dimensional solvmanifolds, see also, e.g., [TO97, Theorems 1.6 and 1.9]. For results concerning four-dimensional solvmanifolds, see also, e.g., [TO97, Theorem 1.10].)

A.2 Five-Dimensional Solvmanifolds

In Table 1.2, we list the 9 isomorphism classes of five-dimensional nilpotent Lie algebras, [Boc09, Table 9], [Mub63a], and the 19 isomorphism classes of five-dimensional solvable non-nilpotent unimodular Lie algebras, [Boc09, Tables 10–14] (a list of the 24, respectively 33 classes of five-dimensional solvable non-nilpotent decomposable, respectively indecomposable Lie algebras has been obtained by G.M. Mubarakzjanov in [Mub63a]).

A.3 Six-Dimensional Nilmanifolds

V.V. Morozov classified in [Mor58] the six-dimensional nilpotent Lie algebras, up to isomorphism, in 34 different classes, see also [Mag86], see also [Boc09, Table 15], see also [Gon98, Sect. 3]. In [Sal01], S.M. Salamon identified the 18 classes of six-dimensional nilpotent Lie algebras admitting a linear integrable complex structure, [Sal01, Theorems 3.1–3.3, Proposition 3.4], and provided an estimate of the moduli spaces of complex structures and symplectic structures, [Sal01, Eqs. (28) and (31)], see [Sal01, Sect. 5]: up to equivalence, the linear integrable non-nilpotent complex structures have been classified by L. Ugarte and R. Villacampa in [UV09], the linear integrable Abelian complex structures have been classified by A. Andrada, M.L. Barberis, and I.G. Dotti in [ABDM11], and lastly the linear integrable nilpotent non-Abelian complex structures have been classified by M. Ceballos, A. Otal, L. Ugarte, and R. Villacampa in [COUV11]. The 26 classes of six-dimensional nilmanifolds admitting left-invariant symplectic structures have been classified by M. Goze and Y. Khakimdjanov, [GK96], see also [Sal01, Sect. 5]. There are 16 classes of nilmanifolds admitting both left-invariant complex structures and left-invariant symplectic structures (only one admitting a Kähler structure). Hence, there are five classes of six-dimensional nilmanifolds admitting neither left-invariant complex structures nor left-invariant symplectic structures, [Sal01, Theorem 5.1]: in fact, G.R. Cavalcanti and M. Gualtieri proved that such classes admit left-invariant generalized complex structures, [CG04, Theorems 4.1 and 4.2], see also [CG04, Table 1]. In Table 1.3, we list the classes of six-dimensional nilpotent Lie algebras following the notation [Boc09, Tables 4 and 15], and [Mag86, CFGU97, CG04, Sal01]. Moreover, in Table 1.4, we recall the classification of linear integrable complex structures on six-dimensional nilpotent Lie algebras up to equivalence by M. Ceballos, A. Otal, L. Ugarte, and R. Villacampa, [COUV11].

Table 1.2 Isomorphism classes of five-dimensional nilpotent Lie algebras [Boc09, Table 9], see also [Mub63a], and isomorphism classes of five-dimensional solvable non-nilpotent unimodular Lie algebras [Boc09, Tables 10–14, 3], see [Boc09, Sect. 7]

Name	Structure equations	Conditions	Type[a]	b_1	b_2	Lattice[b]?	Formal?	Note
$5\mathfrak{g}_1$	$(0,0,0,0,0)$		Ab	5		✓	✓	
$\mathfrak{g}_{3.1} \oplus 2\mathfrak{g}_1$	$(-23,0,0,0,0)$		nil	4		✓	×	
$\mathfrak{g}_{4.1} \oplus \mathfrak{g}_1$	$(-24,-34,0,0,0)$		nil	3		✓	×	
$\mathfrak{g}_{5.1}$	$(-35,-45,0,0,0)$		nil	3		✓	×	
$\mathfrak{g}_{5.2}$	$(-25,-35,-45,0,0)$		nil	2		✓	×	
$\mathfrak{g}_{5.3}$	$(-25,-45,-24,0,0)$		nil	2		✓	×	
$\mathfrak{g}_{5.4}$	$(-24-35,0,0,0,0)$		nil	4		✓	×	
$\mathfrak{g}_{5.5}$	$(-34-25,-35,0,0,0)$		nil	3		✓	×	
$\mathfrak{g}_{5.6}$	$(-34-25,-35,-45,0,0)$		nil	2		✓	×	
$\mathfrak{g}_{3.4}^{-1} \oplus 2\mathfrak{g}_1$	$(-13,23,0,0,0)$		cplt-sol			✓	✓	[Boc09, p. 36]
$\mathfrak{g}_{3.5}^{0} \oplus 2\mathfrak{g}_1$	$(-23,13,0,0,0)$		sol			✓	✓	[Boc09, p. 36]
$\mathfrak{g}_{4.2}^{-2} \oplus \mathfrak{g}_1$	$(\frac{1}{2}14,-24-34,-34,0,0)$		cplt-sol	–	–	×	–	[Boc09, Th 7.11]
$\mathfrak{g}_{4.5}^{p,-p-1} \oplus \mathfrak{g}_1$	$(-14,-p^{-1}24,(p+1)^{-1}34,0,0)$	$-\frac{1}{2} \leq p < 0$	cplt-sol			✓	✓	[Boc09, p. 36]
$\mathfrak{g}_{4.6}^{-2p,p} \oplus \mathfrak{g}_1$	$((2p)^{-1}14,-p^{-1}24-34,24-p^{-1}34,0,0)$	$p > 0$	sol			✓	✓	[Boc09, p. 36]
$\mathfrak{g}_{4.8}^{-1} \oplus \mathfrak{g}_1$	$(-23,-24,34,0,0)$		cplt-sol			✓	✓	[Boc09, p. 36]
$\mathfrak{g}_{4.9}^{0} \oplus \mathfrak{g}_1$	$(-23,-34,24,0,0)$		sol			✓	✓	[Boc09, p. 36]
$\mathfrak{g}_{5.7}^{p,q,r}$	$(-15,-p^{-1}25,-q^{-1}35,-r^{-1}45,0)$	$-1 \leq r \leq q \leq p \leq 1$, $pqr \neq 0$, $p+q+r=-1$	cplt-sol	1	$0,2,4$	✓	✓	[Boc09, Pr 7.2.1]
$\mathfrak{g}_{5.8}^{-1}$	$(-25,0,-35,45,0)$		cplt-sol	2	3	✓	×	[Boc09, Pr 7.2.2, Table 3]
$\mathfrak{g}_{5.9}^{p,-2-p}$	$(-15-25,-25,-p^{-1}35,(2+p)^{-1}45,0)$	$p \geq -1$	cplt-sol	–	–	×	–	[Boc09, Pr 7.2.3]

$\mathfrak{g}_{5.11}^{-3}$	$\left(-15-25, -25-35, -35, \frac{1}{3}45, 0\right)$		cplt-sol	–	–	$\times$	–	[Boc09, Pr 7.2.4] ([Har96])
$\mathfrak{g}_{5.13}^{-1-2q,q,r}$	$\left(-15, (1+2q)^{-1}25, -q^{-1}35 - r^{-1}45, r^{-1}35 - q^{-1}45, 0\right)$	$-1 \leq q \leq 0,\ q \neq -\frac{1}{2},\ r \neq 0$	sol	≥ 1	≥ 0	$\checkmark$	?	[Boc09, Pr 7.2.5-6, Table 3] ([Har96])
$\mathfrak{g}_{5.14}^{0}$	$\left(-25, 0, -45, 35, 0\right)$		sol	≥ 2	≥ 3	$\checkmark$	?	[Boc09, Pr 7.2.7-8, Table 3]
$\mathfrak{g}_{5.15}^{-1}$	$\left(-15-25, -25, 35-45, 45, 0\right)$		cplt-sol	1	2	$\checkmark$	$\times$	[Boc09, Pr 7.2.9, Table 3]
$\mathfrak{g}_{5.16}^{-1,q}$	$\left(-15-25, -25, 35 - q^{-1}45, q^{-1}35 + 45, 0\right)$	$q \neq 0$	sol	–	–	$\times$	–	[Boc09, Pr 7.2.10] ([Har96])
$\mathfrak{g}_{5.17}^{p,-p,r}$	$\left(-p^{-1}15-25, 15-p^{-1}25, p^{-1}35 - r^{-1}45, r^{-1}35 + p^{-1}45, 0\right)$	$r \neq 0$	sol	≥ 1	≥ 0	$\checkmark$	?	[Boc09, Pr 7.2.11-12-13-14, Table 3] ([Har96])
$\mathfrak{g}_{5.18}^{0}$	$\left(-25-35, 15-45, -45, 35, 0\right)$		sol	≥ 1	≥ 2	$\checkmark$	?	[Boc09, Pr 7.2.15, Table 3]
$\mathfrak{g}_{5.19}^{p,-2p-2}$	$\left(-23-(1+p)^{-1}15, -25, -p^{-1}35, (2p+2)^{-1}45, 0\right)$	$p \neq -1$	cplt-sol	–	–	$\times$	–	[Boc09, Pr 7.2.16]
$\mathfrak{g}_{5.20}^{-1}$	$\left(-23-45, -25, 35, 0, 0\right)$		cplt-sol	2	1	$\checkmark$	$\checkmark$	[Boc09, Pr 7.2.16-17, Table 3]
$\mathfrak{g}_{5.23}^{-4}$	$\left(-23-\frac{1}{2}15, -25, -25-35, \frac{1}{4}45, 0\right)$		cplt-sol	–	–	$\times$	–	[Boc09, Pr 7.2.16]

(continued)

Table 1.2 (continued)

Name	Structure equations	Conditions	Type[a]	b_1	b_2	Lattice[b]?	Formal?	Note
$\mathfrak{g}_{5.25}^{p,4p}$	$\left(-23-(2p)^{-1}15, -p^{-1}25+35, -25-p^{-1}35, (4p)^{-1}45, 0\right)$	$p \neq 0$	sol	–	–	×	–	[Boc09, Pr 7.2.16]
$\mathfrak{g}_{5.26}^{0,\varepsilon}$	$\left(-23-\varepsilon^{-1}45, 35, -25, 0, 0\right)$	$\varepsilon \in \{-1, 1\}$	sol	≥2	≥1	✓	?	[Boc09, Pr 7.2.16-18, Table 3]
$\mathfrak{g}_{5.28}^{-\frac{3}{2}}$	$\left(-23+\left(\frac{1}{2}\right)^{-1}15, \frac{2}{3}25, -35, -35-45, 0\right)$		cplt-sol	–	–	×	–	[Boc09, Pr 7.2.16]
$\mathfrak{g}_{5.30}^{-\frac{4}{3}}$	$\left(-24-\frac{3}{2}15, -34+\left(\frac{1}{3}\right)^{-1}25, \frac{3}{4}35, -45, 0\right)$		cplt-sol	–	–	×	–	[Boc09, Pr 7.2.19]
$\mathfrak{g}_{5.33}^{-1,-1}$	$\left(-14, -25, 34+35, 0, 0\right)$		cplt-sol	2	1	✓	✓	[DF09, Theorem 3.1, Claim 3, Table 3]
$\mathfrak{g}_{5.35}^{-2,0}$	$\left(\frac{1}{2}14, -24-35, -34+25, 0, 0\right)$		sol	≥2	≥1	✓	?	[Boc09, Pr 7.2.20, Table 3]

[a] "Ab" means Abelian; "nil" means nilpotent; "cplt-sol" means completely-solvable; "sol" means solvable
[b] "✓" means that there exists a lattice at least for some values of the parameters; "×" means that there is no lattice for any value of the parameters

Table 1.3 Isomorphism classes of six-dimensional nilpotent Lie algebras [Boc09, Tables 4, 15, and 16], see also [Sal01, Sect. 5], [CG04, Table 1], see [Boc09, Sect. 8]

♯ [Boc09]	Name	Structure equations [Boc09]	b_1	b_2	b_3	nilstep	Type 3 (complex)	Type 2	Type 1	Type 0 (symplectic)	Other lists [Mag86]	[CFGU97]	[CG04]	[Sal01]	Structure equations [Sal01]
B01	$6\mathfrak{g}_1$	$(0,0,0,0,0,0)$	6	15	20	1	✓	✓	✓	✓	–	CFGU01	CG34	S34	$(0,0,0,0,0,0)$
B02	$\mathfrak{g}_{3.1} \oplus 3\mathfrak{g}_1$	$(-23,0,0,0,0,0)$	5	11	14	2	✓	✓	✓	✓	–	CFGU8	CG33	S33	$(0,0,0,0,0,12)$
B03	$\mathfrak{g}_{5.4} \oplus \mathfrak{g}_1$	$(-24-35,0,0,0,0,0)$	5	9	10	2	✓	✓	✓	×	–	CFGU03	CG32	S32	$(0,0,0,0,0,12+34)$
B04	$\mathfrak{g}_{5.1} \oplus \mathfrak{g}_1$	$(-35,-45,0,0,0,0)$	4	9	12	2	✓	✓	✓	✓	–	CFGU06	CG30	S31	$(0,0,0,0,12,13)$
B05	$2\mathfrak{g}_{3.1}$	$(-23,0,0,-56,0,0)$	4	8	10	2	✓	✓	✓	✓	–	CFGU02	CG29	S30	$(0,0,0,0,12,34)$
B06	$\mathfrak{g}_{6N.4}$	$(0,0,0,0,-12,-13-24)$	4	8	10	2	✓	✓	✓	✓	M04	CFGU04	CG28	S29	$(0,0,0,0,12,14+23)$
B07	$\mathfrak{g}_{6N.5}$	$(0,0,0,0,-13-24,-14+23)$	4	8	10	2	✓	✓	✓	✓	M05	CFGU05	CG31	S28	$(0,0,0,0,13+42,14+23)$
B08	$\mathfrak{g}_{5.5} \oplus \mathfrak{g}_1$	$(-34-25,-35,0,0,0,0)$	4	7	8	3	✓	✓	✓	✓	–	CFGU09	CG27	S27	$(0,0,0,0,12,14+25)$
B09	$\mathfrak{g}_{4.1} \oplus 2\mathfrak{g}_1$	$(-24,-34,0,0,0,0)$	4	7	8	3	×	✓	✓	✓	–	CFGU17	CG26	S26	$(0,0,0,0,12,15)$
B10	$\mathfrak{g}_{6.N12}$	$(0,0,0,-13,0,-14-25)$	4	6	6	3	×	✓	✓	×	M12	CFGU20	CG25	S25	$(0,0,0,0,12,15+34)$
B11	$\mathfrak{g}_{6.N3}$	$(0,0,0,-13,-23,-12)$	3	8	12	2	✓	✓	✓	✓	M03	CFGU07	CG13	S24	$(0,0,0,12,13,23)$
B12	$\mathfrak{g}_{6.N1}$	$(0,0,-12,-13,0,-15)$	3	6	8	3	✓	✓	✓	✓	M01	CFGU10	CG12	S23	$(0,0,0,12,13,14)$
B13	$\mathfrak{g}_{6.N6}$	$(0,0,0,-13,-14-23,-12)$	3	6	8	3	✓	✓	✓	✓	M06	CFGU11	CG10	S21	$(0,0,0,12,13,14+23)$
B14	$\mathfrak{g}_{6.N7}$	$(0,0,0,-13,-14,-23)$	3	6	8	3	✓	✓	✓	✓	M07	CFGU12	CG11	S22	$(0,0,0,12,13,24)$
B15	$\mathfrak{g}_{5.2} \oplus \mathfrak{g}_1$	$(-25,-35,-45,0,0,0)$	3	5	6	4	×	×	✓	✓	–	CFGU21	CG17	S13	$(0,0,0,12,14,15)$
B16	$\mathfrak{g}_{5.3} \oplus \mathfrak{g}_1$	$(-25,-45,-24,0,0,0)$	3	5	6	3	✓	✓	✓	×	–	CFGU16	CG18	S17	$(0,0,0,12,14,24)$
B17	$\mathfrak{g}_{5.6} \oplus \mathfrak{g}_1$	$(-34-25,-35,-45,0,0,0)$	3	5	6	4	×	×	✓	✓	–	CFGU22	CG16	S12	$(0,0,0,12,14,15+24)$
B18	$\mathfrak{g}_{6.N8}$	$(0,0,-12,-13,-12,-25)$	3	5	6	3	✓	×	✓	✓	M08	CFGU13	CG20[a]	S20	$(0,0,0,12,13+14,24)$
B19	$\mathfrak{g}_{6.N9}$	$(0,0,-12,-13,-23,-15)$	3	5	6	3	✓	×	✓	✓	M09	CFGU14	CG19	S19	$(0,0,0,12,14,13+42)$
B20	$\mathfrak{g}_{6.N10}$	$(0,0,-12,0,-13-24,-14+23)$	3	5	6	3	✓	✓	✓	✓	M10	CFGU15	CG24[b]	S18	$(0,0,0,12,13+42,14+23)$
B21	$\mathfrak{g}_{6.N13}$	$(0,0,0,-13,-12,-14-25)$	3	5	6	3	×	✓	✓	×	M13	CFGU18	CG09	S14	$(0,0,0,12,13,14+35)$
B22	$\mathfrak{g}^1_{6.N14}$	$(0,0,0,-13,-23,-14-25)$	3	5	6	3	✓	✓	✓	×	M14	CFGU19	CG22	S16	$(0,0,0,12,23,14-35)$

(continued)

Table 1.3 (continued)

♯ [Boc09]	Name	Structure equations [Boc09]	b_1	b_2	b_3	nilstep	Type 3 (complex)	Type 2	Type 1	Type 0 (symplectic)	Other lists [Mag86]	[CFGU97]	[CG04]	[Sal01]	Structure equations [Sal01]
B23	$\mathfrak{g}_{6.N14}^{-1}$	$(0, 0, 0, -13, -23, -14 + 25)$	3	5	6	3	×	✓	✓	×	M14	CFGU19	CG21	S15	$(0, 0, 0, 12, 23, 14 + 35)$
B24	$\mathfrak{g}_{6.N15}$	$(0, 0, -12, -13, -12, -14 - 25)$	3	5	6	4	×	×	✓	✓	M15	CFGU24	CG15	S11	$(0, 0, 0, 12, 14, 15 + 23 + 24)$
B25	$\mathfrak{g}_{6.N17}$	$(0, 0, -12, -13, 0, -14 - 25)$	3	5	6	4	×	×	✓	✓	M17	CFGU25	CG14	S10	$(0, 0, 0, 12, 14, 15 + 23)$
B26	$\mathfrak{g}_{6.N16}$	$(0, 0, 0, -13, -14 - 23, -15 - 24)$	3	4	4	4	×	✓	✓	✓	M16	CFGU27	CG23	S09	$(0, 0, 0, 12, 14 - 23, 15 + 34)$
B27	$\mathfrak{g}_{6.N11}$	$(0, 0, -12, -13, -14, -23)$	2	4	6	4	×	×	×	✓	M11	CFGU23	CG04	S06	$(0, 0, 12, 13, 23, 14)$
B28	$\mathfrak{g}_{6.N18}^{1}$	$(0, 0, -12, -13, -23, -14 - 25)$	2	4	6	4	✓	✓	✓	✓	M18	CFGU26	CG06	S08	$(0, 0, 12, 13, 23, 14 + 25)$
B29	$\mathfrak{g}_{6.N18}^{-1}$	$(0, 0, -12, -13, -23, -14 + 25)$	2	4	6	4	×	×	×	✓	M18	CFGU26	CG05	S07	$(0, 0, 12, 13, 23, 14 - 25)$
B30	$\mathfrak{g}_{6.N2}$	$(0, 0, -12, -13, -14, -15)$	2	3	4	5	×	×	✓	✓	M02	CFGU28	CG01	S03	$(0, 0, 12, 13, 14, 15)$
B31	$\mathfrak{g}_{6.N19}$	$(0, 0, -12, -13, -14, -15 - 23)$	2	3	4	5	×	×	✓	✓	M19	CFGU29	CG03	S05	$(0, 0, 12, 13, 14, 23 + 15)$
B32	$\mathfrak{g}_{6.N20}$	$(0, 0, -12, -13, -14 - 23, -15 - 24)$	2	3	4	5	×	×	✓	✓	M20	CFGU30	CG08	S04	$(0, 0, 12, 13, 14 + 23, 24 + 15)$
B33	$\mathfrak{g}_{6.N21}$	$(0, 0, -12, -23, -24, -15 - 34)$	2	2	2	5	×	×	✓	×	M21	CFGU31	CG02	S02	$(0, 0, 12, 13, 14, 34 + 52)$
B34	$\mathfrak{g}_{6.N22}$	$(0, 0, -12, -23, -24 + 13, -15 - 34)$	2	2	2	5	×	×	✓	×	M22	CFGU32	CG07	S01	$(0, 0, 12, 13, 14 + 23, 34 + 52)$

[a] The structure equations in [CG04] are written as (0, 0, 0, 12, 14, 23 + 24)
[b] The structure equations in [CG04] are written as (0, 0, 0, 12, 14 + 23, 13 + 42)

Table 1.4 M. Ceballos, A. Otal, L. Ugarte, and R. Villacampa's classification of linear integrable complex structures on six-dimensional nilpotent Lie algebras up to equivalence [COUV11]

♯	Algebra	b_1	b_2	b_3	Complex structure(s)	Conditions ($\lambda \geq 0,\ c \geq 0,\ B \in \mathbb{C},\ D \in \mathbb{C}$) $\left(\mathcal{S}(B,c) := c^4 - 2\left(\lvert B\rvert^2+1\right)c^2 + \left(\lvert B\rvert^2-1\right)^2\right)$
00	$\mathfrak{h}_1 := (0,\ 0,\ 0,\ 0,\ 0,\ 0)$	6	15	20	$J := (0,\ 0,\ 0)$	
01	$\mathfrak{h}_2 := (0,\ 0,\ 0,\ 0,\ 12,\ 34)$	4	8	10	$J_1^D := \left(0,\ 0,\ \omega^{1\bar{1}} + D\,\omega^{2\bar{2}}\right)$,	$\mathfrak{Im}\,D = 1$
02					$J_2^D := \left(0,\ 0,\ \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}}\right)$,	$\mathfrak{Im}\,D > 0$
03	$\mathfrak{h}_3 := (0,\ 0,\ 0,\ 0,\ 0,\ 12+34)$	5	9	10	$J_1 := \left(0,\ 0,\ \omega^{1\bar{1}} + \omega^{2\bar{2}}\right)$	
04					$J_2 := \left(0,\ 0,\ \omega^{1\bar{1}} - \omega^{2\bar{2}}\right)$	
05	$\mathfrak{h}_4 := (0,\ 0,\ 0,\ 0,\ 12,\ 14+23)$	4	8	10	$J_1 := \left(0,\ 0,\ \omega^{1\bar{1}} + \omega^{1\bar{2}} + \frac{1}{4}\,\omega^{2\bar{2}}\right)$	
06					$J_2^D := \left(0,\ 0,\ \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}}\right)$,	$D \in \mathbb{R} \setminus \{0\}$
07	$\mathfrak{h}_5 := (0,\ 0,\ 0,\ 0,\ 13+42,\ 14+23)$	4	8	10	$J_1^D := \left(0,\ 0,\ \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}}\right)$,	$D \in \left[0,\ \frac{1}{4}\right)$
08					$J_2 := \left(0,\ 0,\ \omega^{12}\right)$	
09					$J_3^{(\lambda,\ D)} := \left(0,\ 0,\ \omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}}\right)$,	$(\lambda,\ D) \in \left\{(0,\ x+\mathrm{i}\,y) \in \mathbb{R}\times\mathbb{C}\ :\ y \geq 0\ , 4y^2 < 1+4x\right\}$ $\cup\left\{(\lambda,\ \mathrm{i}\,y) \in \mathbb{R}\times\mathbb{C}\ :\ 0 < \lambda^2 < \frac{1}{2},\ 0 \leq y < \frac{\lambda^2}{2}\right\}$ $\cup\left\{(\lambda,\ \mathrm{i}\,y) \in \mathbb{R}\times\mathbb{C}\ :\ \frac{1}{2} \leq \lambda^2 < 1,\ 0 \leq y < \frac{1-\lambda^2}{2}\right\}$ $\cup\left\{(\lambda,\ \mathrm{i}\,y) \in \mathbb{R}\times\mathbb{C}\ :\ \lambda^2 > 1,\ 0 \leq y < \frac{\lambda^2-1}{2}\right\}$
10	$\mathfrak{h}_6 := (0,\ 0,\ 0,\ 0,\ 12,\ 13)$	4	9	12	$J := \left(0,\ 0,\ \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}}\right)$	
11	$\mathfrak{h}_7 := (0,\ 0,\ 0,\ 12,\ 13,\ 23)$	3	8	12	$J := \left(0,\ \omega^{1\bar{1}},\ \omega^{12} + \omega^{1\bar{2}}\right)$	
12	$\mathfrak{h}_8 := (0,\ 0,\ 0,\ 0,\ 0,\ 12)$	5	11	14	$J := \left(0,\ 0,\ \omega^{1\bar{1}}\right)$	
13	$\mathfrak{h}_9 := (0,\ 0,\ 0,\ 0,\ 12,\ 14+25)$	4	7	8	$J := \left(0,\ \omega^{1\bar{1}},\ \omega^{1\bar{2}} + \omega^{2\bar{1}}\right)$	
14	$\mathfrak{h}_{10} := (0,\ 0,\ 0,\ 12,\ 13,\ 14)$	3	6	8	$J := \left(0,\ \omega^{1\bar{1}},\ \omega^{12} + \omega^{2\bar{1}}\right)$	

(continued)

Table 1.4 (continued)

♯	Algebra	b_1	b_2	b_3	Complex structure(s)	Conditions ($\lambda \geq 0,\ c \geq 0,\ B \in \mathbb{C},\ D \in \mathbb{C}$) $\left(\mathcal{S}(B,c) := c^4 - 2\left(\lvert B\rvert^2+1\right)c^2 + \left(\lvert B\rvert^2-1\right)^2\right)$
15	$\mathfrak{h}_{11} := (0,\ 0,\ 0,\ 12,\ 13,\ 14+23)$	3	6	8	$J^B := \left(0,\ \omega^{1\bar{1}},\ \omega^{12} + B\,\omega^{1\bar{2}} + \lvert B-1\rvert\,\omega^{2\bar{1}}\right)$,	$B \in \mathbb{R} \setminus \{0,\ 1\}$
16	$\mathfrak{h}_{12} := (0,\ 0,\ 0,\ 12,\ 13,\ 24)$	3	6	8	$J^B := \left(0,\ \omega^{1\bar{1}},\ \omega^{12} + B\,\omega^{1\bar{2}} + \lvert B-1\rvert\,\omega^{2\bar{1}}\right)$,	$\mathfrak{Im}\,B \neq 0$
17	$\mathfrak{h}_{13} := (0,\ 0,\ 0,\ 12,\ 13+14,\ 24)$	3	5	6	$J^{(B,c)} := \left(0,\ \omega^{1\bar{1}},\ \omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}}\right)$,	$c \neq \lvert B-1\rvert,\ (c,\ \lvert B\rvert) \neq (0,\ 1),\ \mathcal{S}(B,c) < 0$
18	$\mathfrak{h}_{14} := (0,\ 0,\ 0,\ 12,\ 14,\ 13+42)$	3	5	6	$J^{(B,c)} := \left(0,\ \omega^{1\bar{1}},\ \omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}}\right)$,	$c \neq \lvert B-1\rvert,\ (c,\ \lvert B\rvert) \neq (0,\ 1),\ \mathcal{S}(B,c) = 0$
19	$\mathfrak{h}_{15} := (0,\ 0,\ 0,\ 12,\ 13+42,\ 14+23)$	3	5	6	$J_1 := \left(0,\ \omega^{1\bar{1}},\ \omega^{2\bar{1}}\right)$	
20					$J_2^c := \left(0,\ \omega^{1\bar{1}},\ \omega^{1\bar{2}} + c\,\omega^{2\bar{1}}\right)$,	$c \neq 1$
21					$J_3^{(B,c)} := \left(0,\ \omega^{1\bar{1}},\ \omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}}\right)$,	$c \neq \lvert B-1\rvert,\ (c,\ \lvert B\rvert) \neq (0,\ 1),\ \mathcal{S}(B,c) > 0$
22	$\mathfrak{h}_{16} := (0,\ 0,\ 0,\ 12,\ 14,\ 24)$	3	5	6	$J^B := \left(0,\ \omega^{1\bar{1}},\ \omega^{12} + B\,\omega^{1\bar{2}}\right)$,	$\lvert B\rvert = 1,\ B \neq 1$
23	$\mathfrak{h}_{19}^{-} := (0,\ 0,\ 0,\ 12,\ 23,\ 14-35)$	3	5	6	$J_1 := \left(0,\ \omega^{13} + \omega^{1\bar{3}},\ \mathrm{i}\left(\omega^{1\bar{2}} - \omega^{2\bar{1}}\right)\right)$	
24					$J_2 := \left(0,\ \omega^{13} + \omega^{1\bar{3}},\ -\mathrm{i}\left(\omega^{1\bar{2}} - \omega^{2\bar{1}}\right)\right)$	
25	$\mathfrak{h}_{26}^{+} := (0,\ 0,\ 12,\ 13,\ 23,\ 14+25)$	2	4	6	$J_1 := \left(0,\ \omega^{13} + \omega^{1\bar{3}},\ \mathrm{i}\,\omega^{1\bar{1}} + \mathrm{i}\left(\omega^{1\bar{2}} - \omega^{2\bar{1}}\right)\right)$	
26					$J_2 := \left(0,\ \omega^{13} + \omega^{1\bar{3}},\ \mathrm{i}\,\omega^{1\bar{1}} - \mathrm{i}\left(\omega^{1\bar{2}} - \omega^{2\bar{1}}\right)\right)$	

A.4 Six-Dimensional Solvmanifolds

The six-dimensional solvable non-decomposable Lie algebras were classified by G.M. Mubarakzjanov, [Mub63b, CS05], and by P. Turkowski, [Tur90]; see also [Boc09, Sect. 8]. In addition to [Boc09, Table 15], which lists the six-dimensional nilpotent non-decomposable Lie algebras, [Mor58], we refer to [Boc09, Tables 17/33], and also [Mac13, Table A.1], for a list of the six-dimensional unimodular solvable non-nilpotent non-decomposable Lie algebras; see [Boc09, Sect. 8]. The Betti numbers of six-dimensional unimodular solvable non-nilpotent Lie algebras can be found in [Mac13, Table 1] (see also [Boc09, Tables 34/36] for the first Betti numbers). See [Boc09, Tables 5–6] for a list of decomposable non-nilpotent solvmanifolds, together with their Betti numbers and results concerning formality and symplectic structures. In [CM12], six-dimensional non-completely-solvable almost-Abelian solvmanifolds for which Mostow condition does not hold are investigated; in particular, see [CM12, Table 1]. The six-dimensional unimodular solvable non-nilpotent Lie algebras admitting a symplectic structure are listed in [Mac13, Theorem 1], and their symplectic structures are listed in [Mac13, Table B.1]. In [Boc09, Sect. 9] and [Mac13], the Hard Lefschetz Condition on solvmanifolds has been investigated; in particular, [Boc09, Table 7] summarizes the relations between formality, Hard Lefschetz Condition, and even odd-degree Betti numbers, providing some known examples, see also [IRTU03, Theorem 3.1, Table 1] (see also [FMS03, dBT06, Kas13a, Kas09, Kas11, Kas12d] for examples and results on cohomologically-Kähler manifolds); furthermore, in [Mac13, Theorem 4], M. Macrì provided three solvmanifolds endowed with symplectic structures that satisfy the Hard Lefschetz Condition, more precisely, $\Gamma\Big\backslash\left(G_{5.7}^{p,-1,-1}\times\mathbb{R}\right)$ (see also [Boc09, Theorem 9.1]), and $\Gamma\backslash\left(G_{3.4}\times\mathbb{R}^3\right)$, and $\Gamma\backslash\left(G_{3.4}\times G_{3.4}\right)$.

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