Aftermath of Finding the Memristor

R. Stanley Williams

Abstract In this paper, a personal guide to the set of Leon Chua's papers that I have found most helpful in my research will be provided in the hope that this will encourage others to study them and answer the questions they have. Then, the paper will be finished with some observations and comments about Prof. Leon Chua's definition of memristor in mathematics.

Observing the response to our paper "The missing memristor found" [1] over the past four years in both the popular press and the scientific literature has been fascinating. A significant part of the scientific process is to vet descriptions of new ideas or objects, and the bigger the potential impact of a concept, the more rigorous that scrutiny should be. However, intertwined with this process are many human issues of desire for recognition and priority of discovery, as well as an often strong bias to reject anything new without actually understanding it. There are a lot of misconceptions about memristors floating around that are difficult to correct with only a few explanatory pages. Real understanding requires a great deal of hard work, and the resources essential to achieve that understanding already exist in the literature. However, for the vast majority of us, skimming over a few papers is completely insufficient; I spent years reading and re-reading Prof. Leon Chua's papers before I started to really get an appreciation of what he was saying. I have several copies of many different papers completely covered with highlighter of many colors and with my scrawled notes—each time I read one of his paper, and I continually refer back to them, I learn something new and my appreciation deepens. Although he has written some wonderful tutorials, most of Prof. Chua's writings are formal and dense with information, and thus can be intimidating; they absolutely require a level of mathematical sophistication to comprehend, but to those who persevere, they are marvels of rigor and, eventually, clarity. Here I will provide a personal guide to the set of papers that I have found most helpful in my research in the hope that this will encourage others to study them and answer the questions they have, and then I will finish with some observations and comments.

R. Stanley Williams (✉)
Hewlett-Packard Labs, 1501 Page Mill Road, Palo Alto, CA 94304, USA
e-mail: stan.williams@hp.com

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During the 1960's, Prof. Chua established the mathematical foundations for nonlinear circuit theory, which was the basis for his classic 1969 textbook *Introduction to Nonlinear Network Theory* as well as a large number of papers in refereed journals. As a result of this work, Prof. Chua made an interesting observation that led to his discovery of the memristor as a mathematical entity, reported in 1971 [2]. For completely linear circuits (which is highly restrictive, since real physical systems will display nonlinearity beyond some operating range), there are only three independent two-terminal passive circuit elements: the resistor R, the capacitor C and the inductor L. However, when he generalized the mathematical relations to be nonlinear, there was another independent differential relationship that in principle coupled the charge q that flowed through a circuit and the flux ϕ in the circuit, $d\phi = M\,dq$, that was mathematically different from the nonlinear resistance that coupled the voltage v to the current i, $dv = R\,di$. He mathematically explored the properties of this new model nonlinear circuit element, and found that it was essentially a resistor with memory—it was a device that changed its resistance depending on the amount of charge that flowed through the device, and thus he called this hypothetical circuit element M a memristor. This conclusion was independent of any physical mechanism that might couple the flux and charge, and none was postulated. Moreover, the memristor definition did not require causality. In other words, the mathematical relationship between flux and charge could be the result of some other cause—any mechanism that led to the constraint embodied by the equation $d\phi = M\,dq$ would lead to a device with the circuit properties of a memristor. This prediction of the properties of a new circuit element from symmetry principles was totally unique and revolutionary, and did not depend on any experimental observation. He published these initial findings essentially as a curiosity—it was not obvious at that time that a physical analog of such a circuit element existed, and thus he called it the "missing element".

In 1976, with his then student Sung Mo Kang, he published a critical generalization of the original memristor concept [3], but this has not been cited with the frequency of the 1971 paper, so fewer people seem to be aware of it. Chua and Kang introduced the fact that a 'memristive device' has a state variable (or variables), indicated by w, that describes the physical properties of the device at any time. A memristive system is characterized by two equations, the 'quasi-static' conduction equation that relates the voltage across the device to the current through it at any particular time via a generalized resistance,

$$v = R(w, i)i$$

and the dynamical equation, which explicitly asserts that the derivative of the state variable w is a function f of itself and the current through the device,

$$dw/dt = f(w, i).$$

Neither the flux ϕ nor the charge q explicitly appears in either of these two equations, but if $w = q$, $R(w, i) = R(w)$ and $f(w, i) = i$, the two equations reduce to the original definition of a memristor. Furthermore, the quasi-static conduction equation places a requirement on the current-voltage characteristic of the device—if a memristive system is driven with some type of cyclic excitation, such as a sinusoidal current, the plot of the voltage vs. the current will be a Lissajous curve for which the voltage is always equal to zero when the current is zero, and vice versa. Chua called this curve a 'pinched hysteresis loop', and it has an important physical interpretation—neither a memristor nor a memristive system stores either charge or energy (like a capacitor, for example), but they do 'remember' their history because of their changing resistance. This 1976 paper showed many other properties of the generalized memristor and also discussed possible examples, but again this was a mathematical exercise that was independent of any physical mechanism known at the time. The importance to real systems is that if one can identify the state variable with a physical property of a device and experimentally determine the dynamical and quasistatic equations, then one has a useful model for the element that can be used for designing a circuit that would utilize the device.

There is another pair of papers that are critical for not only understanding how memristors stand as independent devices, but how to appropriately understand a nonlinear circuit element model, how to construct one from a physics-based mechanism or black-box electrical measurements, and how the model relates to an actual nonlinear circuit [4, 5]. The two papers are best read together; the 1980 paper [4] is mathematically thorough, broad in coverage and filled with deep insights, whereas the 1984 paper is more tutorial and descriptive. I often find myself going back and forth between the two for the complementary viewpoints they express. We learn that no circuit model is an exact equivalent because no physical device can be exactly mimicked by a mathematical equation. A particular physical device may be best described by different models depending on the operating range, with the 'best' model being the simplest one that produces realistic results. There are several properties that a realistic model should have, including well-posedness (no mathematical artifacts that cause an unphysical situation), the capability to be simulated, qualitative similarity to the physical system (e.g. same initial and asymptotic behavior), the ability to predict previously unexplored operating modes, and structural resilience (stability under small perturbations of the model parameters). These concepts are made clear through mathematical definitions and examples. Thus, one needs an appropriate set of models (I think of them as basis functions) that are as complete as possible to describe a real system. Creating a device model is an art that can utilize a wide range of inputs and insights—there is no unique way to define the best possible model; was it useful in enabling a circuit to be designed and did it predict the properties of the circuit to within some desired accuracy? If there is a physical device for which the properties are well described by a particular model, then we can call that device by the name of the model, understanding that a more complete description may require some attribute of a different model. For example, all physical inductors have an intrinsic resistance, which is usually described as a model resistance in series with a model inductance.

The final two papers are both tutorials and are written in a much more informal style [6, 7]. They are very useful for people who just want to get a light overview of memristors without digging into the mathematical details, but also contain significant new insights and are therefore valuable even for people who have mastered the first four papers. However, no one should read only [6, 7] and think that they comprehend the subject—any word description can be misconstrued or misrepresented; the actual definitions are all mathematical. In his 2002 publication [6], Prof. Chua correctly realized that as electronic device dimensions shrink into the nanometer scale, their properties will become more nonlinear and thus the issues for understanding nonlinear circuits are becoming increasingly more relevant and critical. He used a fascinating analogy working up from the 'Laws' of motion postulated by Aristotle, Newton and Einstein to illustrate the necessity of choosing the right variables for a model in the first place and then what happens when the model progresses from an initial linear approximation to a more realistic nonlinear formulation. There follows several completely worked out examples to illustrate nonlinear circuit element modeling from his previous papers, including memristors. In the final paper [7], he describes memristors and memristive systems, and makes the observation that in fact the latter are a relatively straightforward generalization of the former, and recommends that from now on to simplify the nomenclature that both should be called memristors. By creating memristor models for the pinched i–v hysteresis loops of each, he shows that specific physical examples of memristors include several devices that are the subject of contemporary research: bipolar and unipolar resistive switches, often called RRAM or ReRAM; 'atomic switches'; spin-transfer torque (STT) RAM devices; and phase-change memory devices; which are based on a wide variety of materials and physical mechanisms [7].

An important issue to understand is that the discovery of the memristor mathematical model does not conflict with nor compete for priority against the various realizations of physical devices that exhibit this circuit behavior. It is complementary, in that it provides a mathematical framework for designing and actually using the devices in circuits. It also provides an important mathematical constraint for those who are interested in the physics of their devices—any mechanism proposed for how the device operates needs to be in agreement with the memristor equations or it is not valid. Thus, researchers who are working on various types of resistive switching devices need not fear the memristor, but rather should embrace it. It is a high-level mathematical model that can be used to predict the circuit behavior of a wide variety of physical devices, it provides a unifying framework to put the circuit properties of all the devices into context, and therefore provides insight into how the various devices may substitute for each other in a wide variety of (especially nonmemory) applications originally developed for a different device.

Another issue is that no matter how careful one tries to be, any word description of a mathematical model will likely be incomplete, just as the model itself is only an approximation of the properties of a physical system. Thus, papers or discussions that argue about the meaning of a particular word or phrase often miss the point, since words can be ambiguous and interpreted (or twisted) in different ways. That is why we use mathematics in science—when a question arises about the specific meaning of a concept, we must go back to the defining equations. This is where Prof. Leon Chua's work stands out—precise, complete, insightful and totally rigorous.

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