Chapter 2
Wick Polynomials

In this chapter we consider the so-called Wick polynomials, a multi-dimensional generalization of Hermite polynomials. They are closely related to multiple Wiener–Itô integrals.

Let X_t, $t \in T$, be a set of jointly Gaussian random variables indexed by a parameter set T. Let $EX_t = 0$ for all $t \in T$. We define the real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}$ in the following way: A square integrable random variable is in $\mathscr{H}$ if and only if it is measurable with respect to the σ-algebra $\mathscr{B} = \mathscr{B}(X_t,\ t \in T)$, and the scalar product in $\mathscr{H}$ is defined as $(\xi, \eta) = E\xi\eta$, $\xi,\ \eta \in \mathscr{H}$. The Hilbert space $\mathscr{H}_1 \subset \mathscr{H}$ is the subspace of $\mathscr{H}$ generated by the finite linear combinations $\sum c_j X_{t_j}$, $t_j \in T$. We consider only such sets of Gaussian random variables X_t for which $\mathscr{H}_1$ is separable. Otherwise X_t, $t \in T$, can be arbitrary, but the most interesting case for us is when $T = \mathscr{S}_\nu$ or $\mathbb{Z}_\nu$, and X_t, $t \in T$, is a stationary Gaussian field.

Let $Y_1, Y_2, \dots$ be an orthonormal basis in $\mathscr{H}_1$. The uncorrelated random variables $Y_1, Y_2, \dots$ are independent, since they are (jointly) Gaussian. Moreover,

$$\mathscr{B}(Y_1, Y_2, \dots) = \mathscr{B}(X_t,\ t \in T).$$

Let $H_n(x)$ denote the n-th Hermite polynomial with leading coefficient 1, i.e. let $H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2})$. We recall the following results from analysis and measure theory.

Theorem 2A. *The Hermite polynomials $H_n(x)$, $n = 0, 1, 2, \dots$, form a complete orthogonal system in $L_2\left(R, \mathscr{B}, \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right)$. (Here $\mathscr{B}$ denotes the Borel σ-algebra on the real line.)*

Let $(X_j, \mathscr{X}_j, \mu_j)$, $j = 1, 2, \dots$, be countably many independent copies of a probability space $(X, \mathscr{X}, \mu)$. (We denote the points of X_j by x_j.) Let $(X^\infty, \mathscr{X}^\infty, \mu^\infty) = \prod\limits_{j=1}^{\infty} (X_j, \mathscr{X}_j, \mu_j)$. With such a notation the following result holds.

P. Major, *Multiple Wiener-Itô Integrals*, Lecture Notes in Mathematics 849, DOI 10.1007/978-3-319-02642-8_2,

Theorem 2B. *Let $\varphi_0, \varphi_1, \dots, \varphi_0(x) \equiv 1$, be a complete orthonormal system in the Hilbert space $L_2(X, \mathscr{X}, \mu)$. Then the functions $\prod_{j=1}^{\infty} \varphi_{k_j}(x_j)$, where only finitely many indices k_j differ from 0, form a complete orthonormal basis in $L_2(X^\infty, \mathscr{X}^\infty, \mu^\infty)$.*

Theorem 2C. *Let $Y_1, Y_2, \dots$ be random variables on a probability space $(\Omega, \mathscr{A}, P)$ taking values in a measurable space $(X, \mathscr{X})$. Let ξ be a real valued random variable measurable with respect to the σ-algebra $\mathscr{B}(Y_1, Y_2, \dots)$, and let $(X^\infty, \mathscr{X}^\infty)$ denote the infinite product $(X \times X \times \cdots, \mathscr{X} \times \mathscr{X} \times \cdots)$ of the space $(X, \mathscr{X})$ with itself. Then there exists a real valued, measurable function f on the space $(X^\infty, \mathscr{X}^\infty)$ such that $\xi = f(Y_1, Y_2, \dots)$.*

Remark. Let us have a stationary random field $X_n(\omega)$, $n \in \mathbb{Z}_\nu$. Theorem 2C enables us to extend the shift transformation T_m, defined as $T_m X_n(\omega) = X_{n+m}(\omega)$, $n, m \in \mathbb{Z}_\nu$, for all random variables $\xi(\omega)$, measurable with respect to the σ-algebra $\mathscr{B}(X_n(\omega), n \in \mathbb{Z}_\nu)$. Indeed, by Theorem 2C we can write $\xi(\omega) = f(X_n(\omega), n \in \mathbb{Z}_\nu)$, and define $T_m\xi(\omega) = f(X_{n+m}(\omega), n \in \mathbb{Z}_\nu)$. We still have to understand, that although the function f is not unique in the representation of the random variable $\xi(\omega)$, the above definition of $T_m\xi(\omega)$ is meaningful. To see this we have to observe that if $f_1(X_n(\omega), n \in \mathbb{Z}_\nu) = f_2(X_n(\omega), n \in \mathbb{Z}_\nu)$ for two functions f_1 and f_2 with probability 1, then also $f_1(X_{n+m}(\omega), n \in \mathbb{Z}_\nu) = f_2(X_{n+m}(\omega), n \in \mathbb{Z}_\nu)$ with probability 1 because of the stationarity of the random field $X_n(\omega), n \in \mathbb{Z}_\nu$. Let us also observe that $\xi(\omega) \stackrel{\Delta}{=} T_m\xi(\omega)$ for all $m \in \mathbb{Z}_\nu$. Besides, T_m is a linear operator on the linear space of random variables, measurable with respect to the σ-algebras $\mathscr{B}(X_n, n \in \mathbb{Z}_\nu)$. If we restrict it to the space of square integrable random variables, then T_m is a unitary operator, and the operators T_m, $m \in \mathbb{Z}_\nu$, constitute a unitary group.

Let a stationary generalized random field $X = \{X(\varphi), \varphi \in \mathscr{S}\}$ be given. The shift $T_t\xi$ of a random variable ξ, measurable with respect to the σ-algebra $\mathscr{B}(X(\varphi), \varphi \in \mathscr{S})$ can be defined for all $t \in R^\nu$ similarly to the discrete case with the help of Theorem 2C and the following result. If $\xi \in \mathscr{B}(X(\varphi), \varphi \in \mathscr{S})$ for a random variable ξ, then there exists such a countable subset $\{\varphi_1, \varphi_2, \dots\} \subset \mathscr{S}$ (depending on the random variable ξ) for which ξ is $\mathscr{B}(X(\varphi_1), X(\varphi_2), \dots)$ measurable. (We write $\xi(\omega) = f(X(\varphi_1)(\omega), X(\varphi_2)(\omega), \dots)$ with appropriate functions f, and $\varphi_1 \in \mathscr{S}$, $\varphi_2 \in \mathscr{S}$,..., and define the shift $T_t\xi$ as $T_t\xi(\omega) = f(X(T_t\varphi_1)(\omega), X(T_t\varphi_2)(\omega), \dots)$, where $T_t\varphi(x) = \varphi(x-t)$ for $\varphi \in \mathscr{S}$.) The transformations T_t, $t \in R^\nu$, are linear operators over the space of random variables measurable with respect to the σ-algebra $\mathscr{B}(X(\varphi), \varphi \in \mathscr{S})$ with similar properties as their discrete counterpart.

Theorems 2A–2C have the following important consequence.

Theorem 2.1. *Let $Y_1, Y_2, \dots$ be an orthonormal basis in the Hilbert space $\mathscr{H}_1$ defined above with the help of a set of Gaussian random variables X_t, $t \in T$. Then the set of all possible finite products $H_{j_1}(Y_{l_1}) \cdots H_{j_k}(Y_{l_k})$ is a complete orthogonal* *system in the Hilbert space $\mathscr{H}$ defined above. (Here $H_j(\cdot)$ denotes the j-th Hermite polynomial.)*

Proof of Theorem 2.1. By Theorems 2A and 2B the set of all possible products $\prod_{j=1}^{\infty} H_{k_j}(x_j)$, where only finitely many indices k_j differ from 0, is a complete orthonormal system in $L_2\left(R^\infty, \mathscr{B}^\infty, \prod_{j=1}^{\infty} \frac{e^{-x_j^2/2}}{\sqrt{2\pi}}\,dx_j\right)$. Since $\mathscr{B}(X_t,\ t \in T) = \mathscr{B}(Y_1, Y_2, \dots)$, Theorem 2C implies that the mapping $f(x_1, x_2, \dots,) \to f(Y_1, Y_2, \dots)$ is a unitary transformation from $L_2\left(R^\infty, \mathscr{B}^\infty, \prod_{j=1}^{\infty} \frac{e^{-x_j^2/2}}{\sqrt{2\pi}}\,dx_j\right)$ to $\mathscr{H}$. (We call a transformation from a Hilbert space to another Hilbert space unitary if it is norm preserving and invertible.) Since the image of a complete orthogonal system under a unitary transformation is again a complete orthogonal system, Theorem 2.1 is proved. $\square$

Let $\mathscr{H}_{\le n} \subset \mathscr{H}$, $n = 1, 2, \dots,$ (with the previously introduced Hilbert space $\mathscr{H}$) denote the Hilbert space which is the closure of the linear space consisting of the elements $P_n(X_{t_1}, \dots, X_{t_m})$, where P_n runs through all polynomials of degree less than or equal to n, and the integer m and indices $t_1, \dots, t_m \in T$ are arbitrary. Let $\mathscr{H}_0 = \mathscr{H}_{\le 0}$ consist of the constant functions, and let $\mathscr{H}_n = \mathscr{H}_{\le n} \ominus \mathscr{H}_{\le n-1}$, $n = 1, 2, \dots,$ where $\ominus$ denotes orthogonal completion. It is clear that the Hilbert space $\mathscr{H}_1$ given in this definition agrees with the previously defined Hilbert space $\mathscr{H}_1$. If $\xi_1, \dots, \xi_m \in \mathscr{H}_1$, and $P_n(x_1, \dots, x_m)$ is a polynomial of degree n, then $P_n(\xi_1, \dots, \xi_m) \in \mathscr{H}_{\le n}$. Hence Theorem 2.1 implies that

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1 + \mathscr{H}_2 + \cdots, \tag{2.1}$$

where $+$ denotes direct sum. Now we introduce the following

Definition of Wick Polynomials. *Given a polynomial $P(x_1, \dots, x_m)$ of degree n and a set of (jointly) Gaussian random variables $\xi_1, \dots, \xi_m \in \mathscr{H}_1$, the Wick polynomial $:\!P(\xi_1, \dots, \xi_m)\!:$ is the orthogonal projection of the random variable $P(\xi_1, \dots, \xi_m)$ to the above defined subspace $\mathscr{H}_n$ of the Hilbert space $\mathscr{H}$.*

It is clear that Wick polynomials of different degree are orthogonal. Given some $\xi_1, \dots, \xi_m \in \mathscr{H}_1$ define the subspaces $\mathscr{H}_{\le n}(\xi_1, \dots, \xi_m) \subset \mathscr{H}_{\le n}$, $n = 1, 2, \dots,$ as the set of all polynomials of the random variables $\xi_1, \dots, \xi_m$ with degree less than or equal to n. Let $\mathscr{H}_{\le 0}(\xi_1, \dots, \xi_m) = \mathscr{H}_0(\xi_1, \dots, \xi_m) = \mathscr{H}_0$, and $\mathscr{H}_n(\xi_1, \dots, \xi_m) = \mathscr{H}_{\le n}(\xi_1, \dots, \xi_m) \ominus \mathscr{H}_{\le n-1}(\xi_1, \dots, \xi_m)$. With the help of this notation we formulate the following

Proposition 2.2. *Let $P(x_1, \dots, x_m)$ be a polynomial of degree n. Then the random polynomial $:P(\xi_1, \dots, \xi_m):$ equals the orthogonal projection of $P(\xi_1, \dots, \xi_m)$ to $\mathscr{H}_n(\xi_1, \dots, \xi_m)$.*

Proof of Proposition 2.2. Let $:\bar{P}(\xi_1,\dots,\xi_m):$ denote the projection of the random polynomial $P(\xi_1,\dots,\xi_m)$ to $\mathcal{H}_n(\xi_1,\dots,\xi_m)$. Obviously

$$P(\xi_1,\dots,\xi_m) - :\bar{P}(\xi_1,\dots,\xi_m): \in \mathcal{H}_{\le n-1}(\xi_1,\dots,\xi_m) \subseteq \mathcal{H}_{\le n-1}.$$

Hence in order to prove Proposition 2.2 it is enough to show that for all $\eta \in \mathcal{H}_{\le n-1}$

$$E:\bar{P}(\xi_1,\dots,\xi_m):\eta = 0, \tag{2.2}$$

since this means that $:\bar{P}(\xi_1,\dots,\xi_m):$ is the orthogonal projection of $P(\xi_1,\dots,\xi_m) \in \mathcal{H}_{\le n}$ to $\mathcal{H}_{\le n-1}$.

Let $\varepsilon_1,\varepsilon_2,\dots$ be an orthonormal system in $\mathcal{H}_1$, also orthonormal to $\xi_1,\dots,\xi_m$, and such that $\xi_1,\dots,\xi_m,\varepsilon_1,\varepsilon_2,\dots$ form a basis in $\mathcal{H}_1$. If $\eta = \prod_{i=1}^{m} \xi_i^{l_i} \prod_{j=1}^{\infty} \varepsilon_j^{k_j}$ with such exponents l_i and k_j that $\sum l_i + \sum k_j \le n-1$, then (2.2) holds for this random variable η because of the independence of the random variables ξ_i and ε_j. Since the linear combinations of such η are dense in $\mathcal{H}_{\le n-1}$, formula (2.2) and Proposition 2.2 are proved. □

Corollary 2.3. *Let $\xi_1,\dots,\xi_m$ be an orthonormal system in $\mathcal{H}_1$, and let*

$$P(x_1,\dots,x_m) = \sum c_{j_1,\dots,j_m} x^{j_1} \cdots x_m^{j_m}$$

be a homogeneous polynomial, i.e. let $j_1 + \cdots j_m = n$ with some fixed number n for all sets $(j_1,\dots,j_m)$ appearing in this summation. Then

$$:P(\xi_1,\dots,\xi_m):= \sum c_{j_1,\dots,j_m} H_{j_1}(\xi_1)\cdots H_{j_m}(\xi_m).$$

In particular,

$$:\xi^n:= H_n(\xi) \quad \textit{if } \xi \in \mathcal{H}_1, \textit{ and } E\xi^2 = 1.$$

Remark. Although we have defined the Wick polynomial (of degree n) for all polynomials $P(\xi_1,\dots,\xi_m)$ of degree n, we could have restricted our attention only to homogeneous polynomials of degree n, since the contribution of each terms $c(j_1,\dots j_m)\xi_1^{l_1}\cdots\xi_m^{l_m}$ of the polynomial $P(\xi_1,\dots,\xi_m)$ such that $l_1 + \cdots + l_m < n$ has a zero contribution in the definition of the Wick polynomial $:P(\xi_1,\dots,\xi_m):$.

Proof of Corollary 2.3. Let the degree of the polynomial P be n. Then

$$P(\xi_1,\dots,\xi_m) - \sum c_{j_1,\dots,j_m} H_{j_1}(\xi_1)\cdots H_{j_m}(\xi_m) \in \mathcal{H}_{\le n-1}(\xi_1,\dots,\xi_m), \tag{2.3}$$

since $P(\xi_1,\dots,x_m)-\sum c_{j_1,\dots,j_m}H_{j_1}(\xi_1)\cdots H_{j_m}(\xi_m)$ is a polynomial whose degree is less than n. Let $\eta=\xi_1^{l_1}\cdots\xi_m^{l_m}$, $\sum\limits_{i=1}^{m} l_i\le n-1$. Then

$$E\eta H_{j_1}(\xi_1)\cdots H_{j_m}(\xi_m)=\prod_{i=1}^{m}E\xi_i^{l_i}H_{j_i}(\xi_i)=0,$$

since $l_i<j_i$ for at least one index i. Therefore

$$E\eta\sum c_{j_1,\dots,j_m}H_{j_1}(\xi_1)\cdots H_{j_m}(\xi_m)=0. \tag{2.4}$$

Since every element of $\mathscr{H}_{\le n-1}(\xi_1,\dots,\xi_m)$ can be written as the sum of such elements η, relation (2.4) holds for all $\eta\in\mathscr{H}_{\le n-1}(\xi_1,\dots,\xi_m)$. Relations (2.3) and (2.4) imply Corollary 2.3. □

The following statement is a simple consequence of the previous results.

Corollary 2.4. *Let $\xi_1,\xi_2,\dots$ be an orthonormal basis in $\mathscr{H}_1$. Then the random variables $H_{j_1}(\xi_1)\cdots H_{j_k}(\xi_k)$, $k=1,2,\dots$, $j_1+\dots+j_k=n$, form a complete orthogonal basis in $\mathscr{H}_n$.*

Proof of Corollary 2.4. It follows from Corollary 2.3 that

$$H_{j_1}(\xi_1)\cdots H_{j_k}(\xi_k)=:\xi_1^{j_1}\cdots\xi_k^{j_k}:\in\mathscr{H}_n\quad\text{for all } k=1,2,\dots$$

if $j_1+\dots+j_k=n$. These random variables are orthogonal, and all Wick polynomials $:P(\xi_1,\dots,\xi_m):$ of degree n of the random variables $\xi_1,\xi_2,\dots$ can be represented as the linear combination of such terms. Since these Wick polynomials are dense in $\mathscr{H}_n$, this implies Corollary 2.4. □

The arguments of this chapter exploited heavily some properties of Gaussian random variables. Namely, they exploited that the linear combinations of Gaussian random variables are again Gaussian, and in Gaussian case orthogonality implies independence. This means in particular, that the rotation of a standard normal vector leaves its distribution invariant. We finish this chapter with an observation based on these facts. This may illuminate the content of formula (2.1) from another point of view. We shall not use the results of the subsequent considerations in the rest of this work.

Let U be a unitary transformation over $\mathscr{H}_1$. It can be extended to a unitary transformation $\mathscr{U}$ over $\mathscr{H}$ in a natural way. Fix an orthonormal basis $\xi_1,\xi_2,\dots$ in $\mathscr{H}_1$, and define $\mathscr{U}1=1$, $\mathscr{U}\xi_{j_1}^{l_1}\cdots\xi_{j_k}^{l_k}=(U\xi_{j_1})^{l_1}\cdots(U\xi_{j_k})^{l_k}$. This transformation can be extended to a linear transformation $\mathscr{U}$ over $\mathscr{H}$ in a unique way. The transformation $\mathscr{U}$ is norm preserving, since the joint distributions of $(\xi_{j_1},\xi_{j_2},\dots)$ and $(U\xi_{j_1},U\xi_{j_2},\dots)$ coincide. Moreover, it is unitary, since $U\xi_1,U\xi_2,\dots$ is an orthonormal basis in $\mathscr{H}_1$. It is not difficult to see that if $P(x_1,\dots,x_m)$ is an arbitrary polynomial, and $\eta_1, \eta_2 \dots, \eta_m \in \mathscr{H}_1$, then $\mathscr{U} P(\eta_1, \dots, \eta_m) = P(U\eta_1, \dots, U\eta_m)$. This relation means in particular that the transformation $\mathscr{U}$ does not depend on the choice of the basis in $\mathscr{H}_1$. If the transformations $\mathscr{U}_1$ and $\mathscr{U}_2$ correspond to two unitary transformations U_1 and U_2 on $\mathscr{H}_1$, then the transformation $\mathscr{U}_1\mathscr{U}_2$ corresponds to $U_1 U_2$. The subspaces $\mathscr{H}_{\le n}$ and therefore the subspaces $\mathscr{H}_n$ remain invariant under the transformations $\mathscr{U}$.

The shift transformations of a stationary Gaussian field, and their extensions to $\mathscr{H}$ are the most interesting examples for such unitary transformations U and $\mathscr{U}$. In the terminology of group representations the above facts can be formulated in the following way: The mapping $U \to \mathscr{U}$ is a group representation of $U(\mathscr{H}_1)$ over $\mathscr{H}$, where $U(\mathscr{H}_1)$ denotes the group of unitary transformations over $\mathscr{H}_1$. Formula (2.1) gives a decomposition of $\mathscr{H}$ into orthogonal invariant subspaces of this representation.

Springer

http://www.springer.com/978-3-319-02641-1

Multiple Wiener-Itô Integrals
With Applications to Limit Theorems
Major, P.
2014, XIII, 126 p. 4 illus., Softcover
ISBN: 978-3-319-02641-1