Chapter 2
A Simple Model for Decoherence

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Abstract The meaning of decoherence as a (practically) irreversible process in Quantum Mechanics is discussed. Also a simple two-particle model is introduced consisting of a heavy (the system) and a light (the environment) particle and the decoherence effect is explicitly computed on the heavy particle due to the presence of the light one.

It is generally believed that one of the main distinctive character of Quantum Mechanics is the superposition principle.

From the mathematical point of view, it simply means that if one has two possible states for the system then also any their (normalised) linear combination is a possible state, due to the fact that the state space of the system has a linear structure.

The key point is that a superposition state in general describes entirely new physical properties of the system which cannot be argued from the knowledge of the component states separately.

A typical example considered here is the case of a particle in one dimension described, in the position representation, by the superposition state $\psi_t(x) = \frac{1}{\sqrt{2}}(\psi_t^+(x) + \psi_t^-(x))$, where ψ_t^+, ψ_t^- are two normalised and orthogonal states at time $t \geq 0$. If one computes the probability distribution of the position of the particle one obviously has

$$\left|\psi_t(x)\right|^2 = \frac{1}{2}\left|\psi_t^+(x)\right|^2 + \frac{1}{2}\left|\psi_t^-(x)\right|^2 + \mathrm{Re}\big(\psi_t^+(x)\overline{\psi_t^-}(x)\big) \tag{1}$$

Then if the supports of the two states are not disjoint, the interference term $\mathrm{Re}(\psi_t^+(x)\overline{\psi_t^-}(x))$ is relevant and it is responsible for the interference fringes observed in real experiments involving microscopic objects, e.g. the two slits experiment.

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It is remarkable that, due to the presence of the interference term, one cannot interpret the state ψ_t as a classical statistical mixture of identical particles which are in ψ_t^+ or ψ_t^- with probability one half.

The possibility of producing such interference is one of the most relevant characteristic behaviours of the microscopic world which is accurately described applying the rules of Quantum Mechanics.

On the other hand the Schroedinger equation has universal validity and in particular it can be used to describe systems consisting of a macro object coupled with a micro object. In such a situation it is easily seen that a superposition state of the micro object can be transferred to the macro object as a result of the dynamical evolution.

This means that the theory predicts the existence of superposition states and the highly non-classical interference effects also for macro objects which, of course, are not usually observed in our everyday life.

This apparent paradox can be explained if one realises that superposition states are in fact fragile and then they can be destroyed even by a weak interaction with an environment. Such dynamical and practically irreversible mechanism of suppression is usually called decoherence.

In the last 30 years the phenomenon of decoherence has been described in the physical literature using many different models (see e.g. [4] and references therein).

Nevertheless only few of these results are mathematically proved and then a further analysis in the direction of a rigorous study of simple models in which the approximations used are controlled is required.

Here we shall describe a first attempt of rigorous derivation of the decoherence effect in a two particles system ([2], see also [3] for results in the same direction).

The basic tool for the analysis is the representation of the state by a density matrix, i.e. a positive, trace-class operator ρ_t, with $\operatorname{Tr}\rho_t = 1$, acting on the Hilbert space of the system $\mathcal{H}$.

If in particular $\rho_t^2 = \rho_t$, i.e. ρ_t is a projector on some $\xi_t \in \mathcal{H}$, then one recovers the usual description in terms of the wave function ξ_t and ρ_t is called a pure state.

In the general case $\rho_t^2 \neq \rho_t$ the state is called a mixture.

The difference between pure and mixed states can be understood in terms of the entropy $S(\rho_t) = -\operatorname{Tr}\rho_t \log\rho_t$; for a pure state the entropy vanishes (corresponding to the maximal information available on the system) while it is strictly positive for a mixture (corresponding to our degree of knowledge on the preparation of the state).

If one considers an isolated particle described by the superposition (pure) state ψ_t introduced above, the corresponding density matrix in the position representation is given by the kernel

$$\begin{aligned}\rho_t^p\left(x, x'\right) &= \overline{\psi_t}(x)\psi_t\left(x'\right)\\ &= \frac{1}{2}\overline{\psi_t^+}(x)\psi_t^+\left(x'\right) + \frac{1}{2}\overline{\psi_t^-}(x)\psi_t^-\left(x'\right)\\ &\quad + \frac{1}{2}\overline{\psi_t^+}(x)\psi_t^-\left(x'\right) + \frac{1}{2}\overline{\psi_t^-}(x)\psi_t^+\left(x'\right)\end{aligned} \tag{2}$$

The last two terms in (2) are usually called off-diagonal terms and they are responsible for the interference effects (in fact the probability distribution for the position $\rho_t^p(x,x)$ reduces to (1)).

On the opposite side one can considers the mixed state for the same particle

$$\rho_t^m\big(x,x'\big)=\frac{1}{2}\overline{\psi_t^+}(x)\psi_t^+\big(x'\big)+\frac{1}{2}\overline{\psi_t^-}(x)\psi_t^-\big(x'\big) \tag{3}$$

obtained from (2) by eliminating the off-diagonal terms. In such a case all the interference effects are cancelled and one can say that the particle is in ψ_t^+ or in ψ_t^- with probability one half, i.e. one has a classical statistical mixture of ψ_t^+ and ψ_t^-, corresponding to our ignorance on the preparation of the state.

In this sense we can say that a quantum particle described by ρ_t^m exhibits a classical behaviour.

Notice that $S(\rho_t^m)=\log 2$, which is the entropy associated to a classical bit with two possible levels of probability one half.

Between the two extreme cases ρ_t^p and ρ_t^m one can have an intermediate situation in which the off-diagonal terms are non-vanishing but reduced with respect to the pure case.

If one considers the more general situation of a particle interacting with an environment it is convenient to introduce the notion of reduced density matrix. Let x, y be the coordinates of the particle and the environment, respectively, and let $\rho_t(x,y,x',y')$ the corresponding density matrix in the position representation.

If the environment is considered practically not observable, we can only be interested in the expectation values of (bounded) observables A^x relative to the particle, i.e. operators acting only on the x variable.

Then, applying the standard rules of Quantum Mechanics, one has

$$\langle A^x\rangle_{\rho_t}=\mathrm{Tr}\big(A^x\rho_t\big)=\mathrm{Tr}_x\big(A^x\hat\rho_t\big),\qquad \hat\rho_t\big(x,x'\big)=\int dy\rho_t\big(x,y,x',y\big) \tag{4}$$

where Tr_x denotes the trace with respect to the coordinates of the particle and $\hat\rho_t$ is the reduced density matrix. It is now clear that $\hat\rho_t$ is the basic object for the investigation of the dynamics of the particle in presence of the environment.

More precisely, we shall consider an initial state for the particle plus environment in a product form $\rho_0=\rho_0^p\otimes\rho_0^e$, where ρ_0^p is a superposition (pure) state of the particle of the form (2) and ρ_0^e is a state for the environment.

The reduced density matrix of the system at time zero is $\hat\rho_0=\rho_0^p$ and, clearly, $S(\hat\rho_0)=0$. Due to the interaction between the particle and the environment, at any time $t>0$ the density matrix ρ_t is no longer a product state and the reduced density matrix $\hat\rho_t$ is in general a complicated mixture, with $S(\hat\rho_t)>0$ (i.e. in the transition from $\hat\rho_0$ to $\hat\rho_t$ there is an obvious loss of information since the degrees of freedom of the environment have been neglected).

We shall say that the environment has produced a decoherence effect on the particle if, after some short time t, $\hat\rho_t$ takes a form very close to (3).

Such a kind of result can be proved under suitable condition on the environment.

We shall consider here the extremely simple case of one heavy particle (the system) interacting with a light particle (the environment) via a delta potential in dimension one.

The self-adjoint hamiltonian in $\mathcal{H} = L^2(R^2)$ describing the two particles is

$$H = -\frac{\hbar^2}{2M}\Delta_x - \frac{\hbar^2}{2m}\Delta_y + \alpha_0\delta(x-y), \quad \alpha_0 > 0 \tag{5}$$

and we consider the initial state

$$\rho_0\bigl(x,y,x',y'\bigr) = \rho_0^p\bigl(x,x'\bigr)\rho_0^e\bigl(y,y'\bigr), \qquad \rho_0^e\bigl(y,y'\bigr) = \overline{\phi_0}(y)\phi_0\bigl(y'\bigr) \tag{6}$$

where $\rho_0^p(x,x')$ is given in (2) and

$$\begin{aligned} &\psi_0^{\pm}(x) = \frac{1}{\sqrt{\sigma}} f\left(\frac{x \pm R_0}{\sigma}\right) e^{\pm i \frac{P_0}{\hbar} x}, \qquad \phi_0(y) = \frac{1}{\sqrt{\delta}} g\left(\frac{y}{\delta}\right), \\ &\sigma, \delta, R_0, P_0 > 0, \quad f, g \in C_0^\infty(-1,1) \end{aligned} \tag{7}$$

According to (6), (7), the heavy particle is initially in a superposition of two wave packets, one localised in $-R_0$ with momentum P_0 and the other localised in R_0 with momentum $-P_0$; the light particle is localised in the region around the origin.

The model hamiltonian (5) has been considered for the sake of simplicity and the solution of the corresponding Schroedinger equation can be explicitly computed (see e.g. [6]).

In fact we are interested in the case in which the mass ratio $\varepsilon = \frac{m}{M}$ is small and in such a regime the evolution becomes particularly simple.

Since the dynamics is linear, we can analyse the evolutions of the wave packets ψ_0^+, ψ_0^- separately.

If we consider the wave packet ψ_0^+ coming from the left, we expect that it propagates almost freely and, after a time of order $\tau = \frac{MR_0}{P_0}$, it reaches the origin.

Then, due to the presence of the δ potential, the wave function of the light particle is partly reflected far away to the right and partly is transmitted, i.e. it remains localised around the origin.

Obviously, the wave packet ψ_0^- coming from the right produces an analogous effect, i.e. part of the wave function of the light particle is reflected far away to the left and the remaining part is transmitted.

This means that, after a time of order τ, only the transmitted parts of the wave function of the light particle have a common support.

The result is that in the reduced density matrix of the heavy particle $\hat{\rho}_t$ the diagonal terms are almost unaffected while the off-diagonal terms are reduced and the reduction is stronger if the transmitted wave is smaller.

The above intuitive picture can be proved in a rigorous way. In fact, assuming $\varepsilon \ll 1$ and, moreover, $\delta \ll R_0$, $\sigma \ll \frac{1}{\alpha} \ll R_0$, where $\alpha \equiv \frac{m\alpha_0}{\hbar^2}$, for $t > \tau$ one has

$$\begin{aligned}\hat{\rho}_t\left(x, x'\right) = &\frac{1}{2}\overline{U_t^0\psi_0^+}(x)U_t^0\psi_0^+\left(x'\right) + \frac{1}{2}\overline{U_t^0\psi_0^-}(x)U_t^0\psi_0^-\left(x'\right)\\ &+ \frac{\Lambda}{2}\overline{U_t^0\psi_0^+}(x)U_t^0\psi_0^-\left(x'\right) + \frac{\Lambda}{2}\overline{U_t^0\psi_0^-}(x)U_t^0\psi_0^+\left(x'\right) + \mathcal{E} \qquad (8)\end{aligned}$$

$$U_t^0\psi_0^\pm = e^{-i\frac{t}{\hbar}H_0}\psi_0^\pm, \qquad H_0 = -\frac{\hbar^2}{2M}\Delta, \qquad \Lambda = \int dk\left|\tilde{\phi}_0(k)\right|^2\frac{k^2}{\alpha^2+k^2} \quad (9)$$

and the small error $\mathcal{E}$ can be explicitly estimated, uniformly in $t > \tau$ (see [2] for details).

Notice that the parameter Λ is less than one and it represents the fraction of transmitted wave for a particle initially in ϕ_0 and subject to a point interaction of strength α.

Thus the effect of the light particle is to reduce the off-diagonal terms and this means a (partial) decoherence effect on the heavy particle.

The model considered here is clearly too simple and it can only have the pedagogical meaning to show explicitly a dynamical mechanism producing decoherence.

A more reasonable model of environment would be a gas of N (non-interacting) light particles.

In this more general situation we can expect that the effect of each scattering event is cumulative and then we would get the same expression (8) with Λ replaced by Λ^N which, for N large, means complete decoherence.

A similar argument has been heuristically justified in [5], while a rigorous derivation starting from the Schroedinger equation for the N-particle system is given in [1].

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