Chapter 2
Synthesis of Mathematical Models for Measuring Signals

2.1 Introduction

Signals play a key role in instrumentation and are an essential part of measurements. There is a wide variety of signals. First of all, these are signals for measurement of physical quantities: pressures, temperatures, voltages, currents, etc. These signals depend on a type of physical quantity and measurement conditions, and inevitably change with them. For this reason they are known only with a certain degree of probability. In addition, a lot of measurement control signals, signals for display of measured values, etc. are used in measuring means.

The so called reference or standard signals play a special role in instrumentation. Their basic parameters and waveform must be set with high precision since they have a small degree of uncertainty, i.e. high accuracy. They are used to have a desired effect on measured objects as "stimuli" or reference forces. Special measuring instruments (signal generators, sources and calibrators, which have been discussed in Chap. 1) are developed and produced for reproduction of these signals. Such devices can generate a variety of standard signals, including sine wave, square, triangle, and pulse ones. Their parameters and characteristics are well known. Figure 2.1 below shows their graphic presentations and mathematical description.

When we discuss sine wave generators, we usually idealize a waveform of their output voltage, assuming that it is described by the sine function of time $y_1(t) = y_m \sin [(2\pi/T)\,(t - t_0)]$ with parameters y_m, t_0 and T, and has only one harmonic in its spectrum. The sine function serves as an idealized model of voltages at the output of low-frequency and high-frequency signal generators, including ones of G3 and G4 groups (Russian classification). It is characterized by the known relationships between peak, average rectified (AVG) and root-mean-square (RMS) values.

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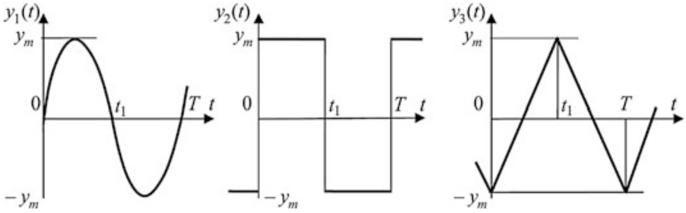

Fig. 2.1 Standard measuring signals

$$y_{\text{rms}} = \sqrt{\frac{1}{T}\int\limits_0^T [y_m \sin(\omega t + \varphi)]^2 dt} = y_m/\sqrt{2};$$

$$y_{|\text{avg}|} = \frac{1}{T}\int\limits_0^T |y_m \sin(\omega t + \varphi)| dt = 2y_m/\pi,$$

where $\omega = 2\pi/T, \ \ \varphi = 2\pi t_0/T.$

This also holds true for square and triangle signals. Their mathematical descriptions can serve as models for output voltages of special waveform generators (G6 group):

$$y_1(t) = y_m \sin(\omega t + \varphi)$$

$$y_2(t) = \begin{cases} y_m \text{ if } 0 \le t < t_1; \\ -y_m \text{ if } t_1 \le t < T; \end{cases}$$

$$y_3(t) = \begin{cases} \dfrac{2y_m}{t_1} t - y_m \text{ if } 0 \le t < t_1; \\ -\dfrac{2y_m}{T - t_1}(t - t_1) + y_m \text{ if } t_1 \le t < T, \end{cases}$$

where y_m and T are the amplitude and the repetition interval, respectively.

They have other relationship between peak, average rectified and root-mean-square values:

$$y_{\text{rms}} = y_{|\text{avg}|} = y_m;$$
$$y_{\text{rms}} = y_m/3, \ \ y_{|\text{avg}|} = y_m/2.$$

The models of square and triangle waveforms will be also considered as models of idealized signals since the first of them has a zero rising/decay time and equal amplitudes of positive and negative half waves, while the second one has an ideal straight line in rise and decay.

Due to the known relationships between parameters and a number of other properties, these signals are of paramount importance in the measuring and metrological practice. For example, a sine wave signal does not change its waveform when passing through linear electric circuits. Therefore, a deviation of its waveform from a sine waveform is a clear evidence of nonlinearity in a circuit. This property is widely used for in-process quality control of resistors and capacitors since high-order current harmonics appearing in an element through a resistor at sine wave voltage may be an indicator of poor workmanship (oxidation of contacts etc.). Moreover, this property is useful for determination of nonlinear distortions in sound signal amplifiers, sampling and holding devices for analog-to-digital and digital-to-analog converters.

The triangle wave voltage generates the square wave current in the ideal capacitor. That is why a deviation of the current waveform from a square one is an indicator of loss resistance in the capacitor, spray inductance, etc.

Nowadays, a lot of Russian and foreign generators of such signals with different accuracy levels are produced and operated. High-precision voltage calibrators are used in metrology for verification and calibration of voltmeters that measure peak, average rectified and root-mean-square values, as well as analog-to-digital converters. Medium-precision sources are designed for exciting resistive, inductive and capacitive sensors of physical quantities in instrumentation, for checking and adjusting sound signal amplifiers, etc.

However, real signals reproduced in generators do not coincide with the models ascribed to them. Therefore, other models should be developed for them to describe them more fully. Development of simple but sufficiently accurate models of real signals is an important theoretical problem.

The other problem is to synthesize signals with specified parameters since there is a continuous need to use signals of other (non-standard) waveforms with specific parameters in the measuring and metrological practice. For example, special test signals with prescribed harmonic distortion factors, amplitude, average values, spectrum or spectral ratio, and probabilistic characteristic are required to calibrate THD measuring instruments, AC voltmeters and spectrum analyzers, as well as to estimate an effect of a waveform on their accuracy. Some of these signals have been already synthesized and put into practice. For example, the so called truncated sine and different-sized sine signals have been legalized in the national and working total THD standards in the CIS countries [1, 2].

An important problem of the measuring signal theory is using different mathematical objects (analytic, piecewise linear functions, etc.) in the mathematical expressions that describe the time models of signals. Discrete and digital signals are described by number sequences, lattice functions and algebraic constructions, in which the continuous time is replaced by a discrete variable. It is needless to say about random signals, which, strictly speaking, cannot be mathematically described as a time function. All this complicates the analysis of electrical circuits and devices exposed to them. Thus, the current situation with mathematical modeling of real signals does not allow us to systematize the description of different (nonperiodic, periodic, continuous, discrete, digital, and random) signals, to make it uniform. Based on the existing models, it is impossible to develop a unified theory of self-oscillation models that reproduce such signals in generators.

Notwithstanding a variety of waveforms, parameters and spectra, all measuring signals have a number of common properties and regularities.

- reproducibility in measuring instruments;
- compatibility with measuring instruments by type and size of informative parameters;
- specified (known) waveforms and parameters;
- standardization of accuracy for basic informative parameters;
- possibility to estimate accuracy of reproduction by computational or experimental methods;
- possibility to extract measurement information in an optimal way.

Far from all signals that are traditionally discussed in the relevant literature [3–5] meet these requirements. For example, signals composed of δ-pulses are not classified as measuring ones because of their incompatibility with measuring means in terms of value. A random signal of "white noise" type cannot be a measuring signal since it is impossible to reproduce it in measuring instruments due to infinite energy.

A very important feature of the measuring signals reproduced in electrical measuring means is the possibility to define them by one or several parameters or characteristics associated with integral or functional dependences with time or frequency description of a signal. For example, average rectified, root-mean-square and peak values are given by the following functionals:

$$y_{|\mathrm{avg}|} = \frac{1}{T}\int_0^T |y(t)|dt; \quad y_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_0^T [y(t)]^2 dt}; \quad y_m = \max y(t).$$

Signals with normalized amplitude factor are defined even by a ratio of functionals that set peak and root-mean-square values. A random signal with a normalized probability density function is defined by a probability functional etc. Certainly, there are infinitely many signals that have a required root-mean-square value, amplitude factor or a given probability density function. From here, it is clear that the mathematical problem of synthesizing a measuring signal model by its parameters or characteristics is incorrect because of ambiguous solutions. Due to such incorrectness, an individual synthesis method is to be developed for almost each new class of signals.

Many studies, e.g. [6–8], are devoted to discussion and application of signals in radio engineering, measuring techniques and telecommunications. They focus on signal generation methods, estimation of signal reproduction accuracy, normalization methods, etc. By now, methods for synthesizing amplitude and frequency-modulated, phase-shift keyed [9], noise-like [10], and random signals have been well developed and become traditional. However, not all of them can be implemented with required metrological parameters and characteristics. The attempts to use these methods for simulation of self-oscillations at the output of signal sources have turned out to be ineffective. Indeed, signals to be reproduced with hardware must be, on the one hand, of a simple waveform, and, on the other hand, have required parameters and characteristics sometimes conflicting with one another.

These and other circumstances allow us to isolate measuring signals to be reproduced in signal sources as a special class of signals that have not been sufficiently studied from a metrological point of view. The industry has been raised the increasingly strict requirements both for metrological performance and functional capabilities of these sources for reproduction of different waveforms, including complex ones. Therefore, there is a real need in analysis and synthesis of signals with prescribed properties.

The term "measuring signal" need to be explained. The word *signal* has been used from the earliest times. It is derived from the Latin *signum*, a sign, and has similar meanings in many other languages. *The Explanatory Dictionary of the Live Great Russian Language* by Vladimir Dal gives the following interpretation to this word: "sign made for a certain purpose, message by sign, conventional mark (neither by mouth nor in writing); the word "beacon" is closely related to this (to make a signal, to give a beacon, to beacon)".[1]

S.I. Ozhegov (*Dictionary of the Russian Language*. Sovetskaya Entsiklopediya, Moscow (1972) interprets this word as follows:

1. Conventional sign for sending certain data or messages at a distance.
2. Something that gives an impetus to a certain action.
3. Warning, message about an undesired situation that may arise.

As defined in the *Great Soviet Encyclopedia*, "signal is a sign, physical process, or phenomenon that carries a certain message or information about a certain event, condition of an object under observation, or sends a control command, an instruction, a notification, etc". This definition combines the signal and its physical carrier (physical process or phenomenon).

The following meaning of the word "signal" is widely used in exact sciences and their applications, e.g. in the communication theory (see *The Electrical Communication Theory: College Textbook* under the editorship of D.D. Klovsky (1998)): "This is a physical process that represent (carries) a message to be sent". This definition focuses on a material carrier—a physical process. L. Franks (*Signal Theory,* 1974) defines a signal as a value that someway represents the condition of a physical system. In this sense, a signal is naturally considered as a result of certain measurements in relation to a physical system under observation.

From the perspective of systems analysis (see, e.g. *Fundamentals of Systems Analysis* (2001) by F.I. Peregudov, F.P. Tarasenko), "a signal is a physical information carrier, a medium for transferring information in space and time". Thus, this definition expressly equates the word "signal" with a physical carrier.

[1] The language style of Vladimir Dal's dictionary and other information sources referred to below is preserved where possible.

There is no uniformity in defining this term in radio engineering textbooks as well. S.I. Baskakov's textbook (*Radio Engineering Circuits and Signals: College Textbook* (1988) Moscow, Vyshaya Shkola), defines the signal as a time change in the physical condition of an object that serves for representing, registering, and sending messages. According to the study guide under the editorship of A.N. Yakovlev [11] (*Radio Engineering Circuits and Signals: Problems and Exercises: Study Guide* (2003), Moscow, INFA-M; Novosibirsk, Novosibirsk State Technical University), the signal is a physical process or a phenomenon that carries a message about a certain event, condition, or operating mode of an object, or sends control commands, etc.

M.M. Ayzinov in his book (*The Selected Issues of Signal and Circuit Theory* (1971). Moscow, Svyaz) gives a definition from A.A. Zheleznov: "The signal is a process capable to propagate, designed or used for control of local power sources or for having an effect on special elements of a specially organized system". This statement focuses on the fact that a physical phenomenon can be a signal in an organized system.

All these definitions underline, to a greater or lesser degree, that the signal has multiple functions (information, carrier, and user). On the one hand, there are not signals without information; on the other hand, signals that cannot be delivered to a user, i.e. when there is no physical process and a carrier (vehicle), are absolutely useless. Finally, the signals delivered to a user must be used for control in an organized system. Some of the definitions given above focus either on one, or two, or all three meanings. But what kind of signals can be classified as measuring ones?

According to P.P. Ornatsky (*Theoretical Basis of Information and Measuring Systems* (1976) Kiev: Vishcha Shkola), the measuring signal is a signal that has informative parameters and contains information on their values. This definition does not mention a carrier and a purpose of the signal.

However, the measuring practice traditionally uses the definitions recommended or legalized by national standards. For example, as defined by RMG-29 (Metrology. Key Terms and Definitions), the measuring signal is a signal that contains qualitative information on a physical quantity to be measured. Moreover, GOST 16465–70 "Measuring Radio Engineering Signals" defines a measuring radio engineering signal as electrical voltage or current that change in time, with previously known characteristics and are used for measuring and monitoring characteristics of radio-technical circuits. It expressly states that the signal is a physical quantity and mentions the existence of an organized system (radio engineering circuit).

Taking into account the fact that many definitions discussed above do not reflect a specific feature of the measuring signal, the author proposes the following definition. The measuring signal is a physical process with known parameters designed for determination of (by way of measurement) metrological performance of measuring means or devices. This definition clearly states that in the field of instrumentation the signal is a physical quantity (mostly voltage or current) that it is characterized by known parameters (a carrier contains some information) and intended to measure characteristics of devices (organized environment). Therefore, from now on, it is unconditionally accepted that a signal, its carrier and an organized system are inseparable since they cannot exist separately from each one. Indeed, it is not possible to measure, e.g. a root-mean-square value of AC voltage (parameter that contains information about signal level) with voltmeter without supplying such voltage to the voltmeter output (electric current is used as signal carrier). Moreover, the operator or processing devices in the information and measurement system read a value of this voltage to carry out some actions in the organized system (e.g. test system, calibration system, etc.) rather than for simple meditation.

Later on, signals are considered only as physical processes in the form of electrical voltage or current, with specified parameters, without referring to the system in which they are used since this system is identified during operation of signal reproduction means.

2.2 Synthesis of Signals Reproducible in Measuring Generators

Mathematical analysis and synthesis problems [3–6] are often solved using a representation of any arbitrary complex function as expansion in terms of the simplest (basis) functions $\{\psi_i(t)\}$:

$$y(t) = \sum_{n=0}^{\infty} a_n \psi_n(t), \tag{2.1}$$

where $\psi_n(t)$ are prescribed time functions, a_n is the expansion coefficient.

This expression is used for expansion of a complex signal to the simplest ones, i.e. for analyzing it (and vice versa, for synthesizing it from a sum of the simplest signals). Let us consider the possibilities of such a synthesis based on the expression (2.1) provided that it is to be implemented with hardware in a measuring signal generator.

For approximating deterministic functions, a_n values are taken as constant [3, 4]; for approximating random—as ransom ones [5, 12, 13]. However, the representation of a signal as series (2.1) based on the simultaneous (parallel) summation of infinitely many elementary functions $\psi_n(t)$ cannot be implemented in an generator due to an infinite number of the functions that form the basis $\{\psi_i(t)\}$. If the signal represented by series (2.1) is to be reproduced by hardware or software, a finite series should be used:

$$x(t) = \sum_{n=0}^{N} c_n \tilde{\psi}_n(t). \tag{2.2}$$

The number of expansion terms in the series (2.2) is limited by the first N terms. The exact values of the coefficient a_n are replaced by its approximate values c_n, while functions $\psi_n(t)$ are replaced by their approximate representations $\tilde{\psi}_n(t)$.

Let us remind that different systems of basic functions are used for expansion to the series (2.2): power, trigonometric, Rademacher, Walsh, Haar functions, Legendre and Jacobi polynomials, Kotelnikov-Shannon sampling series, etc. Let us estimate the possibilities to use the expansion (2.2) for synthesis of signals to be implemented by hardware. Due to a difference of the coefficients c_n from a_n, a limited number of the them, approximate realization of the functions $\tilde{\psi}_n(t)$, and the summation operation, a signal reproduction error arises:

$$\Delta x(t) = x(t) - y(t). \tag{2.3}$$

Obviously, this error is determined by errors of reproducing the functions $\psi_n(t)$, setting the coefficients a_n, and the summation operation. Taking into account that parameters of measuring signals used in means of measuring electric values are expressed through the functionals of $x(t)$, we need to minimize reproduction errors in a metric corresponding to these functionals.

The norm used for estimating the deviation of a real waveform from an idealized one in an uniform metric is defined by the following relationship:

$$\Delta_1 = \max_{x \in X} |x(t) - y(t)|.$$

A deviation of signal parameters is estimated using a quadratic metric with the error norm calculated by the formula:

$$\Delta_2 = \left\{ \frac{1}{T} \int_0^T [x(t) - y(t)]^2 dt \right\}^{1/2}.$$

The norm Δ_2 is known to be minimal if $\{\psi_n(t)\}$ is an orthogonal system of functions.

The values Δ_2 and Δ_1 may be taken as estimation of mean-root-square and maximum absolute error of signal reproduction and calculated for different methods of signal hardware implementation by the formula (2.2).

If signals are reproduced by digital electronics, especially by microprocessor-based hardware, the error is mainly determined by errors of generating the basis functions $\{\psi_i(t)\}$, representing the coefficients c_n and summation. The latter ones depend on bit depth r. Therefore, the round-off error inevitably occurs. It is known to be random, uniformly distributed over the range $[0, 2^{-r}]$ and not depend on the coefficients c_n.

If signals are generated by analog electronics, the error is also determined by errors of reproducing functions and coefficients, as well as summation. If we take into account only one component of the error for both cases, namely the error of setting c_n, the resulting errors can be represented as follows:

$$\Delta_1 = \Delta N; \;\; \Delta_2 = \Delta \sqrt{cN},$$

where Δ is the error of setting the summation factors, N is the number of term of sums.

Thus, the maximum absolute error is directly proportional to the number, while the root-mean-square error of reproducing the measuring signal defined by the series (2.2), which results from inaccuracy of the coefficients, is proportional to the square root of the number of orthogonal functions used for signal representation. In particular, these errors are proportional to the number of harmonics and the square root, respectively. Obviously, no matter how small the error Δ of setting the harmonic amplitudes, i.e. the coefficient c, there will always such a number N at which the errors Δ_1 and Δ_2 exceed any predefined number. A contradiction is evident here: the more accurate a measuring signal is to be, the larger the number of harmonics N, on the one hand, and the proportionally larger the error of reproduction, on the other hand. In other words, the smaller the error to be obtained by summation of series, the more random error. This is also confirmed by the incorrect summation of Fourier series—functions with ill-defined coefficients [14].

When reproducing periodic signals, the error in series coefficients (2.2) results in a significant random error of root-mean-square and, especially peak values of the signal. When generating random signals, dispersion and probability distribution function have a significant deviation.

Let us consider how to calculate the reproduction of a sine wave signal[2] with summation of power functions. Such a signal can be written as a series:

$$x(t) = \sum_{n=0}^{N} (-1)^n \frac{t^{2n+1}}{(2n+1)!}.$$

Figure 2.2a shows that the periodic sine wave signal and the signal represented by the sum of ten first terms of the power series are primarily reproduced only in a finite time interval (slightly more than one period). From then on, the distance between them is increasing, with fast growth of a signal reproduction error. It is necessary to increase a number of series terms for reproduction of a signal within a large time interval.

It follows from Fig. 2.2 that power series can be used only for reproduction of a signal in a limited time interval.

Moreover, the error of power series coefficients results in a higher signal reproduction error even over short time intervals. At a relative error of the series coefficients uniformly distributed over the interval of $+1$ to -1 % (Fig. 2.2b), the signal reproduction error rapidly increases. Here, the acceptable reproduction error can be achieved over the interval that is significantly less than one period.

Or take other example. A periodic signal of complex, triangle waveform is known to be represented as a trigonometric series:

$$y(t) = \sum_{n=1}^{\infty} c_n \sin\left(n \frac{2\pi}{T} t\right).$$

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[2] A sine wave signal can be generated by other, more appropriate methods. We have chosen it only for better illustration.

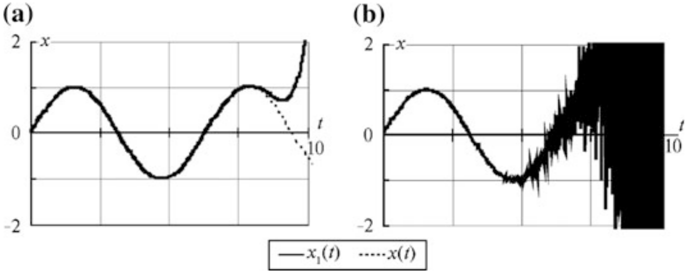


Fig. 2.2 Reproduction of a sine wave signal using power series at accurate (**a**) and inaccurate (**b**) series coefficients

When reproducing the signal with hardware or software, an infinite sum is replaced by a finite one with N terms. Naturally, as the number of terms N increases, the signal reproduction error decreases and can be as small as desired everywhere, except for points of discontinuity. However, when the error of series coefficients occurs (c_n), an increase in the number of its terms does not lead to decreasing in the reproduction error $y(t)$. To illustrate this, let us take an error of the setting coefficients c_n as random and uniformly distributed over the interval from +10 to −10 % at a number of series terms $N_1 = 100$ and $N_2 = 1000$. In this case, the signal *x(t)* to be reproduced will be defined by other sum:

$$x(t) = \sum_{n=1}^{N} \frac{1}{n} [1 + 0,1(2rnd(1) - 1)] \sin\left(n \frac{2\pi}{T} t\right),$$

where $rnd(1)$ are the numbers randomly distributed over the interval from 0 to 1.

The reproduction of a triangle signal is calculated by summing a series of trigonometric sine functions at approximate series coefficients (Fig. 2.3a). This graph shows the type of error. The relative error δ of the root-mean-square value of the triangle signal (Fig. 2.3b) depends on a number of expansion terms at accurate coefficient values (solid line). The random error range ζ at a spread of coefficients

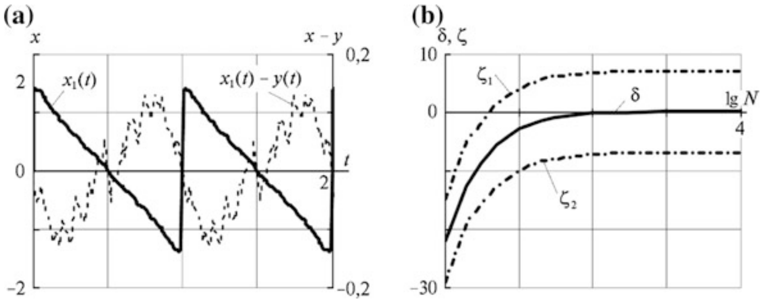


Fig. 2.3 Graphs of the saw tooth signal generated with a trigonometric Fourier series at inaccurate coefficients and $N = 100$ (**a**) and relative errors of a root-mean-square value (**b**)

is within ±10 %. It is seen that an increase in the number of expansion terms makes sense only to a certain value. Here, this value is equal to 6. At the further increase in the number of expansion terms, the random error significantly exceeds the error of method.

Obviously, the reproduction of a signal with a finite sum of Fourier series terms causes a significant method error of reproducing waveform and its root-mean-square value. The number of series terms is known to reduce such an error, but it is not accompanied by decreasing a random error.

Finally, let us give an example of signal representation as the expansion in Daubechies wavelet functions [15]. Figure 2.4 shows optimal square, sine and triangle waveforms $y(i)$ (dotted line) and their representation by a finite sum of five wavelet functions with the inaccurate coefficients $x_1(i)$ (solid line). Here, it is also necessary to increase the number of series terms for higher accuracy of representing source signals, but an acceptable accuracy can be attained at the smaller number of series terms.

However, when a certain error occurs in the coefficients (in Fig. 2.4 it is taken as random and uniformly distributed over the interval from −10 to +10 %), a significant error is also evident in all signal sections. The similar calculations can be performed for other functions of the series (2.2). Consequently, the representation of signals as the series (2.2), which is convenient for expansion (mathematical analysis) of functions in series, is almost not appropriate for hardware-

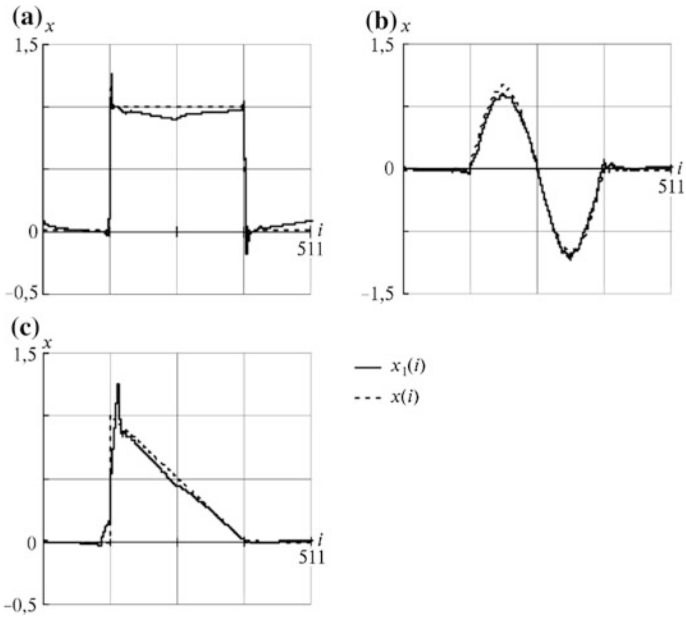


Fig. 2.4 Reproduction of signals using the series from Daubechies wavelet functions: square (**a**), sine (**b**) and triangle (**c**) waveforms at inaccurate coefficients

based reproduction (synthesis) of any complex measuring signals since this requires generating a large number of elementary functions (trigonometric, exponential, wavelet ones) and summing them. Probably, for this reason we know only several implementations of analog signal generators that use the method for reproducing complex signals based on the summation of trigonometric functions. For example, GSS-1, which was commercially produced in the USSR in the 1960s.

It was composed of several driving oscillators with multiple frequencies, output signals of which were summed up with different amplitudes and phase shifts. Today, this method is going through a revival in connection with the development of digital and microprocessor technologies, which make it possible to calculate a sum of the series (2.2) at high accuracy and to obtain an output analog signal at the output of the digital-to-analog converter. However, generation of the sum by means of microprocessors is time consuming and almost not feasible. That is why these operations are performed at high accuracy in advance, using high-power personal computers (PC). Calculations results are stored in memory of the microprocessor as a table of signal values at different times and then read in sequence to be sent to the digital-to-analog converter (DAC). In such an implementation, the summability method is converted into other method, which is called the "tabular" one. It is widely used for constructing digital[3] signal generators.

In addition to the summability method considered above, the methods based on simulation of differential or difference equations have found their application in construction of generators.

For example, it is well known that the solution of the system of i-th ten-order equations $x = f_i(x)$ may be periodic in time at certain constraints on the functions $f_i(x)$. This solution may be the basis for generation of a signal with specified parameters. Properly speaking, such a system of equations simulated by means of the electrical circuit with transistors or operational amplifiers gave rise to the Russian sine wave *RC*- and *LC* generators (G3 and G4 groups). This is the most promising method for constructing such generators.

In recent years, having been driven by the intensive implementation of information technologies in the measuring instrumentation, a new direction in the synthesis of signals has arisen and is dynamically developing. It has not received a common name yet because of diverse forms of its implementation. This is a symbiosis of PC and hardware. In these devices, all computing operations for signal generation are concentrated in PC, while a signal with required parameters is reproduced using the hardware component, which is based on a digital-to-analog converter. This component may be a self-contained device or a board to be inserted into a PCI slot or, finally, a unit to be plugged in the crate of VXI or PXI standard. Each manufacturer of so called arbitrary waveform generators provides their devices with software for reproduction of standard waveforms: sine, square, triangle, sawtooth, etc., as well as with software for synthesis of arbitrary signals.

[3] Let us remind that these oscillators cannot be called analog and digital ones to the full extent. We have discussed this issue in Chap. 1.

The generators produced under the AKIP brand are capable to synthesize signals with software AWG Quick Start and AWG-Navigator. They allow rapidly generating signals of standard waveforms (sine, triangle, pulse) and changing their basic parameters, such as frequency, amplitude, fixed bias, duty ratio or phase.

For example, *Tektronix* supplies its generators of AFG series with ArbExpress software, which uses a set of built-in standard functions or mathematical formulas for synthesizing new signals [16].

Generators 33220 and 33250 of *Agilent* are provided with the software for generating arbitrary waveforms (Signal Creation—Signal Studio) [17].

Tabor Electronics has developed and supplies the software ArbConnection for their generators. It is suitable for generating complex signals.

Brüel & Kjær's generators of 4070*A* type are also provided with software for synthesizing new signals.

National Instruments has achieved much success in this direction. It produces the whole family of signal generators in PXI and SCXI standards and the special software Analog Waveform Editor [18], which enables the synthesis of both periodic and random signals. This program has the following basic features. All signals are designed in PC in digital form on the basis of four types of source data:

- primitives;
- high pass and low pass filters, band-pass and band-rejection filters;
- formulas describing about fourteen functions;
- arithmetic operation with signals: exponentiation, summation, subtraction, multiplication and division of signals.

Primitives include about 25 signals, including ones similar to those shown in Fig. 2.1, as well as sawtooth signals, signals of rising and falling exponents. This group also includes signals of $\sin(x)/x$ type and three types of noise signals. By adding, subtracting or multiplying primitives, a great variety of waveforms for periodic, non-periodic and random signals can be generated.

These software products obviously expand the capabilities of generators for reproduction of complex waveforms by manipulating primitives. However, they have some disadvantages:

- all of them cannot be implemented in stand-alone and portable generators since they require high-power computers to be synthesized, which is not always possible because of higher cost of a device with built-in PC;
- high-quality primitives cannot be reproduced since there is no possibility to synthesize signals with specified parameters. For example, it is difficult if not impossible to generate signals with a given spectrum, harmonic distortion factor etc. Even if there is a possibility to do it, e.g. through summation of primitives on the basis of sine wave functions with multiple frequencies, this is accompanied by increasing the signal reproduction error, as discussed above.

However, these difficulties will be overcome with the development of the integrated-circuit technology and using more and more powerful computing resources.

Table 2.1 Comparison of signal synthesis methods

Hardware implementation	Application of the synthesis method			
	Series summation	Tabular method	Simulation of differential equations	Simulation of difference equations
Analog	–	–	+	+
Digital	+	+	–	–

Table 2.1 demonstrates the currently most known signal synthesis methods that can be implemented by means of analog and digital signal generators.

This table should be explained in detail. As is evident from it, simulation of differential equations is widely used for construction of analog signal generators over a broad frequency band since it can be implemented in the simplest way using electrical and electronic components: capacitors, resistors, inductance coils, transistors and operational amplifiers. Voltages and currents in these components are connected with each one by differential or integral relations, depending upon their physical nature, which makes it possible to easily simulate differential equations. This method is most preferred for generators of periodic sine wave signals and allows us to obtain signals with total harmonic distortion up to 0.0001 % and less, which is not feasible in digital generators. Nevertheless, its potential has not exhausted yet, as shown in Chap. 3 for synthesis of dynamical systems based on differential equations. At the same time, this method is of little use for digital generators over a comparable frequency band since its software implementation will require multiplication and integration operations, the former of which is known to be performed by microprocessors at a relatively low rate. The replacement of differential equations by difference ones does not help to solve this problem of computation speed. The tabular method is the most often used today. It is quite efficient in the so called direct digital synthesis devices.

Moreover, Table 2.1 demonstrates that there is currently no method that would be equally appropriate for reproduction of signals in analog and digital generators and that would help to make such a synthesis universal and to unify designing of these devices.

For these reasons the author proposes a new method of sequential summation to solve the problem of synthesizing measuring signals that can be reproduced both in analog and digital generators.

2.3 The Summation Method of Causal Signals

This method is based on non-simultaneous (sequential) summation of elementary functions, which makes it different from the simultaneous (parallel) summation based on the expression (2.1). Sequential summation may be stable and consequently correct.

Let us divide the signal range $x(t)$ into sections T_1, T_2, …, T_m. We shall choose the system of finite functions, range of which coincides with T_i, as $\{\psi_i(t)\}$, i.e.:

$$\psi(t) = \begin{cases} \psi_i(t) & \text{if } \ t \in T_i, \quad T_i = t_i - t_{i-1}; \\ 0 & \text{if } \ t \notin T_i, \end{cases}$$

Let us also assume that $T_i \cap T_j = 0$ if $i \neq j$ and $\bigcup_{i=1}^{m} T_i = (0, T)$.

Using such a system, the expression for a set of realizable signals can be written as follows:

$$X = \{x(t) : x(t) = \sum_{i=1}^{\infty} a_i \psi_i(t,\ \tau_i,\ T_i)[H(t-\tau_i) - H(t-\tau_{i-1})],\ i = 1,\ \overline{\infty}\}, \tag{2.4}$$

where $\psi_i(t)$ is the non-random time function describing a pulse waveform; a_i, τ_i, T_i are parameters of a pulse waveform (amplitude, starting time and duration, respectively); $H(t - \tau_i)$ is the Heaviside function. On Fig. 2.5 shows examples of composite signals with sinusoidal (a) and rectangular (b) causal signals.

The theoretical basis of the sequential summation method is a known postulate about the duality of signal presentation in a frequency (as spectrum) and time domain. If to consider the expansion into series (2.1) by the system of trigonometric functions as presentation of a signal by an infinite sum of its elementary harmonic components (i.e. its spectrum), the expansion into an infinite series (2.4) can be considered as the expansion of a signal into an infinite number of elementary time functions concentrated on finite intervals of time.

Let us note the advantages of the signal generation based on the expression (2.4). First of all, this is a reduction of the signal reproduction error associated with finite accuracy of the coefficients a_i. Indeed, in this case the root-mean-square error in the summation of series by functions with inaccurately set coefficients (error of setting ε_i) is given by:

$$\Delta_2 = \left\{ \sum_{i=1}^{N} \frac{1}{T} \int_{t_{i-1}}^{t_i} [\varepsilon\, \psi_i(t)]^2 dt \right\}^{1/2}.$$

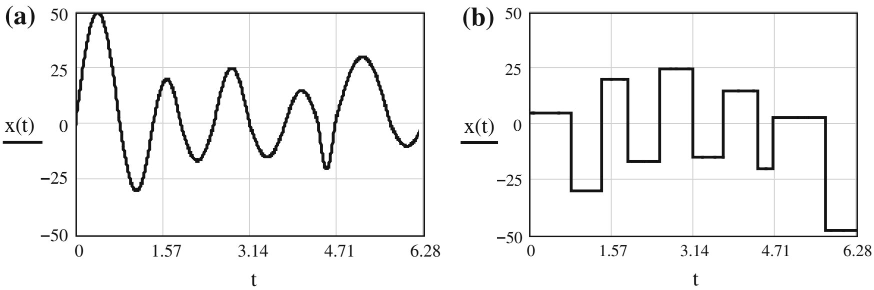


Fig. 2.5 Composite signals: sine (**a**) and square (**b**)

Table 2.2 Comparison of errors for two synthesis methods

Synthesis method	Maximum absolute error Δ_1	Root-mean-square error Δ_2
Simultaneous (parallel) summation by the formula (2.1)	$\Delta_1 = \varepsilon N.$	$\Delta_2 = \varepsilon\sqrt{cN}$
Sequential summation by the formula (2.4)	$\Delta_1 = \varepsilon$	$\Delta_2 = \varepsilon$

Let us find the boundary values of the error on the basis of the integral mean value theorem.

$$\varepsilon\left\{\frac{1}{T}\sum_{i=1}^{N} m_i T_i\right\}^{1/2} \le \Delta_2 \le \varepsilon\left\{\frac{1}{T}\sum_{i=1}^{N} M_i T_i\right\}^{1/2},$$

where m_i and M_i are minimum and maximum values of the function $\psi_i^2(t)$, respectively

Taking the definition ranges of the functions as equal, with $m_i = 0$ and $M_i = 1$, we shall obtain $0 \le \Delta_2 \le \varepsilon$. It follows therefrom that the root-mean-square error of signal representation does not exceed the error of setting the coefficient a_i. It is also obvious that the maximum absolute error $\Delta_1 = \varepsilon$.

The errors of two signal synthesis methods based on (2.1) and (2.4) are compared in Table 2.2. As seen, the signal reproduction error in the proposed sequential summation method does not exceed the error of setting any of the coefficients for a sum of series terms, i.e. this method has an undeniable advantage in terms of the accuracy of hardware-based signal reproduction (with an inevitably inherent implementation error).

The so called time-domain representation of signals [6] is well known in the scientific and academic literature. According to this concept, a signal can be represented as adjacent square wave pulses with the amplitude x_i and the duration Δt:

$$x(t) = \sum_{i=-\infty}^{\infty} x_i[H(t-\tau_i) - H(t-\tau_{i+1})]. \tag{2.5}$$

The comparison of the expressions (2.4) and (2.5) shows that the signal representation method proposed by the author includes this well-known concept as a special case. Indeed, it deals only with square wave pulses and only with such parameters as the amplitude x_i and the duration Δt. The proposed method significantly broadens a range of simulated signals since not only square wave pulses but any other waveforms can be taken as $\psi_i(t)$. Moreover, by controlling a waveform and such parameters as a_i, τ_i, T_i, almost any signals can be easily represented.

Let us consider the models of realizable signals that can be developed on the basis of the expression (2.4). In addition to the function that describes the pulse waveform $\psi_i(t)$, it includes such parameters as a_i, τ_i, T_i, which may take different values. The parameters may be deterministic or random, time-dependent or time-independent, continuous or discrete, etc. The classification presented below (Fig. 2.6) is based on these attributes:

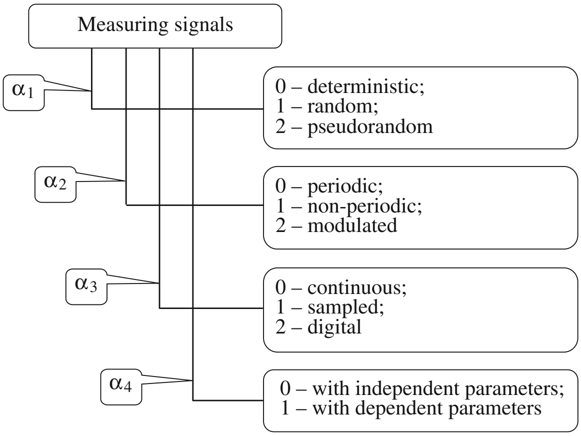


Fig. 2.6 Classification of measuring signals

α_1 is a waveform (deterministic, random, or pseudorandom);
α_2 is a time characteristic of a waveform (periodic, non-periodic, or modulated);
α_3 is a time-dependence characteristic (continuous, discrete, or digital);
α_4 is a relationship between waveform parameters (functionally dependent, statistically dependent, statically and functionally independent).

A number of digital symbols are introduced for each attribute in order to shorten the notation of the model. As seen, a set of realizable signals includes a subset of deterministic and random signals, including a subset of "00" models for deterministic, periodic (sine, square, triangle) signals; a subset of "02" models for deterministic amplitude, phase and frequency-modulated or manipulated signals; a subset of "0010" models for signals with one or several discrete parameters (signals of discrete and digital-to-analog devices); subsets of "110" and "111" models of random signals (including models for random processes of general type "1110", models for discrete random processes "1111", models for pseudorandom discrete sequences "2110").

As shown by the classification, the expression (2.4) may serve as the model for a broad range of realizable measuring signals, which include many known models. In particular, these are models of "truncated sine" and "different-sized sine" signals, which are used in the national THD standard [1]; models of pulse signals used in calibrators for ammeters and voltmeters [2]; models for frequency-modulated signals implemented in the national standard of frequency modulation coefficient [2], etc.

The models of periodic signals are developed on the basis of the expression (2.4) by defining the periodicity condition $\psi_i(t + m\,T) = \psi_i(t)$, where m is the integer; T is the oscillation period. In particular, the sine wave signal model can be defined as follows:

At $\psi_i(t) = \sin \pi/T_i(t - \tau_i),\ \ \tau_i \leq t < \tau_{i-1},\ \ a_i = (-1)^i a,\ \ T_i = \tau_i - \tau_{i-1} = T/2$ we shall have:

$$x(t) = \sum_{i=-\infty}^{\infty} (-1)^i a_i \sin\frac{\pi}{T}(t - \tau_i) = a \sin\frac{2\pi}{T}(t - \tau_0).$$

Defining a sequence of functions $\psi_i(t) = e^{-\alpha_i t} \sin\frac{\pi}{T_i}(t - \tau_i)$, we can write the signal model $x_2(t) = \sum_{i=1}^{\infty} (-1)^i a_i e^{-\alpha_i t} \sin\frac{\pi}{T_i}(t - \tau_i) = ae^{-\alpha t} \sin\frac{2\pi}{T}(t - \tau_i)$ for excitation and steady-state conditions of periodic oscillations in an oscillator. This expression will be used later in Chap. 3 as the model of measuring signals to be reproduced by self-oscillating systems of analog generators.

Selecting

$$\psi_i(t) = \begin{cases} 1 & \text{if } t \in T_i; \\ 0 & \text{if } t \notin T_i, \end{cases}$$

we can easily construct the model of a pulse or pseudorandom signal

$$x(t) = \sum_{i=0}^{\infty} a_i[H(t - \tau_i) - H(t - \tau_{i-1})], \, i = \{0, \overline{\infty}\},$$

generated by self-oscillating system of square wave generators, or the model of a discrete signal to be reproduced by digital-to-analog generlators.

Finally, taking random values as a_i, τ_i, T_i, we can represent the expression (2.4) as the model of a continuous or discrete random process in generators of random signals, synthesis of which is performed in Chap. 4 as well.

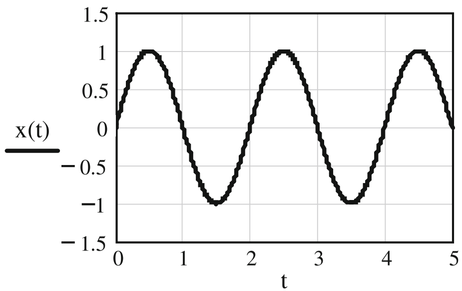
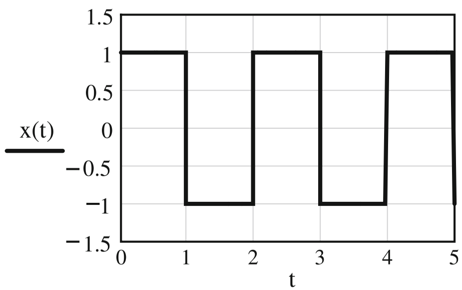
Therefore, by controlling a pulse waveform $\psi_i(t)$ and its parameters a_i, τ_i, T_i, we can construct models for various measuring signals (continuous and discrete, periodic and non-periodic, deterministic and random). In this sense, the expression (2.4) represents a broad range of realizable measuring signals. Table 2.3 illustrates several models of signals composed of sinusoidal segments, straight lines, exponents of different amplitudes, and duration that are constructed on the basis of this method. It should be noted that the summation of signals in time is used for analytical description of signal models.

For the practical implementation of the proposed method, we have to select a system of basis functions $\{\psi_i(t);\, i \in (0,\ \infty)\}$ and parameters a_i, τ_i, T_i. The sequential parametric optimization method developed by the author serves this purpose.

2.4 Sequential Parametric Optimization Method in Synthesis of Measuring Signals

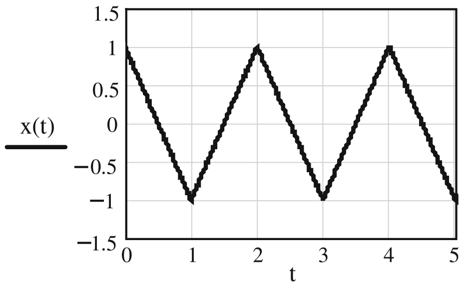
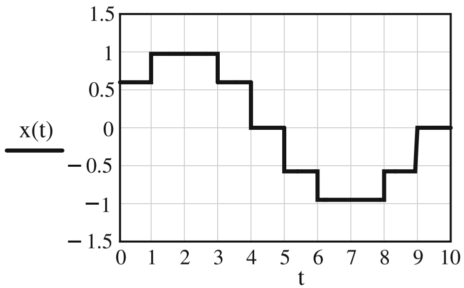
Measuring signals are often to meet different requirements for their parameters and characteristics. Therefore, we have to take into account a lot of criteria in designing the signal. For example, a signal must be described by a required mathematical model as accurately as possible, have a specified spectrum or

Table 2.3 Composite signals

NN п/п	Name of signal	Graphical presentation	Analytical expression of the signal model
1	2	3	4
1	Periodic sine wave signal (0000)		$x(t) = \sum_{i=-\infty}^{\infty} A(-1)^i \sin\left(\frac{\pi}{T}(t - t_0 - iT)\right)[H(t - t_0 - iT) - H(t - t_0 - (i+1)T)]$, where $A = 1$ is the pulse amplitude; T is the pulse duration; H is the Heaviside function; t_0 is the initial time shift (on the graph $t_0 = 0$)
2	Periodic square wave signal (0000)		$x(t) = \sum_{i=-\infty}^{\infty} A(-1)^i[H(t - iT) - H(t - (i+1)T)]$, where A is the pulse amplitude; T is the pulse duration

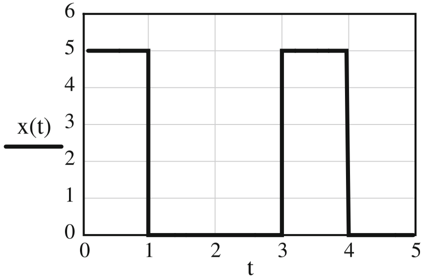
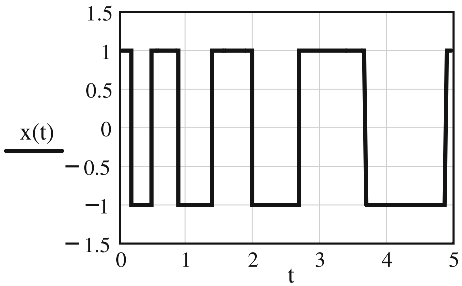
(continued)

Table 2.3 (continued)

NN п/п	Name of signal	Graphical presentation	Analytical expression of the signal model
1	2	3	4
3	Periodic triangle wave signal (0000)		$x(t)=\sum_{i=0}^{\infty}k(-1)^i\left[t-\frac{2i-1}{k}\right][H(t-(i-1)T)-H(t-iT)]$, where T is the pulse duration; $k=2/T$ is the slope coefficient of straight lines
4	Signal quantized by time and level (0010) from the output of a digital-to-analog converter		$x(t)=\sum_{i=1}^{\infty}A_i\sin\left(\frac{2\pi}{T}i\right)[H(t-(i-1)T)-H(t-iT)]$, where A_i is amplitude of signal in the i-th interval; T is the pulse duration

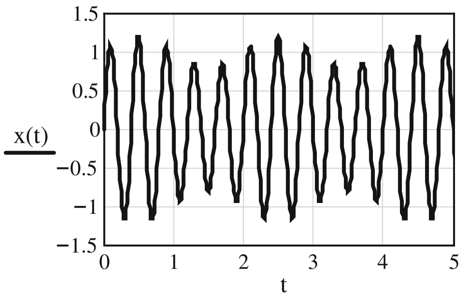
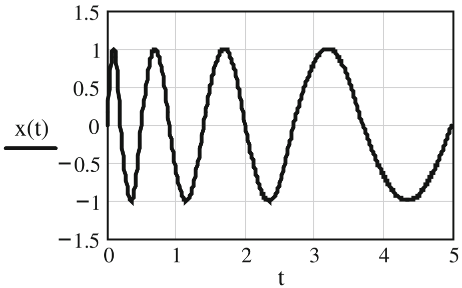
(continued)

Table 2.3 (continued)

NN п/п	Name of signal	Graphical presentation	Analytical expression of the signal model
1	2	3	4
5	Digital signal (0010)		$x(t) = \sum_{i=1}^{\infty} A_k[H(t - iT) - H(t - (i-1)T)]$, where A_k are values of a logic signal (5 or 0 V); T is the duration of the interval
6	Pulse-width modulated signal (0020)		$x(t) = \sum_{i=1}^{\infty} A(-1)^{i+1}[H(t - \tau_{i-1}) - H(t - \tau_i)]$, where $T_i = \tau_i - \tau_{i-1}$is the pulse duration; ti are time moments; $\chi(t)$ is the modulation function $\tau_i = \gamma \cdot \chi(t)$

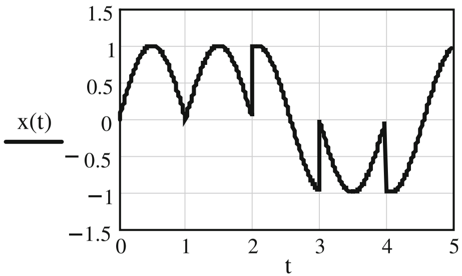
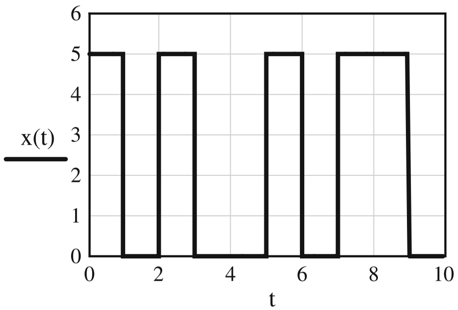
(continued)

Table 2.3 (continued)

NN п/п	Name of signal	Graphical presentation	Analytical expression of the signal model
1	2	3	4
7	Amplitude-modulated signal (0020)		$x(t) = \sum_{i=-\infty}^{\infty} A(-1)^i \sin\left(\frac{\pi}{T}(t - iT)\right)\left[1 + a\sin\left(\frac{2\pi}{T_m}t\right)\right][H(t - iT) - H(t - (i+1)T)]$, where $T_i = \tau_i - \tau_{i-1}$ is the pulse duration, T is the pulse duration, a is coefficient of amplitude modulation
8	Frequency-modulated signal (0020)		$x(t) = \sum_{i=-\infty}^{\infty} A(-1)^i \sin\left(\frac{\pi}{T_i}(t - \tau_i)\right)[H(t - \tau_i) - H(t - \tau_{i+1})]$, where $T_i = \tau_{i+1} - \tau_i$ is the pulse duration

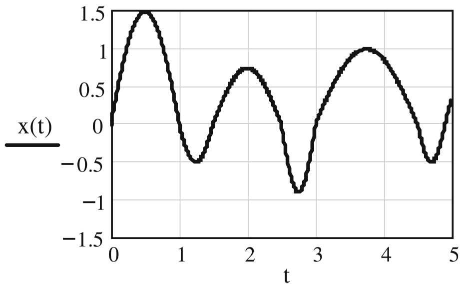
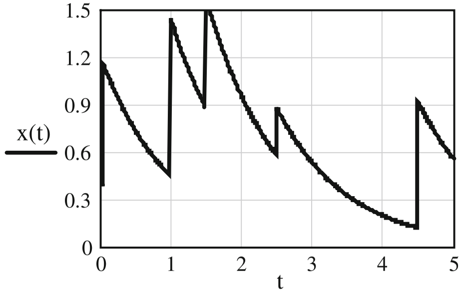
(continued)

Table 2.3 (continued)

NN п/п	Name of signal	Graphical presentation	Analytical expression of the signal model
1	2	3	4
9	Phase-shift keyed signal (0020)		$x(t) = \sum_{i=1}^{\infty} A \sin\left(\pi \frac{\left[t-(i-1)T+\alpha_k {}^{T}/_{4}\right]}{T}\right)[H(t-(i-1)T)-H(t-iT)]$, where T is the pulse duration; $\alpha_k \in (0, 1, 2, 3)$ is a number that depends on a character to be transmitted during phase-difference modulation
10	Pseudorandom digital signal (2010)		$x(t) = \sum_{i=1}^{\infty} A_i[H(t-(i-1)T)-H(t-iT)]$, where $A_i = \begin{cases} 1, \\ 0 \end{cases}$ is a value of logic signal in the i-th interval; T is the pulse duration

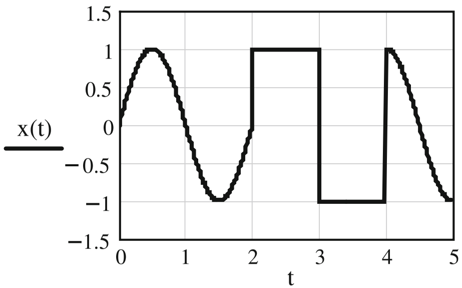
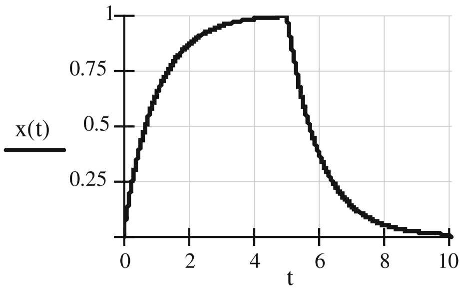
(continued)

Table 2.3 (continued)

NN п/п	Name of signal	Graphical presentation	Analytical expression of the signal model
1	2	3	4
11	Pseudorandom analog signal (1100)		$x(t) = \sum_{i=1}^{\infty} A_i \sin\left(\pi \frac{t-\tau_{i-1}}{T_i}\right)\cdot[H(t-\tau_{i-1}) - H(t-\tau_i)]$, where $T_i = \tau_i - \tau_{i-1}$ is the random duration; τ_i is the random time of pulse occurrence; A_i is the random pulse amplitude
12	Random flicker noise signal (1110)		$x(t) = \sum_{i=1}^{\infty} A_i \exp(-\alpha(t-\tau_{i-1})\cdot[H(t-\tau_{i-1}) - H(t-\tau_i)]$, where τ_i is the random time of pulse occurrence; A_i is the random pulse amplitude

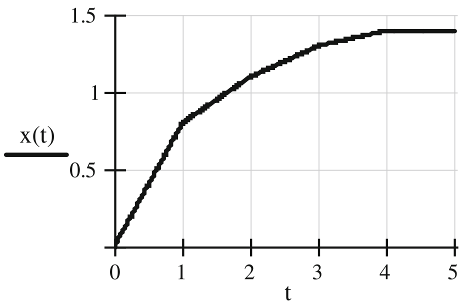
(continued)

Table 2.3 (continued)

NN п/п	Name of signal	Graphical presentation	Analytical expression of the signal model
1	2	3	4
13	Composite signal 1 (0100)		$x(t) = \sum_{i=-\infty}^{\infty} A(-1)^i \psi_i(t)[H(t - t_0 - iT) - H(t - t_0 - (i+1)T)]$, where $A = 1$ is the pulse amplitude; T is the pulse duration; H is the Heaviside function; $\psi_i(t)$—i-th function of pulse
14	Composite signal 2 (0100)		$x(t) = \sum_{i=1}^{2} A_i \exp(-\alpha(t - \tau_{i-1}) \cdot [H(t - \tau_{i-1}) - H(t - \tau_i)]$ where A_i is the pulse amplitude, H is the Heaviside function

(continued)

Table 2.3 (continued)

NN п/п	Name of signal	Graphical presentation	Analytical expression of the signal model
1	2	3	4
15	Composite signal 3 (0100)		$x(t) = \sum_{i=1}^{\infty} [k_i(t - t_i) + b_i T] \cdot [H(t - (i-1)T) - H(t - iT)]$ where T is the pulse duration, k, and $b_i = (k_{i-1} + b_{i-1})$ is parameters of lines

parameters: amplitude, frequency, root-mean-square value, etc. It is rather difficult to take into account all these criteria since many of them conflict with each other: improving one of them may result in deterioration of other. A compromise is usually reached by the following methods: principal factor method, minimax method, additive or multiplicative convolution of criteria with selection of weighting factors for each criteria, etc. Their strengths and weaknesses are commonly known. We will discuss below a new and generally evident method—sequential parametric optimization, which makes it possible to gain optimal or quasioptimal values of all specified criteria at a time.

Let us introduce the following concepts for explanation of this method:

The measuring signal $y(t)$ with a specified set of parameters $Q = \{q_1, q_2, q_3, \ldots, q_m\}$. It is optimal in one sense or another (mathematical model of a signal is presented, e.g. as a sine time function). Let us denote a realizable measuring signal as $x(t)$. It is the mathematical model of a signal represented as a sum of sine function of time and its harmonics. $x(t) \in X$, where X is a set of realizable measuring signals. The operators $Fq_i[x(t)]$, $Fq_i[y(t)]$ are maps of the signals $y(t)$ and $x(t)$ into the parameter region q_i, with $q_i^x = Fq_i[x(t)]$, $q_i^y = Fq_i[y(t)]$. Let us also introduce the distance between i-th parameters of the signals $\rho(q_i^x - q_i^y)$ and the distance between the signals $\xi[y(t), x(t)]$.

Within the framework of these notations, the problem of synthesizing measuring signals can be defined and solved as construction of a model for the signal $y(t)$, which is optimal for achieving a minimum of a specific target function, and the signal $x(t)$, which coincides with or is close to $y(t)$ in a certain sense for all specified parameters. The optimal signal is further considered as an idealized one, while the realizable one—as suitable for hardware and software-based reproduction.

The values presented in Fig. 2.1 can be taken as optimal signals of sine, square and triangle waveforms. It is clear that these signals cannot be implemented with hardware, i.e. in oscillators, at a high accuracy since a sine wave signal is usually accompanied by high harmonic components; a square wave signal has a finite duration of rising and falling edges, etc. For this reason real output signals in oscillators are different from optimal ones. For example, it is reasonable to use the following trigonometric series as a mathematical model for a realizable sine wave signal:

$$x_1(t) = \sum_{n=1}^{N} x_{m_n} \sin[(n2\pi/T)t + \varphi_n]$$

with the dominant first harmonic and small high-order harmonics. In this case, a set of realizable signals X is a set of harmonics with different amplitudes and initial phases.

Naturally, in the general case, a distance between the sine wave signal is $y_1(t) = y_m \sin[(2\pi/T)(t - t_0)]$ and the model of realizable signal $x_1(t)$ occurs in uniform and root-mean-square metrics. Also, a distance between their parameters, i.e. a difference between their root-mean-squares, occurs during the transition from

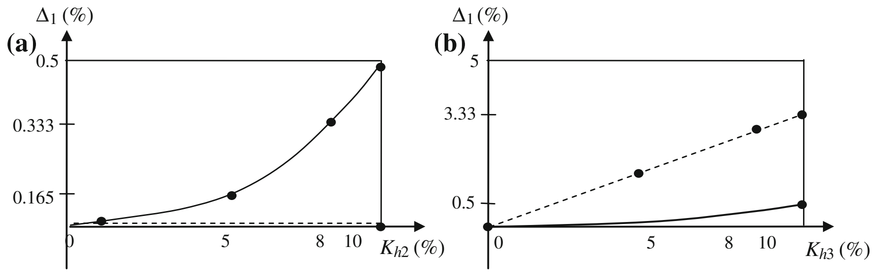


Fig. 2.7 Error of an average-rectified value in case of the second (**a**) and third (**b**) harmonics and different phase shifts

$y_1(t)$ to $x_1(t)$. Therefore, standard values that describe these distances as errors are to be specified in technical manuals for generators. One of the most important parameters for sine wave oscillators is the total harmonic distortion K^x_{THD} of the signal $x_1(t)$. Let us consider a specific case that shows the implications from the incorrect interpretation of this parameter. When calibrating a voltmeter, we apply a sine wave voltage to its input from a reference source (calibrator). In addition to the first harmonic, this voltage contains high-order harmonics that introduce an error to the calibration result.

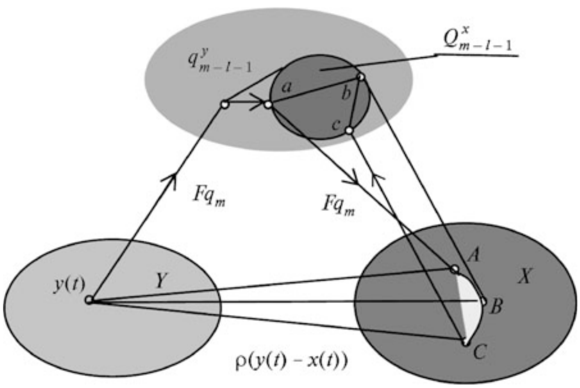
Let us estimate this error. The graphs (Fig. 2.7) demonstrate the relative errors of average-rectified values of the signal $x_1(t)$ in case of only second (Fig. 2.7a) and only third (Fig. 2.7b) harmonics with different phase shifts (in addition to the first harmonic). It is clear that, firstly, errors of average-rectified voltage are different at equal values of harmonic distortion factors. The third harmonic at a equal distortion introduces an error by an order of magnitude greater as compared with the second harmonic. Secondly, the smallest error is introduced by the second harmonic with a phase shift equal to zero (dotted line in Fig. 2.6a) and the third harmonic with a phase shift equal to 90° (solid line in Fig. 2.7b). In the reality, the second harmonic in an oscillator results from the quadratic term that is present in the expansion of a nonlinear response of an amplifier into Taylor series. The third harmonic is a result of the cubic term in the expansion. Both the quadratic and cubic terms cause adverse phase shifts of harmonics. For this reason error in real generators will be maximum and correspond to the top lines in both cases. Unfortunately, manufacturers are not required to disclose amplitude and phase spectrum of output voltage in technical manuals for their generators. This example demonstrates how important it is to choose the real-signal model and to represent it in technical manuals. A wrong choice of the signal model or its parameters may result in incorrect results of implementation or invalid interpretation. But let us return to the synthesis of signals.

Optimal measuring signals will be synthesized further, in the sections devoted to a particular measurement problem. For the synthesis of realizable signals, we will consider that the optimal signal is already known and have m parameters.

Let the signal $x(t)$ is to be realized on a set of X. It can be expected that the signals $x(t)$ belonging to this set include such signals that partially coincides with the optimal signal by a set of parameters. Let us assume that l of m parameters are equal, i.e. $q_i^x = q_i^y$ at $i = 1, \ldots, l$. In this case, we can say that l parameters or l respective coordinates in m-dimensional parameter space of the optimal signal and the realizable signal coincide with each other, while *others* do not. Let us isolate this $(m - l)$-th subspace of coincident parameters. Within this subset, we can assign to the signal $y(t)$ a certain point, the other $(m - l - 1)$-th coordinate of which is equal to parameter q_{m-l-1}^y, and assign to X signals a certain domain Q_{m-l-1}^x of possible variation in q_{m-l-1}^x. The distance $\rho(q_{m-l-1}^x - q_{m-l-1}^y)$. between the point q_{m-l-1}^y and the domain Q_{m-l-1}^x can be determined. It is clear that there is a minimum distance $\rho_{\min}$ at the continuous dependence $q_m^x = Fq_m[x(t)]$. This distance is derived from solving the variational problem of minimizing the functional $\rho_{q_m} = \min\limits_{x(t) \in X} \rho(q_m^x - q_m^y)$. The signal $x^*(t)$ affording a minimum to the functional corresponds to the distance $\rho_{\min}$. Let us call this signal as the quasi-optimal realizable one since it completely coincides with the optimal signal by $(m - l)$ parameters and is different from it by $\rho_{\min}$. If $(m - l - 2)$-th parameters are not coincident, the minimum distance should be determined by $(m - l - 1)$-th and then by $(m - l - 2)$-th parameter. The sequential optimization can be performed for a series of non-coincident parameters provided that a subset of X signals containing more than one element is generated after each intermediate minimization stage. If a subset includes only one element, the further optimization process is interrupted. As a result, we shall get the signal $x^*(t)$ that coincides with $y(t)$ by $(m - l)$-th parameter and is close to it by $(m - l - 1)$-th parameter.

If necessary, the optimization process may be repeated on another set of realizable signals. Let us give the visual geometrical interpretation to the method described above. For this purpose, we shall demonstrate the optimization process in the space (Fig. 2.8). The lower domain Y in the figure below corresponds to a set of ideal signals, in which the optimal signal is marked with the point $y(t)$. The domain X is a subset of realizable signals $x(t)$. The line ABC in the region X shows

Fig. 2.8 Synthesis of quasioptimal signals



the signals that are coincident with the optimal signal by $(m - l)$ parameters; therefore, distances ρ for any of these parameters from any point in the line *ABC* to the point of the optimal signal are equal. The top region represents the subset of non-coincident signal parameters. The plane of $(m - l - 1)$-th parameter is isolated on this subset. $(m - l - 1)$-th parameter of the optimal signal is marked with point q^y_{m-l-1}; the subset of q^x realizable signals is marked with the subregion Q^x_{m-l-1}. The line *abc* shows the subset of parameters Q with coincident $(m - l)$ parameters. The lines with arrows stand for the mapping of the signals into the parameter plane q^x, and vice versa, using the operator Fq_i.

Based on this figure, it is easily to demonstrate a sequence of the signal synthesis process:

(1) to determine the optimal signal $y(t)$;
(2) to define the set X of realizable signals $x(t)$;
(3) to find q^y_m by the signal $y(t)$ using the transformation Fq_m (moving up the line);
(4) to isolate the subset of signals with equal $(m – l)$ parameters in the region X;
(5) in a similar way, to determine q^x_m by the signal $x(t)$ using the transformation Fq_m;
(6) to determine the distances $\rho(q^x_m - q^y_m)$in a chosen metric;
(7) to find the minimum distance, to which the quasioptimal realizable signal $x^*(t)$ correspond, using the appropriate minimization algorithm.

The movement along the zigzag line from the point c corresponds to the minimization process. The minimum difference of the parameters q^x_{m-l-1} and q^y_{m-l-1} is achieved at the point a, which corresponds to the quasioptimal realizable signal $x^*(t)$.

Discussing in a similar way, we can consider the optimization process of $(m - l - 2)$-th parameter etc. It is helpful to assign the flow graph (Fig. 2.9) to the synthesis procedure. Here, the input vertex corresponds to the optimal signal $y(t)$, the output vertex—to the realizable signal $x^*(t)$, while other vertexes—to intermediate variables. The ribs of the graphs show the operators of functional transformations *F, H, G.* The operator Fq_m corresponds to the transformation of the

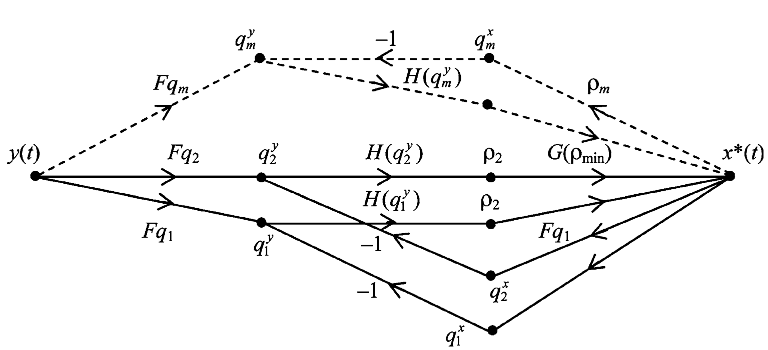


Fig. 2.9 Flow Graph: Construction of a Measuring signal

signal $x^*(t)$ into the number q_m^x, which is its parameter. $H(\rho)$ is responsible for determination of the distance. $G(\rho_{\min})$ represents the minimization procedure. The relationship between the adjacent vortexes of the graph is described by the following system of equations:

$$q_m^y = Fq_m[y(t)];\ q_m^x = Fq_m[x(t)],\ \rho = H[q_m^x - q_m^y];$$

$$G(\rho) = \min_{x(t)\in x}\left\{H[q_m^x - q_m^y]\right\},$$

which can be written as a single synthesis equation:

$$x*(t) = \min_{x(t)\in X}\{G[H(Fq_m[x(t)] - Fq_m[y(t)])]\},$$

It makes it possible to synthesize the quasioptimal signal with specified parameters.

The sequential optimization method is outstanding due to the fact that it allows us to determine not only a signal, but also a structure of the source generating this signal. Indeed, the following transformation flowchart (Fig. 2.10) can be assigned to the transformation graph.

In fact, this flow chart contains almost all elements and all structural links of a measuring instrument (generator) that reproduces the quasioptimal signal $x^*(t)$ with specified q_i^x $(i = \overline{1, m})$ parameters. The units F represent the transformation of the signal *x*(t)* into its parameters (amplitude, frequency, harmonic distortion factor, etc.). In the units Σ, the specified parameters q_i^y are compared with the parameters q_i^x of the output signal. The H units generate a signal proportional to the difference of these parameters, while the output unit G generates the output quasioptimal signal *x*(t)*, which corresponds to the minimum differences.

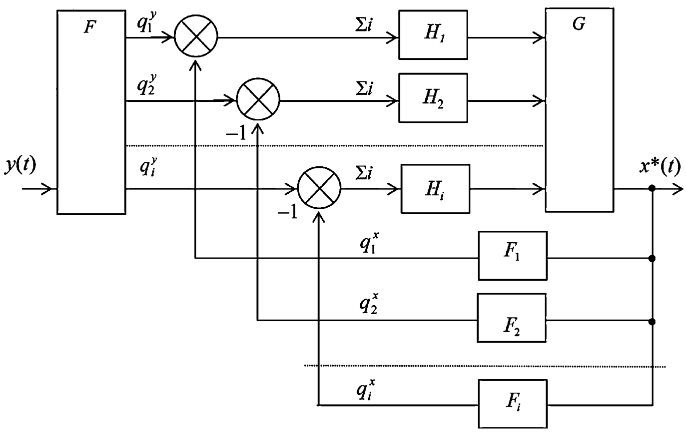


Fig. 2.10 Flowchart: Generation of realizable measuring signals

Real structures may have not some units or feedbacks, but this does not diminish the advantages of the generalized structure, which serves as the basis for the further synthesizing structures for specific measuring means.

In order to solve this problem, we have to choose a set of realizable signals X, i.e. the function basis. The selection of the basis that is adequate to this problem is considered in [5] as the central problem of signal synthesis since a well-matched basis allows us to represent a class of signals to be synthesized in the most simplest and accurate way.

Later on, the synthesis of periodic models (Sect. 2.4), random and sampled signals (Sects. 2.5 and 2.6, respectively) will be performed on the basis of the expression (2.4).

2.5 Synthesis of Mathematical Models for Periodic Signals

The measuring means that reproduce periodic signals have the following numerical metrological characteristics (parameters): amplitude A, frequency ω or period T, average value X_{mean}, root-mean-square value X_{rms}, average-rectified value X_{arv}, root-mean-square harmonic distortion factor K_{h}, amplitude of n-th harmonic A_n, initial phase of n-th harmonic φ_n, n-th harmonic distortion factor $K_{\text{h}n}$, total harmonic distortion K_{thd}, averaging ratio K_{av}, amplitude factor K_{a}, etc.

These are the parameters of the function that describes a waveform. They are connected with it by the known functional relationships.

Almost all the characteristics listed above are determined by a power spectrum of a periodic signal (amplitude spectrum). Knowing it, we can define a root-mean-square value of a signal, root-mean-square harmonic distortion factor, total harmonic distortion, and n-th harmonic distortion factor. A phase spectrum clearly determines amplitude factor, averaging ratio, etc. Obviously, a spectrum is the most meaningful characteristic of a signal. Therefore, we shall begin the synthesis of a signal model from the synthesis of a signal with a specified spectrum.

2.5.1 Synthesis of Models for Measuring Signals with a Specified Spectrum

Let us formulate the problem of synthesizing a signal with a specified spectrum as the problem of synthesizing the model of a measuring signal with a specified constant component and N components of harmonic component in the form of truncated Fourier series:

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(n\omega_0 t) + \sum_{n=1}^{N} b_n \sin(n\omega_0 t) = \frac{a_0}{2} + \sum_{n=1}^{N} A_n \sin(n\omega_0 t + \varphi_n).$$

According to the model proposed above, the synthesis problem is to select such a waveform of the pulses $\psi_i(t)$, their amplitudes A_i and times T_i on each interval, at which reproduction of a signal with prescribed parameters causes a minimum error. Certainly, a signal spectrum will not be limited by N-th harmonic; it will be wider. However, we may impose some constraints on values of secondary components.

Thus, let us detail the problem of synthesizing a periodic signal. In order to make the synthesis feasible, we must determine such a waveform of the pulses $\psi_i(t)$, their amplitude A_i and time T_i, at which a constant component, amplitude and phases of spectral components of the first N harmonics of the signal $x(t)$ are equal to values of spectral components of the signal $y(t)$, while others do not exceed the specified values. Such a statement of the synthesis problem has some specific features as compared with the classical one.

Recall that the classical synthesis problem is formulated as the approximation problem in the following way: Given the function $y(t)$ that describes a measuring signal and, for certain reasons, is hard to implement at different t. For the sake of simplicity, the function $y(t)$ is replaced by the approximate one $y^*(t)$, which is represented as a sum of basis (usually orthogonal) functions $\{\psi_i(t)\}$ taken with the coefficients a_n. The synthesis problem is to determine a_i in accordance with a certain closeness criterion. As compared with the classical statement, our problem is to approximate the function $y(t)$ by a spectrum of the function $x(t)$ rather than to approximate the function $y(t)$ by a certain function $y^*(t)$. Note that functions with overlapping amplitude spectra may significantly differ from each other in a uniform metric. This is the first feature.

The second feature becomes apparent during the synthesis process. As shown below, being linear independent and orthogonal in the classical statement, the functions $\psi_i(t)$ $[H(t-\tau_i)-H(t-\tau_{i-1})]$ that form $x(t)$ become linear dependent in case of approximating the amplitude spectrum. Due to this circumstance, the problem synthesis can be assigned to the class of degenerate problems that do not have regular solutions. In fact, it becomes incorrect. We can avoid this using the sequential optimization method described above.

Let us consider the signal with a prescribed spectrum $y(t)$ as optimal one. Let the key parameters, such as average value Y_{av} and frequency of the first harmonic ω_0, be defined for it, along with the functional description. We shall synthesize the signal $x^*(t)$ with such an average value and frequency that are equal to respective parameters of the optimal signal, and in such a way that the spectrum of the signal $x^*(t)$ is close to the spectrum of the optimal signal $y(t)$.

Let us isolate a subset of signals X with parameters $X_{\mathrm{av}} = Y_{\mathrm{av}}$, $\omega_0^x = \omega_0^y$ on a set of realizable signals. Obviously, this subset will contain periodic signals with the frequency

$$\omega_0^x = \frac{2\pi}{T} = \frac{2\pi}{\sum\limits_{i=0}^{I} T_i},$$

where T_i is the pulse duration; I is the number of pulses within a period; as well as with an average value

$$X_{\text{avg}} = \frac{1}{T}\sum_{i=1}^{I} A_i \int_{\tau_i}^{\tau_{i+1}} \psi_i(t-\tau_i)dt.$$

Let us select a subset of signals with the equal intervals T_i and the equal waveform $\psi_i(t)$. Then, we can write the constraints on coincident parameters:

$$\omega_0^x = \frac{2\pi}{T}, \quad X_{\text{avg}} = \psi_{\text{avg}}\sum_{i=1}^{I} A_i, \quad \psi_{\text{avg}} = \frac{1}{T}\int_{\tau_i}^{\tau_{i+1}} \psi_i dt.$$

Thus, a subset of realizable signals can be defined as:

$$X = \left\{ x(t) : x(t) = \sum_{i=-\infty}^{\infty} A_i\psi_i(t)[H(t-\tau_i) - H(t-\tau_{i-1})];\ T_i I = T;\ \sum_{i=-\infty}^{\infty} A_i = \frac{X_{\text{avg}}}{\psi_{\text{avg}}} \right\}.$$

Obviously, at $T_i \neq 0$, $A_i \neq 0$ the equations have more than one solution.

Let us introduce the optimizable parameter—a spectrum of the signal $q_y = \{a_0, a_1, a_2, a_3, \ldots, a_n; b_1, b_2, b_3, \ldots, b_n\}$, where the coefficients a_i, b_i are coefficients of Fourier expansion.

For approximation of the spectrum $x(t)$ to the spectrum $y(t)$, let us map $x(t)$ into the parameter space (spectrum of the signal $y(t)$), using Fourier transformation. It is known that the spectrum of any time-limited function, including $\varphi_i(t)$, has an infinite number of spectral components; each of them, in its turn, is a function of the number of these functions:

$$x(t) = \sum_{i=1}^{I} c_i + \sum_{i=1}^{I}\sum_{j=1}^{\infty} c_{ij}\cos(j\omega_0 t) + \sum_{i=1}^{I}\sum_{j=1}^{\infty} d_{ij}\sin(j\omega_0 t).$$

That is why this series is defined through double series and depend on the number of the functions $\psi_i(t)$. Let us introduce the notations $\sum_{i=1}^{I} c_i = c_0$, $\sum_{i=1}^{I} c_{ij} = c_k$, $\sum_{i=1}^{I} d_{ij} = d_k$, then the reference signal is given by $x(t) = c_0 + \sum_{k=1}^{\infty} c_k\cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k\sin(k\omega_0 t)$. The spectrum of the signal $x(t)$ to be optimized is $q_x = \{c_0, c_1, c_2, c_3, \ldots, c_n; d_1, d_2, d_3, \ldots, d_n\}$. Recall that these coefficients are defined by the relations:

$$c_i = \frac{2}{T}\int_{\tau_i}^{\tau_{i+1}} A_j\psi\left(\frac{t-\tau_i}{T_i}\right)dt;$$

$$c_{ij} = \frac{2}{T}\int_{\tau_i}^{\tau_{i+1}} A_i\psi\left(\frac{t-\tau_i}{T_i}\right)\cos(j\omega_0 t)dt;$$

$$d_{ij} = \frac{2}{T}\int_{\tau_i}^{\tau_{i+1}} A_i\psi\left(\frac{t-\tau_i}{T_i}\right)\sin(j\omega_0 t)dt;\;\; \omega_0 = \frac{2\pi}{\sum_{i=1}^{I} Ti}.$$

Based on the integral mean value theorem, let us represent c_{ij}, d_{ij} as:

$$c_{ij} = \mu_i A_i \int_{\tau_i}^{\tau_{i+1}} \cos(j\omega_0 t)dt = \mu_i A_i[\sin(j\omega_0\tau_{i+1}) - \sin(j\omega_0\tau_i)];$$

$$d_{ij} = \mu_i A_i \int_{\tau_i}^{\tau_{i+1}} \sin\,(j\omega_0 t)dt = \mu_i A_i[\cos(j\omega_0\tau_i) - \cos(j\omega_0\tau_{i+1})],$$

where μ_i are real numbers, with $m \le \mu \le M$ (m and M are, respectively, minimum and maximum values of the function $\psi_i(t)$, i.e. 0 and 1).

Taking into account the relationships we have derived above, it is easy to set up a system of the equations that connect the spectrum of the signal to be optimized with the parameters of pulses:

$$\begin{aligned}
&c_0 = \mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3 + \ldots + \mu_i A_i + \ldots + \mu_I A_I;\\
&c_1 = \mu_1 A_1 c_{11} + \mu_2 A_2 c_{21} + \mu_3 A_3 c_{31} + \ldots + \mu_i A_i c_{i1} + \ldots + \mu_I A_I c_{I1};\\
&c_2 = \mu_1 A_1 c_{12} + \mu_2 A_2 c_{22} + \mu_3 A_3 c_{32} + \ldots + \mu_i A_i c_{i2} + \ldots + \mu_I A_I c_{I2};\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&c_n = \mu_1 A_1 c_{1n} + \mu_2 A_2 c_{2n} + \mu_3 A_3 c_{32} + \ldots + \mu_i A_i c_{i2} + \ldots + \mu_I A_I c_{In};\\
&d_1 = \mu_1 A_1 d_{11} + \mu_2 A_2 d_{21} + \mu_3 A_3 d_{31} + \ldots + \mu_i A_i d_{i1} + \ldots + \mu_I A_I d_{I1};\\
&d_2 = \mu_1 A_1 d_{12} + \mu_2 A_2 d_{22} + \mu_3 A_3 d_{32} + \ldots + \mu_i A_i d_{i2} + \ldots + \mu_I A_I d_{I2};\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&d_n = \mu_1 A_1 d_{1n} + \mu_2 A_2 d_{2n} + \mu_3 A_3 d_{32} + \ldots + \mu_i A_i d_{i2} + \ldots + \mu_I A_I d_{In};
\end{aligned} \tag{2.6}$$

where $c_{ij} = [\sin(j\omega_0\tau_{i+1}) - \sin(j\omega_0\tau i)]; d_{ij} = \left[\cos\;(j\omega_0\tau i_{+1}) - \cos\;(j\omega_0\tau_{i)}\right];$ $\mu = \{\mu_1, \mu_2, \mu_3, \ldots, \mu_n\}^T$.

The systems of Eq. (2.6) at $c_i + jd_i = q_i$ can be represented in the matrix format:

$$ReQ = CA, ImQ = DA, \tag{2.7}$$

where $c = \{c_0, c_1, c_2, c_3, \ldots, c_n\}$; $d = \{d_0, d_1, d_2, d_3, \ldots, d_n\}$; $A = \{A_1, A_2, A_3, \ldots, A_n\}$; $Q = \{q_1, q_2, q_3, \ldots, q_n\}$; $C = \{c_{ij}\}$; $D = \{d_{ij}\}$. Here, we take $c = ReQ$, $d = ImQ$.

The elements of the matrices C and D in the Eq.(2.7) satisfy the following constraints:

$$\sin(j\omega_0\tau_i) - \sin(j\omega_0\tau_{I-i}) = 0;$$
$$\cos(j\omega_0\tau_i) - \cos(j\omega_0\tau_{I-i}) = 0.$$

Thus, a sum of elements in each line is equal to zero, which makes the matrix C degenerate.

The Eq. (2.7) are featured by multi-variant solutions. Indeed, the left parts of the equations contain $2n + 1$ of prescribed values of coefficients, while the right ones include $3I$ of unknown, including I of unknown functions $\psi_i(t)$, I of intervals T_i and I of amplitudes A_i. In accordance with the sampling theorem, it can be supposed that the number of division intervals I must be at least two times higher than the number of harmonics specified for the synthesis. In this case we have at least $6n$ unknown quantities at a number of known quantities $2n + 1$. It is clear that we have to impose some constraints on $4n$ unknown parameters for a single-valued solution of the system (2.7).

Either a waveform of pulses, or their duration, or amplitude may be unknown quantities.

Three main possible versions of the synthesis problem can be formulated.

- a waveform and duration of pulses are prescribed, amplitude is to be found;
- a waveform and amplitude of pulses are prescribed, their duration is to be found;
- duration and amplitude of pulses are prescribed; the functions describing a waveform of pulses are unknown parameters to be found.

These versions have different degrees of complexity. Let us remind that the synthesis of a signal model is aimed at deriving such an expression, to which the simplest pattern of the dynamical system corresponds. The first version, which is reduced to the solution of the linear system of matrix equations, is the most simplest for synthesis. On the contrary, the second version implies the solution of the nonlinear matrix equation since the elements of matrices $C = \{c_{ij}\}$ and $D = \{d_{ij}\}$ depend on time intervals through trigonometric functions. The third version requires setting l of different pulse waveforms, which makes much more difficult to find them by expansion coefficients and, naturally, to construct and implement a model of the dynamic system, without any expected advantages. Therefore, the first and second versions are the most feasible for the synthesis: a pulse waveform is predefined, while amplitude or duration are sought quantities.

Let us consider the first version of the synthesis problem, which, in its special case, is of practical value. We shall divide the period T into equal time segments. Then, the set of Eq. (2.7) takes on the following form:

$$c = CA;\ d = DA, \tag{2.8}$$

where $c_{ij} = \cos\left[j\frac{\pi}{I}(2j - 1 - 2\frac{t_0}{T}I)\right]$; $d_{ij} = \sin\left[j\frac{\pi}{I}(2j - 1 - 2\frac{t_0}{T}I)\right]$; $q = \{2\mu_i \cos(j\frac{\pi}{I})\}^T$ is a spectral envelope of the pulse.

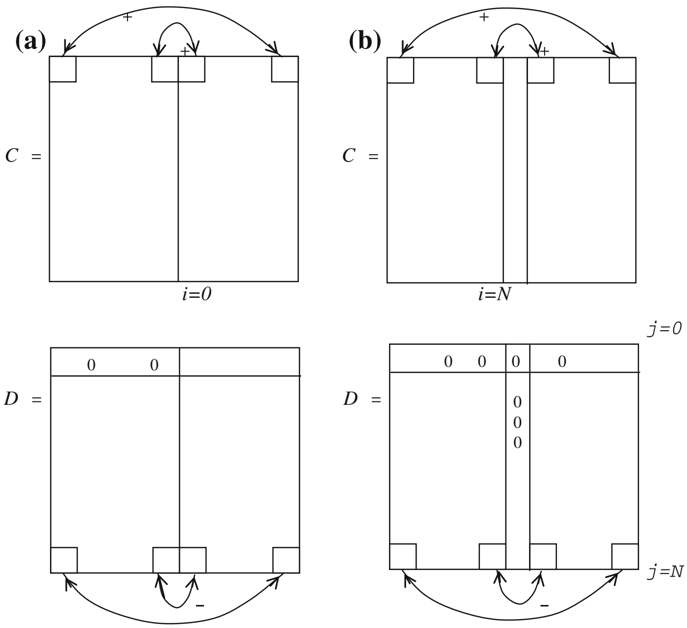


Fig. 2.11 Form of the matrices C и D at even (**a**) and odd (**b**) number of intervals I

Analyzing the coefficients c_{ij} and d_{ij}, it is easy to notice that there is a certain relationship between them. For example,

$$c_{ij} = c_{(2k-1)I/2j};\ \ d_{ij} = d_{kIj};\ c_{ij} = c_{(2k-1)I/2j};\ \ d_{ij} = d_{(kI-i)j}.$$

In this case, the matrices C and D can be represented as block matrices (Fig. 2.11). The arrows mark equivalent elements, while signs on them show whether signs of the coefficients coincide or do not coincide.

The blockness property of the matrices may be used for reducing amount of calculation at least by two times in the synthesis of measuring signals that contain only cosine and sine components of spectral components. As seen from the system of equalities (2.8), zero and linearly dependent rows appear in the matrices C and D. Hence, the matrix rank $r = I/2$ at even I and $r = (I - 1)/2$ at odd I.

Using the derived values, we can determine the lower boundary I—the number of division intervals. Since $2\,N + 1$ of the first Fourier coefficients are set for the synthesis, it is necessary to satisfy the condition $N < I/2$, at which a number of linearly independent rows is not less than a number of sought variables.

This condition is well-matched with the known sampling theorem, according to which a number of pulses I in the interval T must be considered as the number of samples, while N—as the upper boundary of the periodic oscillation spectrum. Thus, when we have N first harmonics of expansion into trigonometric Fourier series, the required number of division intervals I must exceed 2 N. This condition is not essential at low N values.

Let us generate a new matrix Eq. (2.6) from linearly independent rows of the matrix Eq. (2.6), in which we take $I = 2\ N$:

$$\begin{gathered} q_x = CA; \\ q_x = \{c_0, c_1, c_2, c_3, \ldots, c_n\, ;\, d_1, d_2, d_3, \ldots, d_n\}^T_{1\times 2_n};\ G = \{q_{ij}\}, \\ q_{ij} = c_{ij}\ if\ j = \overline{1, n}\ \text{and}\ g_{ij} = d_{ij}\ if\ j = \overline{n+1, 2_n}; \end{gathered} \tag{2.9}$$

where $A = \{A_1, A_2, A_3, \ldots, A_n\}^T$.

I is an even number; I is an odd number.

The Eq. (2.9) defines a spectrum of a sought quantity. Let us find the distance between spectra of optimal and realizable signals. Let us introduce the distances between spectra in the uniform metric $\rho_q = |q_x - q_y|$. Obviously, it will be minimum if $q_x = q_y$. Then, $\rho_q = \rho_{q\ \min} = 0$, so the problem of distance minimization can be reduced to the solution of the equation:

$$q_y = \mu GA. \tag{2.10}$$

The matrix G in the Eq. (2.10) is nonsingular; therefore, it can be solved in relation to the vector A (vector of unknown amplitudes) using linear algebra methods, Gaussian elimination method, inverse matrix method, etc.

In order to solve the equation by the inverse matrix method, let us multiply the left and right sides of the Eq. (2.10) by the matrix $G^{-1}\mu^{-1}$. As a result, we shall obtain the equation $A = G^{-1}\mu^{-1}q_y$, which have a unique solution represented as the vector of unknown amplitudes A.

This step completes the solution of our general problem—synthesis of a measuring signal with specified spectral components of N first harmonics.

Consider a specific example: synthesis of a signal by its spectrum that contains five first harmonics of the Fourier series expansion. Let us take the following equation as an idealized signal

$$y(t) = \sum_{n=1}^{5} \sin(n\omega_0 t),$$

where ω_0 is the frequency of the fundamental harmonic.

Five amplitudes (taken as equal to 1), one frequency and one waveform of a signal harmonic are known for the idealized signal. We do not yet know waveform of the pulses, amplitude and duration (i.e. at least ten parameters), as well as five functions that describe a waveform of the pulses for the synthesis of a realizable signal by the formula (2.4). Therefore, the number of unknowns exceed the number of known quantities, which makes it impossible to solve the synthesis problem. Let us define some unknowns for a single-valued solution. For all pulses we shall select a sine waveform that is the best suitable for implementation by analog electronics. Pulse durations will be taken as equal time intervals between zero crossing moments of the idealized signal $y(t)$. In this case, five pulse amplitudes remain unknown. Let us find them.

Since the idealized signal is described by an odd function, let us synthesize a realizable odd signal with zero values of the coefficient $c_{i,j}$. Let us write the basic equation only for the coefficients d_{ij}:

$$d_{ij} = \frac{2}{\pi} \int\limits_{t_{i-1}}^{t_i} \sin\left(\frac{\pi}{t_i - t_{i-1}} t\right) \sin(jt) \text{ at } i = [1, 2, \ldots, 5],\ j = [1, 2, \ldots, 5]$$

and zero crossing moments of the idealized signal $t_0 = 0; t_1 = 1.047; t_2 = 1.2567; t_3 = 2.0944; t_4 = 2.5133; t_5 = \pi$.

As a result, we shall obtain a matrix of partial Fourier expansion coefficients.

$$D = \begin{bmatrix} 0.207 & 0.077 & 0.332 & 0.126 & 0.078 \\ 0.331 & 0.063 & -0.066 & -0.166 & -0.144 \\ 0.333 & -0.026 & -0.277 & 0.096 & 0.189 \\ 0.236 & -0.083 & 0.105 & 0.033 & -0.208 \\ 0.103 & -0.041 & 0.189 & -0.132 & 0.2 \end{bmatrix}$$

Taking into account a vector of spectrum amplitudes of harmonic components specified for the synthesis $q_y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, substituting the matrices into the equation (2.10) and solving it, we shall obtain a matrix of amplitudes:

$$A = \begin{bmatrix} 3.69 \\ -2.26 \\ 1.341 \\ -0.786 \\ 0.837 \end{bmatrix}.$$

Figure 2.12a shows that the idealized signal $y(t)$ composed of five first harmonics with the amplitude equal to 1 and the realizable signal $x(t)$ composed of ten sine wave pulses with the amplitudes derived above are close to each other within a major part of the period. A difference between them becomes evident on the second and ninth pulses.

The spectrum of the realizable signal (Fig. 2.12b) shows that its first five harmonics coincide with the first five harmonics of the set signal by amplitude since their amplitudes are equal to 1. Nevertheless, due to the deviation of pulse waveform and non-optimal duration, the spectrum of the realizable signal contains the sixth, seventh, and other high-order harmonics. However, the power of these components is low, while the error of root-mean-square value of the realizable signal does not exceed 2 %.

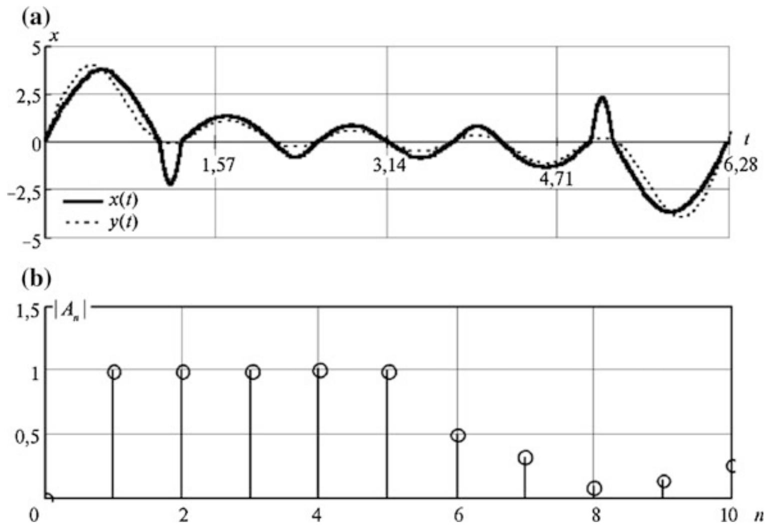


Fig. 2.12 Idealized (*dotted line*) and realizable (*solid line*) signals (**a**) and the spectrum of the realizable signal (**b**)

It is possible to reduce amplitudes of these high-order harmonics if we take into account that the analog implementation of the oscillating system that generates a realizable signal has the second minimum order. Evidently, it will have some signals that reproduce not only a signal itself but also its first and second derivatives. Therefore, we can set a problem not only to synthesize a signal itself but also its first or second derivatives, which are often much easier to implement. For example, the second derivative of the signal in the example discussed above is

$$s(t) = \frac{d^2}{dt^2}y(t) = -\sum_{n=1}^{5} n^2(\omega_0)^2 \sin(n\omega_0 t).$$

Analyzing the graph for the idealized signal (Fig. 2.13a), we can notice that positive and negative waves of its second derivative are close to sine waveform, while values of their duration have a less spread that the idealized signal itself. This fact certainly simplifies the synthesis of the realizable signal.

Performing the calculations similar to those made above and twice integrating the derived function, we shall generate the realizable signal.

Its spectrum (Fig. 2.13b) has a lower level of high-order harmonics at the same inequality between the amplitudes of the harmonics defined for the synthesis.

In recent years, due to the rapid development of digital generation methods, square wave pulses are becoming increasingly popular for realizing measuring signals.

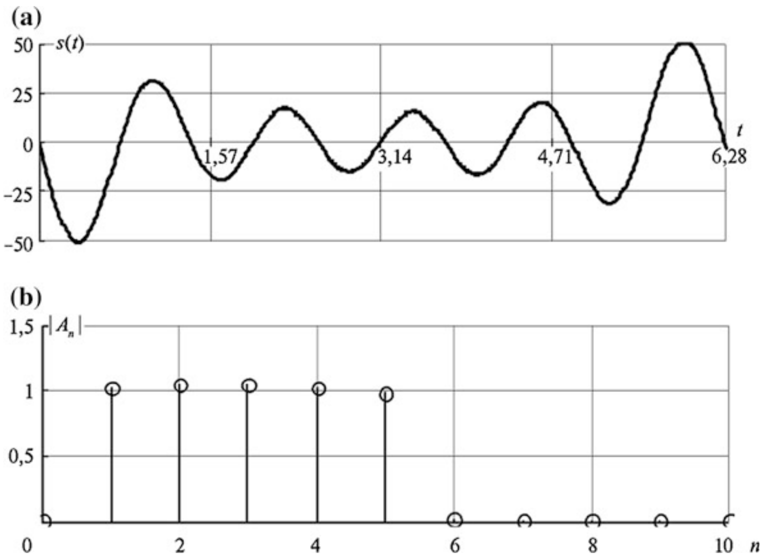


Fig. 2.13 The second derivative (**a**) of the idealized signal $s(t)$ and the spectrum (**b**) of the realizable signal generated through the synthesis by the second derivative

The square waveform of a pulse is the most feasible for implementation by discrete and digital electronics. The respective equations for the coefficient take the following form:

$$\begin{aligned} c_i &= \sum_{i=0}^{I-1} \frac{2A_i}{\pi i} \sin\frac{\pi i}{I} \cos\frac{\pi i}{I}(2j-1); \\ d_i &= \sum_{i=0}^{I-1} \frac{2A_i}{\pi i} \sin\frac{\pi i}{I} \sin\frac{\pi i}{I}(2j-1), \end{aligned} \tag{2.11}$$

where $i = \overline{1,\ I}$, $j = \overline{1,\ N}$.

They also have at least $(I-1)/2$ of linearly independent equations, which can be used for construction of the matrix.

Thus, the method discussed above makes it possible to synthesize the mathematical model of the realizable signal by N prescribed coefficients of expansion into the trigonometric Fourier series.

Moreover, the proposed synthesis method gives a possibility to estimate the level of harmonic components with a number of more than $N/2$. For this purpose, taking into account the equations (2.11), we have to write infinite sets of matrix equations. Due to the linear dependence between the rows of the matrices C and D, it is easy to establish a relationship between specified coefficients and any other ones in these equations:

$$
\begin{aligned}
c_{kI-j} &= c_j \frac{g_{kI-j}}{g_j};\\
d_{kI-j} &= d_j \frac{g_{kI-j}}{g_j}.
\end{aligned} \tag{2.12}
$$

The derived formulas (2.12) make it possible to estimate the power of all harmonics with a number of more than N, i.e. to find a root-mean-square error of approximation:

$$
\varepsilon = \alpha \sqrt{\sum_{j=N+1}^{\infty} \left(c_j^2 + d_j^2\right)} = \alpha \sqrt{\sum_{k=1}^{\infty} \sum_{j=1}^{N} \left(c_j^2 + d_j^2\right) \frac{g_{kI-j}^2}{g_j^2}}.
$$

At the square waveform of a pulse, this formula is simplified:

$$
\varepsilon = \alpha \sqrt{\sum_{j=N+1}^{\infty} \left(c_j^2 + d_j^2\right)} = \alpha \sqrt{\sum_{k=1}^{\infty} \sum_{j=1}^{N} \left(c_j^2 + d_j^2\right) \left[\left(\frac{j}{kI - j}\right)^2 + \left(\frac{j}{kI + j}\right)^2\right]}.
$$

Obviously, the root-mean-square error of approximation will be less, as I is greater and N is lower.

If, e.g. the signal $y(t) = \cos t$ containing only one harmonic is set for the synthesis, while I intervals are taken for the approximation by square wave pulses, it will be easy to estimate the error:

$$
\varepsilon = \frac{1}{\sqrt{2}} \sqrt{\sum_{k=1}^{\infty} \left[\left(\frac{1}{kI - 1}\right)^2 + \left(\frac{1}{kI + 1}\right)^2\right]} \cong \frac{1,8}{I}.
$$

The method presented above is used for synthesis of signals with specified amplitude factor and total harmonic distortion. It can significantly simplify the synthesis of signals with specified spectral ratio that are used in measuring means [19, 20] designed for calibration of digital and selective voltmeters, spectrum analyzers, etc.

2.5.2 *Synthesis of Models for Measuring Signals with a Specified Crest Factor*

The amplitude factor K_a is one of the most important parameters of a signal that characterizes its change in time. It highly affects an error of converters and instruments for measuring root-mean-square voltage. Therefore, as required by GOST 14014-82 “Digital Instruments and Measuring Transducers of Voltage, Current and Resistance”, its limit values at which the rated metrological characteristics remain unchanged are to be disclosed in instruction manuals. For this reason, a permissible value of amplitude factor must be experimentally determined for all digital alternating voltage and current converters during production and putting them in operation. As a consequence, the generation of signals with prescribed K_a is of special importance.

They include pulse signals of square waveform with duty ratio, trapezoidal and exponential waveforms, radio-frequency pulse, etc. The application of them in measuring means used for calibration of voltmeters and AC voltage converters has a number of distinct features. A high pulse rise rate results in a component error, which is associated with the effect of finite pulse rise rate at input stages of a voltmeter to be calibrated. A broad spectrum of pulse and radio-frequency pulse signals causes an error due to a finite width of an operating frequency band.

The disadvantages listed above can be eliminated only if the parameters of measuring signals are well matched with time and frequency parameters of input modules of instruments to be calibrated. Indeed, as required by GOST 14014-82 mentioned above, specifications of a measuring instrument must include a frequency band of harmonic components of an input signal, at which rated values of metrological characteristics remain unchanged. Therefore, models of measuring signals are further synthesized with a bounded spectrum in a normal or operating frequency band. It is clear from this that the synthesis of a measuring signal with a given amplitude factor is closely related to the synthesis of a signal with a given spectrum.

The crest factor is known to be calculated by the formula $K_a = \frac{A}{x_{\text{CK3}}}$, where A is the amplitude, x_{rms} is a root-mean-square value of a signal.

Let us state the following problem: to synthesize a model of measuring signal with a prescribed amplitude factor; its spectrum completely falls within the operating frequency band of a measuring instrument, and its rise rate does not exceed limit values. It is obvious that this requirement limits this spectrum by boundaries of the operating frequency band. Therefore, we can represent a sought signal as

$$y(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t + \varphi_n), \tag{2.13}$$

in which frequencies of spectral components satisfy the inequalities:

$$\omega_1 \geq \omega_l; \omega_N \leq \omega_u;$$

$$\left|\frac{dy}{dt}\right|_{\max} \leq V_{\max},$$

where ω_l, ω_u are, respectively, the lower and upper boundary of the operating frequency band of measuring means; $V_{\max}$ is the permissible rate of signal rise.

The proposed model may be a periodic time function at different relationships ω_l. In order to construct it, these frequencies must have the highest common divisor Ω, which is a signal repetition frequency. It is clear that the frequency Ω must satisfy the inequality $\Omega \geq \omega_1$.

Let us determine a minimum number N, at which a specified harmonic distortion factor is achieved. Obviously, this value depends on a distribution of amplitudes and phases of harmonic components. In fact, the problem is reduced to finding a distribution of amplitudes and phases of the signal $y(t)$, at which K_a achieves a maximum value at a given N. Let us call the signal with such a distribution as optimal one.

The amplitude factor takes the maximum value when its nominator and denominator attain maximum and minimum values, respectively.

The necessary and sufficient conditions, at which $y(t)$ takes a maximum value, are given by:

$$\frac{\partial y}{\partial \varphi_n} = -a_n \sin(\omega_n t + \varphi_n) = 0;$$
$$\Lambda_{ij} = \left\{ \frac{\partial^2 y}{\partial \varphi_i \cdot \partial \varphi_j} \right\} (-1)^n \geq 0, \forall\, i = \overline{1, N};\ j = \overline{1, N},$$

where Λ_{ij} is the determinant of a matrix of partial derivatives.

From the first equation we shall obtain $\varphi_n = \varphi_1 \frac{\omega_n}{\omega_1}$. In this case $y_m = \sum_{n=1}^{N} a_n$. Let us assume that

$$\sum_{n=1}^{N} a_n^2 = C, \tag{2.14}$$

where C is a real number.

Then, K_a achieves a maximum possible (at a given N) value if

$$C = \min_{a \in A} f(a) = \min\left(\sum_{n=1}^{N} a_n^2 \right). \tag{2.15}$$

The function $f(a)$ takes a minimum value at a point $M(a_1, a_2, \ldots, a_N)$ if all partial derivatives at this point become zero.

Let us insert the expression (2.14) in the formula (2.15) to obtain:

$$f(a) = (C - \sum_{n=2}^{N} a_n)^2 + \sum_{n=2}^{N} a_n^2.$$

Finding partial derivatives by a_n, write the system as:

$$\frac{\partial f}{\partial a_n} = 2(a_n - a_1) = 0; \quad n = \overline{2, N},$$

which has a unique solution $a_1 = a_2 = \ldots = a_N$.

Therefore, the maximum amplitude factor of the signal $y(t)$, at a given number of harmonics, is attained at equal amplitudes of harmonics and their phase shifts $\varphi_n = \varphi_1 \frac{\omega_n}{\omega_1}$. The optimal measuring signal takes the form:

$$y(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t + \varphi_n) \;\; if \;\; a_n = 1.$$

The amplitude factor of this signal:

$$K_{\mathrm{a}} = \sqrt{2N}.$$

This value is maximum at a given number of harmonics. Other values can be obtained by changing the number of harmonics or values of amplitudes a_n. For example, varying amplitude of one of harmonics within $0 < a_n < 1$, we can change a value of amplitude factor within

$$\sqrt{2(N-1)} \le K_{\mathrm{a}} \le \sqrt{2N}.$$

The model (2.13) does not define values of frequencies ω_n. From practical considerations, it is convenient to choose the following versions:

$$\begin{aligned}
&\omega_n = n\omega_0 \,\text{at}\, n = \overline{1,N},\\
&\omega_n = \omega_0(2n-1) \text{ at } n = \overline{1,N},\\
&\omega_n = \omega_0 + n\Omega \text{ at } n = \overline{1,N},\\
&\omega_n = \omega_0 + (2n-1)\Omega \text{ at } n = \overline{1,N},
\end{aligned}$$

The first and third versions generate asymmetric signals, while the second and fourth ones—symmetric signals.

Let us construct the model of a realizable measuring signal with specified amplitude coefficient—the approximating optimal signal $y(t)$. For this purpose let us determine a set of realizable signals with such an amplitude factor and root-mean-square value that are equal to respective values of the optimal signal in the form (2.4):

$$X = \{x(t) : x(t) = \sum_{i=0}^{\infty} a_i \psi_i(t, \tau_i, T_i)[H(t-\tau_i) - H(t-\tau_{i-1})]; \;\; i = 1, \overline{\infty}\}.$$

Here, a subset of coincident parameters contains two parameters, while signal spectrum is a parameter to be optimized. Taking into account the a waveform of the optimal signal $y(t)$ on time segments $\tau_i \le t < \tau_{i+1}$, especially at $N \gg 1$, is close to sine waveform, it is reasonable to choose pulses of only a cosine waveform, duration of which is equal to the duration of respective segments of the optimal signal. The expression of closed form is known for the function (2.13) that describes a waveform of the optimal signal:

$$y_{\mathrm{opt}}(t) = \sum_{n=1}^{N} \cos nt = \cos \frac{N+1}{2} t \frac{\sin \frac{N}{2} t}{\sin \frac{t}{2}},$$

It allows us to determine these moments.

This expression implies that zeros of the function $y_{\mathrm{opt}}(t)$ are defined by zeros of the functions $\cos \frac{N+1}{2} t$ и $\sin \frac{N+1}{2} t$, i.e. with relationships

$$\tau_i = \frac{2k+1}{N+1}\pi \text{ at } 0 < k \leq \frac{2N+1}{2} \quad \tau_i = \frac{2k}{N}\pi \text{ at } 0 < k \leq N.$$

Hence, the duration of the first and last pulses is $T_1 = T_i = \frac{2\pi}{N+1}$,, while the duration of the second and subsequent ones is $T_2 = \frac{\pi}{N+1} - \frac{2\pi}{N(N+1)}$.

The distance between spectra of the optimal signal and the realizable signal is given by

$$\rho = \sum_{i=1}^{I} \left[A_i - \left(\sum_{j=1}^{N} c_{j,i} A_{j,i} \right) \right]^2. \tag{2.16}$$

Besides, the following constraints on peak and root-mean-square values must be fulfilled: $A_1 = y_{\max}, \ \sum_{i=1}^{I} c_{j,i} A_i^2 = y_{\mathrm{rms}}; \ c_{j,i} = T_i \Big/ 2 \sum_{j=1}^{N} T_j$.

Solving the variational problem of minimizing the distance (2.16), we shall find the vector of amplitude pulses $\{A\}$. The special amplitude calculation software was developed for numerical solution of this problem. The calculation results are presented in Fig. 2.14 and Table 2.4, which compare the models of optimal (2.13), quasioptimal, square wave and other signals.

The following conclusions can be made from the comparison:

- optimal signal (2.13) has a lower maximum rate of rise that a square wave signal;
- maximum rate of rise of the optimal signal at $K_a \gg 1$ is four times lower than one of the radio-frequency signal;
- the quasioptimal (realizable) and optimal signals have almost identical rates of rise and frequency bands;
- the whole spectrum power of the optimal signal and 98 % of the spectrum power of the quasioptimal signal are concentrated in the operating frequency band of a measuring instrument to be calibrated.

Thus, optimal and quasioptimal signals have the definite advantages over other signals.

Quasioptimal signals are implemented in a programmable measuring generator, which is discussed in Chap. 7.

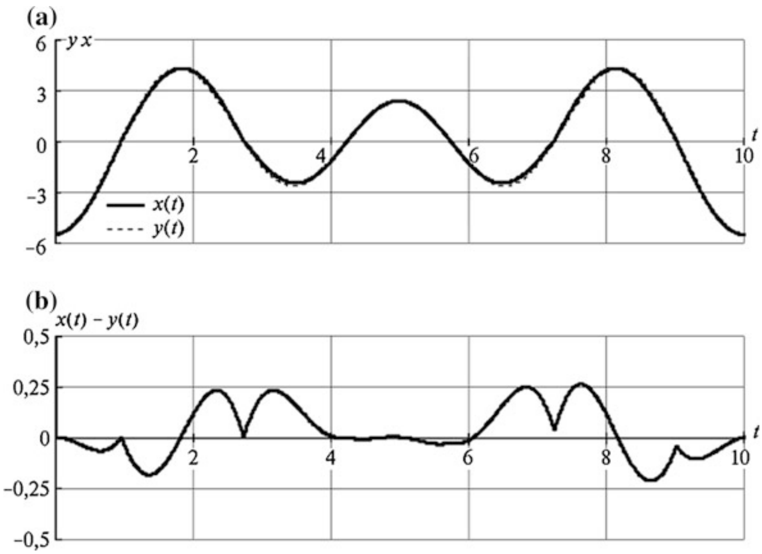


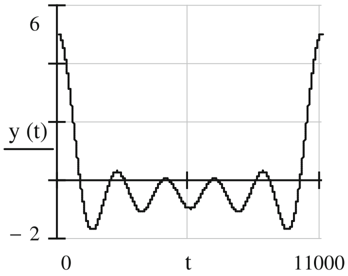
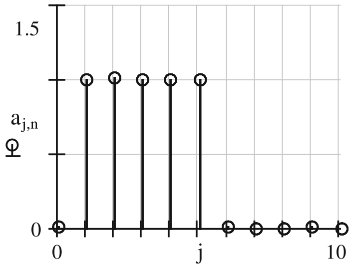
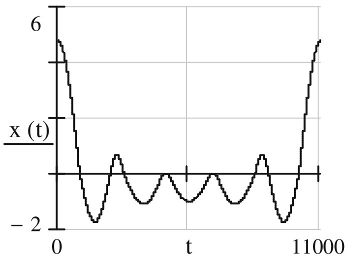
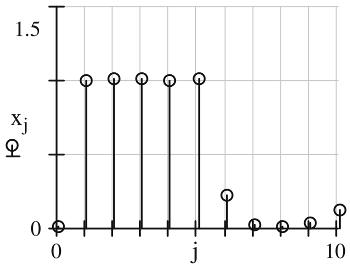
Fig. 2.14 The second derivative of optimal and quasioptimal realizable signals with a specified amplitude factor (**a**) and error (**b**)

2.5.3 Analysis and Synthesis of Models for a Measuring Signal with Specified Total Harmonic Distortion

Measuring signals with normalized harmonic distortion factor K_h are used in calibrators of nonlinear distortion measuring instruments [1], as well as for determining signal distortion errors of digital voltmeters and alternating voltage and current converters, phase and frequency meters. As per GOST 14014-82, specifications for digital converters of root-mean-square values responding to an average-rectified value must contain such values of harmonic distortion factor, at which rated metrological characteristics remain unchanged.

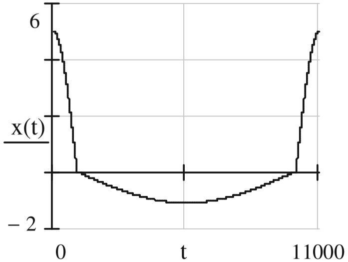
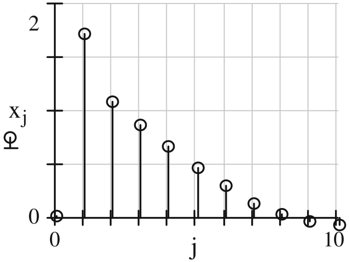
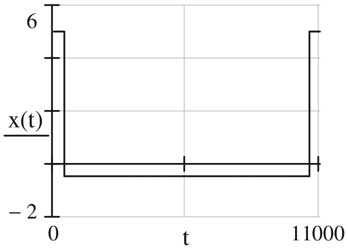
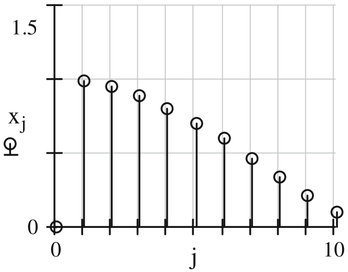
Being an integral parameter of the signal, harmonic distortion factor is invariant to its waveform provided that amplitudes of the first harmonics and root-mean-square values of high-order harmonics are equal. On the one hand, it gives a possibility to use square, triangle or sine wave signals, as well as "truncated sine" [1], "different-sized sine" [2] signals, etc. On the other hand, it causes an uncertainty in selection of a measuring signal and gives rise to ongoing discussions about its best waveform. Obviously, there is no single answer to this question because the waveform is not reduced to defining total harmonic distortion. The author is convinced that a waveform that is optimal in one sense or another must be searched for depending upon a purpose of a measuring signal, e.g. taking into account that a signal spectrum must be matched with an operating frequency band of a measuring instrument to be calibrated. A failure to fulfill this requirement causes some measurement errors that are impossible or rather difficult to estimate.

Table 2.4 Signals with specified Ka

NN 1	Name and analytical expression of signal 2	Graphical presentation 3	Signal spectrum 4	Basic signal parameters 5
1	Optimal signal with $Ka = \sqrt{10}$ with a spectrum of five harmonics $y(t) = \sum_{n=1}^{5} \cos n\omega_0 t$			$Ka = 3{,}162$, $V\max \approx 12,5 \cdot \omega_0$ ω max $\approx 5 \cdot \omega_0$
2	Quasi-optimal (realizable) signal $x(t) = \sum_{i=1}^{11} A_i \sin\left(\pi \frac{t-\tau_{i-1}}{T_i}\right)[H(t-\tau_{i-1}) - H(t-\tau_i)]$ where $T_i = \tau_i - \tau_{i-1}$ is the duration of the i-th interval; τ_i is the time of occurrence of the i-th pulse			$Ka = 3{,}162$, $V\max \approx 12,5 \cdot \omega_0$ ωmax $\approx 5 \cdot \omega_0$

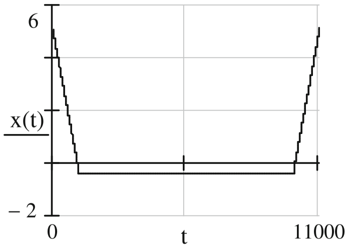
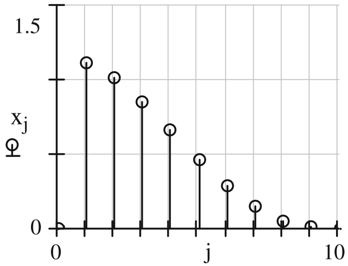
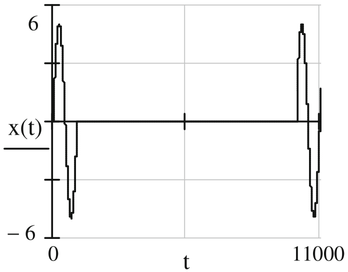
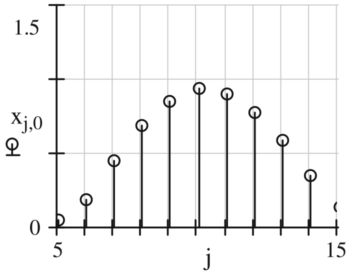
(continued)

Table 2.4 (continued)

NN 1	Name and analytical expression of signal 2	Graphical presentation 3	Signal spectrum 4	Basic signalparameters 5
3	Quasi-optimal signal $x(t) = \sum_{i=1}^{3} A_i \sin\left(\pi \frac{t - \tau_{i-1}}{T_i}\right)[H(t - \tau_{i-1}) - H(t - \tau_i)]$ where $T_i = \tau_i - \tau_{i-1}$ is the duration of the i-th interval; τ_i is the time of occurrence of the i-th pulse			$Ka = 3{,}162$, $V\max \approx 12{,}5 \cdot \omega_0$ $\omega\max \approx 10 \cdot \omega_0$
4	Square wave signal $x(t) = \sum_{i=1}^{3} A_i[H(t - \tau_{i-1}) - H(t - \tau_i)]$, where A_i is the amplitude of the i-th interval; τ_i is the time of occurrence of the i-th pulse			$Ka = 3{,}162$, $V\max \approx \infty$ $\omega\max \approx 55 \cdot \omega_0$

(continued)

Table 2.4 (continued)

NN 1	Name and analytical expression of signal 2	Graphical presentation 3	Signal spectrum 4	Basic signalparameters 5
5	Triangle wave signal $x(t) = \sum_{i=1}^{3} K_i(t - \tau_{i-1})[H(t - \tau_{i-1}) - H(t - \tau_i)]$, where K_i is the slope coefficient of the i-th pulse; τ_i is the time of occurrence of the i-th pulse			$K'a = 3{,}162$, $\nu\max \approx 8,75 \cdot \omega_0 \ \omega\max \approx 5 \cdot \omega_0$
6	RF pulse signal $x(t) = \begin{cases} 5\sin\omega t, & npu \quad 0 < t \le T/11, \quad (10/11) \\ 0, & npu \quad (1/11)T < t \le (10/11)T. \end{cases}$ where где $\omega = 2\pi/\mathrm{T}$			$K'a = 3{,}162$, $\nu\max \approx 50 \cdot \omega_0 \ \omega\max \approx 20 \cdot \omega_0$

Hence, the optimal signal will be deemed such a measuring signal, spectrum of which completely lies within the operating frequency band of a measuring instrument to be calibrated. Certainly, it is a whole class of optimal measuring signals rather than one signal since a number of harmonics and their relationship may be different. For example, sometimes it is recommended to use only two harmonics [21] in a measuring signal; however, a nonlinear distortion calibrator with three harmonics is known [19]. Moreover, it is proposed to use a greater number of them (up to 50) [22].

In addition to optimal signals, other signals that may be generated by simpler means are widely used in the measuring practice. For example, measuring "truncated sine" and "different-sized sine" signals are used in the national standard of total harmonic distortion [2]. They were proposed and studied by N.B. Petrov. However, the known measuring signals have a common disadvantage: their harmonic distortion factor is defined e.g. by a ratio, e.g. of root-mean-square voltage values of the second harmonic to the first one, or by a ratio of limiting level of a sine wave signal to its amplitude value [1]. An accurate definition of relationships between AC voltages is known to depend on a dividing error of dividers that are used for this purpose. For example, it may be an inductive voltage divider [23] or an accurate attenuator [2]. It needs no explanation that these devices are very labour-intensive to manufacture, bulky and inconvenient for automation. That is why the author proposes to define harmonic distortion factor [19] through a ratio of time intervals rather than through a voltage ratio. An error of measurement and reproduction of time intervals is known to have reached fractions of picoseconds. For this reason, there is no principal limitation on the further reduction of it.

Let us explain the essence of the proposed method using the model of measuring signal:

$$x(t) = \sum_{i=-\infty}^{\infty} a_i \sin\left(\frac{t-\tau_i}{T_i}\right)[H(t-\tau_i) - H(t-\tau_{i+1})], \tag{2.17}$$

where $T_i = \tau_{i+1} - \tau_i$, $a_i = a(-1)^i$.

This signal is a special case described by the expression (2.4) and represents a periodic sequence of sine wave pulses of alternating polarity, duration of which can be changed in the period T (Fig. 2.15). At $T_{-i} = \ldots = T_{-1} = T_0 = T_1 = T_2 = \ldots = T_i = T/2$, the expression (2.17) can be used the model of a strictly sine wave periodic signal, for which $K_\mathrm{h} = 0$.

Changing a ratio of times of the positive and negative half waves, we can obtain different values of K_h. The remarkable feature of this signal is a unique relationship between harmonic distortion factor and ratio of times, as well as the stability of its amplitude, average-rectified and root-mean-square values upon changing in half wave times. Using such a signal in measuring means for calibration of total harmonic distortion measuring devices, we can significantly improve their performance due to reduced time of calibration by a root-mean-square value and

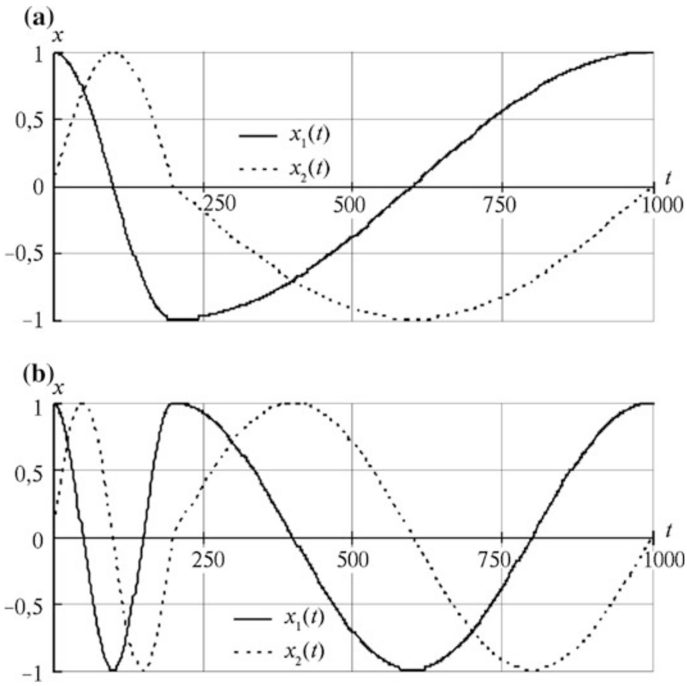


Fig. 2.15 Measuring signals with specified total harmonic distortion

higher level of automation. These improvements were implemented in one generator protected by the certificate of the patent [19].

Carrying out a simple harmonic analysis, we shall derive an expression for the total harmonic distortion: for the signal presented in Fig. 2.15a, and total harmonic distortion, presented in Fig. 2.16a.

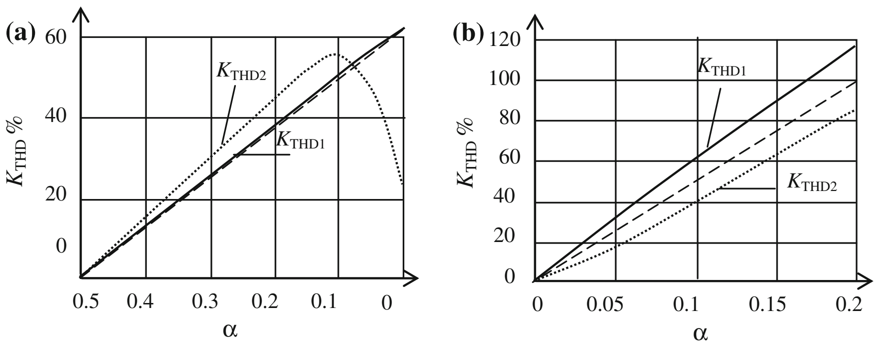
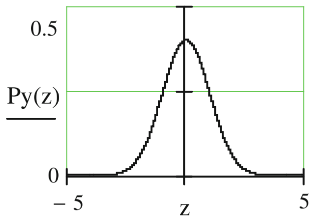
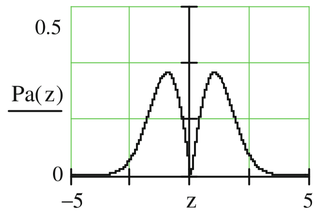
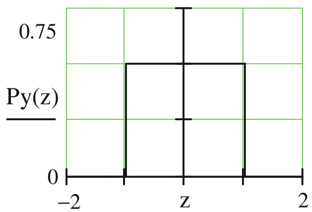
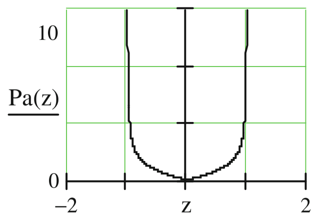
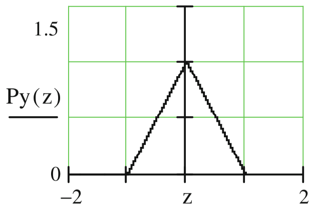
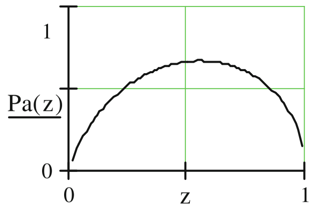


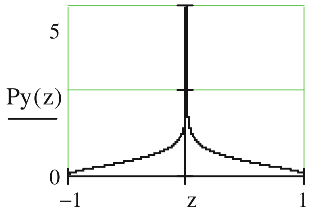
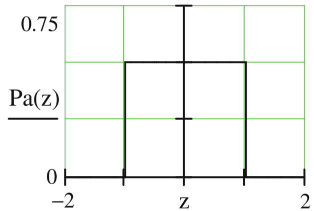
Fig. 2.16 Graphs of K_{THD} as a function of ratio of durations of half-periods (**a**) and half waves (**b**)

Table 2.5 Signals with a defined probability density function at sine waveform of a pulse

NN	Probability density function defined for synthesis		Probability density of signal amplitudes	
	Name and analytical expression	Graphical presentation	Analytical expression	Graphical presentation
1	Normal $P_{y(t)}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}}$		$P_A(z) = \frac{\lvert z\rvert}{2\sigma^2} e^{-\frac{z^2}{2\sigma^2}}$	
2	Uniform $P_{y(t)}(z) = \begin{cases} 1/2, & \lvert z\rvert \le 1 \\ 0, & \lvert z\rvert > 1 \end{cases}$		$P_A(z) = \frac{\lvert z\rvert}{\sqrt{1-z^2}}, \ 0 \le z < 1$	
3	Simpson $P_{y(t)}(z) = \begin{cases} 1+z, & -1 < z \le 0 \\ 1-z, & 0 < z \le 1 \end{cases}$		$P_A(z) \approx 3z(1-z)$	

(continued)

Table 2.5 (continued)

NN	Probability density function defined for synthesis		Probability density of signal amplitudes	
	Name and analytical expression	Graphical presentation	Analytical expression	Graphical presentation
4	Hyperbolic cosine $P_{y(t)}(z) = \frac{1}{2} a \cos \text{h} \frac{1}{z}$		$P_{y(t)}(z) = \begin{cases} 1/2, & 0 \leq \lvert z \rvert < 1 \\ 0, & \lvert z \rvert < 0 \end{cases}$	

The graphs show the dependence of K_{THD} on value $\alpha = t_1/T$, where α is a ratio of the first half period duration (Fig. 2.15a) or the first half wave duration (Fig. 2.15b) to period. Thus, changing the coefficient α, it is easy to generate measuring signals with specified harmonic distortion factor, value of which is determined by a ratio of these durations and lies within 1 до 0. They may be obtained by sine wave oscillators with the possibility to control both half-periods. Note that the generation of sine wave signals at $K_{\mathrm{THD}} = 0$ % is a special case for equal half waves ($\alpha = 1$), or equal half periods ($\alpha = 0$). It should be also noted that two oscillators are generally used in modern equipment for calibration of a total harmonic distortion measuring instrument.

The described signals can be generated by one dynamical system and implemented in one measuring instrument—a sine wave oscillator. Table 2.5 compares different signals with a specified value of total harmonic distortion. Naturally, the generation principle proposed by the author is more promising. The oscillator systems reproducing these measuring signals and their reproduction errors will be considered in Chap. 3.

2.6 Synthesis of Models for Random Measuring Signals

Many studied, e.g. [4, 6, 7, 10] are devoted to the synthesis of random processes. Let us analyze some of them on the assumption that the proposed models are designed for hardware implementation. For example, the signal model considered on the basis of the canonical decomposition method is represented as a sum of deterministic functions of random amplitude.

$$x(t) = \sum_{k=1}^{\infty} V_k f_k(t),$$

where $f_k(t)$ are prescribed deterministic time functions; V_k are random values.

The model of such type has the following disadvantages:

- a large number of series terms is required for higher accuracy of representing a random process;
- a higher amount of computational effort, which is necessary for multiple integration of set of equations.
- a higher reproduction error resulting from the error of coefficients V_k;
- simple error estimates are not applicable.

When using the canonical decomposition method, some other disadvantages become apparent. In particular, a sum of terms is to be limited during the hardware or software implementation of a random process in line with the formula given above. As a result, it should be expected that a series with limited number of expansion terms will differ from an initial series in a probability sense.

Here, as in the case of expansion of deterministic functions, there is a contradiction between an attempt to increase accuracy and resulting errors. Therefore, the models based on the canonical decomposition are unsuitable for implementation of oscillators since it is necessary to generate a large number of random elementary processes and impossible to achieve a high reproduction accuracy.

The models proposed by V.I. Chernetsky [13] are more promising for implementation of the model, in particular:

$$x(t) = \varphi(t, \lambda_1, \lambda_2, \ldots, \lambda_n),$$

where λ_1, λ_2, …, λ_n are independent random values; $\varphi(t, \lambda_1, \lambda_2, \ldots, \lambda_n)$ is the deterministic nonlinear function of time and random values.

Such models are called non-canonical. For instance, the following class of parametric models [13] was studied.

$$y_\mu(t) = A(t, \mu) \cos[\omega(t, \mu), \varphi],$$

where $A(t, \mu)$, $\omega(t, \mu)$ are, respectively, the random amplitude and harmonic representation frequency of a random process; φ is the random phase.

In fact, the random process in these models is an amplitude, frequency and phase-modulated process. By controlling random processes, we can change characteristics of the process. This method is free of the disadvantages that are inherent to the canonical decomposition. For example, it allows us to go beyond the correlation theory. The method is characterized by a higher accuracy of representing random processes and a wider class of random processes. However, it has a number of inherent disadvantages, which are associated with the fact that the class of processes to be simulated is limited by nonergodic processes. This property is of principal value and does not allow us to select a given solution for hardware implementation.

The model of the process can be considered as a sequence of pulses [24]

$$\xi(t) = \sum_{i=-\infty}^{\infty} A_i h(t - t_i, \tau_i); \quad -\infty < t < \infty,$$

where $\{A_i\}$, $\{t_i\}$, $\{\tau_i\}$ are, respectively, the random amplitude, time of occurrence and duration of a pulse described by the function $h(t - t_i, \tau_i)$.

This model makes it possible to simulate general random processes with well-studied parameters and characteristics [12, 21].

Thus, let us formulate the basic requirement for the random signal model that is suitable for hardware implementation. It must be filled with the physical content inherent to this class of physical phenomena so that the transition from the model of a signal to the model of a dynamical system generating it is natural and as formalized as possible. The following model best meets these criteria (2.4).

For this reason, we shall synthesize random signal models, as well as deterministic ones, on the basis of the generalized model (2.4),

$$X = \left\{ x(t) : x(t) = \sum_{i=0}^{\infty} a_i \psi_i(t, \tau_i, T_i)[H(t - \tau_i) - H(t - \tau_{i-1})]; \; i = 1, \overline{\infty} \right\},$$

where $\psi(z)$ is the random function; A_i is the random amplitude distributed by the law $P_A(x)$, with $A_i = \text{const}$ in the section $(\tau_{i\,-\,1}, \tau_i)$; τ_i is the random moment of occurrence of the i-th pulse; $\tau_i = \sum_{k=1}^{i} T_k$, T_i is the random duration of the i-th pulse with the frequency distribution $P_T(x)$ at $T_i = \tau_i - -\tau_{i--1}, \forall i; H(z)$ is the Heaviside function.

According to the expression (2.4), the mathematical random process model is presented as a product of a sequence of random numbers A_i and two deterministic functions $\psi(z)$ and $H(z)$ of random arguments T_i and τ_i It is a generalization of the known model for random processes in the form of pulse sequence [9, 12]. However, despite that these models are well-known, the effects of static and especially functional relationships between homogeneous and heterogeneous quantities on process characteristics have not been given sufficient attention in the literature. Therefore, based on the expression (2.4), we shall synthesize below some specific random process models with specified probabilistic characteristics, in particular for statistically independent, statistically dependent and functionally related random values A_i and T_i. In this case the synthesis is understood to be an identification of deterministic and probabilistic characteristics of controlled quantities $\psi(z), A_i, T_i : P_A(x), P_T(x), R_A(\tau), R_T(\tau)$, depending upon specified parameters and characteristics $M[x(t)], D[x(t)], P_x(y), S(\omega), R_x(\tau)$, where $M[x(t)]$ is an expected value; $D[x(t)]$ is the dispersion; $P_x(y)$ is the probability density; $S(\omega)$ is the spectral density; $R_x(\tau)$ is the correlation function of the process $x(t)$.

Due to their evident limitations, the measuring means designed for reproduction of random measuring signals generate only one implementation rather than an ensemble of implementations. Therefore, as per the classification of E.I. Tsvetkov [25], the random measuring signal model must be a stationary or random non-stationary ergodic process. Hence, we shall synthesize below random stationary ergodic processes. The results presented here were obtained in collaboration with postgraduate student A.L. Baranovsky under the scientific supervision of the author.

It is known [12] that, in a restricted sense, the stationary process means a process for which a joint probability density function does not depend on time, i.e.

$$P_x(\xi_1,\ \xi_2,\ \ldots,\ \xi_n,\ t_1,\ t_2,\ \ldots,\ t_n) = P_x(\xi_1,\ \xi_2,\ \ldots,\ \xi_n,\ t_1 + \tau,\ t_2 + \tau,\ \ldots,\ t_n + \tau).$$

Since the characteristics listed above are determined on the basis of a density function of order not higher than the second, hereinafter we shall consider models of stationary processes in a general sense, for which $P_x(\xi, t) = P_x(\xi), P_x(\xi_1, \xi_2, t_1, t_2) = P_x(\xi_1, \xi_2, t_2 - -t_1)$, and models of periodically stationary processes, for which

$$P_x(\xi, t) = P_x(\xi, t + T),$$
$$P_x(\xi_1, \xi_2, t_1, t_2) = P_x(\xi_1, \xi_2, t_2 - -t_1 - -T),$$

where T is the period of stationarity.

We shall discuss below the synthesis of signal models with statistically independent homogeneous and heterogeneous controlled parameters, i.e.

$$P_X(A_i,\ A_{i+1},\ \ldots,\ A_j) = P_X(A_i)P_X(A_{i+1})\cdots P_X(A_j);\ \ \forall i;\ \ j \in \overline{1, \infty};$$

$$P_X(T_i,\ T_{i+1},\ \ldots,\ T_j) = P_X(T_i)P_X(T_{i+1})\cdots P_X(T_j);\ \ \forall i;\ \ j \in \overline{1, \infty};$$

$$P_X(A_i,\ T_j) = P_X(A_i)P_X(T_j);\ \ \forall i;\ \ j \in \overline{1, \infty}.$$

They are equivalent to the models of 1100 type, as per the classification presented in Sect. 2.1.

2.6.1 Synthesis of Signal Models with a Given Probability Density Function

Let us assume that the random signal model is to be synthesized with the probability density function $P_y(z)$. Choose a stationary random process as the optimal random signal. We shall use the sequential optimization method (see Fig. 2.7) and, for this purpose, define a set of realizable signals, following the agreed notations of the random process theory $X(t) = x(t) : P_A(x), M[x(t)], M[x^2(t)] = \vartheta^2\}$.

Let us present the transformation F_{qm} as follows. Since $x(t)$ is a stationary random process, in which ψ and A are independent, we shall have:

$$P_x(z) = \int\limits_0^1 P_\psi(u) P_A(\frac{z}{u}) P_x(T_j)\,\frac{du}{u},$$

but

$$P\{\psi(t) \le u\} = P\big\{0 \le \eta(t) \le \psi^{-1}(u)\big\} + P\big\{[1 - \psi^{-1}(u)] \le \eta(t) \le 1\big\} = 2\psi^{-1}(u),$$

then, the relationship can be reduced to

$$P_x(z) = 2\int\limits_0^{-1} \psi^{-1}(u) P_A\left(\frac{z}{u}\right)\,\frac{du}{u}. \tag{2.18}$$

The latter equation is a functional transformation F_{qm}, according to which the probability density function $P_x(z)$ corresponds to the process $x(t)$ with the function $\psi(t)$ and the amplitude distribution $P_A(z)$ in cases where the inverse function

$\psi^{-1}(u)$ exists. It shows that the probability density function can be controlled either changing the function ψ or the amplitude distribution $P_A(z)$.

If we define ψ or $P_A(z)$, the expression (2.18) will be transformed to an integral equation in relation to the second unknown function. For example, in case of uniform amplitude distribution

$$P_A(x) = \begin{cases} 1/2, |x| \leq 1; \\ 0, |x| > 1 \end{cases}$$

the solution of the expression (2.18) will be:

$$\psi^{-1}(y) = -\int_0^y z\dot{P}_{x(t)}(z)dz. \tag{2.19}$$

If we choose a certain function $\psi(t)$ and substitute the variable $y = z/u$, the equation will be reduced to the integral Volterra equation of the first kind in relation to $P_A(y)$:

$$P_x(y) = 2\int_0^y \frac{1}{z}\psi^{-1}\left(\frac{y}{z}\right)P_A(z)dz,\ y \geq 0. \tag{2.20}$$

Let us introduce the distance between $P_y(z)$ and $P_x(z)$ for consideration. As is known [23], the distance between densities may be determined in a uniform and quadratic metrics using different criteria: module, root-mean-square, Bernstein, correlation, Bhattacharya, Kullback divergence, etc.

Let us use the most sensitive [23] modulus criterion:

$$\rho = \int_{-\infty}^{\infty} \left|P_x(z) - P_y(z)\right|dz.$$

If $\rho = 0$, then $P_y(z) = P_x(z)$. In this case

$$P_{y(t)}(y) = 2\int_v^\infty \frac{1}{v}\psi^{-1}\left(\frac{v}{z}\right)P_A(z)\,dv,\ \ v \geq 0. \tag{2.21}$$

The left side of the equation (2.21) represents the probability density function defined for the synthesis, while the right side—the functional relationship that connects this function with the function describing a pulse waveform and density of its amplitudes. This formula gives a possibility to determine one unknown quantity, when the other is specified, and, therefore, to synthesize the random signal model with a required density function.

The case with a sine wave pulse is of special practical interest: $\psi(t) = \sin \pi t$, $0 \leq t < 1$. Then, the expression (2.21) is reduced to the Abelian equation relative to $P_A(x)$

$$P_y(v) = \frac{2}{\pi} \int_{v}^{\infty} \frac{P_A(z)}{\sqrt{v^2 - z^2}} dv, \; v \geq 0, \tag{2.22}$$

for which the inversion formula is known [26]

$$P_A(z) = -z \int_{v}^{\infty} \frac{\dot{P}_y(v)}{\sqrt{v^2 - z^2}} dv, \; z > 0. \tag{2.23}$$

The derived relationship (2.23) coincides with the known one [12] for sine wave signals with a random amplitude and phase [12]. Let us illustrate the synthesis of a random process with a defined probability density function:

$$P_y(v) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-v^2}{2\sigma^2}}.$$

This expression describes a probability density function of the stationary centered Gaussian process with the dispersion σ. Let us select such a process as the optimal signal model. Substituting the density function defined for the synthesis into the equation (2.23) and using the expressions for standard integrals, we shall obtain:

$$P_A(z) = \frac{|z|}{2\sigma^2} e^{\frac{-z^2}{2\sigma^2}}, \; |z| \leq \infty.$$

Hence, the process $x(t)$ will have the Gaussian distribution at $\psi(t) = \sin \pi t$, $0 \leq t < 1$ if the density of pulse amplitude values is distributed by the law $P_A(z) = 1/2\ P_R(|z|)$, where $P_R(|z|)$ is the Rayleigh distribution law. The final expression for the synthesized process is as follows:

$$x(t) = \sum_{i=0}^{\infty} A_i \sin\left(\pi \frac{t - \tau_{i-1}}{T_i}\right) [H(t - \tau_{i-1}) - H(t - \tau_i)], \; P_A(z) = \frac{1}{2} P_R(z).$$

Note that the distribution of pulse durations T_i and moment τ_0 may be arbitrary. Table 2.5 presents the standard probability densities and respective amplitude densities for random sine- and triangle wave signals calculated on the basis of the relationships presented above.

Signals with a square waveform are subject to $P_{x(t)}(y) = P_A(|z|),$, as it can be easily seen by making a differentiation in a generalized sense, i.e. density of the process coincides with density of pulse amplitudes.

2.6.2 Synthesis of Signal Models with a Defined Correlation Function

The correlation function of a random stationary ergodic process is known as

$$K_X(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)x(t-\tau)dt. \tag{2.24}$$

Taking into account the imposed constraints, let us substitute the expression for $x(t)$ into this equation by integrating products of infinite sums of pulses of the process and its shifted copy. We shall obtain

$$K_X(\tau) = \frac{M[A^2]}{M[T]} \int_{y>\tau} y \int_0^{1-\frac{\tau}{y}} \psi(z)\psi(z+\frac{\tau}{y})dzP_T(y)dy, \tag{2.25}$$

where $z = (t - \tau_{i-1})/T_i$.

From the equation (2.25) it is easily to find the dispersion that determines the average power of the process:

$$K_X(0) = \sigma^2_{x(t)} = \sigma^2_a \int_0^1 \psi^2(z)dz.$$

This equation also demonstrates the asymptotic properties of the correlation function $\lim_{\tau\to\infty} K_x(z) = 0$, which confirms the stationarity of the process. If we limit the space Ω_T of possible values of a random value T that characterizes the pulse duration, e.g. $\Omega_T = (0, b)$, then $K_x(\tau) = 0$ at $\tau \geq b$. The derived equation (2.25) is an initial equation for solution of the synthetics problem. It contains unknown functions $\psi(z)$ and $P_T(y)$. Naturally, the synthesis problem can be solved by defining one of these functions. If the function $\psi(z)$, is defined, the expression (2.25) goes to the Volterra equation of the first kind relative to the unknown probability density function of a random value T. For example, at $\psi(t) = \sin \pi t/T$, $\Omega_T = (0, \infty)$ we shall have the following equation:

$$K_X(\tau) = \frac{M[A^2]}{2M[T]} \int_\tau^\infty \left[(y-\tau)\cos\frac{\pi}{y}\tau + \frac{y}{\pi}\sin\frac{\pi}{y}\tau\right] P_T(y)dy.$$

If we take that the correlation functions defined for the synthesis—$K_y(\tau)$ and $K_x(\tau)$—coincide in a uniform metric, we shall obtain the following equation:

$$K_y(\tau) = \frac{M[A^2]}{2M[T]} \int_{\tau}^{\infty} \left[(y-\tau)\cos\frac{\pi}{y}\tau + \frac{y}{\pi}\sin\frac{\pi}{y}\tau \right] P_T(y)dy \tag{2.26}$$

with unknown $P_T(y)$, which can be solved by the known methods [12, 13].

2.6.3 Synthesis of Signal Models with a Specified Spectral Density

According to [12, 13], the spectral density of the stationary random process is

$$S_X(\omega) = \frac{2M[A^2]}{M[T]} \int_0^{\infty} x^2 |g(\omega x)| P_T(x)dx, \tag{2.27}$$

where $g(y) = \int_0^1 \psi(t)e^{-jyt}dt$ is the pulse spectrum $\psi(t)$; $M[A^2]$ is the amplitude dispersion; $M[T]$ is the expected value of pulse duration.

The equation (2.27), at a spectral density and a waveform or duration probability density defined for the synthesis using the Mellin transformation [26] can be used to find either duration probability density, or a waveform, respectively. Let us denote a spectral density of the optimal signal as $S_{y(t)}(\omega)$, introduce the notations $\widehat{P}_T(x) = x^2 P_T(x), \quad K(y) = \left|g(y)^2\right|$ and perform the Mellin transformation. Then, multiplying the left and right side of this expression by $\omega^{\,p-1}$ and integrating it over ω between 0 and ∞, we shall obtain:

$$S(p) = \frac{2M[A^2]}{M[T]} \int_0^{\infty} \widehat{P}_T(x) \int_0^{\infty} \omega^{p-1} K(\omega x)\, d\omega\, dx.$$

However, $\int_0^{\infty} \omega^{p-1} K(\omega x)\; dx = \int_0^{\infty} \left[(\omega x)^{p-1} K(\omega x)/x^p \right] d(\omega x) = K(p)x^{-p}.$

From this we can set up the following operator equations:

$$\begin{aligned} S(p) &= \frac{2M[A^2]}{M[T]} K(p) \int_0^{\infty} \widehat{P}_T(x)\, x^{-p}\, dx = \frac{2M[A^2]}{M[T]} K(p)\widehat{P}(1-p); \\ \widehat{P}(p) &= \frac{S(1-p)}{K(1-p)} \frac{M[T]}{2M[A^2]}, \end{aligned} \tag{2.28}$$

where $\widehat{P}(p)$, $S(p)$, $K(p)$ are images of $P_T(x)$, $S_y(\omega)$, $K(y)$.

Solving the operator equations (2.28) by converting the Mellin transformation [26], we shall derive a desired unknown function: either $P_T(x)$, or $K(y)$. In this way we can determine either waveform of a pulse or distribution of its duration.

It should be noted that a real solution of the problem cannot be found for each spectral density defined for the synthesis. Let us find constraints on the class of realizable spectral densities.

Let us assume that $S_y(\omega)$ and $P_T(x) = 1$, $x \in (0, 1)$. Then, denoting $\lambda = 2M[A^2]/M[T]$, we shall obtain the relationship:

$$S_y(\omega) = \lambda \int_0^{\infty} x^2 |g(\omega x)|^2 P_T(x) dx = \frac{\lambda}{\omega^3} \int_0^{\omega} y^2 |g(y)|^2 dy, \tag{2.29}$$

Differentiating it, let us find the condition that limits the class of spectral densities defined for the synthesis:

$$3S_y(\omega) + \omega \dot{S}_y(\omega) = \lambda |g(y)|^2. \tag{2.30}$$

The right side of the equation (2.30) is a positive frequency function. Hence, the left side must be strictly positive. Therefore, the physical realizability of the process with the spectral density $S_{y(t)}(\omega)$ and uniform probability density $P_T(x)$ will be the inequality $3S_y(\omega) + \omega \dot{S}_y(\omega) \geq 0$, $\forall \omega \in \Omega$. The amplitude spectrum of a pulse and, therefore, its waveform can be found at the spectral densities that satisfy this inequality.

$$g(\omega) = \frac{1}{\sqrt{\lambda}} \sqrt{3S_y(\omega) + \omega \dot{S}_y(\omega)}\,. \tag{2.31}$$

If a prescribed spectral density does not satisfy the equation (2.30), we can impose a new constraint that is satisfied by it by choosing other pulse duration distribution law, except for a uniform one.

Adding a phase spectrum to the amplitude spectrum of a pulse and using the Fourier inversion, we shall obtain a pulse waveform. However, of adding a phase spectrum to the amplitude spectrum is not an unequivocal operation. Therefore, we shall seek a pulse form among describable even or odd functions. Then, the relationship

$$\psi_c(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(\omega) \cos \omega t \, d\omega$$

allows us to uniquely determine a pulse waveform as even function and the relationship

$$\psi_c(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(\omega) \cos \omega t \, d\omega$$

as uneven time function.

In cases where the generation of a prescribed spectral density through changing either pulse waveform or distribution of its density is complicated (e.g. random signals are generated from periodic oscillations by stochastization of its amplitude), we can control a spectral density by adding a stochastic dependence of homogeneous random values, e.g. pulse amplitudes. Taking into account this dependence, the spectral density expression will be:

$$S_{X(t)}(\omega) = 2T\left|g(\omega)^2\right|\left\{M[A^2]\left(1 + 2\sum_{p=0}^{\infty}(-1)^p R_A(p)\cos p\omega T\right)\right\}, \tag{2.32}$$

where T is the pulse repetition interval; $R_A(p)$ are the correlation coefficients of amplitude sequences $\{A_i\}$; they differ from the known quantity [12] by a multiplier $(-1)^p$.

T, $g(\omega)$, $M[A^2]$ are assumed to be known, while $R_A(p)$—unknown. Let us determine the correlation coefficients. Let it be required to satisfy

$$S_y(\omega) = S_x(\omega),\ \forall\omega \in \overline{0,\omega_n}, \tag{2.33}$$

where ω_{u} is a prescribed upper frequency of the spectral density.

Let us insert the expression (2.33) in the equation (2.32) to obtain:

$$G(\omega) = \sum_{p=1}^{\infty} R_A(p)\cos p\omega T,\ \forall\omega \in \overline{0,\omega_{\mathrm{B}}},$$

where

$$G(\omega) = \frac{S_{y(t)}(\omega)}{4T|g(\omega)|^2 M[A^2]} - \frac{1}{2}.$$

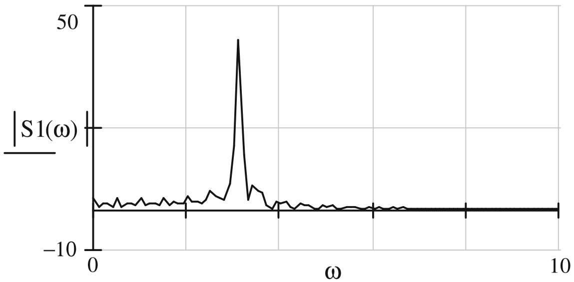
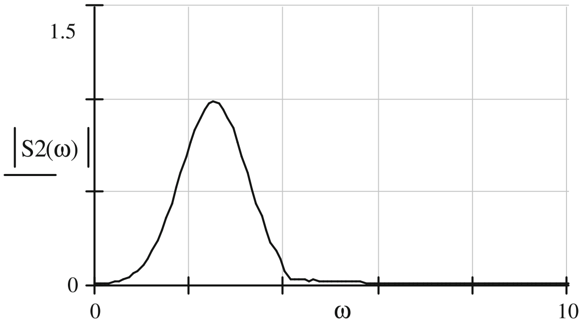
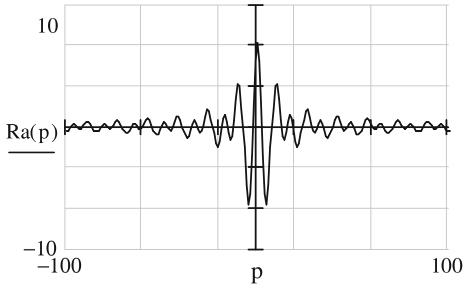
This equation is easy to solve in relation to unknown $R_A(p)$:

$$R_A(p) = \frac{2}{\omega_{\mathrm{B}}}\int_0^{\omega_{\mathrm{B}}} G(\omega)\cos p\omega T d\omega \tag{2.34}$$

In order to approximate the spectral density $S_{x(t)}(\omega)$ to a predefined $S_{y(t)}(\omega)$ within a frequency band $\overline{0,\omega_{\mathrm{B}}}$, it is enough to determine a series of correlation coefficients, i.e. to solve the equation (2.34). Based on the derived values, a sequence of amplitude values $\{A_i\}$ with required correlation properties can be generated by the known methods, e.g. the moving summation method. If the correlation coefficients satisfy the equality $R_A(p) = [R_A(1)]^p$, a required sequence of amplitudes is generated on the basis of the recurrent procedures considered below.

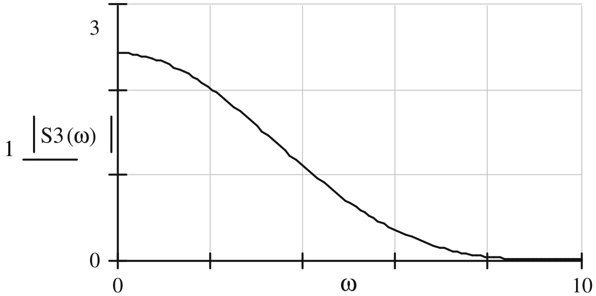
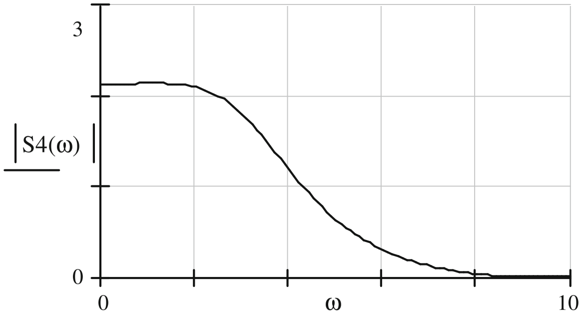
Table 2.6 presents standard spectral densities and the respective coefficients $R_A(p)$ derived on the basis of analytic calculations. The first spectral density graph corresponds to a periodic sine wave signal. Its spectral density is known to be a unit impulse on a frequency of a periodic signal. It is clear that in case of numerical calculations and limitation of p (i.e. at $p = \{-100, +100\}$) the spectral

Table 2.6 Spectral density of random signals

NN	Spectral density	Analytical expression of spectral density, expression or graphs of $R_a(p)$
1		$S_1(\omega) = 2T\left\|g(\omega)^2\right\|\left\{M[A^2]\left(1 + 2\sum_{p=-100}^{100}(-1)^p R_a(p)\cos p\omega T\right)\right\}$, $R_a(p) = 1\ M(A) = 1,\ T = 1$
2		

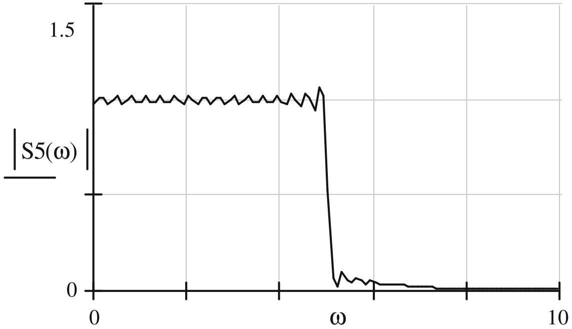
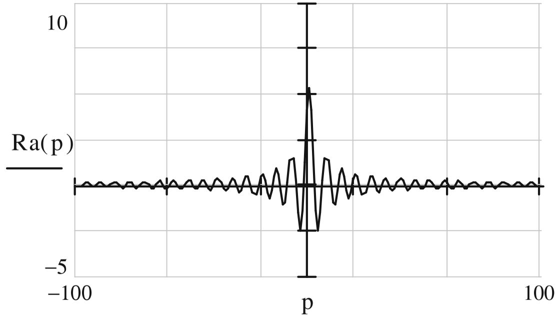
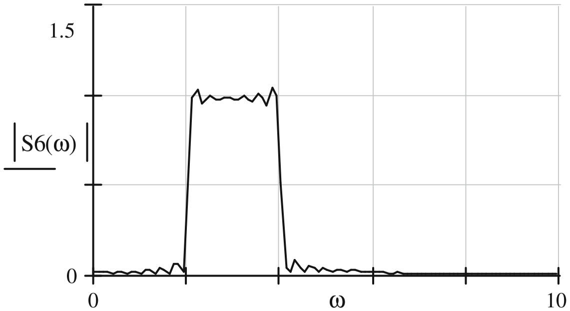
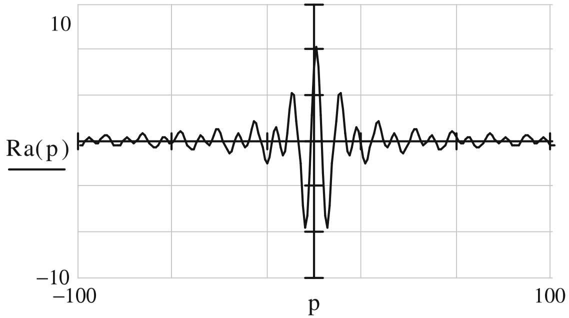
(continued)

Table 2.6 (continued)

NN	Spectral density	Analytical expression of spectral density, expression or graphs of $R_a(p)$
3		$R_a(p) = \begin{vmatrix} 1 & \text{if } p = 0 \\ 0 & \text{otherwise} \end{vmatrix}, M(A) = 1,\ T = 1$
4		$R_a(p) = \exp(-\lvert 2.3p \rvert), M(A) = 1, T = 1$

(continued)

Table 2.6 (continued)

NN	Spectral density	Analytical expression of spectral density, expression or graphs of $R_a(p)$
5		
6		

density graph only tends to a spectrum of unit impulse. Graphs 5 and 6 demonstrate the possibility to implement quasi-white and pink noise.

The analysis considered above allows us to make the following conclusion. At statistical independence of homogeneous and heterogeneous values of A and T, the probability density of the process $x(t)$ is invariant in relation to a change in duration and determined only by a waveform and distribution of pulse amplitudes.

The spectral density of the process $x(t)$ is invariant in relation to amplitude distribution and depends only on a waveform and distribution of pulse durations. It makes it possible to synthesize the process with probability density and spectral density, whether they are defined separately or simultaneously.

Summarizing this study, let us outline the possible ways of simulating a random measuring signal with prescribed probabilistic characteristics.

Based on the probability density defined for the synthesis $P_y(z)$ and using the equation (2.21), we can find a required waveform of pulses, with their amplitudes uniformly distributed over the interval $[-1, 1]$. If a waveform is predefined, the probability density of amplitudes is determined by solving the solution (2.21) using the well-known methods [12]. The solution for a sine waveform can be found by the equation (2.23).

Based on a specified spectral density and using the expression (2.29), we can find a probability density of pulse duration. In case of the defined probability density function $P_y(x)$, a pulse waveform can be determined from the expression (2.23).

Based on a specified spectral density with statistically independent random pulse amplitudes and applying the formula (2.34), we can find required correlation coefficients and use these values for generating a specified sequence of pulse amplitudes by the known procedures.

Based on the defined correlation function $R_y(\tau)$, the probability density function $P_T(x)$ can be derived from the equation (2.26).

Thus, the problem of synthesizing measuring signal models of "1000" type (models of stationary ergodic random processes) stated in the beginning of this chapter can be solved for all probabilistic characteristics listed above on the basis of relatively simple procedures. This is an important advantage of the proposed method.

The models of "1000" type will be used in Chap. 3 for the synthesis of a dynamical system generating the respective random signals.

2.6.4 Synthesis of Random Signals with Functionally Dependent Controlled Parameters

Two statistically independent values A and T must be reproduced for generation of a signal on the basis of the model (2.4). In case of hardware implementation, this makes it necessary to construct two independent random number oscillators. Such a circumstance complicates a random signal source. Therefore, it is reasonable to consider the possibility of synthesizing signal models with functionally related random values. Let us suppose that homogeneous random values are statistically independent, while heterogeneous ones are related by the equation $A = \chi(T)$ or $T = \chi^{-1}(A)$. Obviously, in this version, two random values can be generated using one oscillator of random independent numbers and nonlinear functional oscillator.

Generally speaking, the introduction of the functional connection between A and T makes the process $x(t)$ non-stationary, as opposed to the case with independent values A and T. Here, one question naturally comes to mind: Does any function χ result in nonstationarity? The author has established that uncorrelatedness between A and T is a sufficient condition for stationarity and ergodicity of a random process $x(t)$. The condition of uncorrelatedness between A and T is given by $\mathrm{cov}(A, T) = 0$. Taking into account the symmetry of the amplitude distribution law ($M[A] = 0$), we shall at $M[A, T] = 0$

$$\int_{\Omega_A} x\chi^{-1}(x)P_A(x)dx = 0. \tag{2.35}$$

Obviously, the identical fulfillment of the equation (2.35) is possible only if $\chi^{-1}(x)$ is an even symmetrical function since the expression under the integral sign is an uneven function. Considering that random values T are non-negative, the function $\chi^{-1}(x)$ mapping the frequency Ω_A into the frequency Ω_T, must be non-negative, i.e. $\chi^{-1}(x) \geq 0,\ \forall x \in \Omega_A$. Such functions just ensure the stationarity and ergodicity properties of the random process $x(t)$.

2.6.5 Synthesis of Models for Signals with a Specified Probability Density Function at Functional Relationship of Parameters

Taking into account the stationarity condition of a process to be synthesized, let us use the Parzen–Rosenblatt window method [27] to determine the probability density. As applied to our case, it will take the form:

$$P_x^*(z) = \frac{1}{T^*h}\int_0^{T^*} K\left(\frac{z - x(t)}{h}\right) dt, \tag{2.36}$$

At $T^* \to \infty$, $h \to 0$

$$P_x(z) = \lim_{T^*\to\infty}\lim_{h\to 0}\frac{1}{T^*h}\int_0^{T^*} K\left(\frac{z - x(t)}{h}\right) dt, \tag{2.37}$$

the estimation is consistent and unconfounded. Let us assume, without loss of generality, that $T^* = \tau_L$ i.e. the observation time covers an integral number of pulses L. Then the equation (2.36) can be written as:

$$P_x^*(z) = \frac{1}{\tau_L h} \sum_{i=1}^{L} \int_0^{T^*} K\left(\frac{z - A_i \psi\left(\frac{t - \tau_{i-1}}{T_i}\right)}{h} \right) dt. \tag{2.38}$$

Since $K(t, \tau, T, h)$ is an indicator function, it will be sufficient to integrate the formula (2.38) in the range of such values t, at which

$$-1 \le v = \frac{z - A_i \psi\left(\frac{t - \tau_{i-1}}{T_i}\right)}{h} < 1.$$

It follows from here that

$$P_x^*(z) = \int_{\tau_{i-1}}^{\tau_i^*} K(v)dt = \operatorname{sign} z T_i \left\{ \psi^{-1}\left(\frac{z+h}{A_i}\right) - \psi^{-1}\left(\frac{z-h}{A_i}\right) \right\} \Lambda\left(\frac{z + h \operatorname{sign} z}{A_i}\right),$$

where $\Lambda(w) = \begin{cases} 1, & \text{if } x \in (0,1); \\ 0, & \text{if } x \notin (0,1). \end{cases}$

Since $\tau_L = \sum_{i=1}^{L} T_i$, in case of infinite increase in a number of summands T_i, the value τ_L/L converges to $\frac{1}{L}\sum_{i=1}^{L} M[T_i] = M[T]$ in compliance with the Fisher–Tippett–Gnedenko theorem for a sum of dependent random variables. Taking into account the asymptotic unbiasedness of the estimate (2.38), we shall have:

$$\begin{aligned} P_{X(t)}(z) &= \lim_{T^* \to \infty} \lim_{h \to 0} \frac{\operatorname{sign} z}{hM[T]} M\left\{ \psi^{-1}\left(\frac{z+h}{A_i}\right) - \psi^{-1}\left(\frac{z-h}{A_i}\right) \right\} \Lambda\left(\frac{z + h \operatorname{sign} z}{A_i}\right) \\ &= \lim_{h \to 0} \frac{\operatorname{sign} z}{hM[T]} \int_{\Omega_A} \int_{\Omega_T} y \left\{ \psi^{-1}\left(\frac{z+h}{A_i}\right) - \psi^{-1}\left(\frac{z-h}{A_i}\right) \right\} \Lambda\left(\frac{z + h \operatorname{sign} z}{A_i}\right) P_{A,T}(x, y)\, dxdy, \end{aligned} \tag{2.39}$$

where $P_{A,\ T}(x, y)$ is a joint probability density function of A and T.

The expression under the integral sign contains an indicator function Λ. Hence, according to the expression (2.39), the integration over the whole space in the range of Ω_A goes to the integration of its part defined by the range $\{x : |x| \ge z + h \operatorname{sign} z\}$. It implies that

$$
\begin{aligned}
P_{X(t)}(z) &= \frac{\operatorname{sign} z}{M[T]} \lim_{h\to 0} \frac{1}{h} \\
&\times \int\limits_{|x| > z+h \operatorname{sign} z} \int\limits_{\Omega_T} y\left\{\psi^{-1}\left(\frac{z+h}{A_i}\right) - \psi^{-1}\left(\frac{z-h}{A_i}\right)\right\}\Lambda\left(\frac{z+h \operatorname{sign} z}{A_i}\right) P_{A,T}(x,y)dxdy \\
&= 2\frac{\operatorname{sign} z}{M[T]} \lim_{h\to 0} \frac{1}{h} \int\limits_{|x| > z} \int\limits_{\Omega_T} y\left\{\dot{\psi}^{-1}\left(\frac{z}{x}\right)\right\} \frac{1}{x} P_{A,T}(x,y)dxdy.
\end{aligned}
\tag{2.40}
$$

This expression connects two ransom variables, A and T, through the density of their joint distribution and the deterministic function $\psi^{-1}(v)$ with the distribution density of the process to be synthesized. It is more general than the equation (2.18). In particular, in case of independence between random amplitude and duration, i.e. at

$$P_{A,T}(x,y) = P_A(x)P_T(y)$$

the expression (2.40) changes to the equation (2.26). Indeed,

$$
\begin{aligned}
P_{X(t)}(z) &= 2\frac{\operatorname{sign} z}{M[T]} \lim_{h\to 0} \int\limits_{\Omega_T} yP_T(y)dy \int\limits_{|x>z|} \left\{\dot{\psi}^{-1}\left(\frac{z}{x}\right)\right\} \frac{1}{x} P_{A,T}(x,y)dx \\
&= 2 \operatorname{sign} z \int\limits_{|x>z|} \frac{1}{x}\, \dot{\psi}^{-1}\left(\frac{z}{x}\right) P_A(x)dx
\end{aligned}
\tag{2.41}
$$

coincides with the formula (2.18), i.e. with the density of the process $x(t)$ calculated for a set of implementations rather than for only of them.

At the functional relationship between A and T, the probability density takes the form

$$
\begin{aligned}
P_{X(t)}(z) &= 2\frac{\operatorname{sign} z}{M[T]} \int\limits_{|x| > z} \int\limits_{\Omega_T} y\psi^{-1}\left(\frac{z}{x}\right) \frac{1}{x} \delta[\chi(y-x)P_T(y)dydx \\
&= 2\frac{\operatorname{sign} z}{M[T]} \int\limits_{|x| > z} \frac{1}{x} y\dot{\psi}^{-1}\left(\frac{z}{x}\right) \chi^{-1}(x)P_A(x)dx.
\end{aligned}
\tag{2.42}
$$

By analyzing the equation (2.41), let us note than in case of symmetry of $P_A(x)$ the probability density $P_{X(t)}(y)$ will be symmetrical only if $\psi^{-1}(x)$ is an even symmetric function, i.e. when a random amplitude and duration are not correlated. Let us note other important properties of the function $\psi^{-1}(x)$. Its range of definition coincides with the range Ω_A of random variable A, while its range of values coincides with the range of random variable T. For definiteness we shall assume that $\Omega_A = (-\infty,\ \infty)$, $\Omega_T = (0,\ \infty)$; then it is obvious that $\chi^{-1}(\infty, -\infty) = 0$; $\chi^{-1}(0) = \infty$; $\chi^{-1}(x) \geq 0,\ \forall x \in \Omega_A$.

The expression (2.41) allows us to synthesize the process by a prescribed density function $P_{X(t)}(z)$. For example, at a prescribed pulse waveform $\psi(t)$ it changes to the integral Volterra equation of the first kind relative to the unknown distribution density. In particular, at $\psi(t) = \sin \pi t$ (2.41) it, as well as the expression (2.32), changes to the Abelian equation. Solving it, we shall get

$$\frac{\chi^{-1}(z)}{M[T]} P_A(z) = -z \int_z^{\infty} \frac{\dot{P}_x(y)}{\sqrt{y^2 - z^2}} y, \quad z \geq 0. \tag{2.43}$$

According to the latter expression, the selection of the function $P_A(z)$ is predetermined by the properties of the function $\chi^{-1}(z)$. Let us explain this using the following example. Let

$$P_x(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-z^2/2\sigma^2}, \; z \in (-\infty, \infty),$$

then

$$\chi^{-1}(x) = M[T] \frac{|x|}{2\sigma^2} e^{-x^2/2\sigma^2} \frac{1}{P_A(x)}.$$

The asymptotic properties of the function $\chi^{-1}(x)$ imply that the distribution density $P_A(z)$ must satisfy the following limit relationships:

$$\lim_{x \to 0} M[T] \frac{|x|}{2\sigma^2} e^{-x^2/2\sigma^2} \frac{1}{P_A(x)} = +\infty; \quad \lim_{x \to \infty} M[T] \frac{|x|}{2\sigma^2} e^{-x^2/2\sigma^2} \frac{1}{P_A(x)} = 0.$$

Let us take, e.g. the equation $P_A(x) = \frac{1}{2} P_M(|x|) = \frac{1}{2\sigma^2} \sqrt{\frac{2}{\pi}} x^2 e^{-x^2/2\sigma^2}$, where $P_M(|x|)$ is the Maxwell–Boltzmann distribution that satisfies these constraints. Then the function $\chi^{-1}(x)$ takes a simple form:

$$\chi^{-1}(x) = \sqrt{\frac{\pi}{2}} \sigma M[T] \frac{1}{|x|} = \frac{a}{|x|}, \quad |x| \leq \infty.$$

2.7 Synthesis of Sampled and Digital Measuring Signals

Sampled and digital signals are widely used in the measuring practice, e.g. in automated data acquisition and processing systems etc. They are generated through the transformation of analog measuring signals to be sent to digital devices. In recent years, digital and sampled signals are generated in measuring generators based on the principle of direct digital synthesis. We have to consider the synthesis of these signals in a separate section due to their discontinuity and resolution of variation in time and level, as well as the specific mathematical tools

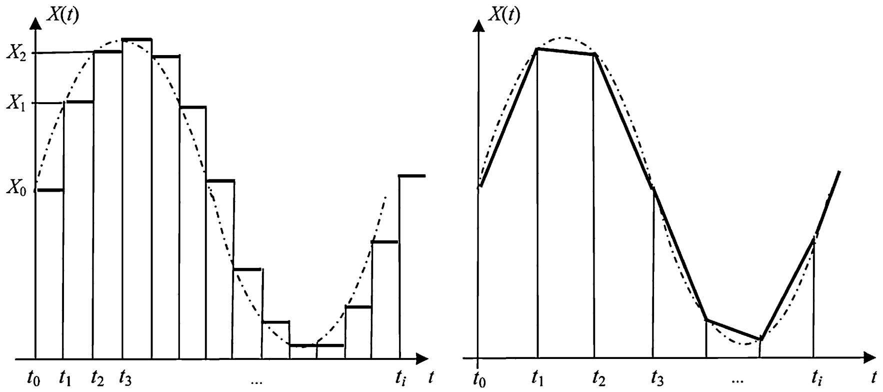


Fig. 2.17 Sampled signals composed of pulses of square and trapezoidal waveforms

we need to study them. It is clear from Fig. 2.17 that sampled signals can be considered as composed of pulses of a certain waveform. Each of them has its own parameters: amplitude, duration, angle of slope, etc. These pulses are summed up in time by the sequential summation method we have proposed above. Therefore, the mathematical description and the synthesis methods discussed above are applicable to sampled signals.

2.7.1 Synthesis of Sequences of Pseudorandom Numbers with Improved Spectral Characteristics

Lattice functions [28], sequences of pseudorandom numbers [29], etc. are usually used as mathematical models of sampled and digital signals. Such models have been well studied in the research of automatic systems [13], in digital radio communication, digital image processing, etc. However, the specific features of their application as measuring signals have not been yet addressed in the literature in detail. Also, the issues associated with estimating a reproduction error of such signals have been poorly studied.

Let us consider the problems of improving the signals composed of sequences of pseudorandom numbers, which can be described by lattice functions and generated from solutions of primitive irreducible polynomials [29].

In a simple case, pseudorandom sequences are known to be generated using a feedback shift register. The principles of constructing and operating such oscillators are widely known and described in the literature [29]. A n-digit code is generated at the i-th clock step of the shift register. Let us denote a value of the binary M-sequence in the k-th digit of the code z_i as x_k. In this case, the condition $P\left\{x_k^{(i)} = 0\right\} \cong P\left\{x_k^{(i)} = 1\right\} \cong \frac{1}{2}, \quad \forall k = \overline{1, n}, \quad \forall i = \overline{1, \infty},$ is fulfilled, where

$P\{A\}$ is the probability of the event A. The properties of these sequences have been studied well enough. Their probabilistic, spectral and correlation characteristics are known.

Let us study new sequences of random numbers to improve parameters of signals to be generated, in particular to increase uniformity of spectral density, to normalize the probability density law, etc. New sequences can be generated using different summation circuits $x_k^{(i)}$ of signals from outputs of the n-digit shift register multiplied by properly selected weighting factors α_k. Let us assume the following procedure for generation of pseudorandom numbers

$$z_i = \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} + \ldots + \alpha_n x_n^{(i)} = \sum_{k=1}^{n} \alpha_k x_k^{(i)}. \tag{2.44}$$

If values z_i are considered as amplitudes of square wave pulses, a sequence of these values can be associated with a continuous process—a sequence of contiguous square wave pulses with random amplitude. It can serve as a random process model with defined probabilistic characteristics of such type as (2.4)

$$Z = \left\{ z(t) = \sum_{i=0}^{\infty} z_i [H(t - \tau_i) - H(t - \tau_{i-1})],\ i = 1, \overline{\infty} \right\}, \tag{2.45}$$

where z_i, τ_i, T_i are parameters of a pulse waveform (amplitude, starting time and duration, respectively); $H(t - \tau_i)$ is the Heaviside function.

Let us study the correlations of the sequence z_i demonstrated in Fig. 2.18 of the random sampled process. Let us introduce the following notations: $K_z(p)$ is the correlation function, $M_1(z)$ is the expected value. Then

$$K_z(p) = M\left(z_i z_{i+p}\right) - (M_1(z))^2. \tag{2.46}$$

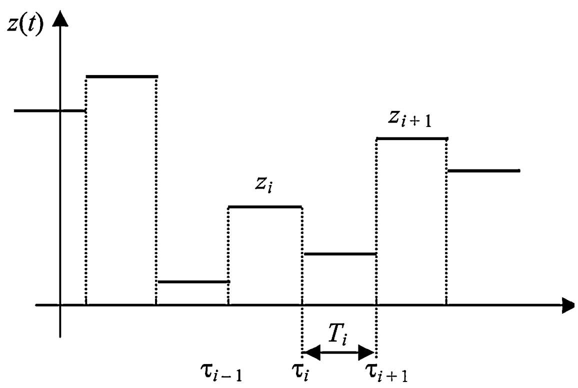


Fig. 2.18 Implementation of the random process

Let us find the expected value $z(t)$ by the formula

$$M_1(z) = M_1\left(\sum_{k=1}^{n} \alpha_k x_k^{(i)}\right)$$
$$= \sum_{k=1}^{n} \alpha_k M_1(x_k^{(i)}) = \sum_{k=1}^{n} \alpha_k \left(P\left\{x_k^{(i)} = 0\right\} + P\left\{x_k^{(i)} = 1\right\}\right) = \frac{1}{2}\sum_{k=1}^{n} \alpha_k.$$

The pseudorandom M-sequences $x_k^{(i)}$ produced by the oscillator have the non-correlatedness property, which is expressed by the formula

$$M(x_{k_1}^{(i)} x_{k_2}^{i+p}) = \begin{cases} \frac{1}{2}, & k_2 - k_1 = p; \\ \frac{1}{4}, & k_2 - k_1 \neq p. \end{cases} \tag{2.47}$$

Let us calculate $K_z(p)$ using the expression (2.46) and the formula (2.47):

$$K_z(p) = M\left(\sum_{k=1}^{n} \alpha_{k_1} x_{k_1}^{(i)} M \sum_{k=1}^{n} \alpha_{k_2} x_{k_2}^{(i+p)}\right) - \frac{1}{4}\sum_{k=1}^{n} \alpha_{k_1} \sum_{k=1}^{n} \alpha_{k_2},$$

where

$$M\left(\sum_{k=1}^{n} \alpha_{k_1} x_{k_1}^{(i)} M \sum_{k=1}^{n} \alpha_{k_2} x_{k_2}^{(i+p)}\right) = \sum_{k_1=1}^{n}\sum_{k_2=1}^{n} \alpha_{k_1}\alpha_{k_2} \begin{cases} \frac{1}{2}, & k_2 - k_1 = p \\ \frac{1}{4}, & k_2 - k_1 \neq p. \end{cases}$$

Assuming that $p = 0$, let us find the dispersion z $(\sigma_z^2 = K_z(0))$.

$$\begin{aligned} \sigma_z^2 = & \frac{1}{2}\sum_{k_1,k_2}^{n} \sum_{:k_1=k_2}^{n} \alpha_{k2}\alpha_{k1} + \frac{1}{4}\sum_{k_1,k_2:}^{n} \sum_{:k_1\neq k_2}^{n} \alpha_{k_1}\alpha_{k_2} - \frac{1}{4}\sum_{k_1,k_2:}^{n} \sum_{:k_1=k_2}^{n} \alpha_{k_1}\alpha_{k_2} \\ & - \frac{1}{4}\sum_{k_1,k_2:}^{n} \sum_{:k_1\neq k_2}^{n} \alpha_{k1}\alpha_{k2} = \frac{1}{4}\sum_{k_1=1}^{n} \alpha_{k_1}^2; \quad \sigma_z^2 = \frac{1}{4}\sum_{k_1=1}^{n} \alpha_{k_1}^2. \end{aligned} \tag{2.48}$$

We shall obtain the following relationship for the arbitrary p:

$$\begin{aligned} K_z(p) = & \sum_{k_1=1}^{n-p} \alpha_{k_1}\alpha_{k_1} + \frac{p}{2} + \frac{1}{4}\sum_{k_1,k_2:}^{n} \sum_{:k_1-k_2\neq p}^{n} \alpha_{k_1}\alpha_{k_2} - \frac{1}{4}\sum_{k_1=1}^{n-p} \alpha_{k_1}\alpha_{k_1+p} - \frac{1}{4}\sum_{k_1,k_2}^{n} \sum_{:k_1-k_2\neq p}^{n} \alpha_{k_1}\alpha_{k_2} \\ & = \frac{1}{4}\sum_{k_1=1}^{n-p} \alpha_{k_1}\alpha_{k_1+p} = \frac{1}{4}\sum_{i=1}^{n-p} \alpha_i\alpha_{i+p}, \quad K_z(p) = 0, \quad \forall p \geq n. \end{aligned} \tag{2.49}$$

According to the expression (2.49), the correlation function is fully determined by weighting factors and, therefore, can be determined by changing the value α_i.

We can assign such coefficients that allow us to generate a sequence of numbers with a prescribed correlation function and consequently with a prescribed spectral density. Let us state the problem of finding the coefficients α_I, $i = \overline{1, n}$, that ensure minimum non-uniformity of spectral density on the maximum frequency bandwidth. Since among z_i values the random square wave process $z(t)$ is generated in an oscillator (see Fig. 2.17), its spectral density is known [12]. Taking into account (2.48), (2.49), we shall obtain

$$S_z(\omega) = 2T \frac{\sin^2\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)^2} \left[\sigma_z^2 + 2\frac{1}{4}\sum_{i=1}^{n-p} \alpha_i \alpha_{i+p} \sum_{p=1}^{n-1} \cos p\omega T\right]. \tag{2.50}$$

Let us proceed to the solution of the optimization problem with the following simplifying assumption: correlation function ix given by

$$K_z(p) = \sigma_z^2 (R_1)^p, \tag{2.51}$$

where $R(p) = \frac{K_z(p)}{\sigma_z^2} = (R_1)^p$ is the correlation factor.

In its turn,

$$R_1 = \frac{1}{\sigma_z^2} \frac{1}{4} \sum_{i=1}^{n-1} \alpha_i \alpha_{i+1} = \frac{\sum_{i=1}^{n-1} \alpha_i \alpha_{i+1}}{\sum_{i=1}^{n} \alpha_i^2}. \tag{2.52}$$

In this case the expression for spectral density will be significantly simplified:

$$S_z(\omega) = 2T \frac{\sin^2\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)^2} \sigma_z^2 \frac{1 - R_1^2}{1 - 2R_1 \cos \omega T + R_1^2}. \tag{2.53}$$

The equation (2.53) was derived on the basis of the known closed formula for the series

$$\sum_{p=1}^{n-1} R_1^p \cos p\omega T.$$

Let us choose a certain window Δ for non-uniformity of spectral density. Then, using the relationship (2.53), we will be able to determine such a value of R_1 and, therefore, the whole function $K_z(p)$, at which the frequency ω^* is maximized in such a way that

$$|S_z(\omega) - S_z(0)| \leq \Delta \text{ at } \omega \in [0, \ \omega^*] \tag{2.54}$$

This problem was solved by the sequential summation method, which makes it possible to determine required correlation factors.

It is seen from the maximum frequency of spectral density at prescribed flatness of 2 and 1 dB (Fig. 2.19) that

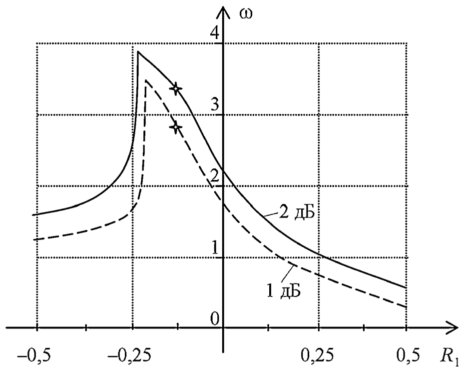


Fig. 2.19 Dependence of the maximum spectrum frequency on the correlation factor R_1 at a prescribed flatness

$$R_1 = R_1^{\max} \approx -1/4,4 \tag{2.55}$$

a frequency band with flatness of spectral density δS not exceeding 2 dB has a maximum width (up to $\omega^* = 3.75$ radians per second). The points on the graph marked with starts correspond to the maximally flat spectral density, where its flatness is unilateral.

Here, at

$$R_1 = R_1^{\text{opt}} \approx -1/8 \tag{2.56}$$

a frequency band of 0–3.3 radian per seconds is provided. Note that this value reaches more a half of sampling rate $\omega_s = 2\pi/T = 6.28$ radians per second. In other words, a flat frequency band of spectral density covers more than a half of sampling rate.

The graph implies that the optimal correlation functions corresponding to the expression (2.54) must be alternating and rapidly decreasing (damped). Now the problem is to ensure this optimal correlation function by a respective selection of α_i. It is obvious that the weighting factor α_i must satisfy the system of nonlinear algebraic equations that can be written as

$$R(p) = \frac{\sum_{i=1}^{n-p} \alpha_i \alpha_{i+p}}{\sum_{i=1}^{n} \alpha_i^2}. \tag{2.57}$$

The sum in the numerator of the expression (2.57) may not contain less than one term; therefore, the number n of unknown values of the factor α_i is always one unit more than the number of equations p, which is equal to the number of known values of the correlation function. Therefore, the system of equations does not have a single-valued solution in relation to n. In order to solve it, let us add another equation to the expression (2.54), i.e. take a zero mean value of the process $z(t)$:

$$M_1(z) = \frac{1}{2}\sum_{i=1}^{n} \alpha_i = 0. \tag{2.58}$$

In this case we can derive the following system from the equations (2.57) and (2.58):

$$\begin{aligned}
&\alpha_1\alpha_1 + \alpha_2\alpha_2 + \alpha_3\alpha_3 + \alpha_4\alpha_4 + \ldots + \alpha_{n-1}\alpha_{n-1} + \alpha_{n-1}\alpha_{n-1} + \alpha_n\alpha_n = 4\sigma^2 R_0;\\
&\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_4\alpha_5 + \ldots + \alpha_{n-2}\alpha_{n-1} + \alpha_{n-1}\alpha_n = 4\sigma^2 R_1;\\
&\alpha_1\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_5 + \alpha_4\alpha_6 + \ldots + \alpha_{n-2}\alpha_n = 4\sigma^2 R_2;\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\alpha_1\alpha_{n-2} + \alpha_2\alpha_{n-1} + \alpha_3\alpha_n = 4\sigma^2 R_{n-3};\\
&\alpha_1\alpha_{n-1} + \alpha_2\alpha_n = 4\sigma^2 R_{n-2};\\
&\alpha_1\alpha_n = 4\sigma^2 R_{n-1}.
\end{aligned} \tag{2.59}$$

This system contains n of equations with n of unknown and may have a single-valued solution represented as a vector of the factor $\{\alpha_i\}$. The set of equations can be solved by known methods, convergence of which significantly depends on a degree of closeness between the vector of initial values $\alpha_i,\ i = \overline{1,n}$ and the solution vector. For this reason, it is recommended to solve the set of equations in two stages: to set an initial value of the vector $\alpha_i,\ i = \overline{1,n}$, at the first stage and to arrive at a solution of the set (2.59), e.g. by the gradient descent method, at the second stage. Finally, we shall determine the factor α_i on the basis of correlation coefficients. At $R(p) = (R_1)^p$, the right side of the equations (2.59) must contain power factors $\{R_1^0,\ R_1^1,\ R_1^2,\ \ldots,\ R_1^{n-1}\}$. The system can be also solved using any software for solving systems of equations, e.g. *MathCad*, with operators *Given* and *Find*. At $R(p) = (-1/4{,}4)^p$, the coefficient vector has the following values (Fig. 2.20).

(a)

Find $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{10})$ =

α_1	–0.074
α_2	0.569
α_3	–0.057
α_4	0.022
α_5	$-3.842 \cdot 10^{-3}$
α_6	$1.039 \cdot 10^{-3}$
α_7	$-2.143 \cdot 10^{-4}$
α_8	$5.159 \cdot 10^{-5}$
α_9	$-1.135 \cdot 10^{-5}$
α_{10}	$2.77 \cdot 10^{-6}$

(b)

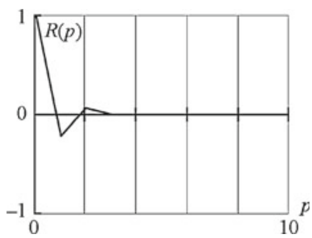


Fig. 2.20 Vector of factor α_i (here, the numeration of factors is displaced by a unit) (**a**) and normalized correlation function at $R_1 = R_{1\max} \approx -1/4.4$ (**b**)

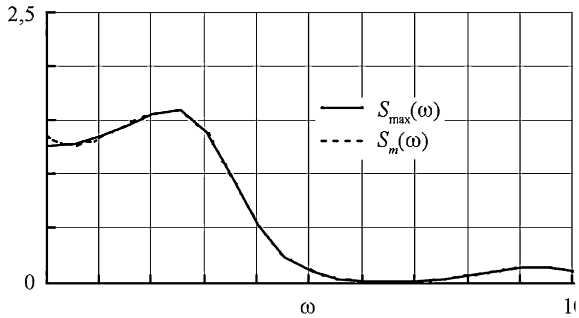


Fig. 2.21 Spectral densities corresponding to the theoretical correlation function ($|S_{max}(\omega)|$) and the realized function with weighting factors ($|S_m(\omega)|$)

Figure 2.21 compares spectral densities of processes with the correlation function that maximizes a frequency band $R_1 = R_{1\max} \approx -1/4.4$, and the correlation function of the process with weighting factors (see Fig. 2.19b). It is clear that they are almost identical.

A slight deviation is evident in the low-frequency region. It is associated with the fact that the factors $\alpha_5, \ldots, \alpha_{10}$ are discarded because of their smallness during the implementation of a random signal. The performed analysis confirms that the process $z(t)$ is possible to be implemented with a maximally wide band of the uniform spectral density.

An analytical approximate solution can be proposed for $R_1(p)$ that are close to zero even at small values of p. Note than it results in alternating signs with a step of the correlation function R_1^p, which can be realized according to the expression (2.49) at the same alternating signs of weighting factors. Indeed, if we choose signs of weighting factors in the following way: $\operatorname{sign}\alpha_i = (-1)^i,\ i = \overline{1,n}$ and using the expression (2.49), we shall obtain sign $K_z(1) = -1$, sign $K_z(2) = +1, \ldots,$ sign $K_z(p) = (-1)^p$. Then, taking into account the identified pattern of correlation damping, let us determine the weighting factor α_i on a qualitative level by the formula $\alpha_i = (-1)^i f(i)$, where $f(i)$ is a certain function, e.g. as shown in Fig. 2.22.

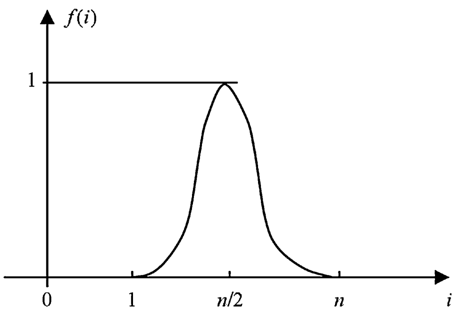


Fig. 2.22 Shape of the auxiliary function

After making the corrections, let us determine:

$$K_z(p) = \frac{1}{4}\left(\alpha_1\alpha_{1+p} + \alpha_{\frac{n}{2}-p}\alpha_{\frac{n}{2}} + \alpha_{\frac{n}{2}-p+1}\alpha_{\frac{n}{2}+1} + \ldots + \alpha_{\frac{n}{2}}\alpha_{\frac{n}{2}+p} + \ldots + \alpha_{n=p}\alpha_n\right)$$
$$\approx \frac{1}{4}\left(\alpha_{\frac{n}{2}}\left(\alpha_{\frac{n}{2}-p} + \alpha_{\frac{n}{2}+p}\right)\right), \quad \alpha_{\frac{n}{2}} = 1. \tag{2.60}$$

Let us simplify the equation (2.60) by neglecting squares of numbers less than 1 due to their smallness. Then

$$K_z(p) = \frac{1}{2}\alpha_{\frac{n}{2}-p}. \tag{2.61}$$

If $K_z(p) = \sigma_z^2 R_1^p$, the solution can be written as the system

$$K_z(p) = \begin{cases} \alpha_{\frac{n}{2}-p} = \frac{1}{2}R_1^p, & p = \overline{1, \frac{n}{2} - 1}; \\ \alpha_{\frac{n}{2}+p} = \alpha_{\frac{n}{2}-p}, & \alpha_{\frac{n}{2}} = 1, \quad \alpha_n = \alpha_1 R_1. \end{cases} \tag{2.62}$$

For example, by selecting $R_1 = -\frac{1}{8}$ and substituting this value into the expression (2.62), i.e. by solving a direct problem, it is easy to prove that the solution is correct (2.62).

Figure 2.23 shows that at $R_1 \to -1$ the random process $z(t)$ degenerates to a square wave signal with a frequency of π rad/s, while at $R_1 \to +1$ it tends to a time constant equal to 1, i.e. the random process degenerates to a deterministic one at the boundaries of the variation range $-1 \le R_1 \le +1$.

Let us determine the constraint on α_i, $i = \overline{1, n}$, taking into account the condition of reproducing the process $z(t)$ by a specified probability distribution law.

Since random values of $z = \sum_{k=1}^{n} \alpha_k x_k$ are not correlated and have a Bernoulli distribution with parameter 1/2, while the value n is sufficiently large, we can study a convergence to a Gaussian probability density function. Let us denote each summand by $\alpha_k x_k = \xi_k$.

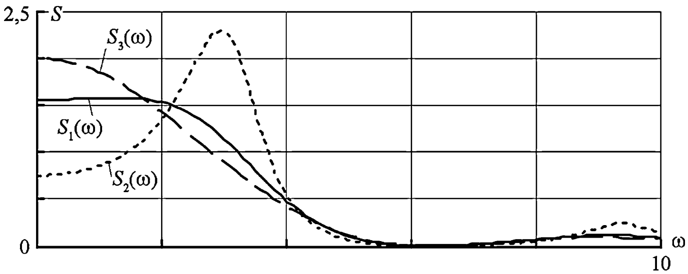


Fig. 2.23 Graphs of spectral densities: $|S_1(\omega)|$ at $R_1 = -1/8$; $|S_2(\omega)|$ at $R_1 = -1/2$; $|S_3(\omega)|$ at $R_1 = 0$

Then

$$P\{\xi_k = 0\} = P\{\xi_k = \alpha_k\} = \frac{1}{2}$$

and, therefore, $M_1(\xi_k) = \frac{\alpha_k}{2} = a_k,\ \ D(\xi_k) = M_2(\xi_k) - [M_1(\xi_k)]^2 = \frac{\alpha_k^2}{4}$.

In this case $\sigma_z^2 = D_z = \sum\limits_{k=1}^{n} D(\xi_k) = \frac{1}{4}\sum\limits_{k=1}^{n} \alpha_k^2 = \sum\limits_{k=1}^{n} a_k^2$.

Note also that despite the fact that random values of ξ_k are generally only uncorrelated rather than independent ones, the practice shows that it is quite correct to apply limiting theorems to this case.

The Lindeberg condition is a sufficient condition for convergence of the sum distribution function (2.60) to the normal law. If a sequence of mutually independent random variables ξ_1, ξ_2, …, ξ_k satisfied the Lindeberg condition at any constant $\tau > 0$

$$\lim_{n\to\infty} \frac{1}{\tau^2} \sum_{k=1}^{n} \int\limits_{|x-a_k|\,>\,\tau\sigma_z} (x - a_k)^2 dF_k(x) = 0, \tag{2.63}$$

then at $n \to \infty$

$$P\left\{\frac{1}{\sigma_k}\sum_{k=1}^{n}(\xi_k - a_k) < x\right\} \to \frac{1}{2\pi}\int\limits_{-\infty}^{x} e^{-\frac{z^2}{2}} dz. \tag{2.64}$$

It is obvious that the distribution function of random variables ξ_k

$$F_k(x) = P\{\xi_k < x\} = \frac{1}{2}H(x) + \frac{1}{2}H(x - \alpha_k), \tag{2.65}$$

where $H(x)$ is the Heaviside function.

Let us denote in the expression (2.63)

$$\varphi\ (\tau, k) = \int\limits_{|x-a_k|\,>\,\tau\sigma_z} (x - a_k)^2 dF_k(x) = 0. \tag{2.66}$$

Then the condition of the theorem is reduced to the verification of the relationship $\lim\limits_{n\to\infty} \frac{1}{\tau^2} \sum\limits_{k=1}^{n} \varphi\ (\tau, k) = 0$. Let us find a value of the function $\varphi(\tau, k)$ for different τ. According to the equations (2.65), (2.66)

$$
\begin{aligned}
\varphi(\tau,k) &= \int\limits_{|y>\tau\sigma_k|} y^2\left(\frac{1}{2}\delta(y+a_k)+\frac{1}{2}\delta(y-a_k)\right)dy \\
&= \frac{1}{2}\left[\int\limits_{-\infty}^{-\tau\sigma_k} y^2\delta(y+a_k)dy + \int\limits_{\tau\sigma_k}^{\infty} y^2\delta(y+a_k)dy + \int\limits_{-\infty}^{-\tau\sigma_k} y^2\delta(y-a_k)dy + \int\limits_{\tau\sigma_k}^{\infty} y^2\delta(y-a_k)dy\right] \\
&= \begin{cases} a_k^2, & |a_k|>\tau\sigma_k; \\ 0, & |a_k|\le\tau\sigma_k \end{cases} = \begin{cases} a_k^2 \text{ if } 0<\tau<|a_k|\Big/\sqrt{\sum\limits_{i=1}^{n} a_i^2}; \\ 0 \text{ otherwise.} \end{cases}
\end{aligned}
$$

It is obvious that $|a_k|\Big/\sqrt{\sum\limits_{i=1}^{n} a_i^2}<1$. Then let us define the auxiliary function

$$
v(\tau,n)=\frac{1}{\sigma_z^2}\sum_{k=1}^{n}\varphi(\tau,k).
$$

If we introduce the parameters

$$
\tau_1=\min_{\{k;k\in N\}}\frac{|a_k|}{\sigma_k}=\frac{|a_{k1}|}{\sigma_k};\ \tau_2=\min_{\{k\in N\backslash\{k1\}\}}\frac{|a_k|}{\sigma_k}=\frac{|a_{k2}|}{\sigma_k};\ \tau_3=\min_{\{k;k\in N\backslash\{k1,k2\}\}}\frac{|a_k|}{\sigma_k}=\frac{|a_{k3}|}{\sigma_k},\ \ldots,
$$
$$
\tau_n=\min_{\{k\in N\backslash\{k1,\,k2,\,\ldots,\,kn-1\}\}}\frac{|a_k|}{\sigma_k}=\frac{|a_{kn}|}{\sigma_k}=\max_{\{k\in N\}}\frac{|a_k|}{\sigma_k}=\frac{|a_{kn}|}{\sigma_k},\ N=\{1,\ 2,\ 3,\ \ldots,\ n\},
$$

it is easy to notice that $0\le\tau_1\le\tau_2\le\tau_3\le\cdots\le\tau_n<1$. Then

$$
v(\tau,n)=\begin{cases}
1, \ 0<\tau<\tau_1; \\
\frac{1}{\sigma_z^2}(\sigma_z^2-a_{k1}^2);\ \ \tau_1<\tau<\tau_2; \\
\frac{1}{\sigma_z^2}(\sigma_z^2-a_{k1}^2-a_{k2}^2);\ \ \tau_2<\tau<\tau_3; \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\frac{1}{\sigma_z^2}(\sigma_z^2-\sum\limits_{i=1}^{n-1}a_{ki}^2;\ \ \tau_{n-1}<\tau<\tau_n; \\
0, \quad \tau>\tau_n,
\end{cases} \tag{2.67}
$$

which is graphically shown in Fig. 2.24.

Hence, the condition $\tau_n \underset{n\to\infty}{\to} 0$ is sufficient for fulfillment of $v(\tau,n)\underset{n\to\infty}{\to} 0$. Hence follows the constraints on values of the weighting factor α_k : $0<|\alpha_k|<\infty,\ \forall k=1,2,\ldots,$ at which the value z has the Gaussian probability distribution.

Indeed, otherwise $a_k \underset{k\to\infty}{\to} \infty$ or $a_k \underset{k\to\infty}{\to} 0$, $\left(a_k=\frac{\alpha_k}{2}\right)\to\tau_n\underset{n\to\infty}{\to}0.$

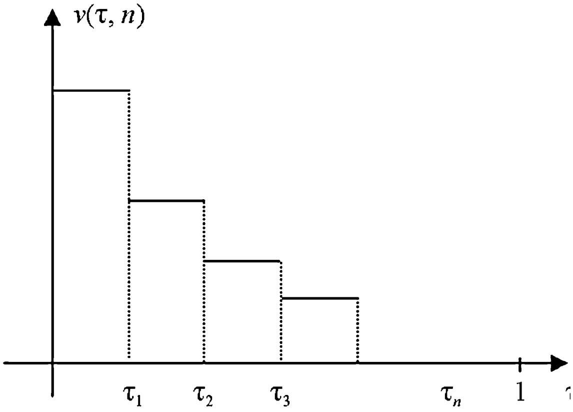


Fig. 2.24 Function $v(\tau, n)$

At a finite n, which is the case for the random process $z(t)$ generated by the register (e.g. in G2-57 $n = 32$), a value of τ_n will be a kind of indicator of $z(t)$ in the following natural sense: the closer τ_n is to zero, the more accurate $z(t)$ is described by the Gaussian distribution law. In such a context, a connection of τ_n with the statistical goodness-of-fit test can be seen. We mean that by setting a certain goodness-of-fit test a critical point τ^* of the function v can be set so that at $\tau_n < \tau^*$ the hypothesis for Gauss distribution hypothesis of $z(t)$ is accepted. Otherwise, this hypothesis is rejected. Calculating the critical points for different criteria and selecting the minimum one among them (let us denote it as T^*), we can conclude that in case of finite n the process $z(t)$ is Gaussian provided that

$$\tau_n = \max_{k\in N} \frac{|a_k|}{\sigma_k} < T * . \tag{2.68}$$

The quantity T^* is of theoretical rather than practical interest. In the practice it is sufficient to verify one or two classical criteria.

Thus, all the foregoing constitutes the proof of the following theorem developed in collaboration with A.L. Baranovsky:

If weighting factors $\alpha_i,\ i = \overline{1,n}$ are those at which $0 < |\alpha_i| < \infty,\ \forall i = \overline{1,n}$ and $\max\limits_{i\in N} \frac{|a_i|}{\sqrt{\sum\limits_{k=1}^{n} a_k^2}} \le T^*$, a random variable $z = \sum\limits_{i=1}^{n} \alpha_i x_i$, where x_i values are not correlated and distributed according to the Bernoulli law with the parameter 1/2, have the distribution function

$$F_z(y) = \frac{1}{\sqrt{2\pi}\sigma_z} \int\limits_{-\infty}^{y} e^{-\frac{(y-M_1(z))^2}{2\sigma_z^2}} dy,$$

The analysis allows us to infer that the weighting factor calculation law must have a certain parameter β, varying which we can broaden a spectrum without affecting the normal probability distribution law. In other words, there is a certain bifurcation point β^*, in which the normal probability density $z(t)$ is invariant in relation to varying β.

Thus, let us select a certain law of generating the factors

$$\alpha_i = f(\beta, i) \tag{2.69}$$

in the following way:

$$f(\beta, i) = \sin\left(\beta\,\Delta\left|\frac{n-1}{2} - i\right|\right) \bigg/ \left(\beta\,\Delta\left|\frac{n+1}{2} - i\right|\right), \quad i = \overline{1,n}. \tag{2.70}$$

At $\beta = 1$ we shall obtain the well-known α_i-generating circuit, which was implemented in the Russian generator G2-57 and the 3722A of Hewlett-Packard. According to the arguments given above, we should increase β for a higher variability of α_I and, therefore, select $\beta^* > 1$.

Selecting χ^2 (Pearson criterion) with a significance level $\rho = 0.1$ as goodness-of-fit test, let us set the bifurcation parameter $\beta^* \approx 3.5$.

Therefore, choosing $\beta = \beta^*$ as optimal parameter and substituting the α_i generation law (2.70) into the expression (2.69), we shall obtain the Gaussian process $z(t)$ with the spectrum broadened by three times as compared with the spectrum of the process $z(t)$ at $\beta = 1$ (at the output of the generator G2-57).

Let us discuss the possibility to improve spectral properties of a random signal with uniform probability distribution density. If $\alpha_j = 2^{j-1}$, $z(t)$ is known to be an uniformly distributed random value in the interval of $0.2^n - 1$. Let us rewrite z as x. Then

$$x_i = \sum_{jk=1}^{n} 2^{j-1} x_j^{(i)}, \tag{2.71}$$

where x_i is a decimal ransom number.

It is also known that x_i are trajectory points of the one-dimensional dynamical system $x_{i+1} = g(x_i)$ described by the map g—the Bernoulli shift (Fig. 2.25).

Obviously, the spectral density of the process has the same form as one corresponding to the expression (2.50) but at $\alpha_j = 2^{j-1}$. Unfortunately, the Bernoulli map has limited capabilities for signal generation. Further, we shall consider the general approach to the analysis of one-dimensional Poincare maps $G(x_i)$ with prescribed spectral and correlation properties.

2.7.2 *Synthesis of One-Dimensional Maps with Specified Probabilistic Characteristics*

Let us consider the more general case of generating the uniformly distributed numbers using the map

$$x_{i+1} = G(x_i). \tag{2.72}$$

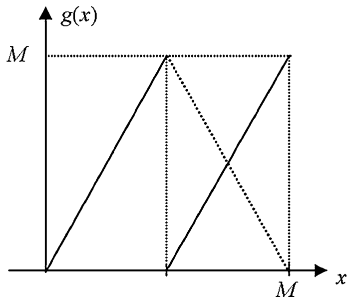


Fig. 2.25 Bernoulli map

We shall confine ourselves to considering piecewise linear maps of the following kind:

$$G(x) = \begin{cases} C_1^1 x + C_2^1, & -1 = b_0 \le x < b_1; \\ C_1^2 x + C_2^2, & b_1 \le x < b_2; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ C_1^m x + C_2^m, & b_{m-1} \le x < b_m = +1, \end{cases} \quad (2.73)$$

where $C_1^i = \frac{\pm 2}{b_i - b_{i-1}}$ is the line slope coefficient; $C_2^i = -\frac{1}{2} C_1^i (b_i - b_{i-1})$ is the shear coefficient, b_i is a variable that characterizes a partition of the section $[-1, 1]$. The Bernoulli function g is a special case $G(x)$ at $m = 2$.

Let us determine the probability density law. In the general case, the relationship for probability density may be written as:

$$P_{n+1}(x) = \sum_{i=1}^{n} \left| \frac{dG_i^{-1}(x)}{dx} \right| P_n\left(G_i^{-1}(x)\right),$$

where $G_i^{-1}(x)$ is an inverse function to $G_i(x) = C_1^i x + C_2^i$, which is equal to $(x - C_2^i)/C_1^i$.

Then

$$P_{n+1}(x) = \sum_{i=1}^{m} \frac{b_i - b_{i-1}}{2} P_n\left(\pm \frac{b_i - b_{i-1}}{2} x + \frac{b_i + b_{i-1}}{2} \right), \quad |x| \le 1. \quad (2.74)$$

It should be noted that the argument $P_n(v)$ does not exceed 1 at any partition of a section $[-1, 1]$ and arbitrary setting of x within $|x| \le 1$. Hence, at $P_0(x) = \begin{cases} 1/2, & \text{if } |x| \le 1 \\ 0, & \text{if } |x| > 1 \end{cases}$, using the expression (2.74), we shall obtain $P_1(x) = \sum_{i=1}^{m} \frac{b_i - b_{i-1}}{2} \frac{1}{2} = \frac{1}{2}$ and in the general case $P_n(x) = \frac{1}{2}$. Therefore, the probability density law will remain uniform at any partition of a section $[-1, 1]$ by straight lines. The essence of the proposed method is the possibility to change correlation properties of the sequence by varying slope coefficients C_1^i and shear coefficients C_2^i and not affecting the uniformity of the probability density law.

At an expected value of zero, the correlation function will take the form:

$$K_j(x) = \frac{1}{N-j}\sum_{i=1}^{N-j} x_i x_{i+j},\ j = 0,\ 1,\ 2,\ \ldots$$

It follows from the expression (2.72) that

$$x_{i+j} = G\left(x_j\right) = G\left(G\left(x_{i+j-2}\right)\right) = G(G(G(\ldots G(x_i)))), \tag{2.75}$$

the composition j of embedded maps is generated on j-th stroke. Denoting $G(G(G(\ldots G(x_1)))) = G^{\,j}(x_i)$, we shall rewrite the correlation function as follows:

$$K_j(x) = \frac{1}{N-j}\sum_{i=1}^{N-j} x_i G^j(x_i),\ j = 0,\ 1,\ 2,\ \ldots \tag{2.76}$$

Due to linearity, $G^j(x_i)$ in the expression (2.76) is a linear function as well. It can be easily represented if we construct it in a stepwise manner, i.e. starting from $G(x_i)$, then $G^2(x_i)$, etc. Beginning from $G^2(x_i)$, it becomes apparent that a number of straight lines (segments) is increased by times m, i.e. each segment $[b_i,\ b_{i-1}),\ \forall i = \overline{1,m}$ is partitioned into m segments proportionally to the initial partition of the segment $[-1,\ 1]$ so that their slope increases. At positive and negative slopes of the segments of the initial map, all subsequent maps are exact replicas of the initial segment embedded into the intervals that are equal to the length of the respective segment. If slopes are positive, they are embedded in the same order. If slopes are negative, a sign of slope and a sequence order of segments change at each step, i.e. the first left segment in a previous map becomes the first right one. From this it is clear that after arithmetic transformations we shall obtain the following function at j-th step:

$$G^j(x_i) = \begin{cases} D_1^1 x + D_2^1, & b_0^{(j)} \le x < b_1^{(j)}; \\ D_1^2 x + D_2^2, & b_1^{(j)} \le x < b_2^{(j)}; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ D_1^m x + D_2^m, & b_{m^j-1}^{(j)} \le x < b_{m^j}^{(j)}, \end{cases}$$

where

$$D_1^k = \frac{2}{b_k^{(j)} - b_{k-1}^{(j)}}\,\mathrm{sign}\ C_1^{\,k \bmod m};\ D_2^k = -\frac{1}{2}D_1^k\left(b_k^{(j)} + b_{k-1}^{(j)}\right);\ b_k^{(j)} = b_{i_j-1}^{(j)}$$
$$= \sum_{n=1}^{j}\prod_{l=1}^{j-n}\frac{b_{i_l} - b_{i_l-1}}{2}\left(b_{i_{j-(n-1)}} - b_0\right) + b_{i_1-1}.$$

In its turn, $i_l - 1 = \left\{ \left(k - \sum_{p=1}^{l-1} (i_p - 1) m^{j-p} \right) \Big/ m^{j-l} \right\}$, where the right part is integer; $l = \overline{1, j-1}$; $k = \overline{1, m^j}$.

Now it is obvious that the expression (2.76) is transformed into

$$K_j(x) = \frac{1}{N-j} \left[D_1^1 \sum_{i=1}^{n_1} x_{1i}^2 + D_2^1 \sum_{l=1}^{n_1} x_{1i}^2 + D_1^2 \sum_{l=1}^{n_2} x_{2i}^2 + D_2^2 \sum_{l=1}^{n_2} x_{2i}^2 + \ldots + D_1^{m^j} \sum_{l=1}^{n_{m_j}} x_{m_i^j}^2 + D_2^{m^j} \sum_{l=1}^{n_{m_j}} x_{m_i^j}^2 \right],$$

where n_p is the number of digits that fall within the segment $[b_{p-1}^{(j)}, \; b_p^{(j)})$.

In this expression p-th summand is equal to

$$\frac{1}{N-j} D_1^p \sum_{l=1}^{n_p} x_{p_i}^2 = \frac{1}{N-j} D_1^p \frac{1}{\Delta x} \sum_{l=1}^{n_p} x_{p_i}^2 \Delta x = \underset{\Delta x = 0}{\longrightarrow} \int_{b_{p-1}^{(j)}}^{b_p^{(j)}} x^2 \frac{1}{2} \, dx D_1^p,$$

since $\Delta x = \frac{b_p^{(j)} - b_{p-1}^{(j)}}{n_p} = \frac{2}{N-j}$. This implies that

$$\begin{aligned} K_j(x) &= \sum_{i=1}^{m^j} \frac{D_1^i}{2} \int_{b_{p-1}^{(j)}}^{b_p^{(j)}} x^2 \, dx + \frac{D_2^i}{2} \int_{b_{p-1}^{(j)}}^{b_p^{(j)}} x \, dx \\ &= \sum_{i=1}^{m^j} \frac{D_1^i}{6} \left((b_i^{(j)})^3 - (b_{i-1}^{(j)})^3 \right) + \frac{D_2^i}{4} \left((b_i^{(j)})^2 - (b_{i-1}^{(j)})^2 \right). \end{aligned} \tag{2.77}$$

The derived correlation function allows us to study correlation properties of pseudorandom sequences without the need to resort to the direct statistical analysis.

In case of uniform partition, the correlation function $b_i = -1 + (2/m)i$, $i = \overline{1, m}$ will take a simpler form:

$$K_j(x) = \begin{cases} 1/3, \;\; j = 0 \\ \dfrac{m - 2i}{3m^{j+1}} \operatorname{sign}\,(i_+ - i_-) \end{cases}, \;\; j \geq 1, \;\; i_+ = 0, \; 1, \; 2, \; \ldots, \; \left\{ \frac{m}{2} \right\}, \tag{2.78}$$

where i_+, i_- is the number of positive or negative signs in the whole set of m. For example, at alternation of signs +, −, +, −, …, + , − and, in the general case, at equality between numbers of positive and negative slopes (i.e. at $i_+ = i_-$) $K_j = 0$, $\forall j \geq 1$, which corresponds to the δ-correlated sequence of pseudorandom numbers. At any other sequence of signs, the correlation functions will be steadily decreasing.

At an arbitrary partition of the segment $[-1, \, 1]$, the function K_j will take the form:

$$K_j(x) = \frac{1}{3} m^{-jf(b, b_2, \ldots, b_{m-1})} \operatorname{sign}\left(\sum_{\left\{i:\operatorname{sign} C_1^i > 0\right\}} (b_i - b_{i-1})^2 - \sum_{\left\{i:\operatorname{sign} C_1^i < 0\right\}} (b_i - b_{i-1})^2 \right), \tag{2.79}$$

where $f(b_1, b_2, \ldots, b_{m-1})$ can be found from the condition $K_1 = 1/3 m^{-f(b, b_2, \ldots, b_{m-1})}$ at K_1 derived according to the equation (2.78). Therefore,

$$f(b_1, b_2, \ldots, b_{m-1}) = -\frac{1}{\ln m} \ln\left(\frac{1}{4} \left| \sum_{i=1}^{m} \operatorname{sign} C_1^i (b_i - b_{i-1})^2 \right| \right). \tag{2.80}$$

In case of uniform partition, the relationships (2.79) and (2.80) are transformed to the equation (2.76).

Let us define the inverse problem—to find the map (2.73) by the known correlation function, i.e. the mapping synthesis problem. It is obvious that in this case a selection of the correlation function is limited by the functions that satisfy the relationship $K_j = 1/3(3K1)^j, \ \forall j \geq 1$, i.e. only the first value of K_1 can be arbitrary. Certainly, this value cannot exceed the dispersion.

$$K_1 = \frac{1}{12} \sum_{i=1}^{m} (b_i - b_{i-1})^2 = \frac{1}{6} \left[1 + \sum_{i=1}^{m} b_i^2 - \sum_{i=1}^{m} b_i b_{i-1} \right]. \tag{2.81}$$

The partition parameters b_i and the sequence of signs are known [30].

We can set a problem of synthesizing the optimal mapping, i.e. to find such a relationship G, at which a random process is implemented with an optimal correlation function, and a uniform part of the spectral density is significantly broadened at a minimum deviation Δ according to the formula (2.54).

Therefore, the spectrum optimization problem is to synthesize maps with a defined correlation function. It was solved by the author for the class of piecewise linear maps such as (2.73), e.g. [30].

$$G_{\text{opt}}(x) = -8x + 2k - 9; \quad -1 + \frac{1}{4}(k-1) \leq x < -1 + \frac{1}{4}k \tag{2.82}$$

at $m = 8, \ k = \overline{1, \ 8}$, which generates a process with the correlation function that maximizes a band of the flat part of the spectral density (2.56):

$$R_{\text{opt}}(p) = \left(-\frac{1}{8} \right)^p.$$

It is possible to broaden the class of admissible functions by introducing a shear of the map (2.73). The resulting constraints on the correlation functions are considered in detail in the thesis of A.L. Baranovsky [30] written under the author's supervision.

2.8 Conclusions

The parallel summation method (expansion into the generalized Fourier series, which is efficient for analyzing signals) is inappropriate for synthesis of signals since a random component error increases with increasing a number of expansion terms.

The proposed method of sequential parametric optimization allows us to generate efficient models both for deterministic and random signals.

The sequential summation of signals in time makes it possible to create models of deterministic, random and digital signals.

Based on this method, different models of periodic signals with specified harmonic distortion factors, amplitude factors and spectrum, as well as models of random signals with a prescribed probability density function, correlation function and spectral density, and models of sampled signals were developed. All of them are suitable for reproduction in measuring instruments and have a number of advantages over the known methods.

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