Chapter 1
Introduction

Abstract In this chapter we introduce mappings of finite distortion as a generalization of mappings of bounded distortion. We show that this class is natural for the regularity of inverse mappings and for models in nonlinear elasticity.

In 1981, Ball [9] established an invertibility property for Sobolev mappings and used this to show that the solution of the displacement boundary value problem of nonlinear elastostatics does not violate the principle of interpenetration of matter. He posed an open problem that in a simplified form asks the following:

Suppose that a continuous mapping $f : B(0,1) \to B(0,1)$, in $\mathbf{R}^n$ for some $n \geq 2$, belongs to the Sobolev class $W^{1,n}(B(0,1), \mathbf{R}^n)$, satisfies $J_f(x) := \det Df(x) > 0$ almost everywhere, f is a homeomorphism from $B(0,1) \setminus \overline{B(0,1-\delta)}$ onto $B(0,1) \setminus f(\overline{B(0,1-\delta)})$, and f additionally satisfies

$$\int_{B(0,1)} |(Df(x))^{-1}|^n J_f(x)\, dx < \infty. \tag{1.1}$$

Does it follow that the mapping f has an inverse $f^{-1} : B(0,1) \to B(0,1)$ with $f^{-1} \in W^{1,n}(B(0,1), \mathbf{R}^n)$? Above and in what follows, $|A|$ for a matrix A refers to the operator norm of A.

The original idea of Ball [8,9] was to establish a class of mappings that can serve as a class of deformations in nonlinear elasticity. Nowadays the whole theory is very rich and we recommend the monographs [4, 67] for history, references and further motivation.

We can view a domain $\Omega \subset \mathbf{R}^n$ as a solid body in space and our mapping $f : \Omega \to \mathbf{R}^n$ as a deformation of the body Ω to $f(\Omega)$. There are several natural questions one can ask.

- Is f continuous? (Does the material break or are there any cavities created during the deformation?)

S. Hencl and P. Koskela, *Lectures on Mappings of Finite Distortion*, Lecture Notes in Mathematics 2096, DOI 10.1007/978-3-319-03173-6_1,

- Does f map sets of measure zero to sets of measure zero? (Is new material created from "nothing"? Is some material "lost" during the deformation?)
- Is the mapping one-to-one? Does the inverse map f^{-1} exist? (Is there no interpenetration of the matter, i.e. can we map two parts of the body to the same place? Can we deform the body back to its original state?)
- What are the properties of f^{-1}? (Is the reverse deformation reasonable?)

In the planar setting, basic linear algebra transforms (1.1) to the requirements that $J_f(x) > 0$ almost everywhere and that the quantity

$$K_f(x) := \frac{|Df(x)|^2}{J_f(x)}$$

is integrable over $B(0, 1)$. The latter condition is a relaxation of the definition of quasiregularity (or quasiconformality) that requires that infinitesimal circles be mapped to infinitesimal ellipses whose eccentricities $K_f(x)$ are uniformly bounded. Thus the study of mappings as in (1.1) can be viewed as a generalization of the study of quasiregular mappings, also called mappings of bounded distortion. In a sense, one relaxes the boundedness of the distortion to integrability of the distortion. We will give the relevant definitions in Sect. 1.2 below. Mappings of bounded distortion are continuous, map sets of measure zero to sets of measure zero, and they are either constant or locally bounded to one. Also, a mapping of bounded distortion that is injective close to the boundary is necessarily a homeomorphism and the inverse is also of bounded distortion. Thus this class of mappings has the properties that are desirable from the point of view of nonlinear elasticity.

In the case of bounded distortion in the complex plane, one has the associated Beltrami equation

$$\overline{\partial} f(z) = \mu(z)\partial f(z),$$

where one assumes that $\|\mu\|_{L^\infty} < 1$. Each mapping of bounded distortion is a solution to this equation and each such an equation with $\|\mu\|_{L^\infty} < 1$ has a homeomorphic solution of bounded distortion. See Sect. 7.3 for more details. It is possible to show the existence of solutions under weaker assumptions (see [4] for exact statements and proofs), like for those compactly supported μ with $\exp(\frac{p}{1-|\mu(z)|})$ integrable for some p; this corresponds to the class of mappings whose distortion is not necessarily bounded but $\exp(\lambda K_f(x)) \in L^1$ for some $\lambda > 0$. Actually O. Lehto proved existence theorems for certain Beltrami coefficients that correspond to suitably integrable distortions K_f already in 1976. We will briefly touch this issue in Sect. 7.3 and we recommend the excellent monograph [4] for the interested reader.

Going back to the question posed by Ball, the following natural issue arises. Given a Sobolev homeomorphism f when do we have that also f^{-1} is a Sobolev mapping or more generally $f^{-1} \in W^{1,p}_{\mathrm{loc}}$? For simplicity, we call a mapping f such that both $f \in W^{1,1}_{\text{loc}}$ and $f^{-1} \in W^{1,1}_{\text{loc}}$ a bi-Sobolev mapping . We will show below that, for a planar Sobolev homeomorphism $f : \Omega \to \mathbf{R}^2$ we have that

$$\int_{\Omega} K_f(x)\, dx = \int_{f(\Omega)} |Df^{-1}(y)|^2\, dy\,.$$

This statement means that the finiteness of the left-hand side guarantees that $f^{-1} \in W^{1,2}(f(\Omega), \mathbf{R}^2)$. Consequently, the planar minimization problem for $\int K_f$ corresponds to the minimization problem for the harmonic energy of f^{-1}. We will discuss this important issue at the end of Sect. 1.1.

We begin with a discussion on bi-Sobolev mappings that results in the definition of our main concept, a mapping of finite distortion, in the planar, homeomorphic setting. After this, in Sect. 1.2, we recall the definition and basic properties of mappings of bounded distortion. These properties are relevant for the discussion on the deformations in the fourth paragraph of this introduction. We then give the definition of a mapping of finite distortion in the general case. In the following chapters we establish optimal conditions for a mapping of finite distortion to enjoy analogs of the basic properties of mappings of bounded distortion. Especially, we answer J.M.Ball's question described in paragraph two of this introduction (see Sect. 7.2) in the affirmative.

At the end of each chapter and of some of the sections we offer a Remark where we recall the sources of the involved ideas and we also point out the reader's attention to some additional references and results.

1.1 Planar Bi-Sobolev Mappings

Let us first look at the single variable case in order to gain intuition on necessary and sufficient conditions for a homeomorphism to be a bi-Sobolev mapping.

The one dimensional case has a simple solution. Indeed, let $u : \mathbf{R} \to \mathbf{R}$ be an increasing homeomorphism that belongs to $W^{1,1}_{\text{loc}}(\mathbf{R})$; recall that a function v belongs to the Sobolev space $W^{1,1}_{\text{loc}}(\mathbf{R})$ if and only if the restriction of v to each compact interval is absolutely continuous. First of all, if $u^{-1} \in W^{1,1}_{\text{loc}}(\mathbf{R})$, then u^{-1} is absolutely continuous on each compact interval, and hence $|u^{-1}(E)| = 0$ whenever $|E| = 0$. On the other hand,

$$|u(A)| = \int_A u'(x)dx \tag{1.2}$$

for each measurable set A by the (local) absolute continuity of u and the assumption that u is both injective and increasing. Using these observations for the sets $A = \{x : u'(x) = 0\}$ and $E = u(A)$ we thus conclude that necessarily $u'(x) > 0$ almost everywhere. Conversely, suppose that $u \in W^{1,1}_{\text{loc}}(\mathbf{R})$ is an increasing homeomorphism with $u(\mathbf{R}) = \mathbf{R}$ and $u'(x) > 0$ almost everywhere. Then, given a compact interval I, clearly u^{-1} is of bounded variation on I (since u^{-1} is continuous and increasing). Moreover, (1.2) together with the assumption that $u'(x) > 0$ almost everywhere guarantee that u^{-1} maps sets of Lebesgue measure zero to sets of Lebesgue measure zero. Hence u^{-1} is continuous, has bounded variation and maps sets of measure zero to sets of measure zero. It follows that u^{-1} is absolutely continuous on I and thus $u^{-1} \in W^{1,1}_{\mathrm{loc}}(\mathbf{R})$.

Let us move to dimension two. Let $\Omega \subset \mathbf{R}^2$ be a domain and consider a homeomorphism $f : \Omega \to \mathbf{R}^2$ that belongs to $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^2)$. This is the usual Sobolev class consisting of all mappings $f : \Omega \to \mathbf{R}^2$ for which both component functions belong to $W^{1,1}_{\mathrm{loc}}(\Omega)$. For the sake of completeness, the Appendix below contains a brief introduction to Sobolev spaces. When do we also have that $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbf{R}^2)$?

This is not always the case as shown by the following example.

Example 1.1. There is a homeomorphism $f : \mathbf{R}^2 \to \mathbf{R}^2$ such that f is Lipschitz, but $f^{-1} \notin W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2, \mathbf{R}^2)$.

Proof. Indeed, let u be the usual Cantor ternary function (see e.g. [13]) on the interval $(0, 1)$. Then u is continuous, non-decreasing, constant on each complementary interval of the ternary Cantor set and fails to be absolutely continuous. Let now $v(x) = x + u(x)$ on $(0, 1)$ and extend v to negative reals as $v(x) = x$ and to $x \geq 1$ as $v(x) = x + 1$. Then also v fails to be absolutely continuous but v^{-1} is Lipschitz. The mapping g defined simply by $g([x_1, x_2]) = [v(x_1), x_2]$ is clearly a homeomorphism, but it is not absolutely continuous on almost all lines parallel to coordinate axes as v is not absolutely continuous. It follows that g does not satisfy the ACL-condition and hence $g \notin W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2, \mathbf{R}^2)$, see Theorem A.15. It is easy to check that $f = g^{-1}$ is Lipschitz continuous and thus f has the desired properties. □

Notice that, in the above construction, J_f vanishes in a set of positive area. Based on this and the above discussion on the single variable setting, one could expect that, in the orientation preserving case, the answer should be "if and only if $J_f(x) := \det Df(x) > 0$ almost everywhere." This turns out not to be the correct answer.

Indeed, it is not hard to construct a homeomorphism $f : \mathbf{R}^2 \to \mathbf{R}^2$ that belongs to $W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2, \mathbf{R}^2)$ so that also $f^{-1} \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2, \mathbf{R}^2)$ but $J_f(x) = 0$ in a set of positive area. This can be done by mapping a product Cantor-set E of positive area onto a product Cantor-set of area zero via a suitable Lipschitz homeomorphism $f : \mathbf{R}^2 \to \mathbf{R}^2$, see the proof of Theorem 4.15 below for the details. In this construction, one maps squerical frames surrounding E onto substantially smaller squerical frames surrounding the second product Cantor-set in a canonical manner set and then $Df(x) = 0$ (the zero matrix) almost everywhere in the above Cantor-set E and hence almost everywhere in the zero set of the Jacobian of f. We will show below that this phenomenon is both necessary and sufficient for $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbf{R}^2)$. Towards this end, we begin with a simple oscillation estimate.

Lemma 1.2. *Let $B(y,2r) \subset\subset f(\Omega)$ and suppose that $f \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbf{R}^2)$ is a homeomorphism. Then*

$$r \operatorname{diam} f^{-1}(B(y,r)) \leq \int_{f^{-1}(B(y,2r))} |Df|. \tag{1.3}$$

Proof. Set $d = \operatorname{diam} f^{-1}(B(y,r))$ and pick $a, b \in \overline{f^{-1}(B(y,r))}$ such that $|a - b| = d$. Without loss of generality, we will suppose that $a = [0,0]$ and $b = [d,0]$. For $t \in [0,d]$ we denote

$$L_t = \big\{s \in \mathbf{R} : [t,s] \in f^{-1}(B(y,2r))\big\}.$$

Since f is absolutely continuous on almost every line parallel to the y-axis and $\operatorname{diam} f(\{t\} \times L_t) \geq r$, we obtain

$$\int_{L_t} |Df(t,s)|ds \geq r$$

for almost every $t \in [0,d]$. By integrating this inequality over $[0,d]$ we obtain (1.3). □

Lemma 1.3. *Let $\Omega \subset \mathbf{R}^2$ be a domain and let $f : \Omega \to f(\Omega) \subset \mathbf{R}^2$ be a homeomorphism. Suppose that both $f \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbf{R}^2)$ and $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega),\mathbf{R}^2)$. Then $Df^{-1}(x) = 0$ almost everywhere in the set $\{x : J_{f^{-1}}(x) = 0\}$.*

Proof. Suppose that we can find a measurable set M and an open set A such that $M \subset A \subset\subset f(\Omega)$, $|A| < \infty$, $|M| > 0$, and

$$\text{for every } x \in M \text{ we have } |Df^{-1}(x)| > 0 \text{ and } J_{f^{-1}}(x) = 0. \tag{1.4}$$

Then there exists $k \in \mathbf{Z}$ such that for

$$\tilde{M} = \{x \in M : 2^k < |Df^{-1}(x)| \leq 2^{k+1}\} \text{ we have } |\tilde{M}| > 0. \tag{1.5}$$

In view of Lemma A.28, we may moreover assume that f^{-1} is differentiable at every point of $\tilde{M}$. Let $\eta > 0$. From (1.4) and (1.5) we obtain that for every $x \in \tilde{M}$ we can pick a disk $B(x,r(x))$ such that

$$\begin{aligned} &B(x,2r(x)) \subset A,\ r(x) < 1,\\ &\operatorname{diam} f^{-1}(B(x,r(x))) > 2^k r(x) \text{ and}\\ &|f^{-1}(B(x,2r(x)))| < \eta |B(x,2r(x))|. \end{aligned} \tag{1.6}$$

We use the Vitali covering theorem, Theorem A.1, for the family $\{B(x, 2r(x))\}_{x\in\tilde{M}}$ to obtain disks $B_i := B(x_i,r_i)$ such that

$$\tilde{M} \subset \bigcup_i 10B_i \text{ and } 2B_i \text{ are pairwise disjoint.} \tag{1.7}$$

Hence (1.7), (1.6) and Lemma 1.2 give us

$$\begin{aligned} |\tilde{M}| &\le 10^2 \sum_i |B_i| \le C \sum_i r_i^2 \\ &\le C2^{-k} \sum_i r_i \operatorname{diam}(f^{-1}(B(x_i, r_i))) \\ &\le C(k) \sum_i \int_{f^{-1}(2B_i)} |Df| = C(k) \int_{\bigcup_i f^{-1}(2B_i)} |Df|. \end{aligned} \tag{1.8}$$

By (1.6) we conclude that

$$|\bigcup_i f^{-1}(2B_i)| \le \eta \sum_i |2B_i| \le \eta |A| \overset{\eta \to 0+}{\to} 0.$$

Using this fact, (1.8) and the absolute continuity of the integral of $|Df| \in L^1_{\mathrm{loc}}(\Omega)$ we obtain a contradiction with $|\tilde{M}| > 0$. □

Lemma 1.4. *Let $\Omega \subset \mathbf{R}^2$ be a domain. Suppose that $f : \Omega \to f(\Omega) \subset \mathbf{R}^2$ is a homeomorphism that also belongs to $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^2)$. If $|Df(x)| = 0$ almost everywhere in the set $\{x : J_f(x) = 0\}$, then also f^{-1} belongs to $W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbf{R}^2)$.*

Proof. Let $A \subset\subset f(\Omega)$ be a fixed domain. First we will construct approximations to f^{-1}. We fix $0 < \varepsilon < \frac{1}{4} \operatorname{dist}(A, \partial f(\Omega))$ and we denote the standard ε-grid in $\mathbf{R}^2$ by $G_\varepsilon = (\varepsilon \mathbf{Z}) \times (\varepsilon \mathbf{Z})$. Pick a partition of unity $\{\phi_z\}_{z \in G_\varepsilon}$ such that

$$\begin{aligned} &\text{each } \phi_z : \mathbf{R}^2 \to \mathbf{R} \text{ is continuously differentiable;} \\ &\operatorname{spt} \phi_z \subset B(z, 2\varepsilon) \text{ and } |\nabla \phi_z| \le \frac{C}{\varepsilon}; \\ &\sum_{z \in G_\varepsilon} \phi_z(y) = 1 \text{ for every } y \in \mathbf{R}^2. \end{aligned} \tag{1.9}$$

Now we set

$$g_\varepsilon(y) = \sum_{z \in G_\varepsilon} \phi_z(y) f^{-1}(z) \text{ for every } y \in A.$$

The supports of ϕ_z have bounded overlap and hence this approximation to f^{-1} clearly satisfies $g_\varepsilon \in C^1(A, \mathbf{R}^2)$. Next we show that

$$|Dg_\varepsilon(y)| \le \frac{C}{\varepsilon} \operatorname{diam} f^{-1}(B(y, 2\varepsilon)).$$

Indeed, for a fixed $y \in A$, choose z_0 so that $y \in B(z_0, 2\varepsilon)$. Then

$$Dg_\varepsilon(y) = D \sum_{z \in G_\varepsilon} \phi_z(y)(f^{-1}(z) - f^{-1}(z_0)),$$

by (1.9), and the asserted estimate follows. Together with Lemma 1.2 this implies that for every $y \in A$ we have

$$|Dg_\varepsilon(y)| \le \frac{C}{\varepsilon^2} \int_{f^{-1}(B(y,4\varepsilon))} |Df(x)| dx. \tag{1.10}$$

Denote $\tilde{G} = \{x \in \Omega : f \text{ is differentiable at } x \text{ and } J_f(x) > 0\}$. By Lemma A.28 we know that f is differentiable a.e. and hence we may use our assumptions to conclude that $Df(x) = 0$ a.e. in $\Omega \setminus \tilde{G}$. Pick a Borel set $G \subset \tilde{G}$ such that $|G| = |\tilde{G}|$. From (1.10) and the Area Formula, Corollary A.36 (a), we now have

$$\begin{aligned} |Dg_\varepsilon(y)| &\le \frac{C}{\varepsilon^2} \int_{f^{-1}(B(y,4\varepsilon)) \cap G} |Df(x)| dx \\ &\le \frac{C}{\varepsilon^2} \int_{B(y,4\varepsilon) \cap f(G)} \frac{|(Df)(f^{-1}(z))|}{J_f(f^{-1}(z))} dz. \end{aligned} \tag{1.11}$$

We claim that

$$F(z) := \frac{|(Df)(f^{-1}(z))|}{J_f(f^{-1}(z))} \chi_{f(G)}(z) \in L^1(A). \tag{1.12}$$

Note that $f(G)$ is a Borel set (as a preimage of a Borel set under the continuous map f^{-1}) and hence F is measurable. For every $z \in f(G)$ we know that f is differentiable at $f^{-1}(z)$ and that $J_f(f^{-1}(z)) > 0$. Therefore f^{-1} is differentiable at z and $J_{f^{-1}}(z) = 1/J_f(f^{-1}(z))$. It follows from the Area Formula, Corollary A.36 (a), for $g = f^{-1}$, that

$$\int_A F(z) dz = \int_{A \cap f(G)} |(Df)(f^{-1}(z))| J_{f^{-1}}(z) dz \le \int_{f^{-1}(A) \cap G} |Df| < \infty.$$

Since $⨍_{B(y,4\varepsilon)} F \to F(y)$ in $L^1(A)$ as $\varepsilon \to 0$, there is a subsequence $\varepsilon_j \to 0$ such that $⨍_{B(y,4\varepsilon_j)} F$ has a majorant $H \in L^1(A)$. From this, (1.11) and characterization of weak compactness in L^1, Lemma A.3, we obtain that there is a subsequence $\varepsilon_i \to 0$ and $g \in L^1(A, \mathbf{R}^2)$ such that $Dg_{\varepsilon_i} \to g$ weakly in $L^1(A)$. Clearly

$$\int_A Dg_{\varepsilon_i}(y)\varphi(y) dy = -\int_A g_{\varepsilon_i}(y) D\varphi(y) dy$$

for every test function $\varphi \in C_c^\infty(A, \mathbf{R}^2)$. Since $g_\varepsilon \to f^{-1}$ locally uniformly as $\varepsilon \to 0$, we obtain, after passing to a limit, that

$$\int_A g(y)\varphi(y)dy = -\int_A f^{-1}(y)D\varphi(y)dy$$

which means that g is a weak gradient of f^{-1} in A and therefore $f^{-1} \in W^{1,1}(A, \mathbf{R}^2)$. $\square$

Let us formulate the above necessary and sufficient condition for the Sobolev regularity of f^{-1} as a definition. First of all notice that, assuming that $J_f(x) \geq 0$ almost everywhere, the requirement that $Df(x) = 0$ almost everywhere in the set $\{x : J_f(x) = 0\}$ is equivalent to requiring the existence of a function $K(x) \geq 1$ so that

$$|Df(x)|^2 \leq K(x)J_f(x)$$

almost everywhere. Here one necessarily has $K(x) \geq 1$ by Hadamard's inequality for matrices. Also the almost everywhere differentiability of a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2, \mathbf{R}^2)$ guarantees the local integrability of J_f; see Corollary A.36 (a).

Definition 1.5. We say that a homeomorphism $f : \Omega \to f(\Omega) \subset \mathbf{R}^2$ on an open set $\Omega \subset \mathbf{R}^2$ has finite distortion if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^2)$ and there is a function $K : \Omega \to [1, \infty]$ with $K(x) < \infty$ almost everywhere such that

$$|Df(x)|^2 \leq K(x)J_f(x) \qquad \text{for almost all } x \in \Omega.$$

For mappings of finite distortion we can define the optimal distortion function as

$$K_f(x) := \begin{cases} \frac{|Df(x)|^2}{J_f(x)} & \text{for all } x \in \{J_f > 0\}, \\ 1 & \text{for all } x \in \{J_f = 0\}. \end{cases}$$

Using this definition, we can formulate the outcome of Lemmas 1.3 and 1.4 as the following result.

Theorem 1.6. *Let $\Omega \subset \mathbf{R}^2$ be a domain and let $f : \Omega \to f(\Omega) \subset \mathbf{R}^2$ be a homeomorphism with $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^2)$ and assume that $J_f(x) \geq 0$ almost everywhere. Then the following conditions are equivalent:*

(i) $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbf{R}^2)$,

(ii) *f has finite distortion,*

(iii) *f^{-1} has finite distortion.*

Let us recall the question from our introduction regarding the existence and regularity of an inverse of a Sobolev mapping. In our planar setting, the regularity question asks if the inverse of a homeomorphism $f : \Omega \to f(\Omega)$ that belongs to $W^{1,2}(\Omega, \mathbf{R}^2)$ also belongs to $W^{1,2}(f(\Omega), \mathbf{R}^2)$ under the assumptions that $J_f(x) > 0$ almost everywhere and

$$\int_\Omega |(Df(x))^{-1}|^2 J_f(x)\, dx < \infty.$$

Since $J_f(x) > 0$, our homeomorphism has finite distortion and the above integrability condition guarantees that $K_f \in L^1(\Omega)$. The following theorem completely solves this modified problem.

Theorem 1.7. *Let $\Omega \subset \mathbf{R}^2$ be a domain and let $f : \Omega \to f(\Omega) \subset \mathbf{R}^2$ be a homeomorphism with $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^2)$ and assume that $J_f(x) \geq 0$ almost everywhere. Then the following conditions are equivalent:*

(*i*) $J_f(x) > 0$ *almost everywhere and* $\int_\Omega |(Df(x))^{-1}|^2 J_f(x)\, dx < \infty$,

(*ii*) f *has finite distortion and* $K_f \in L^1(\Omega)$,

(*iii*) $f^{-1} \in W^{1,2}_{\mathrm{loc}}(f(\Omega), \mathbf{R}^2)$ *and* $|Df^{-1}| \in L^2(f(\Omega))$.

For the proof of this theorem we need the following result from [114] (see also [89]) which is of independent interest.

Theorem 1.8. *Let $\Omega \subset \mathbf{R}^2$ be a domain and let $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{R}^2)$ be a homeomorphism. Then $|f(E)| = 0$ for every set $E \subset \Omega$ with $|E| = 0$.*

Proof. Let $x \in \Omega$ and $B(x,r) \subset\subset \Omega$. Since f is a homeomorphism we get

$$\operatorname{diam}(f(B(x,r))) = \operatorname{diam}(f(S^1(x,r)))\,.$$

Each Sobolev mapping (or function) f is absolutely continuous on almost all lines parallel to coordinate axes, Theorem A.15. By a change of variables it follows that f is absolutely continuous on almost all circles. Hence for a.e. $r > 0$ we get by the fundamental theorem of calculus and Hölder's inequality that

$$\operatorname{diam}(f(S^1(x,r))) \leq \int_{S^1(x,r)} |Df| \leq \Big(\int_{S^1(x,r)} |Df|^2\Big)^{\frac{1}{2}} \Big(\int_{S^1(x,r)} 1\Big)^{\frac{1}{2}}\,.$$

It follows that

$$\operatorname{diam}^2(f(B(x,r))) = \operatorname{diam}^2(f(S^1(x,r))) \leq Cr \int_{S^1(x,r)} |Df|^2\,. \tag{1.13}$$

Let $E \subset \Omega$ satisfy $|E| = 0$. We cover E by disks and we use the previous estimate on each disk in question. We distinguish if $\omega(r) = \int_{B(x,r)} |Df|^2$ decreases too fast or not. Formally, we set

$$E_1 = \left\{x \in E : \operatorname{essliminf}_{r\to 0+} \frac{r\int_{S^1(x,r)} |Df|^2}{\int_{B(x,r)} |Df|^2} \le 16\right\}.$$

and $E_0 = E \setminus E_1$. For every $x \in E_0$ there is $\delta = \delta_x > 0$ such that for a.e. $r \in (0,\delta)$ we have

$$\int_{B(x,r)} |Df|^2 \le \frac{r}{16}\int_{S^1(x,r)} |Df|^2 .$$

Fix $x \in E_0$ and let $\rho \in (0,\delta/2]$. Integrating the previous inequality over the interval $[\rho, 2\rho]$ we obtain

$$\rho\int_{B(x,\rho)} |Df|^2 \le \frac{\rho}{8}\int_{B(x,2\rho)} |Df|^2 .$$

Write $\omega(\rho) = \int_{B(x,\rho)} |Df|^2$. By iteration of the inequality $\omega(\rho) \le \frac{1}{8}\omega(2\rho)$ we obtain

$$\omega(\rho) = \int_{B(x,\rho)} |Df|^2 \le \rho^2 \tag{1.14}$$

for small enough $\rho > 0$. Indeed, fix $\rho < \delta/2$ and pick $m \in \mathbf{N}$ such that $\delta/2^m \le \rho \le \delta/2^{m-1}$. Now

$$\omega(\rho) \le \omega\Big(\frac{\delta}{2^{m-1}}\Big) \le \frac{1}{8^{m-1}}\omega(\delta) = \frac{8}{\delta^3}\frac{\delta^3}{8^m}\omega(\delta) \le \frac{8}{\delta^3}\omega(\delta)\rho^3$$

and (1.14) follows if we choose ρ sufficiently small. For a fixed $x \in E_0$ we may use (1.14) and the Fubini theorem to find $r \in [\rho/2, \rho]$ such that

$$\int_{S^1(x,r)} |Df|^2 \le 8r .$$

Choose an open set G such that $E \subset G \subset \Omega$. For every $x \in E$ we may use the definition of E_1 and our previous observation to find $r > 0$ such that $B(x,r) \subset G$ and

$$\int_{S^1(x,r)} |Df|^2 \le \begin{cases} \frac{16}{r}\int_{B(x,r)} |Df|^2, & x \in E_1,\\ 8r, & x \in E_0. \end{cases}$$

By (1.13) we obtain

$$\operatorname{diam}^2(f(B(x,r))) \le Cr\int_{S^1(x,r)} |Df|^2 \le C\int_{B(x,r)} (1 + |Df|^2) .$$

Using Besicovitch covering theorem, Theorem A.2, to this collection of disks, we find disks $B_i = B(x_i, r_i)$ that cover E with uniformly bounded overlap. Now

$$|f(E)| \leq \sum_i |f(B_i)| \leq C \sum_i \operatorname{diam}^2(f(B_i)) \leq C \sum_i \int_{B_i} (1 + |Df|^2)$$
$$\leq C \int_G (1 + |Df|^2).$$

Letting $|G| \to 0$ we obtain that $|f(E)| = 0$ by the absolute continuity of the integral. □

Proof (of Theorem 1.7). (*ii*) $\Rightarrow$ (*iii*): From Theorem 1.6 we know that $f^{-1} \in W^{1,1}_{\text{loc}}$ is a mapping of finite distortion and it remains to show that $|Df^{-1}| \in L^2_{\text{loc}}$. Since f^{-1} is a mapping of finite distortion and differentiable a.e. by Lemma A.28, we obtain that

$$\int_{f(\Omega)} |Df^{-1}(y)|^2 dy = \int_A |Df^{-1}(y)|^2 dy$$

where A is a Borel subset of the set

$$G := \{y \in f(\Omega) : \ f^{-1} \text{ is differentiable at } y \text{ and } J_{f^{-1}}(y) > 0\}$$

such that $|A| = |G|$. It is easy to see that f is differentiable at every point $x \in f^{-1}(A)$ and we have $Df^{-1}(f(x)) = (Df(x))^{-1}$ and $J_f(x) = (J_{f^{-1}}(f(x)))^{-1}$ (see Lemma A.29). Applying these facts, the Area formula for f^{-1}, Corollary A.36 (c), and $E \operatorname{adj} E = I \det E$ we arrive at

$$\begin{aligned}\int_{f(\Omega)} |Df^{-1}(y)|^2 dy &= \int_A \frac{|Df^{-1}(y)|^2}{J_{f^{-1}}(y)} J_{f^{-1}}(y) dy = \int_{f^{-1}(A)} \frac{|Df^{-1}(f(x))|^2}{J_{f^{-1}}(f(x))} dx \\ &= \int_{f^{-1}(A)} |(Df(x))^{-1}|^2 J_f(x) dx = \int_{f^{-1}(A)} \frac{|\operatorname{adj} Df(x)|^2}{J_f(x)} dx \\ &= \int_{f^{-1}(A)} \frac{|Df(x)|^2}{J_f(x)} dx \leq \int_\Omega K_f(x)\, dx < \infty\,. \end{aligned} \tag{1.15}$$

(*iii*) $\Rightarrow$ (*i*): Suppose for contrary that $|\{J_f = 0\}| > 0$. Then we may use a.e. differentiability of f, Lemma A.28, to find a Borel subset $\tilde{A} \subset \{J_f = 0\}$ of full measure such that f is differentiable at every point of $\tilde{A}$. By the Area formula, Corollary A.36 (c), we get

$$0 = \int_{\tilde{A}} J_f(x)\, dx = \int_{f(\tilde{A})} 1\, dy$$

and hence $|f(\tilde{A})| = 0$. We may apply Theorem 1.8 to f^{-1} to obtain that $|f^{-1}(E)| = 0$ for every set $E \subset f(\Omega)$ such that $|E| = 0$. This observation for $E = f(\tilde{A})$ gives us the contradiction and hence $J_f > 0$ a.e.

We can find a Borel set $\tilde{A}$ which is a subset of

$$\tilde{G} := \{x \in \Omega : \ f \text{ is differentiable at } x \text{ and } J_f(x) > 0\}$$

with $|\tilde{A}| = |\tilde{G}|$. By differentiability of f a.e. and the Area formula, Corollary A.36 (c), we obtain analogously to (1.15) that

$$\begin{aligned}\int_\Omega |(Df(x))^{-1}|^2 J_f(x)\, dx &= \int_{\tilde{A}} |(Df(x))^{-1}|^2 J_f(x)\, dx \\ &= \int_{\tilde{A}} |Df^{-1}(f(x))|^2 J_f(x)\, dx \\ &= \int_{f(\tilde{A})} |Df^{-1}(y)|^2\, dy < \infty\,.\end{aligned}$$

$(i) \Rightarrow (ii)$: Since $J_f > 0$ a.e. it is immediate that f has finite distortion. Since f is differentiable a.e. by Lemma A.28, we may integrate over the set

$$\tilde{G} := \{x \in \Omega : \ f \text{ is differentiable at } x \text{ and } J_f(x) > 0\}$$

and analogously to (1.15) we obtain

$$\begin{aligned}\int_\Omega K_f(x)\, dx = \int_{\tilde{G}} K_f(x)\, dx &= \int_{\tilde{G}} \frac{|\operatorname{adj} Df(x)|^2}{J_f(x)} dx \\ &= \int_{\tilde{G}} |(Df(x))^{-1}|^2 J_f(x)\, dx < \infty\,.\end{aligned}$$ □

Corollary 1.9. *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, f(\Omega))$ be a homeomorphism in the plane with finite distortion. Then*

$$\int_\Omega K_f(x)\, dx = \int_{f(\Omega)} |Df^{-1}(y)|^2\, dy\,.$$

Proof. We may clearly assume that one and hence, by Theorem 1.7, both of these integrals are finite. By the statement of Theorem 1.7 and the first part of its proof it is enough to show that we have an identity in (1.15). This amounts to the claim that

$$\int_{\Omega \setminus f^{-1}(A)} K_f(x)\, dx = 0\,.$$

To prove this claim it suffices to prove that $|\Omega \setminus f^{-1}(A)| = 0$. By the Area formula, Corollary A.36 (c), we get that

$$\left|f^{-1}(S)\right| = 0 \text{ for } S := \{y \in f(\Omega) : f^{-1} \text{ is differentiable at } y \text{ and } J_{f^{-1}}(y) = 0\}.$$

Since $|f(\Omega) \setminus (A \cup S)| = 0$ and $f^{-1} \in W^{1,2}_{\mathrm{loc}}$ we can use Theorem 1.8 to conclude

$$\left|f^{-1}(f(\Omega) \setminus (A \cup S))\right| = 0 \quad \text{and hence} \quad |f^{-1}(f(\Omega) \setminus A)| = 0\,. \qquad \square$$

Remark 1.10. The definition of K_f is based on the operator norm of Df. Another natural choice is the Hilbert-Schmidt norm

$$\|A\|^2 = \frac{1}{2}\operatorname{Trace}(A^T A) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\,.$$

The identity in Corollary 1.9 also holds when the operator norm is replaced by the Hilbert-Schmidt norm and K_f is replaced with the associated distortion $\mathbb{K}_f$. Notice that the Hilbert-Schmidt norm is strictly convex. Hence the minimization problem for $\int_\Omega \mathbb{K}_f$ for planar homeomorphisms $f : \Omega \to \Omega'$, that coincide on the boundary with a fixed homeomorphism $g : \overline{\Omega} \to \overline{\Omega'}$ with $\int_\Omega \mathbb{K}_g < \infty$, assuming that Ω is convex, has a unique (modulo conformal changes of variables) minimizer and its inverse is a harmonic mapping. For this see [5, 6] and [52]. Regarding powers of the distortion function, it was shown in [6] that, apart from some trivial cases, the minimizers of the L^p-norm of the distortion function never exist when $p < 1$.

Open problem 1. Let Ω be a convex planar domain and $g : \overline{\Omega} \to \overline{\Omega'}$ be a homeomorphism of finite distortion with $\int_\Omega \mathbb{K}_g^p < \infty$, where $p > 1$. Does the minimization problem of $\int_\Omega \mathbb{K}_f^p$ for homeomorphisms $f : \Omega \to \Omega'$ that coincide on the boundary with g have a diffeomorphic solution?

The following open problems deal with minimization without given boundary values. In nonlinear elasticity, this corresponds to traction free problems.

Open problem 2. Let Ω, Ω' be bounded doubly connected planar domains. Then the minimization problem for $\int_\Omega \mathbb{K}_f$ for homeomorphisms $f : \Omega \to \Omega'$ does not necessarily have a homeomorphic solution. There is a homeomorphic minimizer if $\operatorname{Mod}\Omega \geq \operatorname{Mod}\Omega'$, where $\operatorname{Mod} G$ refers to the conformal modulus. Prove that the existence of a homeomorphic minimizer implies that necessarily $\operatorname{Mod}\Omega' \leq \log\cosh\operatorname{Mod}\Omega$. See [5, 64, 68] for the definition of the conformal modulus and more details.

Open problem 3. Let Ω, Ω' be multiply connected planar domains such that there exists a homeomorphism $g : \Omega \to \Omega'$ of finite distortion with $\mathbb{K}_g \in L^p(\Omega)$, where $p > 1$. Prove that there is a homeomorphism $f : \Omega \to \Omega'$ that minimizes $\int_\Omega \mathbb{K}_f^p$. For related results see [93].

1.2 Mappings of Bounded and Finite Distortion

Let Ω be an open connected set in $\mathbf{R}^n$ for some $n \geq 2$. Then a mapping $f : \Omega \to \mathbf{R}^n$ is called quasiregular or a mapping of bounded distortion if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ and there is a constant $K \geq 1$ so that

$$|Df(x)|^n \leq KJ_f(x)$$

almost everywhere in Ω. It is then customary to say that f is K-quasiregular. This class of mappings was introduced by Reshetnyak in 1967 [115]. We recommend the monographs [75, 116, 117] for an interested reader.

Let us list some of the basic properties of quasiregular mappings. Let $f : \Omega \to \mathbf{R}^n$ be a K-quasiregular mapping.

- The mapping f has a continuous representative; actually a $1/K$-Hölder continuous one.
- This continuous representative $\hat{f}$ is either constant or both open and discrete: images of open sets are open and the preimage of no point can accumulate in Ω. In the latter case, $\hat{f}$ is locally bounded-to-one, and if Ω is bounded and $\hat{f}$ is injective close to the boundary of Ω, then $\hat{f}$ is a homeomorphism.
- The mapping $\hat{f}$ maps sets of measure zero to sets of measure zero. In the case of a non-constant $\hat{f}$, preimages of sets of measure zero are sets of measure zero and thus the Jacobian determinant is necessarily strictly positive almost everywhere.
- Homeomorphic quasiregular (i.e. quasiconformal) mappings form a group with respect to composition: f^{-1} is K^{n-1}-quasiregular if f is K-quasiregular, and $f_1 \circ f_2$ is K_1K_2-quasiregular whenever defined, if f_1 is K_1-quasiregular and f_2 is K_2-quasiregular.
- Regarding regularity, there is $p = p(n, K) > n$ so that each K-quasiregular mapping f belongs to $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ and so that J_f^{n-p} is locally integrable (unless f is constant). For homeomorphic quasiregular mapping this implies that also $J^{-\varepsilon}_{f^{-1}}$ is locally integrable.

Definition 1.11. We say that a mapping $f : \Omega \to \mathbf{R}^n$ on an open connected set $\Omega \subset \mathbf{R}^n$ has finite distortion if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$, $J_f \in L^1_{\mathrm{loc}}(\Omega)$ and there is a function $K : \Omega \to [1, \infty]$ with $K(x) < \infty$ almost everywhere such that

$$|Df(x)|^n \leq K(x)J_f(x) \qquad \text{for almost all } x \in \Omega. \tag{1.16}$$

For mappings of finite distortion we can define the optimal distortion function as

$$K_f(x) := \begin{cases} \frac{|Df(x)|^n}{J_f(x)} & \text{for all } x \in \{J_f > 0\}, \\ 1 & \text{for all } x \in \{J_f = 0\}. \end{cases}$$

In the next chapters we relax the assumption $K \in L^\infty$ and we prove that mappings of finite distortion have properties similar to those of mappings of bounded distortion. We usually have two kinds of positive results. We assume that f is in the nice Sobolev space $W^{1,n}$ and then we require some mild assumptions on the distortion like integrability or only finiteness almost everywhere. Alternatively, we assume only that $f \in W^{1,1}$ but then we usually need much stronger assumptions on the distortion, like $\exp(\lambda K_f) \in L^1$ for some $\lambda > 0$.

http://www.springer.com/978-3-319-03172-9

Lectures on Mappings of Finite Distortion
Hencl, S.; Koskela, P.
2014, XI, 176 p. 4 illus., Softcover
ISBN: 978-3-319-03172-9