Coherence and Convexity of Euclidean Radial Implicative Fuzzy Systems

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Abstract The chapter discuss a necessary condition for coherence of radial implicative fuzzy systems. We present the general condition in an implicit form. The condition is based on the value of the minima of a certain function. We show that this function is convex. Further an explicit solution for Euclidean systems is provided.

Keywords Fuzzy systems · Coherence · Convexity

1 Introduction

The property of *coherence* of a fuzzy system deals with the consistency of knowledge stored in the rule base the system. For *implicative fuzzy systems* this consistency means that for *any input* there always exists at least one output which is in degree 1 compatible with the given input. The notion of coherence point out the dynamical character of consistency. That is, the rule base is consistent for any input, not only for some special subset. The degree 1 compatibility is considered over the fuzzy relation representing the rule base of the system. If there is no such output, then the rule base is controversial and its rules should be revised. For example, if the system is used as a controller, then under incoherence it could happen that we end with the empty set of relevant actions. Thus the study of coherence represents a serious issue in the area of fuzzy computing [1].

To decide whether a given system is coherent on the basis of the values of its parameters is generally a difficult task. The introduction of *radial fuzzy systems*, which are the systems which employ radial fuzzy sets, allows to tackle the coherence question effectively for the class of combined systems called *radial implicative*

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fuzzy systems [2]. The formulation of conditions of coherence in the form of a sufficient condition was presented in [3]. If this condition is satisfied, then the system is coherent, i.e., safe in the above sense of the risk of obtaining the empty output.

In this chapter we ask for how to detect incoherent systems, i.e., we ask for a necessary condition on coherence for radial implicative fuzzy systems. As a main result we state this condition explicily for systems based on the Euclidean norm. However, in the implicit way the condition is valid for other norms as well and due to the convexity the condition can be stated also explicitly with the help of numerical optimization algorithms.

2 Radial Implicative Fuzzy Systems

The class of radial implicative fuzzy systems (radial I-FSs in short) was introduced in [2]. Let us give a brief review of the class. In radial I-FSs, IF-THEN rules are considered in MISO (multiple-input single-output) configuration. Let a rule base consist of $m \in \mathcal{N}$ IF-THEN rules. The j-th rule, $j = 1, \ldots, m$, writes as

$$A_{j1}(x_1) \star \cdots \star A_{jn}(x_n) \rightarrow B_j(y). \tag{1}$$

In the formula, A_{ji}, $i = 1, \ldots, n$, represent one-dimensional fuzzy sets specified on corresponding dimensions of n-dimensional input space $X \subseteq \mathcal{R}^n$. B_j denotes a one-dimensional fuzzy set specified on a one-dimesional output space $Y \subseteq \mathcal{R}$. The $\star$ symbol denotes a fuzzy conjunction representing AND connective and $\rightarrow$ corresponding residuated implication [4, 5] for THEN connective.

The product and minimum t-norms are typically used as fuzzy conjunctions. Corresponding residuated implications are obtained by the operation of residuation [4, 5]: $x \rightarrow_\star y = \sup\{z \in [0, 1] \,|\, z \star x \leq y\}$. The formula implies that $x \rightarrow_\star y = 1$, whenever $x \leq y$. For product or minimum, we obtain the so-called Goguen ($x \rightarrow_{\mathrm{P}} y = 1$ for $x \leq y$; $x \rightarrow_{\mathrm{P}} y = y/x$ for $x > y$) or Gödel implication ($x \rightarrow_{\mathrm{M}} y = 1$ for $x \leq y$; $x \rightarrow_{\mathrm{M}} y = y$ for $x > y$), respectively.

Individual IF-THEN rule (1) represents a fuzzy relation $R_j(\boldsymbol{x}, y)$ on $\mathcal{R}^{n+1}$ space. In the short notation the rule writes as $R_j(\boldsymbol{x}, y) = A_j(\boldsymbol{x}) \rightarrow B_j(y)$, where $A_j(\boldsymbol{x})$ is the antecedent of the rule and $B_j(y)$ its consequent. The antecedent then read as $A_j(\boldsymbol{x}) = A_{j1}(x_1) \star \cdots \star A_{jn}(x_n)$.

Individual rules are in implicative systems combined into the whole rule base by a fuzzy conjunction. The most common choice for this operation is the minimum t-norm. If the system consists of $m \in \mathcal{N}$ rules, then the whole rule base $RB(\boldsymbol{x}, y)$ forms again a fuzzy relation on $\mathcal{R}^{n+1}$ space, which is specified as

$$RB(\boldsymbol{x}, y) = \bigwedge_{j=1}^{m} R_j(\boldsymbol{x}, y) = \min_j\{A_j(\boldsymbol{x}) \rightarrow B_j(y)\}. \tag{2}$$

In radial systems, A_{ji} and B_j sets are represented by radial membership functions. Radial functions are well known from the theory of radial basis neural networks [6]. A radial function $f : \mathcal{R}^n \to \mathcal{R}$ is determined by its central point $\boldsymbol{a} \in \mathcal{R}^n$ and by a non-increasing shape function $act : \mathcal{R} \to [0, 1]$ which satisfies $act(0) = 1$ and $\lim_{z\to 0} act(z) = 0$. Finally, the application of a norm on the difference between the function's argument and its central point constitutes the final formula for a radial function as $f(x) = act(|x - a|)$ or $f(\boldsymbol{x}) = act(||\boldsymbol{x} - \boldsymbol{a}||)$ in the one-dimensional or the multi-dimensional case, respectively. In the context of radial systems, the class of scaled ℓ_p norms is relevant for parameter $p \in [1, \infty)$ (we do not consider here the limit case $p = \infty$). The norms of this class are defined for $\boldsymbol{u} \in \mathcal{R}^n$ and a vector of scaling parameters $\boldsymbol{b} = (b_1, \ldots, b_n)$, $b_i > 0$ as

$$||\boldsymbol{u}||_{\boldsymbol{b}} = \left[\sum_{i=1}^{n}(|u_i|/b_i)^p\right]^{1/p}. \tag{3}$$

The employment of radial functions for representation of membership functions in radial fuzzy systems is performed by the following specification of one-dimensional fuzzy sets

$$A_{ji}(x) = act(|x - a_{ji}|/b_{ji}), \quad B_j(y) = act(\max\{0, |y - c_j| - s_j\}/d_j) \tag{4}$$

where a_{ji}, $c_j \in \mathcal{R}$ are central points, b_{ji}, $d_j > 0$ are scaling parameters and $s_j > 0$ is a kernel's width controlling parameter. We see that the antecedent fuzzy sets are strictly radial. The formula for consequent fuzzy set is enhanced by s parameter which yields its generally trapezoid-like shape. Remark that $B_j(y)$ falls into the introduced framework of general radial functions as we can consider have $act(\max\{0,\ z - s_j\}/d_j)$ as another shape function.

The most prominent example of radial fuzzy sets are Gaussian fuzzy sets

$$A_{ji}(x) = \exp\left[-\frac{(x - a_{ji})^2}{b_{ji}^2}\right], \quad B_j(y) = \exp\left[-\frac{\max\{0, |y - c_j| - s_j\}^2}{d_j^2}\right]. \tag{5}$$

In Fig. 1 there are presented examples of these sets graphically.

In radial systems the combination of individual fuzzy sets by a t-norm retains the shape. That is, the following equality called *the radial property* holds

$$act\left(\frac{|x_i - a_{ji}|}{b_{j1}}\right) \star \ \ldots \ \star \ act\left(\frac{|x_n - a_{jn}|}{b_{jn}}\right) = act(||\boldsymbol{x} - \boldsymbol{a}_j||_{\boldsymbol{b}_j}). \tag{6}$$

In the formula on the right hand side, there is a multi-dimensional radial fuzzy set representing the antecedent of the rule $A_j(\boldsymbol{x})$. For Gaussian fuzzy sets the radial property is exhibited when the product t-norm is used and the corresponding norm is the scaled Euclidean norm ($p = 2$). Generally, shapes (act functions) cannot be

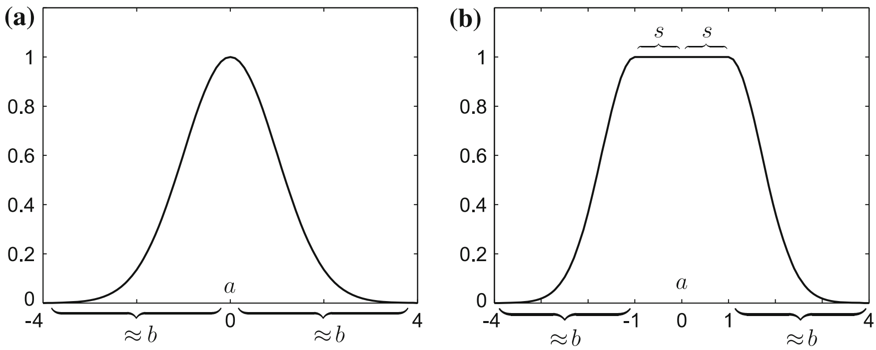

Fig. 1 An example of Gaussian radial fuzzy sets; (**a**) antecedent fuzzy set; (**b**) consequent trapezoid-like fuzzy set

combined freely with t-norms $\star$ when the validity of the radial property is required. In [7] there is proved the theorem stating the conditions on the t-norm $\star$ and the shape function *act* such that the radial property holds.

3 Computational Model

Computational model of radial I-FSs is based on the radial property and the basic property of residuated implications: $x \to y = 1$ iff $x \leq y$ [5].

Consider an input $\boldsymbol{x}^* \in \mathcal{R}^n$ to the system. In degree 1 compatible outputs from the j-th rule are those ys for which $A_j(\boldsymbol{x}^*) \to B_j(y) = 1$ holds. This is equivalent to the validity of inequality $A_j(\boldsymbol{x}^*) \leq B_j(y)$, and these ys are in degree 1 compatible with the fuzzy relation $R_j(\boldsymbol{x}^*, y) = A_j(\boldsymbol{x}^*) \to B_j(y)$. Due to the radial property it can be shown that the set $I_j(\boldsymbol{x}^*)$ of ys satisfiying $A_j(\boldsymbol{x}^*) \leq B_j(y)$ can be stated explicitly as

$$I_j(\boldsymbol{x}^*) = [c_j - d_j||\boldsymbol{x}^* - \boldsymbol{a}_j||_{\boldsymbol{b}_j} - s_j,\ c_j + d_j||\boldsymbol{x}^* - \boldsymbol{a}_j||_{\boldsymbol{b}_j} + s_j]. \tag{7}$$

That is, we obtain the set of ys of interest as the closed interval with limit points specified on the basis of input $\boldsymbol{x}^*$ and parameters of the j-th rule.

Concerning the output from the whole system on the basis of formula (2), we obtain the set of compatible ys evaluating the whole rule base to degree 1 as the intersection of individual intervals $I_j(\boldsymbol{x}^*)$. Denoting the left and the right limit points of $I_j(\boldsymbol{x}^*)$ as $L(I_j(\boldsymbol{x}^*))$ and $R(I_j(\boldsymbol{x}^*))$, respectively, we have

$$I(\boldsymbol{x}^*) = \bigcap_{j=1}^{m} I_j(\boldsymbol{x}^*) = [\max_j\{L(I_j(\boldsymbol{x}^*))\}, \min_j\{R(I_j(\boldsymbol{x}^*))\}], \tag{8}$$

under the condition $\max_j\{L(I_j(\boldsymbol{x}^*))\} \leq \min_j\{R(I_j(\boldsymbol{x}^*))\}$, otherwise $I(\boldsymbol{x}^*) = \emptyset$.

The formula (8) determines the set of in degree 1 compatible outputs of a radial implicative fuzzy system for a given input $\boldsymbol{x}^*$. We see that the output set is again a closed interval. Practically, only one point $y^* \in I(\boldsymbol{x}^*)$ is released as the output of the system. Typically, the middle point of the interval is selected.

Coherence of the systems means that for any $\boldsymbol{x}^* \in \mathcal{R}^n$ we have $I(\boldsymbol{x}^*) \neq \emptyset$. Note that if for some $\boldsymbol{x}^*$ it happens that there exists a pair of rules j, k such that $L(I_j(\boldsymbol{x}^*)) > R(I_k(\boldsymbol{x}^*))$, then $I_j(\boldsymbol{x}^*) \cap I_k(\boldsymbol{x}^*) = \emptyset$ and therefore $I(\boldsymbol{x}^*) = \emptyset$ and the system is incoherent.

4 A Necessary Condition on Coherence of Euclidean Systems

In this section we state the necessary condition on coherence of Euclidean systems, i.e., for systems using $p = 2$ in the specification of the scaled ℓ_p norm (3). The norm occurs in the antecedents' representation formula under the validity of the radial property (6).

We start by the lemma which enables explicit computations in the main theorem. To start define for pairs of rules $j, k \in \{1, \ldots, m\}$ and corresponding scaled ℓ_p norms the following entities

$$\begin{aligned} J_{jk}(\boldsymbol{x}) &= d_j ||\boldsymbol{x} - \boldsymbol{a}_j||_{\boldsymbol{b}_j} + d_k ||\boldsymbol{x} - \boldsymbol{a}_k||_{\boldsymbol{b}_k}, \\ J^{(p)}_{jk}(\boldsymbol{x}) &= d_j^p ||\boldsymbol{x} - \boldsymbol{a}_j||^p_{\boldsymbol{b}_j} + d_k^p ||\boldsymbol{x} - \boldsymbol{a}_k||^p_{\boldsymbol{b}_k}, \\ \boldsymbol{x}^{*(p)}_{jk} &= \operatorname{argmin}_{\boldsymbol{x}} \{J^{(p)}_{jk}(\boldsymbol{x})\}. \end{aligned} \tag{9}$$

The value of p is not restricted here, i.e., $p \in [1, \infty)$. However, for both norms composing J_{jk} and $J^{(p)}_{jk}$, the value of p is the same as the norms are related to the rules of the same fuzzy system.

Lemma 1. *For any $p \in [1, \infty)$,*

$$\min_{\boldsymbol{x}} \{J_{jk}(\boldsymbol{x})\} \leq 2[\min_{\boldsymbol{x}} \{J^{(p)}_{jk}(\boldsymbol{x})\}]^{1/p} = 2[J^{(p)}_{jk}(\boldsymbol{x}^{*(p)}_{jk})]^{1/p}. \tag{10}$$

If $p = 2$, then $\boldsymbol{x}^{*(2)}_{jk} = (x^{*(2)}_{jk1}, \ldots, x^{*(2)}_{jki}, \ldots, x^{*(2)}_{jkn})$ and

$$x^{*(2)}_{jki} = \frac{d_j^2 b_{ki}^2 a_{ji} + d_k^2 b_{ji}^2 a_{ki}}{d_j^2 b_{ki}^2 + d_k^2 b_{ji}^2}. \tag{11}$$

Proof. The first assertion is the corollarly of the following inequality for $u, v \in \mathcal{R}$.

$$\begin{aligned} |u + v|^p &\leq (|u| + |v|)^p \leq 2^p \max\{|u|^p, |v|^p\} \leq 2^p (|u|^p + |v|^p), \\ |u + v| &\leq 2(|u|^p + |v|^p)^{1/p}. \end{aligned} \tag{12}$$

Denoting $u(\boldsymbol{x}) = d_j||\boldsymbol{x} - \boldsymbol{a}_j||_{\boldsymbol{b}_j}$, $v(\boldsymbol{x}) = d_k||\boldsymbol{x} - \boldsymbol{a}_k||_{\boldsymbol{b}_k}$ and taking into account that both $u(\boldsymbol{x}), v(\boldsymbol{x}) \geq 0$ we have $J_{jk}(\boldsymbol{x}) = |u(\boldsymbol{x}) + v(\boldsymbol{x})|$ and $J^{(p)}_{jk}(\boldsymbol{x}) = |u(\boldsymbol{x})|^p + |v(\boldsymbol{x})|^p$. Applying (12) we immediately get $J_{jk}(\boldsymbol{x}) \leq 2[J^{(p)}_{jk}(\boldsymbol{x})]^{1/p}$. Since the p-th root is an increasing function the application of minimum (norms are continuous) writes as $\min_{\boldsymbol{x}}\{J_{jk}(\boldsymbol{x})\} \leq 2[\min_{\boldsymbol{x}}\{J^{(p)}_{jk}(\boldsymbol{x})\}]^{1/p}$.

For the Euclidean norm the localization of the point of minima is based on the standard procedure of computation of partial derivatives and setting them to zero. For $p = 2$ we have

$$J^{(2)}_{jk}(\boldsymbol{x}) = d_j^2 \sum_i \frac{(x_i - a_{ji})^2}{b_{ji}^2} + d_k^2 \sum_i \frac{(x_i - a_{ki})^2}{b_{ki}^2}.$$

Computing partial derivatives $\partial J^{(2)}_{jk}(\boldsymbol{x})/\partial x_i$ and setting them to zero we obtain

$$\begin{aligned} \frac{2d_j^2(x_i - a_{ji})}{b_{ji}^2} + \frac{2d_k^2(x_i - a_{ki})}{b_{ki}^2} &= 0, \\ d_j^2 b_{ki}^2(x_i - a_{ji}) + d_k^2 b_{ji}^2(x_i - a_{ki}) &= 0, \\ x_i(d_j^2 b_{ki}^2 + d_k^2 b_{ji}^2) - (d_j^2 b_{ki}^2 a_{ji} + d_k^2 b_{ji}^2 a_{ki}) &= 0, \\ \frac{d_j^2 b_{ki}^2 a_{ji} + d_k^2 b_{ji}^2 a_{ki}}{d_j^2 b_{ki}^2 + d_k^2 b_{ji}^2} &= x_i. \end{aligned}$$

Using notation $x^{*(2)}_{jki}$ instead of plain x_i we get the second assertion of the lemma. □

Theorem 1. *Let a radial I-FS use the Euclidean scaled norm in the specification of its rules. If the system is coherent, then the following set of inequalities holds for* $j, k \in \{1, \ldots, m\}$

$$|c_j - c_k| - (s_j + s_k) \leq 2\left[d_j^2||\boldsymbol{x}^{*(2)}_{jk} - \boldsymbol{a}_j||^2_{\boldsymbol{b}_j} + d_k^2||\boldsymbol{x}^{*(2)}_{jk} - \boldsymbol{a}_k||^2_{\boldsymbol{b}_k}\right]^{1/2} \tag{13}$$

where $\boldsymbol{x}^{*(2)}_{jk} = (x^{*(2)}_{jk1}, \ldots, x^{*(2)}_{jkn})$ *is given by* (11).

Proof. We prove the inverse implication, i.e., "if some of the above inequalities is violated, then the system is incoherent".

Suppose that for some pair of rules j, k the corresponding inequality (13) is violated, i.e.,

$$|c_j - c_k| - (s_j + s_k) > 2\left[d_j^2||\boldsymbol{x}^{*(2)}_{jk} - \boldsymbol{a}_j||^2_{\boldsymbol{b}_j} + d_k^2||\boldsymbol{x}^{*(2)}_{jk} - \boldsymbol{a}_k||^2_{\boldsymbol{b}_k}\right]^{1/2}.$$

Employing notation (9) and Lemma 1 this writes as

$$|c_j - c_k| - (s_j + s_k) > 2[J_{jk}^{(2)}(\boldsymbol{x}_{jk}^{*(2)})]^{1/2} \quad \geq \quad \min_{\boldsymbol{x}}\{J_{jk}(\boldsymbol{x})\},$$
$$|c_j - c_k| - (s_j + s_k) > d_j||\boldsymbol{x}^* - \boldsymbol{a}_j||_{\boldsymbol{b}_j} + d_k||\boldsymbol{x}^* - \boldsymbol{a}_k||_{\boldsymbol{b}_k},$$

where $\boldsymbol{x}^*$ is the point at which the minimum of $J_{jk}(\boldsymbol{x})$ over $\mathcal{R}^n$ is reached.

Let $c_j - c_k \geq 0$, otherwise switch the labeling of rules. The above then writes

$$c_j - c_k - (s_j + s_k) > d_j||\boldsymbol{x}^* - \boldsymbol{a}_j||_{\boldsymbol{b}_j} + d_k||\boldsymbol{x}^* - \boldsymbol{a}_k||_{\boldsymbol{b}_k},$$
$$c_j - d_j||\boldsymbol{x}^* - \boldsymbol{a}_j||_{\boldsymbol{b}_j} - s_j > c_k + d_k||\boldsymbol{x}^* - \boldsymbol{a}_k||_{\boldsymbol{b}_k} + s_k,$$
$$L(I_j(\boldsymbol{x}^*)) > R(I_k(\boldsymbol{x}^*));$$

and therefore $I_j(\boldsymbol{x}^*) \cap I_k(\boldsymbol{x}^*) = \emptyset$, i.e., there exists an input $\boldsymbol{x}^*$ for which the system is incoherent. □

5 Convexity

By inspection of Lemma 1 and Theorem 1 we see that the core object the necessary condition is based on is the value of the minima of $J_{jk}(\boldsymbol{x})$ function. In the Euclidean systems we are able to state the upper bound on this minimum by stating the explict value of minima of $J_{jk}^{(2)}(\boldsymbol{x})$ which is reached at the point specified by formula (11).

Concerning the very value of minima of J_{jk} over $\mathcal{R}^n$, we can found it numerically by for example the Levenberg-Marquardt algorithm [8]. It helps significantly to know that J_{jk} is convex and therefore any local minimum is also the global minimum [9].

Lemma 2. *For any $p \in [1, \infty)$ the function J_{jk} of (9) is convex, i.e., for any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{R}^n$, $\alpha \in [0, 1]$, $\beta = 1-\alpha$ we have $J_{jk}(\alpha\boldsymbol{x}_1+\beta\boldsymbol{x}_2) \leq J_{jk}(\alpha\boldsymbol{x}_1)+J_{jk}(\beta\boldsymbol{x}_2)$.*

Proof. J_{jk} is specified as the sum of two different scaled norms. The value of p is the same but the scaling parameters $\boldsymbol{b}_j$, $\boldsymbol{b}_k$ generally differ. However, for each of the scaled norms the basic norm's properties hold (a scaled ℓ_p norm is the norm in the standard sense) and we have

$$\begin{aligned} J_{jk}(\alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2) &= ||\alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2 - \boldsymbol{a}_j||_{\boldsymbol{b}_j} + ||\alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2 - \boldsymbol{a}_k||_{\boldsymbol{b}_k} \\ &= ||\alpha(\boldsymbol{x}_1 - \boldsymbol{a}_j) + \beta(\boldsymbol{x}_2 - \boldsymbol{a}_j)||_{\boldsymbol{b}_j} + ||\alpha(\boldsymbol{x}_1 - \boldsymbol{a}_k) + \beta(\boldsymbol{x}_2 - \boldsymbol{a}_k)||_{\boldsymbol{b}_k} \\ &\leq \alpha||(\boldsymbol{x}_1 - \boldsymbol{a}_j)||_{\boldsymbol{b}_j} + \beta||(\boldsymbol{x}_2 - \boldsymbol{a}_j)||_{\boldsymbol{b}_j} + \\ &\qquad + \alpha||(\boldsymbol{x}_1 - \boldsymbol{a}_k)||_{\boldsymbol{b}_k} + \beta||(\boldsymbol{x}_2 - \boldsymbol{a}_k)||_{\boldsymbol{b}_k} \\ &\leq \alpha J_{jk}(\boldsymbol{x}_1) + \beta J_{jk}(\boldsymbol{x}_2). \end{aligned}$$

□

Due to the convexity of J_{jk} it is reasonable to use some procedure of numerical optimization to search for approximation of $J_{jk}^* = \min_{\boldsymbol{x}}\{J_{jk}(\boldsymbol{x})\}$. For any pair of rules $j, k \in \{1, \ldots, m\}$ denote by J_{jk}^{*n} the value of minima found by numerical optimization and by $\boldsymbol{x}_j^{*n}$ the point where J_{jk}^{*n} is reached. Then the following lemma applies.

Lemma 3. *If the radial I-FS is coherent, then* $|c_j - c_k| - (s_j + s_k) \leq J_{jk}^{n*}$ *holds for any pair of rules* $j, k \in \{1, \ldots, m\}$.

Proof. The proof follows the proof of Theorem 1 with the $\min_{\boldsymbol{x}}\{J_{jk}(\boldsymbol{x})\}$ replaced by J_{jk}^{*n} and $\boldsymbol{x}^*$ by $\boldsymbol{x}^{n*}$. □

6 Conclusions

In the paper we have stated an explicit necessary condition for an Euclidean radial implicative system to be coherent. In fact, in implicit form the condition can be extended for non-Euclidean systems, i.e., for other $p \in [1, \infty)$ than $p = 2$. In this case it reads as

$$|c_j - c_k| - (s_j + s_k) \leq J_{jk}^* = \min_{\boldsymbol{x}}\{J_{jk}(\boldsymbol{x})\}. \tag{14}$$

However, here we do not have an explicit formula for computing the value of minima of $J_{jk}(\boldsymbol{x})$ over $\mathcal{R}^n$ or at least its reasonable upper bound as in the Euclidean case. On the other hand, due to the convexity of J_{jk} we can rely on numerical optimization procedures and replace J_{jk}^* by its numerical approximation.

In the future research we aim at the further inspection of convexity of J_{jk} function in order to get more insight whether or not we are able to state the value of J_{jk}^* in some explicit form.

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